Penning, Neil

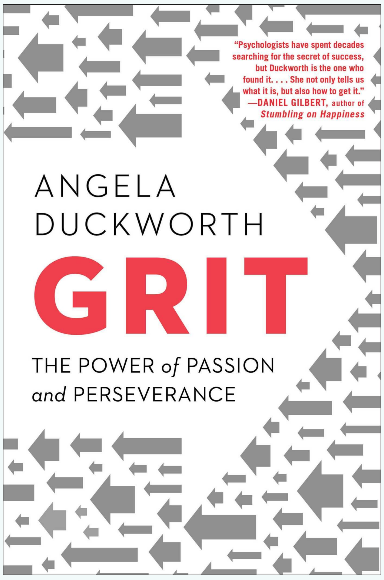
Contents

Chapter 1

Resources

1.1 Books

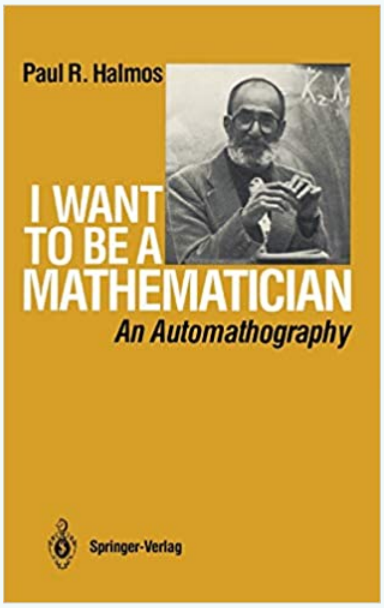
Resource 1.1: Duckworth



Quote 1.2: Terry Tao

Talent is important, but how one develops and nurtures it is even more so.

Resource 1.3:



Quote 1.4: Paul Halmos

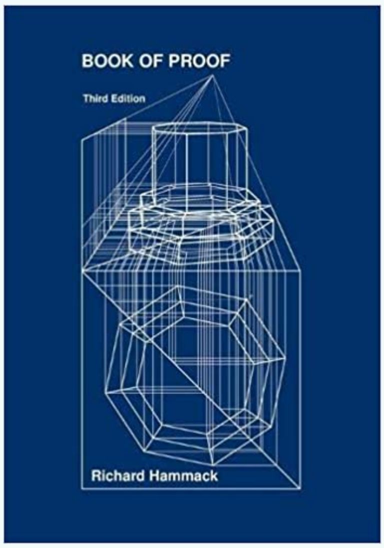
When a student comes and asks, ”Should I become a mathematician?” the answer should be no. If you have to ask, you shouldn’t even ask.

Quote 1.5: Paul Halmos

Don’t just read it; fight it! Ask your own question, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? ... Where does the proof use the hypothesis?

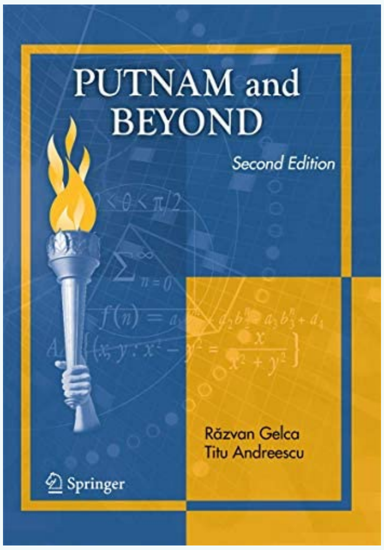
1.2 Textbooks

Resource 1.6: Hammack Book of Proof



Hammack's Book of Proof is a free online textbook that can be found https://www.people.vcu.edu/~rhammack/BookOfProof/. It is also the highest quality introduction to rigorous mathematics that I have found, and scores of colleges seem to agree.

Resource 1.7:



Quote 1.8: Paul Halmos

The heart of mathematics is its problems.

The Putnam Exam takes place the first Saturday of December every year. It is Twelve Questions across two three hour sessions. The exam is famously difficult, with a median score of a 0 or a 1. This book gives an in depth overview, along with hundreds of more-difficult-than-average problems across undergraduate mathematics. I use it to review and push myself in material I have already learned.

Resource 1.9: The Infinitely Large Napkin

Quote 1.10: Evan Chen

The origin of the name "Napkin" comes from the following quote of mine.

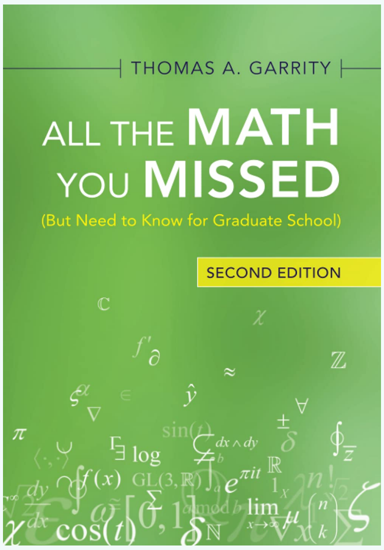
Quote 1.11: Evan Chen

I'll be eating a quick lunch with some friends of mine who are still in high school. They'll ask me what I've been up to the last few weeks, and I'll tell them that I've been learning category theory. They'll ask me what category theory is about. I tell them it's about abstracting things by looking at just the structure-preserving morphisms between them, rather than the objects themselves. I'll try to give them the standard example Grp, but then I'll realize that they don't know what a homomorphism is. So then I'll start trying to explain what a homomorphism is, but then I'll remember that they haven't learned what a group is. So then I'll start trying to explain what a group is, but by the time I finish writing the group axioms on my napkin, they've already forgotten why I was talking about groups in the first place. And then it's 1PM, people need to go places, and I can't help but think: "Man, if I had forty hours instead of forty minutes, I bet I could actually have explained this all".

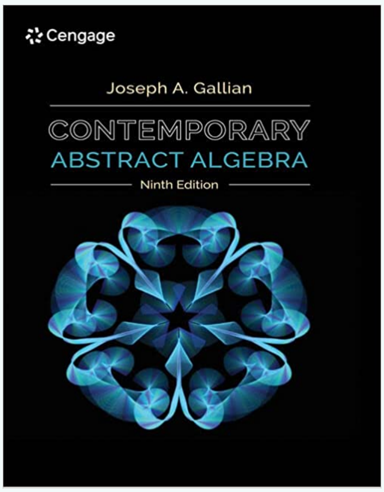
This book was initially my attempt at those forty hours, but has grown considerably since then.

Evan Chen is a Graduate Student at MIT who has been working on the Napkin for several years now. The napkin touches every facet of undergraduate (and many parts of graduate) mathematics. The latest version can be downloaded for free here: https://web.evanchen.cc/napkin.html

Resource 1.12: Garrity

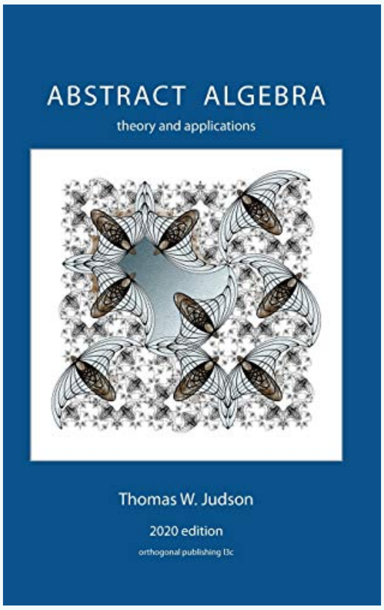


Resource 1.13: Gallian Algebra



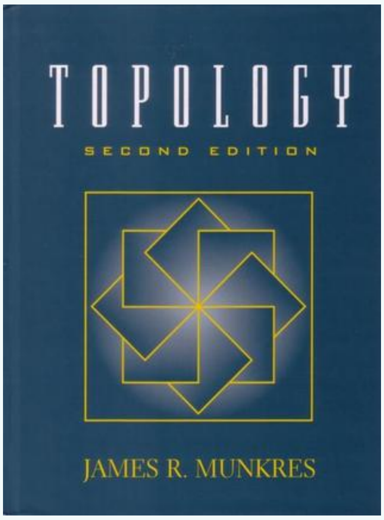
My favorite Algebra Book.

Resource 1.14: Judson Algebra



This free (and open source) book can be found at http://abstract.ups.edu/.

Resource 1.15:



1.3 Online Resources

Quote 1.16: Terry Tao

I don't have any magical ability. I look at a problem, play with it, work out a strategy.

Resource 1.17: Terance Tao's Blog

https://terrytao.wordpress.com/

Terry Tao is a fields medalist, and likely the best mathematician actively researching. Thankfully for us mere mortals, he posts his thoughts and insights on a well-maintained blog.

Resource 1.18: Borcherd's Lectures

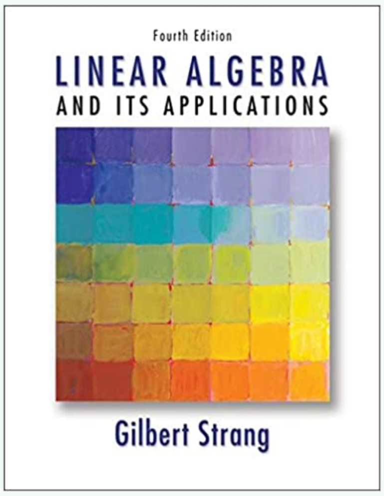
https://www.youtube.com/channel/UCIyDqfi_cbkp-RU20aBF-MQ

Richard Borcherds is a Fields Medalist who, starting during the Pandemic, publishes his lectures online.

Resource 1.19: MIT Open Course Ware

https://ocw.mit.edu/

Resource 1.20: Strang Linear Algebra



This is the best Linear Algebra Book. Along with Strang's accompanying OCW Lectures, it is an indispensable resource for beginning engineers and mathematicians alike.

Resource 1.21: Missing Semester

https://www.youtube.com/c/MissingSemester

The thesis of the video series is that Computer Scientists are expected to use, but are never taught, a variety of tools, including:

- Vim
- Bash
- Git

Mathematicians with a command over computers are sought after, both in Academia and Industry. The series was taught by three Graduate Students at MIT.

Resource 1.22: Sage Math

https://www.sagemath.org/

An open source, mathematical superset of the Python Language.

1.4 Problem Banks

Resource 1.23: Yutsumura

https://yutsumura.com/

A problem bank that tests intuition in

- Linear Algebra
- Group Theory
- Ring Theory
- Field Theory, Galois Theory
- Module Theory

Resource 1.24: Project Euler

https://projecteuler.net/

These problems deal with numbers that are too big to be solved by hand. The first step is to solve the problem, the second is to optimize the code so that it runs as fast as possible. It's a good site to practice the intersection between math and computer science.

Resource 1.25:

https://www.reddit.com/r/math/comments/thto4v/what_are_your_favourite_math_puzzles_to_give/

Fun problems to occasionally chew on.

Chapter 2

Problems

Quote 2.1: Paul Halmos

It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts.

2.1 Group Theory

Problem 2.2:

Let H be a subgroup of G such that $G/H \approx \mathbb{Z}$. Does G have a subgroup N such that $G/N \approx \mathbb{Z}/x\mathbb{Z}$ for any $x \in \mathbb{N}$?

Problem 2.3: A Painting and Two Nails

You want to hang a painting by a string with two nails in such a way that removing either nail will cause the painting to fall indefinitely.

Note 2.4:

This problem was my first exposure to Group Theory, long before I knew what a group was. Best of Luck

Resource 2.5: Painting and Two Nails

The solution, along with related problems, can be found at `https://arxiv.org/pdf/1203.3602.pdf` I highly recommend reading the solution.

2.2 Topology

Problem 2.6: Continuity Questions

Definition 2.7:

In a topological sense, a function $f : A \to B$ is continuous iff for every open $O \subseteq B$ we have $f^{-1}(O)$ is open in A.

Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that if O is open in $\mathbb{R}$ then $f(O)$ is open.

- Is g continuous?
- Let g be surjective, is g continuous?
- Let g be injective, is g continuous?
- Let g be bijective, is g continuous?

2.3 Real Analysis

Problem 2.8:

Let $f : [a, b] \to [a, b]$ be an increasing function. Show that there exists an $\xi \in [a, b]$ such that $f(\xi) = \xi$.

Problem 2.9:

Definition 2.10: Unit Interval

$$I := [0, 1]$$

Let $f : I \to I$ be a function with exactly two fixed points. Show that f is not continuous.

2.4 Graph Theory

Problem 2.11:

Definition 2.12:

A graph is planar if it can be drawn on a plane without any edge crossings.

Show that a graph is planar if and only if it can be drawn on a sphere without any edge crossings.

Problem 2.13:

Convince yourself that K_5 (The Complete Graph on 5 vertices) is non-planar.

Problem 2.14: Toroidal Graphs

Definition 2.15:

A graph is toroidal if it can be drawn on a torus without any edge crossings.

Example 2.16: K5 is Toroidal

Problem 2.17:

Is K_6 toroidal?

Problem 2.18:

What is the maximal n such that K_n is toroidal?

2.5 Logic

Problem 2.19: Gelca Problem 5

Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.

Problem 2.20:

You and a friend are playing a game with a deck of cards. You only care about the color of the card drawn (red or black). You both draw a card and place it on your own forehead, so that each person can see the other person's card, but not their own. On the count of three, you both say a color at the same time. Both players win if either player gets the color of their own card correct. Both players loose if neither person gets the color of the card correct. You're not allowed to communicate after you draw the card, but you are allowed to come up with a strategy beforehand. Can you come up with a strategy where both players always win?

2.6 Set Theory

Problem 2.21:

Construct a bijection from $(0, 1)$ to $[0, 1]$. Before you construct the function, why is this function not continuous?