Numerical Semigroups Notes for Polymath Jr. 2021

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Summer 2021

Chapter 1

Introduction

1.1 Properties of Numerical Semigroups

Definition 1.1: Numerical Semigroup

Note 1.2:

Zero is a generator of every numerical subgroup, it is just not explicitly mentioned.

Example 1.3:

Let $S := \langle 6, 9, 20 \rangle$ Then:

$$S = \{\}$$

TODO 1.4:

Fill in mcnugget set. Fix wording below.

Note 1.5:

This is called the "McNugget Semigroup", as McDonald's Chicken Nuggets used to come in sizes 6, 9, or 20. The question was, "what's the largest number of McNuggets you cannot recieve from McDonalds"?

Theorem 1.6:

Every numerical subgroup has a unique minimal generating set.

Example 1.7:

$$S = \langle 6, 9, 20 \rangle = \langle 6, 9, 18, 20 \rangle$$

Definition 1.8: Multiplicity

Let S be a numerical subgroup. The multiplicity of S, denoted $m(S)$, is the minimal nonzero generator of S. That is:

$$m(S) := \mathrm{Min}(S \setminus \{0\})$$

Definition 1.9: Embedding Dimension

Let S be a numerical subgroup. The embedding dimension of S, denoted $e(S)$, is the number of minimal nonzero generators of S.

Example 1.10:

$$e(\langle 6, 9, 20\rangle = 3$$

Definition 1.11: Frobenius Number

Let S be a numerical subgroup. The Frobenius Number of S, denoted $F(S)$, is the largest non-negative integer not in S. That is:

$$F(S) := \text{Max}(\mathbb{Z}_{\geq 0} \setminus S)$$

Note 1.12:

Often we will require:

$$|\mathbb{N} \setminus S| < \infty$$

So that these properties are well defined.

Definition 1.13:

Example 1.14: The Four Properties

Let $S := \langle 6, 9, 20\rangle$. Then:

$m(S) = 6$	Smallest generator
$e(S) = 3$	Number of generators
$F(S) = 43$	Largest "gap"
$g(S) = 22$	Number of "gaps"

Concept 1.15:

Definition 1.16:

Semigroup isomorphism

TODO 1.17:

?

Example 1.18:

$$S := \langle 4, 6 \rangle$$
$$T := \langle 2, 3 \rangle$$

We claim, without proving:

$$S \approx T$$

We notice $\mathbb{N} \setminus S$ is infinite, but $\mathbb{N} \setminus T$ is not.

Concept 1.19: Factorizations

If $S := \langle 6, 9, 20 \rangle$ then:

$$60 = 4 \cdot 6 + 4 \cdot 9 = 3 \cdot 20$$

This shows factorizations are not unique. So we define the set of all vectors:

Definition 1.20:

$$Z_S(\ell) := \{a \in \mathbb{N}^k \ : \ a_1 n_1 + \cdots a_k n_k = \ell\}$$

Where $\ell \in S$.

Example 1.21:

$$Z(60) = \left\{ \begin{array}{c} (10, 0, 0), \\ (7, 2, 0), \\ (4, 4, 0), \\ (0, 0, 3), \\ (1, 6, 0) \end{array} \right\}$$

1.2 The Apery Set

Definition 1.22:

Let $S \subseteq \mathbb{N}$ be a numerical semigroup.
Let $m := m(S)$ be the multiplicity of S.
Then we define the Apery set:

$$\mathrm{Ap}(S) := \{n \in S \ : \ n - m \notin S$$

Note 1.23:

These can be though of as the element "just barely inside" of S

Note 1.24:

Note the elements of the Apery set are distinct modulo m.

Problem 1.25:

Let:

$$S := \langle 3, 5, 7 \rangle$$

This implies:

$$S = \{0, 3, \underbrace{5, 6, 7}, 8, 9, 10, \cdots\}$$

We want the smallest element in each equivalence class mod 3, which is:

$$\mathrm{Ap}(S) = \{0, 7, 5\}$$

Note 1.26:

We note the following properties of $\mathrm{Ap}(S)$:

- $|\mathrm{Ap}(S)| = m$
- the minimal generators of S are (except $m(S)$ are in $\mathrm{Ap}(S)$.

Definition 1.27: Apery Poset of S

TODO 1.28:

Chapter 2

Chapter 3

Faces of Polyhedra

TODO 3.1:

Definition 3.2: Polytope

Definition 3.3: Polyhedra

Definition 3.4: Cone

Note 3.5:

Here are some properties to test:

Problem 3.6:

For a polyhedra P, $\emptyset$ is a face of P

Problem 3.7:

For a polyhedra P, P itself is a face of P

Problem 3.8:

For any Polytope P, face F is the convex hull of vertices of P it contains.

Note 3.9: V-Description

This is called the V-Description.

Problem 3.10:

Any face F is the intersection of the faces of P it contains.

Note 3.11: H-Description

This is called the H-Description.

3.1 Face Structure of Polyhedra and Polytopes

Definition 3.12: The Face Lattice of a Polytope

This is a poset via containment.

TODO 3.13:

Draw or copy from Thomas' book.

Theorem 3.14:

A k-dimensional simplex has a face lattice that looks like a $(k+1)$-dimensional cube.

Theorem 3.15:

A k dimensional simplex has a face lattice isomorphic to the poset of the powerset on $(k+1)$-elements.

Theorem 3.16:

Note 3.17:

The degree of $\emptyset$ is -1, by convension.

Note 3.18:

A face lattice will always have ”nice” rows, via dimension.

Note 3.19:

Face latices can get complicated

TODO 3.20:

put sage output here.

Note 3.21:

here are some combinatorial shapes

- simpliex
- (?)
- permutathedron
- associahedron

Note 3.22:

Problem 3.23:

does every poset correspond to a face lattice.

Face lattices need bo be graded, even further, they need to be eulerian.
It is an open question: ”what are a list of requirements for a poset to be a face lattice”

3.2 The permutahedron

Concept 3.24:

$R_d \subseteq \mathbb{R}^d$ is the convex hull of the points obtained by permuting the coordinates of:

$$(1, 2, \cdots d)$$

Example 3.25:

For example $d = 3$;

$$\left\{ \begin{array}{c} (1,2,3), \\ (1,3,2), \\ (2,1,3), \\ (2,3,1), \\ (3,1,2), \\ (3,2,1) \end{array} \right\}$$

Theorem 3.26:

The dimension of R_d is $(d-1)$.