Calculus

Calculus is the mathematical study of continuous change. As you should be familiar with functions, you may or may not be aware that we have both continuous functions and discontinuous functions. A discontinuous function is a function that is either not fully defined ie. some data is missing, or one which does not reach all values of x as x approaches infinity eg. an asymptote. When you think about a common continuous function, the parabola for example, assuming the range of possible values for the parabola is the set of all real numbers, in theory the function should continue forever, thus always reaching the next value of x as it approaches infinity. Calculus is only applicable to such functions.

**Limits**

A limit refers to the value of f(x) or y at a particular x coordinate. Limits are represented by the word lim with a subscript denoting the value x approaches. So, for example, lim x->2 would represent x as it approaches 2. The purpose of limits is to figure out whether or not a function is continuous. If a point is indeterminant as I mentioned, ie. it does not have a limit at a given value x, then the function is discontinuous. There are various methods that we must try in order to truly determine if a function has a limit or not. As you may guess, the first method of testing this is to input the value that x approaches into f(x). So for example, the function:  We could substitute 9 into f(x) and see that 9 to the power of 2 equals 9x9 so by subtracting 9 squared by 9 squared we get 0 thus showing that the equation is discontinuous because it is undefined at x=9. Similarly, by substituting 0 we find another undefined point.

The next method we have of testing limits is to use a table. Since a continuous function will yield small changes in f(x) with respect to small changes in x, we can substitute values for x which approach the limit. For example: We can begin by substituting values that approach x from below the limit: = 6.25, = 7.48, = 8.41 and values which approach from above the limit: = 12.25, = 10.56, = 9.61. As you can see, the values of f(x) approach 9 as x comes closer to 3. We can then infer that will have a limit of 9.

The third method we have can be used on functions which are indeterminant. In other words, if the first two methods yield a fraction which has 0 as the denominator, we can try factoring the equation. For example, the equation: Here we can factor x-5 from the numerator which then cancels with the denominator and leaves x-5. If we were to substitute 5 for x into the unfactored equation, or create a table with values approaching 5, we would end up with an indeterminant. By factoring, we can see that f(x) will be 0 at x=5. This means it is defined because 0 is still a valid point.

The fourth method we have will also work on indeterminants but only for very special cases. If the numerator of a function has the format a + or a - , we can use what’s called the conjugate of the equation. We can multiply a fraction by another fraction if the other fraction is equal to 1 because at that point it is the same as multiplying by 1. So what we can do is take the numerator of our fraction, flip the sign from a + or vice versa and then use the fact that (a + b)(a - b) = Here are the steps as to how you would solve an equation like this:

First, multiply by what we call the conjugate (notice the operator changes from – to +):

Then start by multiplying the numerators using foil or the knowledge that (a + b)(a - b) =

We know that the root of a number is actually just

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Once we have simplified, we can now sub the limit of 5 into the equation:

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= 5/6

So then, our limit for this function is 5/6 when x=5.

**The derivative**

The derivative is a concept which describes the steepness of a slope. An average slope between 2 points is called the secant line, or the average rate of change of a function. Derivatives measure the steepness at an instantaneous singularity, also referred to as the tangent line. When we have a constant function ie. a straight line such as y = 5 or x = -2, they will always have a derivative of 0 since there is no steepness to a flat line. As for linear functions such as f(x) = x where we have a certain ratio that corelates x with y (called the slope), the derivative will be 1. If a horizontal stretch is applied such as 4x, then the derivative will be 4(1) = 4. The derivative need not apply to such functions. The purpose of calculus is to analyse functions which change over time eg. exponentially. Linear functions have a constant rate of change and thus already have their slopes defined. When we take the average slope of an exponential function using the secant line between two points, we call these points P and Q. We say that P is our limit, thus Q always approaches P. As Q gets closer and closer, the secant line draws nearer to the tangent line. In other words, P = tangent line which A close up of a map

Description automatically generatedis the same thing as the derivative of a line. We know that secant is the same as the slope of a linear function therefore we say that or the slope between points P and Q = where d simply means delta or “change in”. Since y = f(x) and dy = , we can rewrite the equation as . It is written this way because represents the y coordinate at delta x ie. f(x+dx) and represents y at the first x coordinate. We also tend to define dx as h for simplicity while dividing. In other words, the equation for slope of a secant line is the same as the equation for the tangent line; the only difference being that the second equation works for a single point. An equation of a line will be given, and in order to find the derivative of said function, we substitute it at every point we see f(x). This means that the values for x in the function will either be replaced by (x + dx) or stay x. For example, given the equation find the derivative using the delta method (also referred to as the first principle or “finding the derivative by definition”). We begin by substituting each f() with y (the equation), and each x with the argument in f() (either x+dx or x). This would appear as:

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To test for a limit, you would use the final equation. It should be noted that the tangent line represented by , while technically the same as the derivative, still must be differentiated from it. In other words, to represent a derivative, we say f prime of x or f’(x) or y’ or instead of or . If a derivative exists for x = we say that the function is differentiable at .

A practice example may ask for the velocity of a moving object which will be equal to the derivative of an equation distance over time. dy will be replaced by ds and dx will be replaced by dt. This can then be described by the equation If an equation is given to us such as s = 4.90, then we can plug this in for s(t+dt) and s(t). Let’s say that we want velocity at 4 seconds of the object’s trajectory. We can then infer that dt will be 4 and t will be 0, or whatever the starting time of the object’s trajectory is (if it’s not 0, dt will change accordingly). Our equation ends up looking like:

There arise other problems such as asymptotes which are fractions by nature. The method doesn’t change but it appears different from a more basic equation. For example, take the equation

y= In order to find the derivative for this equation, we use our formula as usual, and I find it easiest to leave h out of the formula until the numerator is written. This would appear as Since h is in the denominator in our equation for the derivative, we can simply multiply the denominator by 1/h ie. Now we cross multiply since we are subtracting the equations, and anytime you add or subtract fractions, you must cross multiply:

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= =Multiply 1 in numerator and h in denominator

= limit of h approaches h -> 0 so remove h

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**Finding Derivative by**

Finding the derivative of a function through the delta method quickly becomes obsolete once you learn the equation . I won’t explain how we arise at this equation, but it relates to what I said about constants becoming 0 and straight lines becoming 1x the horizontal stretch. When we have a term, in order to get the derivative of that term we can multiply the coefficient of the term by the power of x and then subtract 1 from the power of x. For example, 2x becomes 2 because 2 times the power of x (which is 1) is 2, and then we subtract 1 from the power of 1 to get which is 1. Another example would be . In this example we multiply he coefficient, 4, by the power -2 to get -8 and then subtract 1 from the power -2 to get -3. The 2 at the end is a constant so it becomes 0, therefor the derivative of the function f(x) = is . This is known as the power rule, and it is simply a shortcut from finding the derivative by definition. You would receive the same answer by doing