ESQUANT
Statistical Consulting

LeanSigmaInstitute
Sustainable Business Improvement

ISO18532 Training
WELCOME
Day 5

ESQUANT Statistical Consulting
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ESQUANT Statistical Consulting

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Session	Topic	ISO Clauses	Montgomery	Quality Trainer
1	Introduction, Basic Statistical Tools, and Variation	4, 5	Chapter 2 Chapter 3	Section 1 Section 2
2	Conformity Assessment, Samples, and Populations	6, 7, 8	Chapter 4	Section 3
3	Acceptance Sampling	9	Chapter 15 Chapter 16	
4	Statistical Process Control	10	Chapter 6 Chapter 7	Section 4
5	Design of Experiments	12	Chapter 13 Chapter 14	Section 9
6	Process Capability and Measurement Systems	11, 13	Chapter 8	Section 5 Section 8

Session	Topic
7.30am to 9.00am	Project Reviews
9.00am to 10.00am	Introduction to Design of Experiments
10.00am to 10.30am	Morning Tea
10.30am to 12noon	Factorial Experiments
12noon to 12.45pm	Lunch
12.45pm to 2.30pm	Fractional Factorial Experiments
2.30pm to 2.45pm	Afternoon Tea
2.45pm to 4.00pm	Response Surface Methodology, Mixture Designs, EVOP, Wrap-up and setting the scene for Session 6

History and Overview

Experimental Design, as a statistical discipline, began with the pioneering work of R.A.Fisher in the 1920's and 1930's. Key concepts included:

Randomisation Fisher showed that the most important assumption was that of independence and not that of Normality. He developed the method of randomisation to ensure approximate independence, even in highly correlated situations.

Analysis of Variance Fisher's analysis of variance identity is of fundamental importance in the analysis of experiments.

$$\begin{aligned}\text{Total Sum of Squares} &= \text{Sum of Squares between treatments}\\ &+ \text{Sum of squares within treatments}\end{aligned}$$

History and Overview (ctd)

Blocking The development of the Randomised Block Design was due to Fisher. By blocking the experiment more precise and more generalisable conclusions could be made.

Factorial Design Many treatments have a factorial structure. Fisher exploited this with his development of the factorial design where he showed that it was much more efficient to experiment with many factors at a time rather than just one.

Explosives Example 1

In the development of a new explosive the following set of experiments were run. It was desired to achieve a low viscosity.

Gum Age	Thiourea Age	Controlled pH	Aluminium	Viscosity
2 Hrs	2 Hrs	No	Yes	1202
1 Week	2 Hrs	No	Yes	1166
2 Hrs	1 Week	No	Yes	316
1 Week	1 Week	No	Yes	1056
2 Hrs	2 Hrs	Yes	Yes	1160
1 Week	2 Hrs	Yes	Yes	1244
2 Hrs	1 Week	Yes	Yes	213
1 Week	1 Week	Yes	Yes	1204
2 Hrs	2 Hrs	No	No	1162
1 Week	2 Hrs	No	No	1238
2 Hrs	1 Week	No	No	232
1 Week	1 Week	No	No	920
2 Hrs	2 Hrs	Yes	No	1140
1 Week	2 Hrs	Yes	No	1228
2 Hrs	1 Week	Yes	No	276
1 Week	1 Week	Yes	No	1194

Conclusion: The best viscosities are achieved by using gum two hours old and thiourea one week old.

Ball Bearings Example

In an experiment to develop longer life ball bearings the following experiments were run:

Inner Ring Heat Treatment	Outer Ring Osculation	Cage Design	Life (Hrs)
std	std	std	17
mod	std	std	26
std	mod	std	25
mod	mod	std	85
std	std	mod	19
mod	std	mod	16
std	mod	mod	21
mod	mod	mod	128

Conclusion: The effect of changing the inner ring heat treatment is minimal when the standard outer ring osculation is used but very large when the modified outer ring osculation is used. (This is called an interaction between inner ring heat treatment and outer ring osculation).

Reaction Example

In an experiment to increase the percentage reacted the following experiments were conducted:

	Factors	Levels −	Levels +
A	Feed rate (litres/min)	10	15
B	Catalyst	1	2
C	Agitation Rate (rpm)	100	140
D	Temperature (°C)	140	180
E	Concentration (%)	3	6

Reaction Example (ctd)

A	B	C	D	E	Reacted
−	−	−	−	−	61
+	−	−	−	−	53
−	+	−	−	−	63
+	+	−	−	−	61
−	−	+	−	−	53
+	−	+	−	−	56
−	+	+	−	−	54
+	+	+	−	−	61
−	−	−	+	−	69
+	−	−	+	−	61
−	+	−	+	−	94
+	+	−	+	−	93
−	−	+	+	−	66
+	−	+	+	−	60
−	+	+	+	−	95
+	+	+	+	−	98

A	B	C	D	E	Reacted
−	−	−	−	+	56
+	−	−	−	+	63
−	+	−	−	+	70
+	+	−	−	+	65
−	−	+	−	+	59
+	−	+	−	+	55
−	+	+	−	+	67
+	+	+	−	+	65
−	−	−	+	+	44
+	−	−	+	+	45
−	+	−	+	+	78
+	+	−	+	+	77
−	−	+	+	+	49
+	−	+	+	+	42
−	+	+	+	+	81
+	+	+	+	+	82

Conclusion: This is a bit harder to see but only the catalyst, temperature and concentration affect the percentage reacted. The Feed rate and Agitation rate don't seem to have any effect.

Advantages of Factorial Design

The advantages of factorial design are:

- If factors interact then only factorial designs can avoid giving misleading results.
- Even if factors do not interact then factorial designs give more precise estimates of the main effects.
- In a factorial design the effect of a factor is estimated at several levels of the other factors, so therefore conclusions hold over a wider range of conditions.

Apparent Disadvantages of Factorial Designs

- One of the apparent disadvantages of factorial designs is the number of observations that are required if the number of factors is large.
- The development of Fractional Factorial Designs enables the advantages of factorial designs with only a fraction of the observations required for a full factorial design.

Projection Properties of Full Factorial Designs

- Fractional factorial designs have important projection properties.
- A fractional factorial design that involves r factors will be a full (sometimes replicated) factorial for every set of $s < r$ factors.
- Since usually not all experimental factors have effects on the response, then we may have a full factorial in the important factors.

Introduction to Two-Level Factorial Designs

- All factors are only tested at two levels.
- All possible combinations of levels are tested.
- These restrictions will be relaxed later.

How to interpret the results

Example: 2^3 Factorial Design

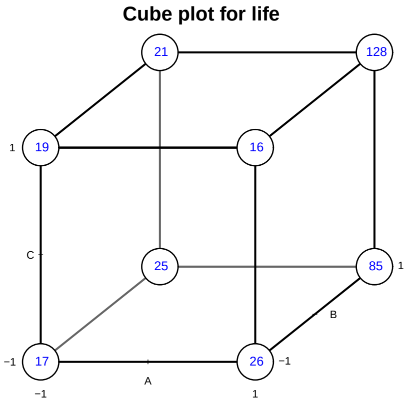
		Levels	
	Factors	$-$	$+$
A	Inner Ring Heat Treatment	std	mod
B	Outer Ring Osculation	std	mod
C	Cage Design	std	mod

Design and Results

A	B	C	Trial Identification	Life (Hrs)
$-$	$-$	$-$	(1)	17
$+$	$-$	$-$	a	26
$-$	$+$	$-$	b	25
$+$	$+$	$-$	ab	85
$-$	$-$	$+$	c	19
$+$	$-$	$+$	ac	16
$-$	$+$	$+$	bc	21
$+$	$+$	$+$	abc	128

Cube Diagram

It is a good idea to plot the results on a cube diagram.



Main Effects

The main effect of a factor is given by the average of all the results at the high level of the factor minus the average of all the results at the low level of the factor.

$$
\begin{aligned}
\hat{B} &= \frac{1}{4}(b + ab + bc + abc) \\
&\quad - \frac{1}{4}((1) + a + c + ac) \\
&= \frac{1}{4}(25 + 85 + 21 + 128) \\
&\quad - \frac{1}{4}(17 + 26 + 19 + 16) \\
&= 64.75 - 19.5 = 45.25
\end{aligned}
$$

Two-factor Interactions

The (two-factor) interaction effect between two factors, say A and B, is half the difference between the effect of B at high A and the effect of B at low A.

Note: An interaction of zero means that factor B has the same effect at both levels of A.

Calculation of two-factor interactions

$$
\begin{aligned}
\text{Effect of } B \text{ at high } A &= \frac{1}{2}(85+128) - \frac{1}{2}(26+16) = 85.5 \\
\text{Effect of } B \text{ at low } A &= \frac{1}{2}(25+21) - \frac{1}{2}(17+19) = 5.0 \\
\text{Interaction Effect} &= \frac{1}{2}(85.5-5.0) = 40.25
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\widehat{AB} &= \frac{1}{2}\left[\frac{1}{2}(abc+ab) - \frac{1}{2}(ac+b) - \left\{\frac{1}{2}(bc+b) - \frac{1}{2}(c+(1))\right\}\right] \\
&= \frac{1}{4}[abc+ab+c+(1)] - \frac{1}{4}[bc+b+ac+a]
\end{aligned}
$$

Alternative Calculation of Two-Factor Interaction

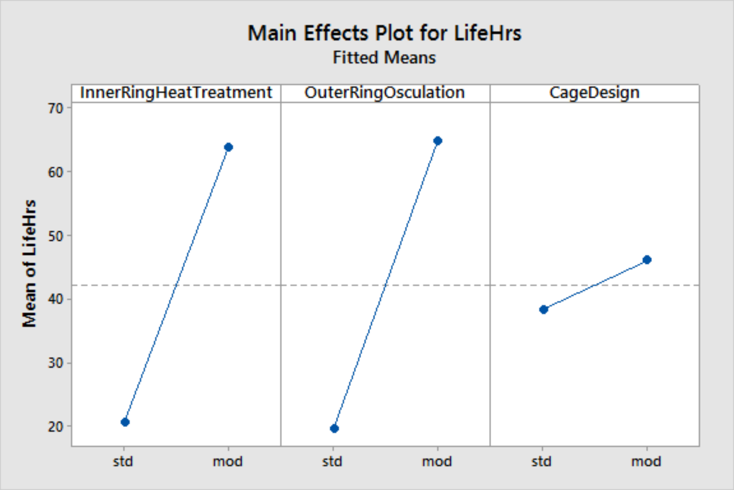
Trial Identification	A	B	C	AB
(1)	$-$	$-$	$-$	$+$
a	$+$	$-$	$-$	$-$
b	$-$	$+$	$-$	$-$
ab	$+$	$+$	$-$	$+$
c	$-$	$-$	$+$	$+$
ac	$+$	$-$	$+$	$-$
bc	$-$	$+$	$+$	$-$
abc	$+$	$+$	$+$	$+$

Three-Factor Interactions

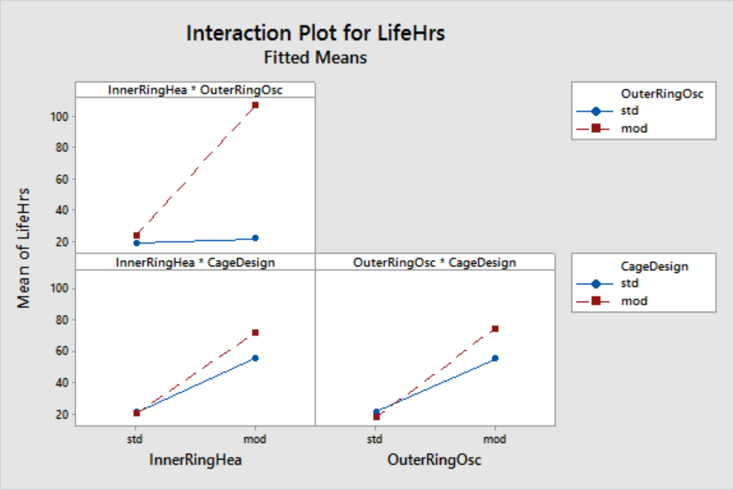
The three-factor interaction effect between A, B, and C is half the difference between the interaction effect between A and B at the high level of C and the interaction effect between A and B at the low level of C.

$$\widehat{ABC} = \frac{1}{4}(a + b + c + abc) - \frac{1}{4}((1) + ab + ac + bc)$$

Analysis in Minitab: Main Effects



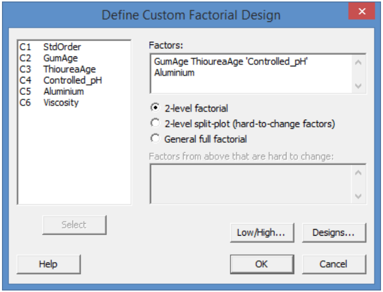
Analysis in Minitab: Interactions

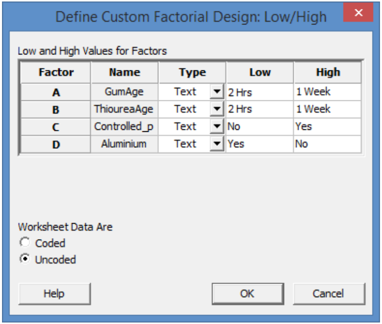


Analysis in R

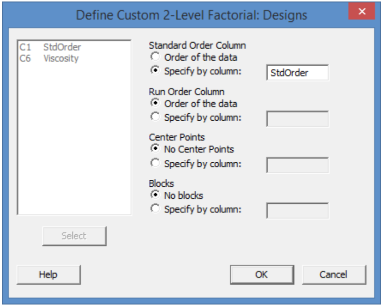
```
library(FrF2)
plan <- FrF2(nruns=8, randomize=FALSE,
             default.levels = c("std","mod"),
             factor.names = c("InnerRingHeatTreatment",
                              "OuterRingOsculation",
                              "CageDesign"))
ballbearings <- read.csv("ballbearings.csv")
LifeHrs <- ballbearings$LifeHrs
plan.resp <- add.response(plan, response=LifeHrs)
MEPlot(plan.resp)
IAPlot(plan.resp)
DanielPlot(plan.resp)
coef(lm(plan.resp, degree=3))
```

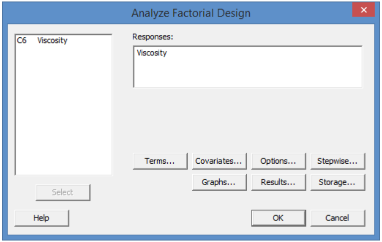
Analysis of Explosives Example in Minitab



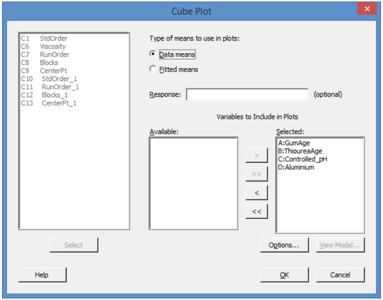


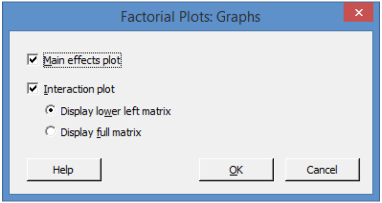
Analysis of Explosives Example in Minitab (ctd)



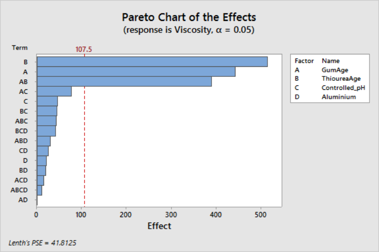


Analysis of Explosives Example in Minitab (ctd)





Analysis of Explosives Example in Minitab (ctd)



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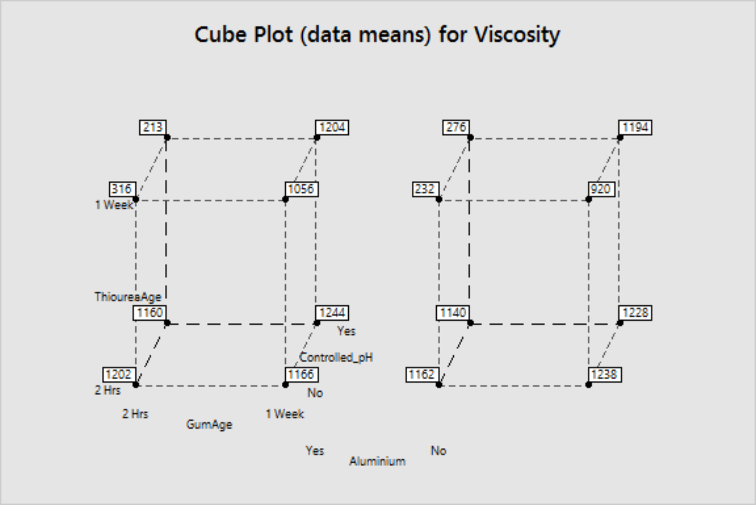
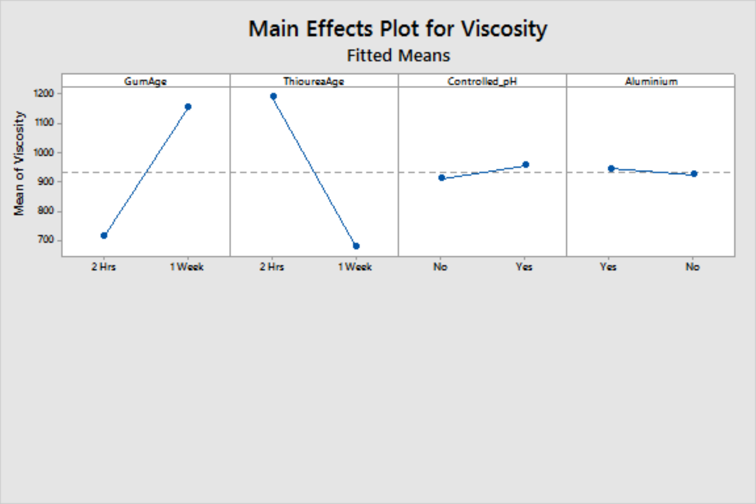
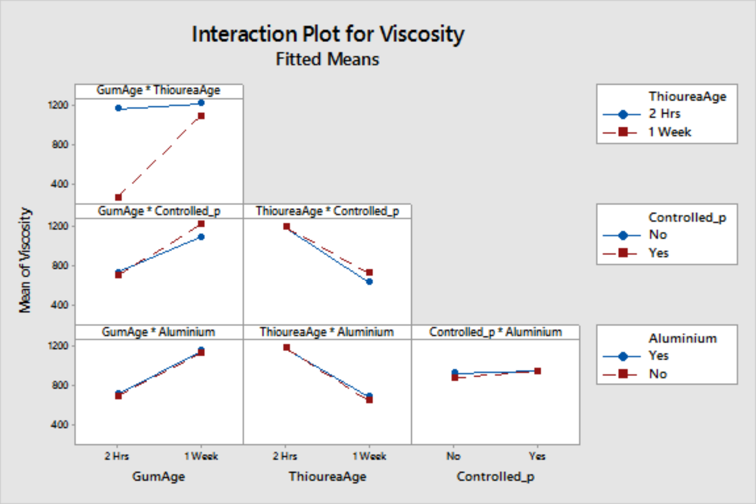
Cube Plot



Main Effects Plot



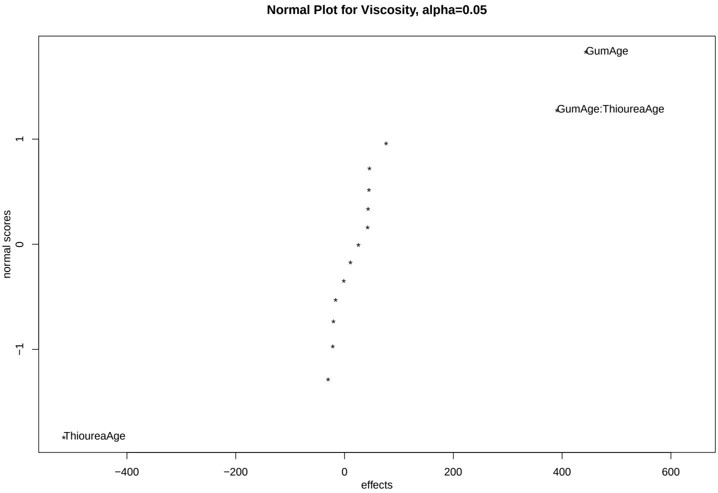
Interactions Plot



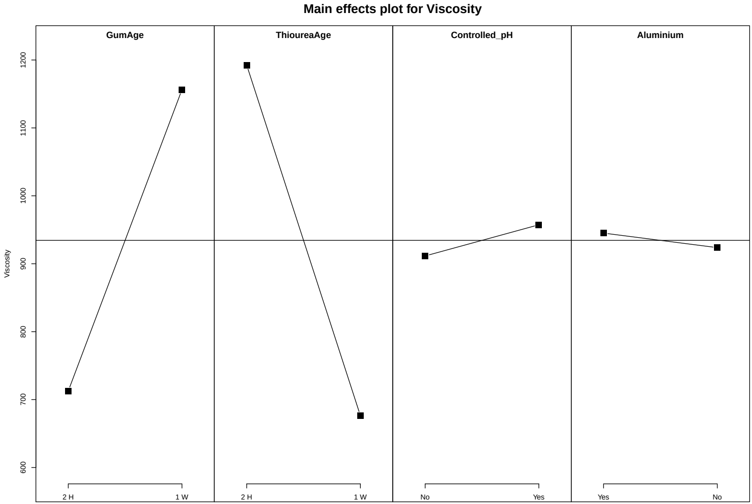
Analysis of Explosives Example in R

```
explosive1 <- FrF2(nruns=16, randomize=FALSE,
    factor.names=list(GumAge=c("2 Hrs", "1 Week"),
                      ThioureaAge=c("2 Hrs", "1 Week"),
                      Controlled_pH=c("No", "Yes"),
                      Aluminium=c("Yes", "No")))
Viscosity <- read.csv("Explosive1.csv")$Viscosity
explosive1.resp <- add.response(explosive1, response=Viscosity)
coef(lm(explosive1.resp, degree=4))
DanielPlot(explosive1.resp)
```

Daniel Plot


Normal Plot for Viscosity, alpha=0.05
GumAge
GumAge:ThioureaAge
ThioureaAge
normal scores
effects
−400
−200
0
200
400
600
−1
0
1

Main Effects Plot



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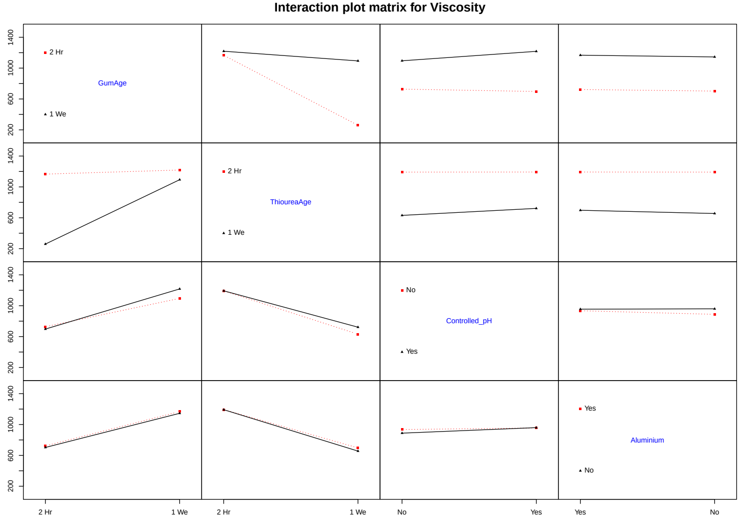
Interactions Plot



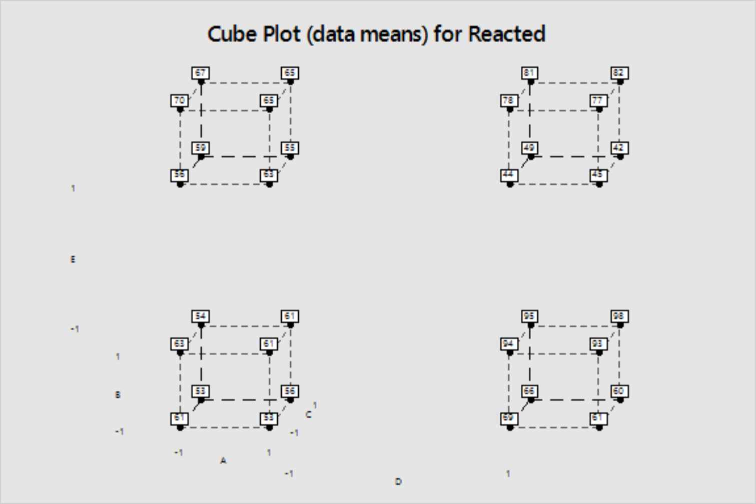
Fractional Factorial Designs

- Two level full factorial designs allow all the interactions to be estimated.
- Very efficient estimates of the main effects.
- Results are more generalisable than for the traditional one factor at a time design.
- One problem is the large number of trials required for these experiments.
- Although 8 and 16 run designs are manageable it is very rare that 32 or 64 runs could be justified in an industrial context.
- Solution is Fractional Factorial designs, (almost) the same information in less runs.

Example 3 with only half the runs

A	B	C	D	E	Reacted
−	−	−	−	+	56
+	−	−	−	−	53
−	+	−	−	−	63
+	+	−	−	+	65
−	−	+	−	−	53
+	−	+	−	+	55
−	+	+	−	+	67
+	+	+	−	−	61
−	−	−	+	−	69
+	−	−	+	+	45
−	+	−	+	+	78
+	+	−	+	−	93
−	−	+	+	+	49
+	−	+	+	−	60
−	+	+	+	−	95
+	+	+	+	+	82

Analysis of original Design-Minitab



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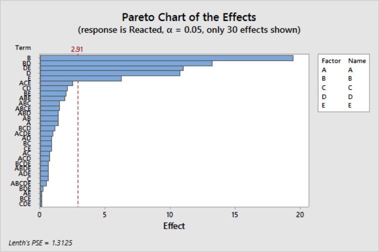
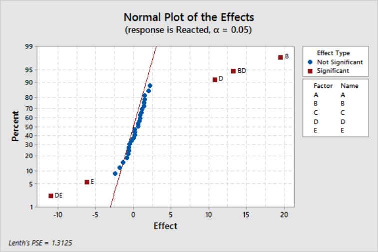
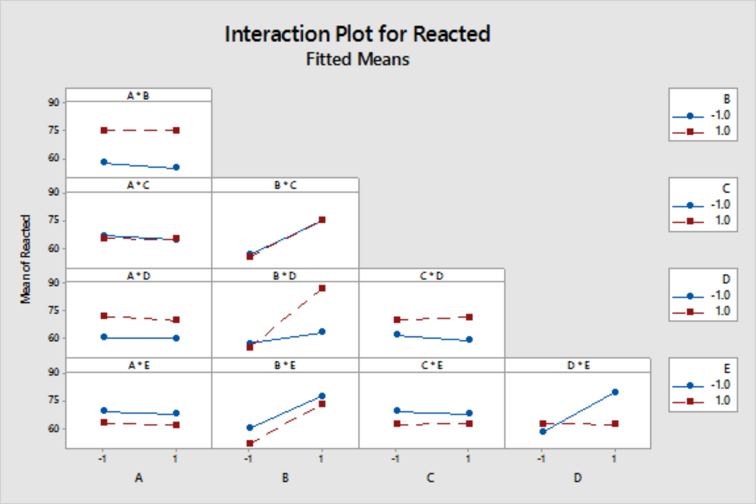
Analysis of Original Design-Minitab (ctd)



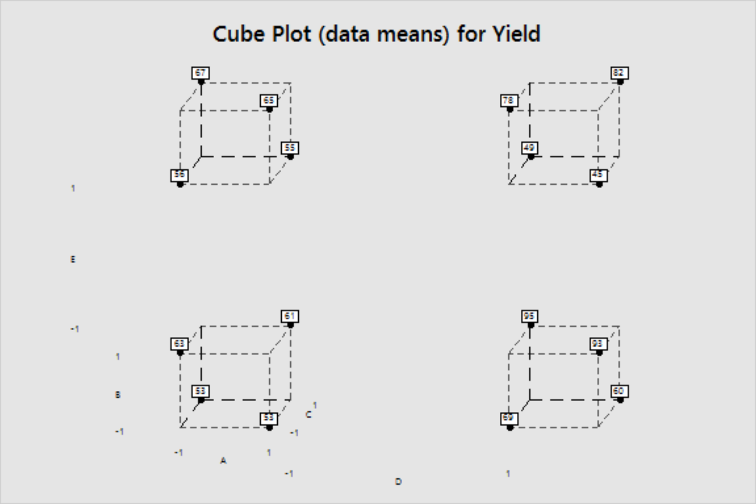


Analysis of Original Design-Minitab (ctd)

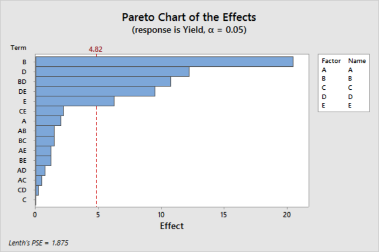
Analysis of Original Design-Minitab (ctd)

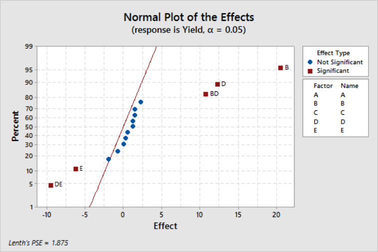


Analysis of Half Design-Minitab

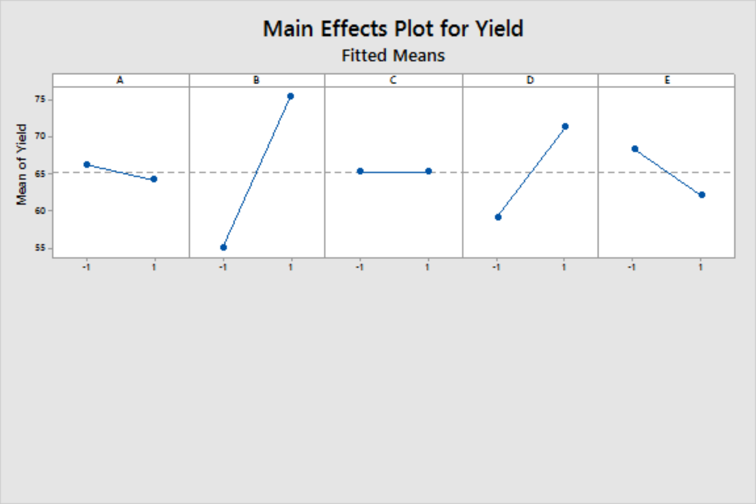


Analysis of Half Design-Minitab (ctd)

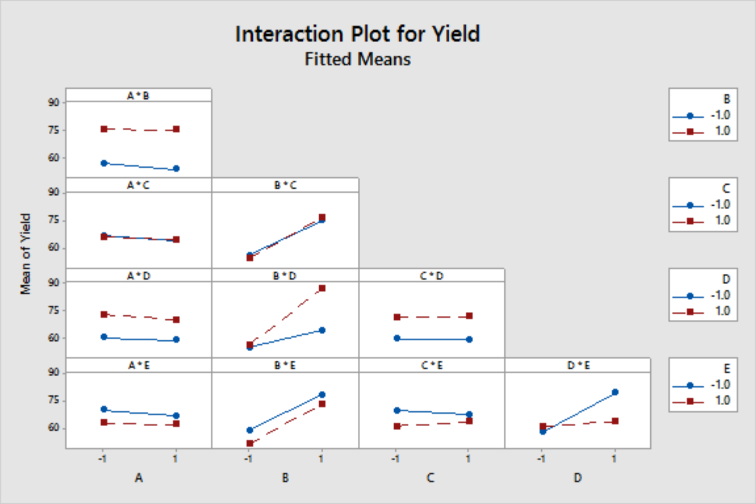




Analysis of Half Design-Minitab (ctd)



Analysis of Half Design-Minitab (ctd)



Analysis-R

```
chemreaction <- FrF2(nruns=32, 5, randomize=FALSE)
reacted <- read.csv("ChemReaction.csv")$Reacted
chemreaction.resp <- add.response(chemreaction,
                                  response=reacted)
coef(lm(chemreaction.resp, degree=5))
DanielPlot(chemreaction.resp, pch=19)
MEPlot(chemreaction.resp)
IAPlot(chemreaction.resp)
chemreactionhalf <- FrF2(nruns=16, 5, randomize=FALSE)
yield <- read.csv("ChemReactionHalf.csv")$Yield
chemreactionhalf.resp <- add.response(chemreactionhalf,
                                   response=yield)
coef(lm(chemreactionhalf.resp, degree=2))
DanielPlot(chemreactionhalf.resp, pch=19)
MEPlot(chemreactionhalf.resp)
IAPlot(chemreactionhalf.resp)
```

The Bush Experiment

- A VU student project conducted by Peter Kostaridis and Nick Condilis.
- An experiment to improve the rubber composition of the bush, an important part of the suspension system, used in a locally manufactured car.
- The levels of six components were varied and the resulting compound was tested to determine the Loss Angle and Dynamic Stiffness.
- The factors have been coded as 301 (A), 302 (B), 303 (C), 304 (D), 308 (E), and 309 (F) for confidentiality reasons and the levels given as $-$ and $+$.

Results of the Bush Experiment

Run	A	B	C	D	E	F	Loss Angle	Dynamic Stiffness
1	$-$	$-$	$-$	$-$	$-$	$-$	5.45	520
2	$+$	$-$	$-$	$+$	$+$	$+$	4.55	501
3	$-$	$+$	$-$	$+$	$+$	$-$	7.23	864
4	$+$	$+$	$-$	$-$	$-$	$+$	6.11	999
5	$-$	$-$	$+$	$-$	$+$	$+$	4.93	573
6	$+$	$-$	$+$	$+$	$-$	$-$	6.37	523
7	$-$	$+$	$+$	$+$	$-$	$+$	5.59	946
8	$+$	$+$	$+$	$-$	$+$	$-$	7.72	686

Conclusion: Factor B increases the Loss Angle while factor F decreases the Loss Angle. For Dynamic Stiffness only Factor B has an effect.

Explosive Development

In the development of a new explosive, the following set of experiments were run involving five factors with A, Age of Gum; B, Age of Thiourea; C, pH controlled; D, Aluminium level; and E, Crosslinker Level. The response was the Gel Strength after 10 minutes.

A	B	C	D	E	Gel Strength (10 Mins)
−	−	−	−	+	469
+	−	−	−	−	330
−	+	−	−	−	266
+	+	−	−	+	351
−	−	+	−	−	316
+	−	+	−	+	522
−	+	+	−	+	357
+	+	+	−	−	430
−	−	−	+	−	293
+	−	−	+	+	708
−	+	−	+	+	267
+	+	−	+	−	341
−	−	+	+	+	502
+	−	+	+	−	453
−	+	+	+	−	197
+	+	+	+	+	568

Conclusion: Only the Age of Gum, the Age of Thiourea, and the Crosslinker level have an effect on the Gel Strength after 10 minutes.

Injection Molding Experiment

In an injection molding experiment eight variables were studied in 16 runs. The response was percentage shrinkage. The variables were:

A	Mold Temperature
B	Moisture Content
C	Holding Pressure
D	Cavity Thickness
E	Booster Pressure
F	Cycle Time
G	Gate Size
H	Screw Speed

Injection Molding Experiment (ctd)

Run	A	B	C	D	E	F	G	H	Shrinkage
1	−	−	−	−	−	−	−	−	20.3
2	+	−	−	−	−	+	+	+	16.8
3	−	+	−	−	+	−	+	+	15.0
4	+	+	−	−	+	+	−	+	15.9
5	−	−	+	−	+	+	+	−	17.5
6	+	−	+	−	+	−	−	+	24.0
7	−	+	+	−	−	+	−	+	27.4
8	+	+	+	−	−	−	+	−	22.3
9	−	−	−	+	+	+	−	−	14.0
10	+	−	−	+	+	−	+	+	16.7
11	−	+	−	+	−	+	+	−	21.9
12	+	+	−	+	−	−	−	+	15.4
13	−	−	+	+	−	−	+	+	27.6
14	+	−	+	+	−	+	−	−	21.5
15	−	+	+	+	+	−	−	−	17.1
16	+	+	+	+	+	+	+	+	22.6

Conclusion: Only the Holding Pressure and Booster Pressure have an effect and, in addition, there is an interaction between Holding Pressure and Screw Speed.

The Concept of Design Fractionation

- The bush example was a $\frac{1}{8}$ fraction of a 2^6 design.

$$\frac{1}{8}2^6 = 2^{6-3}$$

- The explosive example was a $\frac{1}{2}$ fraction of a 2^5 design.

$$\frac{1}{2}2^5 = 2^{5-1}$$

- The injection molding example was a $\frac{1}{16}$ fraction of a 2^8 design.

$$\frac{1}{16}2^8 = 2^{8-4}$$

Design Resolution

- A design of resolution $R = III$ does not confound main effects with each other but does confound main effects with two-factor interactions.
- A design of resolution $R = IV$ does not confound main effects and two-factor interactions but does confound two-factor interactions with other two-factor interactions.
- A design of resolution $R = V$ does not confound main effects and two-factor interactions with each other, but does confound two-factor interactions with three-factor interactions, and so on.
- The bush example was 2^{6-3}_{III}.
- The explosive example was 2^{5-1}_{IV}.
- The injection molding example was 2^{8-4}_{IV}.

Defining Relation

- The explosive example was a 2^{5-1} design.
- In this design we had columns associated with E having the same signs as the column associated with $ABCD$. Hence,

$$\begin{aligned} E &= ABCD \\ E \times E &= ABCD \times E \\ E^2 &= ABCDE \\ I &= ABCDE \end{aligned}$$

- The identity

$$I = ABCDE$$

is called the defining relation of the design.

Confounding Pattern

From the defining relation, we can work out the confounding pattern. Multiplying through by each of the main effects and two-factor interactions we get:

$$
\begin{aligned}
I &= ABCDE \\
A &= BCDE \\
B &= ACDE \\
&\vdots \\
AB &= CDE \\
AC &= BDE \\
&\vdots \\
ABCD &= E
\end{aligned}
$$

Another example: 2^{5-2} design

- In this case the defining relation is given by
$$I = ABD = ACE = BCDE$$
- In the defining relation, ABD, ACE, and $BCDE$ are called words.
- The generators of the design are $D = AB$ and $E = AC$
- The design is as follows:

$$\begin{array}{ccccc} & & & D & E \\ A & B & C & AB & AC \\ - & - & - & + & + \\ + & - & - & - & - \\ - & + & - & - & + \\ + & + & - & + & - \\ - & - & + & + & - \\ + & - & + & - & + \\ - & + & + & - & - \\ + & + & + & + & + \end{array}$$

Confounding Pattern

The confounding pattern is given below:

$$
\begin{array}{ccccccc}
I & = & ABD & = & ACE & & BCDE \\
A & = & BD & = & CE & & ABCDE \\
B & = & AD & = & ABCE & & CDE \\
AB & = & D & = & BCE & & ACDE \\
C & = & ABCD & = & AE & & BDE \\
AC & = & BCD & = & E & & ABDE \\
BC & = & ACD & = & ABE & & DE \\
ABC & = & CD & = & BE & & ADE
\end{array}
$$

How to work out the Design Resolution

- In general, the resolution of a two-level design is the length of the shortest word in the defining relation.
- For example, consider the 2^{5-2} design with defining relation $I = ABD = ACE = BCDE$.
  - The word lengths are 3, 3, and 4.
  - The minimum word length is 3, and therefore the resolution of the design is *III*.
  - This means that the main effects will be confounded with two-factor interactions.

Fractional Factorial Table

2^3_{III} $\pm C = AB$					
2^3	2^{4-1}_{IV} $\pm D = ABC$	2^{5-2}_{III} $\pm D = AB$ $\pm E = AC$	2^{6-3}_{III} $\pm D = AB$ $\pm E = AC$ $\pm F = BC$	2^{7-3}_{III} $\pm D = AB$ $\pm E = AC$ $\pm F = BC$ $\pm G = ABC$	
2×2^3	2^4	2^{5-1}_{V} $\pm E = ABCD$	2^{6-2}_{IV} $\pm E = ABC$ $\pm F = BCD$	2^{7-3}_{IV} $\pm E = ABC$ $\pm F = BCD$ $\pm G = ACD$	2^{8-4}_{IV} $\pm E = BCD$ $\pm F = ACD$ $\pm G = ABC$ $\pm H = ABD$

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Blocking

- Improves the precision of an experiment
- Makes the results more generalisable
- Overcomes the problemd created by differing people, days, raw materials, . . .

An example of Blocking

Consider a factorial design involving eight runs where two batches of raw materials must be used since a batch of raw material is only sufficient for four runs. The problem is that the batch to batch differences in the raw materials may effect the results. This is called a block effect.
A good arrangement is required which will not affect any main effects nor any two factor interactions effects.

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A good blocked design

Run	A	B	C	AB	AC	BC	ABC	Block
(1)	$-$	$-$	$-$	$+$	$+$	$+$	$-$	1
a	$+$	$-$	$-$	$-$	$-$	$+$	$+$	2
b	$-$	$+$	$-$	$-$	$+$	$-$	$+$	2
ab	$+$	$+$	$-$	$+$	$-$	$-$	$-$	1
c	$-$	$-$	$+$	$+$	$-$	$-$	$+$	1
ac	$+$	$-$	$+$	$-$	$+$	$-$	$-$	2
bc	$-$	$+$	$+$	$-$	$-$	$+$	$-$	2
abc	$+$	$+$	$+$	$+$	$+$	$+$	$+$	1

About blocking

- We have deliberately confounded (i.e. confused) the block differences with the three-factor interaction
- We cannot estimate the 3fi separately from the block differences. Usually the 3fi is unimportant.
- If we call the block factor $B_1 = ABC$
- Four blocks are also possible. In this case two block generators are required.
  - You must be careful as to which block generators you choose: $B_1 = AB, B_2 = AC$ is much better than $B_1 = ABC, B_2 = BC$

Aims of Response Surface Methodology

- How is a particular response affected by a given set of input variables over some specified region of interest?
- What settings (if any) of the inputs will give a product that simultaneously satisfies a number of specifications?
- What value of the inputs will yield a maximum for a specific response and what is the response surface like close to this maximum?

A First Order Design

Run	Time	Temp	x_1	x_2	y
1	70	127.5	-1	-1	54.3
2	80	127.5	1	-1	60.3
3	70	132.5	-1	1	64.6
4	80	132.5	1	1	68.0
5	75	130.0	0	0	60.3
6	75	130.0	0	0	64.3
7	75	130.0	0	0	62.3

First order model

With

$$x_1 = \frac{\text{Time}-75}{5} \quad x_2 = \frac{\text{Temp}-130}{2.5}$$

the first order model is

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

Least squares fit:

$$\begin{aligned}
\hat{\beta}_0 &= \frac{1}{7}(54.3 + 60.3 + 64.6 + 68.0 + 60.3 + 64.3 + 62.3) \\
\hat{\beta}_1 &= \frac{1}{4}(-54.3 + 60.3 - 64.6 + 68.0) \\
&= 2.35 \\
\hat{\beta}_2 &= \frac{1}{4}(-54.3 - 60.3 + 64.6 + 68.0) \\
&= 4.50
\end{aligned}$$

Interaction Check and Curvature Check

Second order model is:

$$
\begin{aligned}
\hat{y} &= \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \\
&\quad \hat{\beta}_{11} x_1^2 + \hat{\beta}_{22} x_2^2 + \hat{\beta}_{12} x_1 x_2 \\
\hat{\beta}_{12} &= \frac{1}{2}(54.3 - 60.3 - 64.6 + 68.0) \\
&= -0.65 \\
\widehat{\beta_{11} + \beta_{22}} &= \frac{1}{4}(54.3 + 60.3 + 64.6 + 68.0) \\
&\quad -\frac{1}{3}(60.3 + 64.3 + 62.3) \\
&= -0.50
\end{aligned}
$$

Standard Error of Coefficients

$$
\begin{aligned}
V(\hat{\beta}_1) &= \frac{1}{16}(\sigma^2 + \sigma^2 + \sigma^2 + \sigma^2) = \frac{\sigma^2}{4} \\
sd(\hat{\beta}_1) &= \frac{\sigma}{2}
\end{aligned}
$$

Say σ is known to be 1.5, then $sd(\hat{\beta}_1) = 0.75$. The same applies to $\hat{\beta}_2$ and $\hat{\beta}_{12}$.

$$
\begin{aligned}
V(\widehat{\beta_{11} + \beta_{22}}) &= \frac{1}{16}(4\sigma^2) + \frac{1}{9}(3\sigma^2) = \frac{7\sigma^2}{12} \\
se(\widehat{\beta_{11} + \beta_{22}}) &= 1.15
\end{aligned}
$$

Since both $\hat{\beta}_{12}$ and $\widehat{\beta_{11} + \beta_{22}}$ are smaller then twice their standard errors, the first order model

$$\hat{y} = 62.01 + 2.35x_1 + 4.50x_2$$

is accepted.

Path of Steepest Ascent

- At right angles to the contour lines.
- Move $\hat{\beta}_2 (= 4.5)$ units in x_2 direction for every $\hat{\beta}_1 (= 2.35)$ units in x_1 direction.
- Equivalently

$$x_2 = \frac{\hat{\beta}_2}{\hat{\beta}_1} x_1$$

that is move $(4.5/2.35) = 1.91$ units in x_2 for every one unit in x_1.

Points on the Path of Steepest Ascent

x_1	x_2	Time	Temp	Run	Yield
0	0	75	130	5, 6, 7	62.3
1	1.91	80	134.8	8	73.3
2	3.83	85	139.6		
3	5.74	90	144.4	10	86.8
4	7.66	95	149.1		
5	9.57	100	153.9	9	58.2

A new set of experiments

Now do another set first-order design at the best set of conditions, with a slightly wider range.

Run	Time	Temp	x_1	x_2	Yield
11	80	140	-1	-1	78.8
12	100	140	1	-1	84.5
13	80	150	-1	1	91.2
14	100	150	1	1	77.4
15	90	145	0	0	89.7
16	90	145	0	0	86.8

The coding is given by

$$x_1 = \frac{\text{Time}-90}{10} \qquad x_2 = \frac{\text{Temp}-145}{5}$$

Analysis

The least squares estimate of the first degree equation is

$$y = 84.73 - 2.025x_1 + 1.325x_2$$

To check for interactions and/or curvature we compute

$$\begin{aligned} \hat{\beta}_{12} &= -4.88 \pm 0.75 \\ \widehat{\beta_{11} + \beta_{22}} &= -5.28 \pm 1.15 \end{aligned}$$

Clearly the first degree equation is quite inadequate over the region of experimentation.

Second Order Strategy

Since the first order model is inappropriate now, to fit a second order model it is necessary to augment the initial design with star points to form a central composite design. The star points take the form

$$\begin{array}{ll} (\pm\alpha, 0), (0 \pm \alpha) & \text{for } k = 2 \\ (\pm\alpha, 0, 0), (0, \pm\alpha, 0), (0, 0, \pm\alpha) & \text{for } k = 2 \end{array}$$

Values of α recommended are given in the table below.

k	α
2	$\sqrt{2}$
3	1.68
4	2
5	2.38

Choosing these values of α makes the design rotatable, i.e. the variance of the predicted values depends only on the distance from the centre of the design.

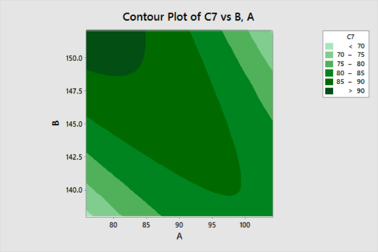
Central Composite Design

The additional runs and results are then given by:

Run	Time	Temp	x_1	x_2	Yield
17	76	145	$-\sqrt{2}$	0	83.3
18	104	145	$\sqrt{2}$	0	81.2
19	90	138	0	$-\sqrt{2}$	81.2
20	90	152	0	$\sqrt{2}$	87.0
21	90	145	0	0	87.0
22	90	145	0	0	86.0

Fitted second-order model

$$\hat{y} = 87.735 + 0.225\text{Block} - 1.384x_1 + 1.688x_2 - 2.612x_1^2 - 1.687x_2^2 - 4.885x_1x_2$$



Mixture Experiments

- Used when response depends on the relative proportions of the factors (ingredients) and not on the absolute amounts.
- Mixture space is a simplex, not a (hyper)cube.
- With no constraints, a good mixture design is a augmented simplex centroid design.
  - For three ingredients, the design consists of the three single-component blends, the three binary blends, and the tertiary blend, augmented by three interior points.
  - The linear model is

$$y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

  - A quadratic model is

$$\begin{aligned} y &= \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \\ &\quad \beta_{12} x_1 x_2 + \beta_{13} x_1 x_2 + \beta_{23} x_2 x_3 \end{aligned}$$

- More complicated with constraints.
  - Use psuedo-components if possible,
  - OR use extreme vertices designs.

Evolutionary Operation (EVOP)

Strategy developed by Box and Draper (1969) to run experiments on a process while in operation. EVOP studies are designed to be run by operators on a full-scale manufacturing process while continuing to produce a product of satisfactory quality.
Important features include:

- A series of sequential experiments.
- Simple factorial experiments with two or three factors that do not interupt operations.
- Small changes to the factor levels so that radical changes in product quality or efficiency do not occur.
- A large number of runs required per factor combination.
- Simple graphical analysis.

Session 6

- Projects
- Process Capability
  - Normally Distributed Data
    - C_p, C_{pk}, and C_{pm}
  - Non-Normal data
  - Process Capability for Attribute Data
- Measurement Systems