Numerical Semigroups Notes for Polymath Jr. 2021

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Chapter 1

Introduction

1.1 Properties of Numerical Semigroups

Definition 1.1: Numerical Semigroup

Note 1.2:

Zero is a generator of every numerical subgroup, it is just not explicitly mentioned.

Example 1.3:

Let $S := \langle 6, 9, 20 \rangle$ Then:

$$S = \{\}$$

TODO 1.4:

Fill in mcnugget set. Fix wording below.

Note 1.5:

This is called the "McNugget Semigroup", as McDonald's Chicken Nuggets used to come in sizes 6, 9, or 20. The question was, "what's the largest number of McNuggets you cannot recieve from McDonalds"?

Theorem 1.6:

Every numerical subgroup has a unique minimal generating set.

Example 1.7:

$$S = \langle 6, 9, 20 \rangle = \langle 6, 9, 18, 20 \rangle$$

Definition 1.8: Multiplicity

Let S be a numerical subgroup. The multiplicity of S, denoted $m(S)$, is the minimal nonzero generator of S. That is:

$$m(S) := \text{Min}(S \setminus \{0\})$$

Definition 1.9: Embedding Dimension

Let S be a numerical subgroup. The embedding dimension of S, denoted $e(S)$, is the number of minimal nonzero generators of S.

Example 1.10:

$$e(\langle 6, 9, 20\rangle = 3$$

Definition 1.11: Frobenius Number

Let S be a numerical subgroup. The Frobenius Number of S, denoted $F(S)$, is the largest non-negative integer not in S. That is:

$$F(S) := \text{Max}(\mathbb{Z}_{\geq 0} \setminus S)$$

Note 1.12:

Often we will require:

$$|\mathbb{N} \setminus S| < \infty$$

So that these properties are well defined.

Definition 1.13:

Example 1.14: The Four Properties

Let $S := \langle 6, 9, 20\rangle$. Then:

$m(S) = 6$	Smallest generator
$e(S) = 3$	Number of generators
$F(S) = 43$	Largest "gap"
$g(S) = 22$	Number of "gaps"

Concept 1.15:

Definition 1.16:

Semigroup isomorphism

TODO 1.17:

?

Example 1.18:

$$S := \langle 4, 6 \rangle$$
$$T := \langle 2, 3 \rangle$$

We claim, without proving:

$$S \approx T$$

We notice $\mathbb{N} \setminus S$ is infinite, but $\mathbb{N} \setminus T$ is not.

Concept 1.19: Factorizations

If $S := \langle 6, 9, 20 \rangle$ then:

$$60 = 4 \cdot 6 + 4 \cdot 9 = 3 \cdot 20$$

This shows factorizations are not unique. So we define the set of all vectors:

Definition 1.20:

$$Z_S(\ell) := \{a \in \mathbb{N}^k \ : \ a_1 n_1 + \cdots a_k n_k = \ell\}$$

Where $\ell \in S$.

Example 1.21:

$$Z(60) = \left\{ \begin{array}{c} (10, 0, 0), \\ (7, 2, 0), \\ (4, 4, 0), \\ (0, 0, 3), \\ (1, 6, 0) \end{array} \right\}$$

1.2 The Apery Set

Definition 1.22:

Let $S \subseteq \mathbb{N}$ be a numerical semigroup.
Let $m := m(S)$ be the multiplicity of S.
Then we define the Apery set:

$$\mathrm{Ap}(S) := \{n \in S \ : \ n - m \notin S$$

Note 1.23:

These can be though of as the element "just barely inside" of S

Note 1.24:

Note the elements of the Apery set are distinct modulo m.

Problem 1.25:

Let:

$$S := \langle 3, 5, 7 \rangle$$

This implies:

$$S = \{0, 3, \underbrace{5, 6, 7}, 8, 9, 10, \cdots\}$$

We want the smallest element in each equivalence class mod 3, which is:

$$\mathrm{Ap}(S) = \{0, 7, 5\}$$

Note 1.26:

We note the following properties of $\mathrm{Ap}(S)$:

- $|\mathrm{Ap}(S)| = m$
- the minimal generators of S are (except $m(S)$ are in $\mathrm{Ap}(S)$.

Definition 1.27: Apery Poset of S

TODO 1.28:

Chapter 2

Polyhedral Polyhedral Geometry

Concept 2.1:

In Geometry and Research, it is harder to figure out what is true rather than how to prove it. Having an intuitive picture of what's going on makes this a lot easier.

2.1 Polyhedra

Definition 2.2: Half Space

This is the solution set to an affine linear inequality:

Definition 2.3: Affine Linear Inequality

An inequality in the terms of:

$$a_1x_1 + a_2x_2 + \cdots + a_dx_d \leq b$$

Where $a_i \in \mathbb{R}$ (for us)

Definition 2.4: Affine

Affine refers to $b \neq 0$. If b were strictly equal to 0, then this would be a strict linear inequality.

Definition 2.5: Polyhedron

The intersection of finitely many half-spaces in $\mathbb{R}^d$.

Note 2.6: Polyhedron aren't nescissarily bounded.

Example 2.7: Simplex in $\mathbb{R}^2$

TODO 2.8:

This is the convex hull of $(0,0),(0,1),(1,0)$

TODO 2.9:

Fill in picture

$$\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 + x_2 \leq 1 \end{cases}$$

Example 2.10: Simplex in $\mathbb{R}^3$

TODO 2.11:

This is the convex hull of $(0,0,0),(0,0,1),(0,1,0),(1,0,0)$

TODO 2.12:

Fill in picture

$$\begin{cases} x_1 & \geq 0 \\ x_2 & \geq 0 \\ x_3 & \geq 0 \\ x_1 + x_2 + x_3 & \leq 1 \end{cases}$$

TODO 2.13: Unbounded Polyhedron Example

Definition 2.14: Simplex in $\mathbb{R}^d$

We gave the examples of

TODO 2.15:

Reference R1 and R2

Definition 2.16: Cube in $\mathbb{R}^d$

TODO 2.17:

Insert cube def

Problem 2.18: Octahedron Problem

TODO 2.19:

Inser octahedron from lecture.

Definition 2.20: Cross Polytope

TODO 2.21:

Later

Note 2.22:

Each facet has in inequality associcated with it.

Theorem 2.23:

There is a unique, irredundent, half spaces to describe any polyhedron.

Note 2.24:

Irrendundent = no extras

2.2 Bounded Polyhedra (Polytope)

Concept 2.25: Polytope

Definition 2.26: Polytope

A polytope is a bounded polyhdron.

TODO 2.27:

Define bounded

Definition 2.28: Convex Hull

The convex hull of a set of points, is given by:

$$\mathrm{Conv}\{v_1, v_2, \cdots, v_n\} = \{\lambda_1 v_1 + \cdots \lambda_n v_n \quad : \quad lambda_i \in \mathbb{R}^{\geq}0,\ \lambda_1 + \cdots + \lambda_n = 1\}$$

Note 2.29: Shrink Wrap

This is the unique smallest convex volume that contains all of v_i.
This is obtained by "shrink wrapping", in a non-rigerous sense, the points of v_i.

Theorem 2.30: All Polytopes can be Expressed as a Convex Hull

Example 2.31:

TODO 2.32:

Give an example

Concept 2.33: V and H descriptions

Definition 2.34: H-Description

The H-description of a polyhedra is in terms of **H**alfspaces.

Definition 2.35: V-Description

The V-description of a polytope is in terms of **V**ectors.
That is, convex hulls. Note that only bounded polyhedra (polytopes) have a V-description.

2.3 Cones

Definition 2.36: Cone

A cone is a polyhedron where every half-space in its H-description has the origin on its boundary.

Theorem 2.37: Alternate Cone Definition

If each halfspace has 0 in its inequality. That is, $b_i = 0$ for all i.

Problem 2.38:

All cones, except one, are unbounded. Which cone is unbounded?

Example 2.39: Example of a cone

TODO 2.40:

Cone

2.4 Dimension of a Polyhedron

Example 2.41: Motivation for Dimension

Let

$$P := \text{Conv}\{(1,2),(2,3)\} \subset \mathbb{R}^2$$

TODO 2.42:

Insert image

If we naively assign $\text{Dim}\, P = 2$, because it sits in $\mathbb{R}^2$, it wouldn't be much help, as P is in reality a line.

Definition 2.43: Dimension of a Polyhedron

The dimension of a polyhedra is:

$$\mathrm{Dim}(P) := \mathrm{Dim}\,\mathrm{Span}\{x - y \;:\; x, y \in P\}$$

This merits some explaination.
We note if $x = x$, then this gives us $x - x = 0$, which is simply the origin.

TODO 2.44:

Explain better

2.5 Faces of a Polyhedron

TODO 2.45: Define Facet

Did we define facets?

Definition 2.46: Face

A Face of P is the intersection of P with the boundary of a halfspace containing P.

TODO 2.47:

Insert exmaples

Example 2.48:

Example 2.49:

Definition 2.50: Facet

Let P be a polyhedron. A face F of P is a facet if:

$$\mathrm{Dim}(F) = \mathrm{Dim}(P) - 1$$

Definition 2.51: Edge

Let P be a polyhedron. A face F of P is an edge if

$$\mathrm{Dim}(F) = 1$$

TODO 2.52:

Definition 2.53: Ridge

Definition 2.54: Co-Dimension

Theorem 2.55:

Every face of a polyhedron is itself a polyhedron.

Example 2.56: The 4-cube

Find all of the faces of the 4-cube.

Chapter 3

Faces of Polyhedra

Note 3.1:

Here are some properties to test:

Problem 3.2:

For a polyhedra P, $\emptyset$ is a face of P

Problem 3.3:

For a polyhedra P, P itself is a face of P

Problem 3.4:

For any Polytope P, face F is the convex hull of vertices of P it contains.

Note 3.5: V-Description

This is called the V-Description.

Problem 3.6:

Any face F is the intersection of the faces of P it contains.

Note 3.7: H-Description

This is called the H-Description.

3.1 Face Structure of Polyhedra and Polytopes

Definition 3.8: The Face Lattice of a Polytope

This is a poset via containment.

TODO 3.9:

Draw or copy from Thomas' book.

Theorem 3.10:

A k-dimensional simplex has a face lattice that looks like a $(k+1)$-dimensional cube.

Theorem 3.11:

A k dimensional simplex has a face lattice isomorphic to the poset of the powerset on $(k+1)$-elements.

Theorem 3.12:

Note 3.13:

The degree of $\emptyset$ is -1, by convension.

Note 3.14:

A face lattice will always have "nice" rows, via dimension.

Note 3.15:

Face latices can get complicated

TODO 3.16:

put sage output here.

Note 3.17:

here are some combinatorial shapes

- simpliex
- (?)
- permutathedron
- associahedron

Note 3.18:

Problem 3.19:

does every poset correspond to a face lattice.

Face lattices need bo be graded, even further, they need to be eulerian.
It is an open question: "what are a list of requirements for a poset to be a face lattice"

3.2 The permutahedron

Concept 3.20:

$R_d \subseteq \mathbb{R}^d$ is the convex hull of the points obtained by permuting the coordinates of:

$$(1, 2, \cdots d)$$

Example 3.21:

For example $d = 3$;

$$\left\{\begin{array}{c}(1,2,3),\\(1,3,2),\\(2,1,3),\\(2,3,1),\\(3,1,2),\\(3,2,1)\end{array}\right\}$$

Theorem 3.22:

The dimension of R_d is $(d-1)$.

TODO 3.23:

draw

Note 3.24:

Edges connect vertices where a single transposition is preformed and the two values being tranposed differ by 1.

Now, we will represent this is a different notation.

$$a|b|c$$

where

- $a, b, c \in \{1, 2, \cdot, d\}$
- a is the first coordinate
- b is the second coordinate
- c is the third coordinate

$$\left\{\begin{array}{c}(1,2,3),\\(1,3,2),\\(2,1,3),\\(2,3,1),\\(3,1,2),\\(3,2,1)\end{array}\right\} = \left\{\begin{array}{c}1|2|3\\1|3|2\\2|1|3\\3|1|2\\2|3|1\\3|2|1\end{array}\right\}$$

Now, we stop for an interlude

3.3 ordered set partitions

TODO 3.25: