TV-Inpainting via Preconditioned Douglas-Rachford Iteration

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Considering an image u on $\Omega \in \mathbb{R}^2$ and its corrupted version u_0 which is only available on $\Omega' \in \Omega$, called the inpainting domain and $\Omega'' = \Omega - \Omega'$. The following TV inpainting model is called the modified ROF model:

$$\min_{u \in U} TV(u) + \frac{\lambda}{2} \|u - f\|_2^2, \tag{1}$$

where U is the discrete image space, $TV(u) = \|\nabla\|$, $f = u_0$ and ∇ is the discrete gradient operator.

We want to adapt the Douglas-Rachford approach to the inpainting problem (1):

Discretization. Consider the image domain $\Omega \in \mathbb{Z}^2$:

$$\Omega = \{(i,j)|i,j \in N, 0 \leq i \leq N_x - 1, 0 \leq j \leq N_y - 1\},$$

where N_x and N_y are the image dimensions. For the task of image inpainting, with the above assumptions, the model (1) can then be defined as

$$\min_{u \in U} \|\nabla u\|_1 + \frac{\lambda}{2} \sum_{(i,j) \in \Omega''} (u_{i,j} - f_{i,j})^2, \tag{2}$$

with $\lambda \in (0, +\infty)$corresponds to joint inpainting and denoising and the choice $\lambda = \infty$ corresponds to pure inpainting.

Using the finite difference for the discretization of the distributional derivative and the adjoint operator $\nabla^* = -div$, we use the notations and definitions in [1], we have

$$(\nabla u) = \begin{pmatrix} \partial_x^+ u \\ \partial_y^+ u \end{pmatrix},$$

where

$$(\partial_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & 0 \leq i < N_x - 1, \\ 0 & i = N_x - 1, \end{cases}$$

and

$$(\partial_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & 0 \leq j < N_y - 1, \\ 0 & i = N_y - 1. \end{cases}$$

With $V = U^2$ with the standard product, the linear operator $\nabla : U \to V$, is considered. Now, considering the negative adjoint of ∇, we can get the discrete divergence, i.e. the unique linear mapping $div : V \to U$, satisfying

$$\langle \nabla u, p \rangle_V = \langle u, \nabla^* p \rangle_U = -\langle u, divp \rangle_U \quad \forall u \in U, p \in V \quad ,$$

$$divp = \partial_x^- p^1 + \partial_y^- p^2,$$

with the backward difference operators

$$(\partial_x^- u)_{i,j} = \begin{cases} u_{0,j} & i = 0, \\ u_{i,j} - u_{i-1,j} & 0 < i < N_x - 1, \\ -u_{N_x-1,j} & i = N_x - 1, \end{cases}$$

and

$$(\partial_y^- u)_{i,j} = \begin{cases} u_{i,0} & j = 0, \\ u_{i,j} - u_{i,j-1} & 0 < j < N_y - 1, \\ -u_{i,N_y-1} & j = N_y - 1, \end{cases}$$

and the discrete form of the L_1 norm for $u \in X$, $v(v_1, v_2) \in Y$, and $1 \leq p < \infty$ is

$$\|u\|_t = \left(\sum_{(i,j)\in\Omega} |u_{i,j}|^t \right)^{1/t}, \quad \|u\|_\infty = \max_{(i,j)\in\Omega} |u_{i,j}| \ ,$$

and

$$\|p\|_t = \left(\sum_{(i,j)\in\Omega} ((p^1_{i,j})^2 + (p^2_{i,j})^2)^{t/2} \right)^{1/t}, \quad \|p\|_\infty = \max_{(i,j)\in\Omega} \sqrt{(p^1_{i,j})^2 + (p^2_{i,j})^2}.$$

Now, let us apply the discrete framework on (2):
We have the following TV-inpainting problem

$$\min_{u\in U} TV(u) + F(u), \tag{3}$$

where $TV(u) = \|\nabla u\|_1$, $F(u) = \frac{\lambda}{2} \sum_{(i,j)\in\Omega''} (u_{i,j} - f_{i,j})^2$. Since both F and $\|.\|$ are continuous and we are in finite dimensions, we will use Fenchel-Rockafellar duality to obtain the saddle-point problem of the type

$$\min_{x\in X} \max_{y\in Y} \langle Kx, y\rangle + F(x) - G(y)$$

by assuming $X = U$, $Y = V$, $K = \nabla$, and $G = \mathrm{I}_C$, while I_C is the indicator function of the set C defined as:

$$\mathrm{I}_C(y) = \begin{cases} 0 & y \in Y \\ \infty & else \end{cases} \quad , here\ C = \{v \in V : \|v\|_\infty \leq 1\}.$$

The operator M needs to be a feasible preconditioner for $T_{PDR} = I + \sigma^2 K^* K = I - \sigma^2 div \nabla = I - \sigma^2 \Delta$.

Now, in order to state the algorithm, the resolvent operators with respect to these functions needs to be evaluated [2]:

$$\begin{aligned} S_\sigma(u, f) = (I + \sigma\partial F)^{-1}(u) \Leftrightarrow S_\sigma(u, f) &= \arg\min_{\bar{u}\in U} \frac{1}{2} \|\bar{u} - u\|_2^2 + \sigma F(u) \\ \Rightarrow S_\sigma(u_{i,j}, f_{i,j}) &= \begin{cases} u_{i,j} & (i,j) \in \Omega', \\ \frac{u_{i,j} + \sigma\lambda f_{i,j}}{1+\sigma\lambda} & else, \end{cases} \end{aligned} \tag{4}$$

and

$$\begin{aligned} P(v) = (I + \tau\partial G)^{-1}(v) \Leftrightarrow P(v) &= \arg\min_{\bar{v}\in Y} \frac{1}{2} \|\bar{v} - v\|_2^2 + \tau G(\bar{v}) \\ &= \arg\min_{\bar{v}\in V} \frac{1}{2} \|\bar{v} - v\|_2^2 + \tau \mathrm{I}_{\{\|v\|_\infty \leq 1\}}(\bar{v}) \\ &= \frac{v}{\max(1, |v|)}, \end{aligned} \tag{5}$$

with $|v| = \sqrt{(v^1)^2 + (v^2)^2}$.

The Preconditioner. Observe that M is needed to be feasible for the operators with the form $T = \lambda I - \mu\Delta$, $\lambda, \mu > 0$, where $\Delta = div\nabla$ is the discrete Laplace operator with homogeneous Neumann boundary conditions. In fact, $Tu = b$ is equal to a discrete version of the BVP bellow:

$$\begin{cases} \lambda u - \mu\Delta u = b & in\ \Omega \\ \frac{\partial u}{\partial v} = 0 & on\ \partial\Omega \end{cases}$$

with the choice of ∇u, we have a finite difference equation with five-point stencils. Following the approach in [1, 3], we use a symmetric Gauss-Seidel update in conjuction with a Red-Black enumeration scheme:

$$\Omega_{\text{Red}} = \{(i,j) \in \Omega | i + j\ even\},$$
$$\Omega_{Black} = \{(i,j) \in \Omega | i + j\ odd\},$$

and $u = (u_{Red}, u_{Black})$.

A Gauss-Seidel update then performs an update according to $Red \to Black$ and the adjoint is $Black \to Red$.

Thus, n steps of symmetric red-Black Gauss-Seidel, $SRBGS^n_{\lambda,\mu}$ is denoted bellow:

$$\begin{array}{rlr} SRBGS^n_{\lambda,\mu}(u^k, b^k) = (u^{k+1}_{red}, u^{k+1}_{black}), & & \\ (u^{k+(v+1/2)/n}_{red})_{i,j} = \frac{1}{\lambda+c_{i,j}\mu}(b^k_{i,j} + \mu \sum\limits_{(i',j')\in N(i,j)} (u^{k+v/n}_{black})_{i',j'}), & (i,j) \in \Omega_{red}, & \\ (u^{k+(v+1)/n}_{black})_{i,j} = \frac{1}{\lambda+c_{i,j}\mu}(b^k_{i,j} + \mu \sum\limits_{(i',j')\in N(i,j)} (u^{k+(v+1/2)/n}_{red})_{i',j'}), & \begin{array}{c}(i,j) \in \Omega_{black} \\ v = 0, ..., n-1\end{array}, & \\ (u^{k+1}_{red})_{i,j} = \frac{1}{\lambda+c_{i,j}\mu}(b^k_{i,j} + \mu \sum\limits_{(i',j')\in N(i,j)} (u^{k+1}_{black})_{i',j'}), & (i,j) \in \Omega_{red}, & \end{array} \tag{6}$$

and $N(i,j) = \{(i',j') \in \Omega \,|\, |i-i'| + |j-j'| = 1\}$ is the set of neighbor pixels of (i,j) in Ω ,and $c_{i,j} = \#N(i,j)$.

Now, we can derive the algorithm for PDR method:

Algorithm 1 PDR Objective: TV-Inpainting $\min_{u_0\in BV(\Omega)} TV(u) + \mathrm{I}_{\{v:v|_{\Omega''}=u_0\}}(u)$

1: Initialize $(u^0, \bar{u}^0, \bar{v}^0) \in U \times U \times V$ *inital guess*,
   $\sigma > 0$ step size, $\tau > 0$ gradient scaling,
   $n \leq 0$ inner iteration for symmetric Gauss-Seidel.

2: Iteration

$$\begin{aligned} u^{k+1} &= SRBGS^n_{\lambda,(\sigma\tau)}(u^k, \bar{u}^k + \sigma\tau div \bar{v}^k), \\ v^{k+1} &= \bar{v}^k + \sigma\tau\nabla u^{k+1}, \\ u^{k+1}_{test} &= S_\sigma(2u^{k+1} - \bar{u}^k, f), \qquad \text{with } (4) \\ \bar{u}^{k+1} &= \bar{u}k + u^{k+1}_{test} - u^{k+1}, \\ v^{k+1}_{test} &= P(2v^{k+1} - \bar{v}^k), \qquad \text{with } (5) \\ \bar{v}^{k+1} &= \bar{v}^k + v^{k+1}_{test} - v^{k+1}. \end{aligned}$$

Figure (1) shows the application of the inpainting model to the recovery of lost region in Lena image. The parameters used in this model are $\sigma = 0.1$, $\tau = 1$ and the inner iterations for symmetric Gauss–Seidel n is equal to one. In (6), $\lambda = 1$ and $\mu = \sigma\tau$ are considered.

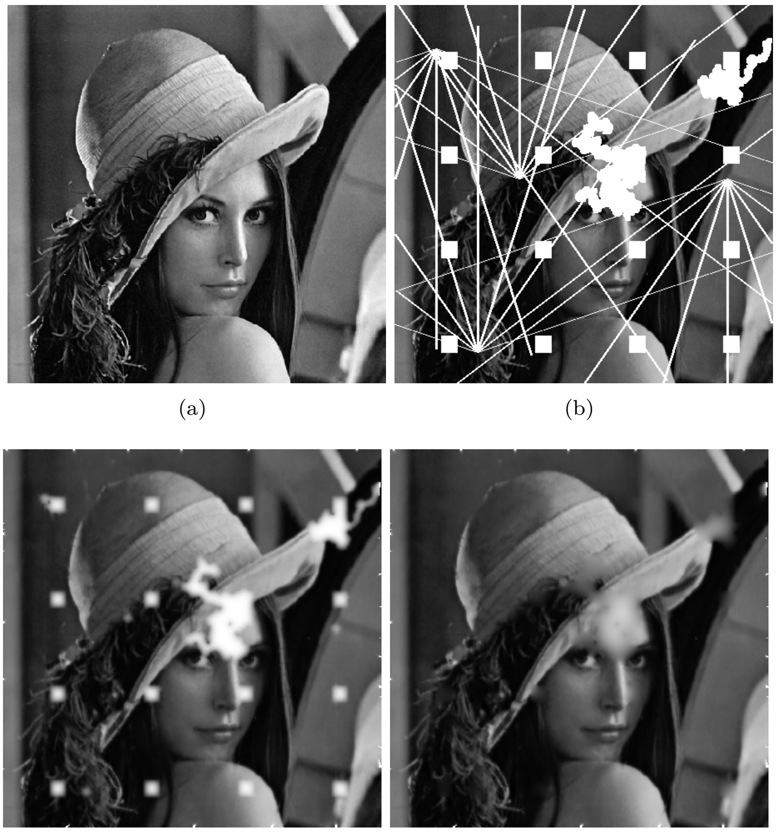
We should note here, this algorithm is sensitive to the values $\tau, \sigma and \lambda$. With the general form of the inpainting problem in mind, an image with both noise and missing parts, the de-noising part of TV model is also considered. In a situation where the image is noise free, we can consider $\sigma = 0$, $\tau = 0.1$ and $\lambda = tau$.

Referências

[1] Bredies K, Sun H. Preconditioned Douglas-Rachford splitting methods for convex-concave saddle-point problems. SIAM Journal on Numerical Analysis. 2015;53(1):421-44.

[2] Bredies K. Recovering piecewise smooth multichannel images by minimization of convex functionals with total generalized variation penalty. InEfficient algorithms for global optimization methods in computer vision 2014 (pp. 44-77). Springer, Berlin, Heidelberg.

[3] Bredies K, Sun HP. Preconditioned Douglas–Rachford algorithms for TV-and TGV-regularized variational imaging problems. Journal of Mathematical Imaging and Vision. 2015 Jul 1;52(3):317-44.



(a) (b)

(c) (d)

Figura 1: Recovery of lost image information. (a) shows the Lena image, (b) the destroyed image (c) the recovered image using PDR model form Algorithm 1 with 100 iterations, (d) the recovered image using PDR model form Algorithm 1 with 1000 iterations.