Numerical Semigroups Notes for Polymath Jr. 2021

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Chapter 1

Introduction

1.1 Properties of Numerical Semigroups

Definition 1.1: Numerical Semigroup

Note 1.2:

Zero is a generator of every numerical subgroup, it is just not explicitly mentioned.

Theorem 1.3:

Every numerical subgroup has a unique minimal generating set.

Example 1.4:

$$S = \langle 6, 9, 20 \rangle = \langle 6, 9, 18, 20 \rangle$$

Definition 1.5: Multiplicity

Let S be a numerical subgroup. The multiplicity of S, denoted $m(S)$, is the minimal nonzero generator of S. That is:

$$m(S) := \text{Min}(S \setminus \{0\})$$

Definition 1.6: Embedding Dimension

Let S be a numerical subgroup. The embedding dimension of S, denoted $e(S)$, is the number of minimal nonzero generators of S.

Example 1.7:

$$e(\langle 6, 9, 20 \rangle = 3$$

Definition 1.8: Frobenius Number

Let S be a numerical subgroup. The Frobenius Number of S, denoted $F(S)$, is the largest non-negative integer not in S. That is:

$$F(S) := \text{Max}(\mathbb{Z}_{\geq 0} \setminus S)$$

Note 1.9:

Often we will require:

$$|\mathbb{N} \setminus S| < \infty$$

So that these properties are well defined.

Definition 1.10:

Example 1.11: The Four Properties

Let $S := \langle 6, 9, 20 \rangle$. Then:

$$\begin{aligned} m(S) &= 6 && \text{Smallest generator} \\ e(S) &= 3 && \text{Number of generators} \\ F(S) &= 43 && \text{Largest ”gap”} \\ g(S) &= 22 && \text{Number of ”gaps”} \end{aligned}$$

Concept 1.12:

Definition 1.13:

Semigroup isomorphism

TODO 1.14:

?

Example 1.15:

$$\begin{aligned} S &:= \langle 4, 6 \rangle \\ T &:= \langle 2, 3 \rangle \end{aligned}$$

We claim, without proving:

$$S \approx T$$

We notice $\mathbb{N} \setminus S$ is infinite, but $\mathbb{N} \setminus T$ is not.

Concept 1.16: Factorizations

If $S := \langle 6, 9, 20 \rangle$ then:

$$60 = 4 \cdot 6 + 4 \cdot 9 = 3 \cdot 20$$

This shows factorizations are not unique. So we define the set of all vectors:

Definition 1.17:

$$Z_S(\ell) := \{a \in \mathbb{N}^k \; : \; a_1 n_1 + \cdots a_k n_k = \ell\}$$

Where $\ell \in S$.

Example 1.18:

$$Z(60) = \left\{ \begin{array}{c} (10,0,0), \\ (7,2,0), \\ (4,4,0), \\ (0,0,3), \\ (1,6,0) \end{array} \right\}$$

1.2 The Apery Set

Definition 1.19:

Let $S \subseteq \mathbb{N}$ be a numerical semigroup. Let $m := m(S)$ be the multiplicity of S.

$$Ap(S) := \{n \in S \; : \; n - m \notin S$$

Note 1.20:

These can be though of as the element "just barely inside" of S