Simplex Method for Standard Minimization Problem

Previously, we learned the simplex method to solve linear programming problems that were labeled as standard maximization problems. However, many problems are not maximization problems. Often we will be asked to minimize the objective function. There are two types of minimization problems.

def: The first type of **standard minimization problem** is one in which

1. the objective function is to be minimized,
2. all variables involved in the problem are nonnegative, and
3. all other linear constraints may be written so that the expression involving the variables is less than or equal to a nonnegative constant.

The process for solving the first type is very similar to the process for solving the standard maximization problem. Given a type one standard minimization problem with some objective function C, we solve the problem by maximimizing $-C$. After changing our objective, the problem becomes exactly the same as before.

ex.) Minimize $C = -2x + 3y$ subject to $3x + 4y \leq 24$, $7x - 4y \leq 16$, and $x, y \geq 0$.

Let $P = -C$. Then $P = 2x - 3y$. We seek to maximimize P subject to the above constraints.

(Step 1) $3x + 4y + s_1 = 24$
$7x - 4y + s_2 = 16$

(Step 2) $-2x + 3y + P = 0$

(Step 3) $\left[\begin{array}{ccccc|c} 3 & 4 & 1 & 0 & 0 & 24 \\ 7 & -4 & 0 & 1 & 0 & 16 \\ -2 & 3 & 0 & 0 & 1 & 0 \end{array}\right]$

(Step 4) The optimal solution has not been reached.

(Step 5) $\left[\begin{array}{ccccc|c} 3 & 4 & 1 & 0 & 0 & 24 \\ 7 & -4 & 0 & 1 & 0 & 16 \\ -2 & 3 & 0 & 0 & 1 & 0 \end{array}\right]$

(Step 6) $\left[\begin{array}{ccccc|c} 3 & 4 & 1 & 0 & 0 & 24 \\ 7 & -4 & 0 & 1 & 0 & 16 \\ -2 & 3 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\frac{1}{7}R_2 \to R_2,\ \ 2R_2 + R_3 \to R_3]{-3R_2 + R_1 \to R_1} \left[\begin{array}{ccccc|c} 0 & \frac{40}{7} & 1 & -\frac{3}{7} & 0 & \frac{120}{7} \\ 1 & -\frac{4}{7} & 0 & \frac{1}{7} & 0 & \frac{16}{7} \\ 0 & \frac{13}{7} & 0 & \frac{2}{7} & 1 & \frac{32}{7} \end{array}\right]$

(Step 4) The optimal solution is reached.

(Step 7) P is maximized at $\frac{32}{7}$ when $x = \frac{16}{7}$ and $y = 0$. Therefore, C is minimized at $-\frac{32}{7}$ when $x = \frac{16}{7}$ and $y = 0$.

The second type of standard minimization problem is similar to the first type, except that the constraints follow a different requirement.

def: The second type of **standard minimization problem** is one in which

1. the objective function is to be minimized,
2. all variables involved in the problem are nonnegative, and
3. all other linear constraints may be written so that the expression involving the variables is greater than or equal to a constant.

Notice that the direction of the inequality in the constraints has changed, and notice that we no longer require the constant involved in the constraint to be nonnegative.

To solve minimization problems of type two, use the following steps.

Suppose we are asked to minimize $C = c_1x_1 + c_2x_2 + ... + c_nx_n$ subject to $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \geq b_1$, $a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \geq b_2$, ..., $a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \geq b_m$ where $x_i \geq 0$ for all i.

(Step 1) Form the augmented matrix for the system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\ c_1 & c_2 & \dots & c_n & 0 \end{array}\right]$$

(Step 2) Form the transpose of this matrix.

$$\left[\begin{array}{cccc|c} a_{11} & a_{21} & \dots & a_{m1} & c_1 \\ a_{12} & a_{22} & \dots & a_{m2} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\ b_1 & b_2 & \dots & b_m & 0 \end{array}\right]$$

(Step 3) Form the dual maximization problem corresponding to the transposed matrix. That is, find the maximum of the objective function $P = b_1y_1 + b_2y_2 + ... + b_my_m$ subject to the constraints $a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m \leq c_1$, $a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m \leq c_2$, ..., $a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m \leq c_n$ where $y_i \geq 0$ for all i.

(Step 4) Apply the simplex method for standard maximization problems. The maximum value of P will be the minimum value of C. Moreover, the values of x_1, x_2, ..., x_n will occur in the bottom row of the final matrix in the columns corresponding to the slack variables.

ex.) Minimize $C = 2x + 5y$ subject to $x + 2y \geq 6$, $3x + 2y \geq 6$, and $x, y \geq 0$.

This is a minimization problem of type two. First, we form the coefficient matrix and take its transpose. Then, we interpret the transpose as a standard maximization problem. We have

$$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 2 & 6 \\ 2 & 5 & C \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 5 \\ 6 & 6 & C \end{bmatrix}.$$

Therefore, we seek to maximize $C = 6u + 6v$ according to $u + 3v \leq 2$, $2u + 2v \leq 5$, and $u, v \geq 0$.

(Step 1) $u + 3v + s_1 = 2$
$2u + 2v + s_2 = 5$

(Step 2) $-6u - 6v + C = 0$

(Step 3) $\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 1 & 0 & 5 \\ -6 & -6 & 0 & 0 & 1 & 0 \end{array}\right]$

(Step 4) The optimal solution has not been reached.

(Step 5) $\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 1 & 0 & 5 \\ -6 & -6 & 0 & 0 & 1 & 0 \end{array}\right]$

(Step 6) $\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 1 & 0 & 5 \\ -6 & -6 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-2R_1+R_2 \to R_2]{6R_1+R_3 \to R_3} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 2 \\ 0 & -4 & -2 & 1 & 0 & 1 \\ 0 & 12 & 6 & 0 & 1 & 12 \end{array}\right]$

(Step 4) The optimal solution has been reached.

(Step 7) Remember we interpret the final matrix in a special way. C is minimized at 12 and it occurs when $x = 6$, $y = 0$, and $z = 0$.