Pre-Algebra

Pre-Cursor:

I have decided to compose an entirely new set of notes diverging from my computer/software engineering notes that will solely focus on mathematics. I aim to cover most areas of mathematics ranging from elementary concepts (such as the topic of this specific document, pre-algebra), to very advanced topics such as differential equations, physics, probability theory, set theory, etc. During the time of writing this document, I recently subscribed to a service called Brilliant (not sponsored because, how could it be...) I will be following the flow and talking points of Brilliants courses to help me with the structure of these documents (at least the ones that are, in fact, covered by Brilliant). I will still be putting things in my own words, and adding additional topics of study as I go. I hope that this will be an intuitive look at mathematics; one that is more engaging than the sort of math class you would take in high school. My aim is to help you learn how to think as a mathmetician, moreso than teaching you how to solve a particular equation. If that interests you, I emplore you to continue reading.

Intro:

We will be starting with pre-algebra. This will be the fundamentals that are necessary for you to grasp our next topic, algebra, and eventually, linear algebra (for which I am most excited). I am confident that most of the things in this document will already be common knowledge to you if you paid any attention in elementary school, but if you are like me and ignored elementary school math, then I hope to focus on definitions and terms that will arise often as we continue onwards, as well as refresh you on certain things that you may have missed or forgotten. Before we begin however, I want to define what algebra is, so that we are on the same page as to why we are learning these concepts. The word ‘Algebra’ is Arabic, because of its origins in Persia. The word algebra means “reunion of broken parts”. This is important to keep in mind because algebra is the study of relationships between quantities. These quantities are often times unknown. This is simply a fact of reality: We often lack data or understanding. Algebra can be a useful tool for solving unkowns given other information that we do know.

Basics:

If you are like me and your forte is in computer science, but not necesarily mathematics, there are perhaps a few things that ought to be addressed, however obvious they may be. I am someone who needs constant reminders, and one such reminder that I often tell myself, is that it is important to distinguish between what the equals sign means in programming vs. what it means in math. In programming, ‘=’ means assignment. You assign a value (often a number) to a variable. But this is not really how we should think about it in mathematics. In mathematics, ‘=’ is better described as “equality”. In other words, whatever is on the left of the equals sign must be equivallent to what is on the right. Another thing to note about math is that we always assume that the equals sign is true. In reality, humans can make errors, and a proposition such as 1 = 3 would be an example of a falsehood. In math, the equations we are given are assumed to be true. A very basic assumption like 2 + 2 = 4 being true is what we call an “axiom” i.e. a given; but we will delve into that more in the section on proofs. For now, we can ignore such axioms and safely assume that the left hand side is always equal to the right hand side in any given equation.

As I mentioned, we often deal with unknowns in algebra. These are numbers that have an unknown quantity, but can be solved for. We refer to such unknowns as “variables”. The term variable means that the unknown quantity can vary/change depending on the constants that are present. We often represent variables with letters ranging from a – z, though variables can be represented by any word or symbol/sequence of symbols if we wanted (letters are simply the convention in math). Sometimes these letters have hidden meaning. They might stand for a particular word, or they might just be traditionally used for some specific purpose (for example x, y, and z are often associated with 3-dimentional axes), and other times, the letters themselves hold no significance at all. Let me give you an example of a variable altering its quantity depending on certain given constants. The expression “x + 3 = 15” means that x must be a quantity, such that, when added to 3, is equal to 15. Whether you do it subconciously or not, in order to solve for the variable x, we find the *difference* between 3 and 15 ie. 15 – 3. In this case x is 12 because 12 + 3 = 15. However, notice that x can vary/change if the term on the right-side of the equals sign were different. For example, “x + 3 = 7” would make x = 4.

Practical Application:

Let’s look at an example of a practical example of algebra (taken from Brilliant)! Unfortunately, it would simply be too much work for me to add imagery with pictographs and such, like they do on Brilliant, but I think a description should be enough for this example. Imagine a coffee shop where they use pictures to represent the items on their menu (to help customers who can’t read english well, naturally). For coffee, they have a picture of a mug, and for ice cream, a picture of an ice cream cone. The first option on their menu shows 2 coffee mugs and 3 ice cream cones which comes to a total of $19 dollars. The second option on the menu is 2 ice cream cones for $6. This can be described as 2 mathematical equations. Let C represent one mug of coffee, and let I represent one ice cream cone. The first option on the menu can be modelled mathematically as: 2 x C + 3 x I = 19 and the second equation can be stated as 2 x I = 6. Let’s say now that you want to buy 3 ice cream cones. In order to figure out how much 3 ice cream cones is worth, it would be easiest to take the second option on the menu, 2 x I = 6, and reason about how much 1 ice cream cone is worth. This can be solved by dividing the price of 2 ice cream cones in half. 6 / 2 = 3, so I = 3. Now that we know the worth of one ice cream cone, in order to find the price of 3 we simply multiply by 3. I x 3 = 9.

Consecutive Integers:

Consecutive integers are integers that are next to each other on the number line. For example, 2, 3, 4, 5, 6 is a group of 5 consecutive integers. If the biggest of three consecutive integers is represented by the variable m, then how can we represent each of the three integers in terms of an equation? The answer is simply m, m – 1, m – 2. We know that the numbers must be consecutive, so if m is the largest of the three, the second largest must be 1 less, and the third largest must be 2 less.

Order of Operations:

In algebra, it is often useful for us to be able to reword/refactor/modify an expression without changing the equality of both sides of the equation. In order to do this, we must consider the order in which calculations are performed. This order is called the order of operations. Sometimes order does not matter. For example, 3 + 5 = 5 + 3. Other times, however, order does matter. For example 2/3 is not the same as 3/2. In elementary school, you are often taught the anagram: BEDMAS. Which stands for “brackets, exponents, division, multiplication, addition, subtraction”. This is, in a nutshell, the order of operations. For example, in the equation 4 + (2 x 3), we multiply first, but in the equation (4 + 2) x 3, we add first (because the brackets take precedence). BEDMAS is a bit deceiving though, because it is not always necesary. For example, division and multiplication actually have the same priority, so in the case that we get something like 2 \* 6 / 9, we can just read it left to right. For a better demonstration of this, 2 x (6 / 9) and (2 x 6) / 9 both yield 1.333, proving that their order does not matter. Let’s look at a tougher example: 2 x 2 + 2 – 2 / 2 – 2 x 2 / 2. In this case we read the equation left to right, always prioritizing operations that are division or multiplication over addition or subtraction.

2×2+2−2÷2−2×2÷2

=4+2−2÷2−2×2÷2

=4+2−2÷2−2×2÷2

=4+2−1−2×2÷2

=4+2−1−2×2÷2

=4+2−1−4÷2

=4+2−1−4÷2

=4+2−1−2

=3

An important observation to make is that we often have *groups* in our equations which essentially have invisible brackets around them, making them top priority. For example, take the equation In this equation, the numerator (top half of the fraction) is a group, the denominator is another group, and everything inside the square root operation is a group. This could be rewritten like so:

Here’s another example using exponents: . In this case, we take the square root operation first since it is a group. goes first according to BEDMAS. The inner expression of the square root group evaluates to 31. Then we go left to right since 12/3 is multiplication and the rest is addition. Here is the written example to better demonstrate:

= 12/3 +

= 12/3 +

= 12/3 + 5.567

= 9.567

Evaluating Expressions:

An expression is a combination of numbers, variables, and operators that represent a quantity. For example, if a = 5, what is a + 7? The answer is 12. “a + 7” is considered an expression for the given quantity, 12.

Note that there are various ways in which we can describe multiplication between two numbers. In elementary school, multiplication is often represented with the letter/symbol ‘x’. In algebra, we try to avoid this, since x is often used to represent variables. Other ways that we can represent multiplication are with a dot in between 2 numbers, an asterisk (\*), or simply by placing two numbers next to each other. For example, (3)(2) means 3 x 2. Note that I added the parenthesis so that we wouldn’t confuse the expression with the number 32. When combining variables and numbers, we don’t need the brackets. For example, 9z is the same as 9 x z.

Let’s skip all the way to physics class and learn a physics formula! The formula for the density (D) of a substance with mass m and volume V is D = m/V. Let’s say that an ice cube has a volume of 27 cm^3 and mass of 24.84 g. If we want to know the density of the ice, we can plug in our given values for m(ass) and V(olume) into our equation. In this case: D = 24.84/27 = 0.92 g/cm^3.

Let’s look at another example with negative integers. It can be easy to forget that a variable is negative since it is represented by a single letter so the negative sign is not visible within the equation. For example, if a = -1 and b = 2, what is the value of b + ba? In this case, the expression expands to 2 + 2 \* -1 = 0.

Creating Equations:

2 expressions that are equal to each other form an **equation**. In other words, an expression, followed by an equals sign, followed by another expression is an equation! We will now learn how to create equations, the central language of algebra.

In 15 years, Agnes will be 25 years old. If a denotes Agnes’ age today, how can we create an equation to describe this statement? We can infer that the right side of the equation will be 25 since we want to find a number, that when added to Agnes’ age (a) will = 25. The equation can thus be modelled as a + 15 = 25.Note the two expressions, a + 15 and 25, and the equality operator (=) between them.

Pavel’s dog is 3kg heavier than his cat. Both pets together weigh 13kg. This equation can be represented as y + (y + 3) = 13. What does y represent in this equation? The answer is the cat’s weight. We can infer this because (y + 3) represents the dog being 3kg heavier than the cat’s weight (y).

Describing Relationships:

When we are given a relationship between two variables, it is important to think carefully about how to represent them. One helpful technique can be to ask yourself which of the variables is larger.

One orange weighs o units and seven oranges weigh n units. What is the relationship between the units o and n? The answer is that o = n / 7 since n is always 7 \* o. This sort of equation can be solved with what I call the triange rule. Any time that you have an equation in the form a = bc, you can think of each variable being in one corner of a triangle. a is situated at the peak, b at the bottom left corner, and c in the bottom right corner. This looks something like:

a

b | c

In order to solve for a, we multiply b \* c. In order to solve for b, we divide a / c. In order to solve for c, we divide a / b. This makes sense if you think of ‘a’ as being the numerator, and b and c as being values that can be multiplied to get a. So in the case of the previous question, we know that n is always 7 \* o ie. n = 7o. In order to solve this in terms of o, we can use the triangle rule, and say that o = n / 7. Take some time to reflect and dwell on the question if that was too confusing for you.

Here is a good brain excersize: Greg bought a total of one hundred chocolates contained in four small and three large boxes. Each large box contains twice as many chocolates as a small box. Both equations below describe this situation:

A. 4x + 3(2x) = 100

B. 4(y/2) + 3y = 100

What is true of the relationship between x and y?

a) x < y

b) x = y

c) x > y

d) x can be smaller or bigger than y

In this case, the answer is a. What gives it away is the second *term* in both equations. In equation A. we see that the large boxes are represented by 3(2x). We can infer that x is the number of chocolates in a small box since there are 3 large boxes, and 2x implies “twice as many”. In equation B, the second term is 3y which would imply that y = 2x. Therefore, y must be greater than x.

Finding Unknowns:

At long last, we will now learn how, given an equation, we can solve for the value of unknown variables. For certain equations, intuition, or guessing and checking is good enough for finding an unknown, but when we get to more complex equations, intuition is not enough, and we will need to learn certain rules that can be applied to help simplify the equation.

This example should be able to be solved with intuition/guess and check: The options below are possible solutions of the equation

A. x = 2

B. x = 3

C. x = 5

D. x = 6

Which values of x make the equations true?

a) A and B

b) A, B, and C

c) B and C

d) B, C and D

The answer is a), A and B are true. All we must do is replace x with each option and test the equality between the two expressions in the equation.

A number k has a value that is 3 less than l. If l equals 7, what is k?

a) -4

b) 3

c) 4

d) 10

The answer is c), 4. Whether you actually wrote an equation or not, this can in fact be modeled with the equation k + 3 = I, or the equation I – 3 = k. The second equation is preferable because k is isolated which means that I – 3 will give us the value of k which is the variable that we are trying to solve for. The first equation must be rearranged in order to find k.

Prime Numbers:

One area of mathematics that I personally ignored during elementary school was prime numbers and factoring. Factoring is a process in which we divide/split numbers into smaller numbers and put them back together. Seeing factors can help us spot which moves to make when working with algebraic expressions. Before factoring however, it is important that we understand the root of all factoring, prime numbers!

Note that each arithmetic operation (addition, subtraction, multiplication, and division) all have words to describe their outcome. When we add two numbers, the outcome is their *sum*, when we subtract, we get the *difference* of the numbers, when we multiply, we get the *product* of the two numbers, and when we divide, we get the *quotient*. When we look at factorization, we want to find all of the prime numbers that, when multiplied, give us the product that we are factoring.

Imagine two rectangles made up of square blocks. Each rectangle has an area of 6 units. The first rectangle is 1 x 6 blocks, and the second is 2 x 3 blocks. Note that if we rotate the blocks 90 degrees, we get two new dimensions, 6 x 1 and 3 x 2. These are still the same rectangles though, just rotated.

How many different ways can 12 unit squares be arranged into a rectangle?

a) 2

b) 3

c) 4

d) 5

The answer is b): We can arrange the rectangles in the following ways: 1 x 12, 2 x 6, and 3 x 4. These are all factorizations of 12! If the product of two or more numbers is equal to some value, then that set of numbers is called a factorization of that value.

How many factorizations are there of 11?

a) 1

b) 2

c) 3

d) 5

The answer is a), 1. The only factorization of 11 is 1 x 11. All numbers have at least one factorization: 1 x the number. This is called the trivial factorization of the number. For example, 1 x 12 is the trivial factorization of 12 because it includes the number being factored, 12.

In math speak, we would say that the set of all **prime numbers** have no non-trivial factorizations. That is to say, no prime number can have a non-trivial factorization, otherwise it is not a prime number. 11 is a prime number because the only way we can factor it is by its trivial factorization, 1 x 11. Also note that we only consider whole numbers as factors. 11 is technically divisible by 2 x 5.5, but 5.5 is a decimal, which is not a whole number.

Composite Numbers:

Numbers that *can* be factored by a number other than themselves or 1 are called composite numbers. These numbers can be written as a product (or composition) of smaller numbers. Composite numbers have more than one factorization. One neat fact to consider is that all even numbers that are greater than 2 are composite. Some odd numbers are also composite, such as 91 = 7 x 13.

Let’s look at one possible factorization of 210. 210 can be the product of 2 x 3 x 5 x 7. This factorization is special; it is called a prime factorization. This is because each of the factors (2, 3, 5, and 7) are prime numbers. Here are some other examples of prime factorizations:

* 20 = 2 x 2 x 5
* 24 = 2 x 2 x 2 x 3
* 33 = 3 x 11
* 35 = 5 x 7

Prime Factorization:

Remember, when we express a composite number as a product of only prime numbers, we call that its prime factorization. Which of the following factorization is not a prime factorization?

a) 36 = 2 x 3 x 6

b) 38 = 2 x 19

c) 40 = 2 x 2 x 2 x 5

d) 42 = 2 x 3 x 7

The (hopefully) obvious answer is a), since it contains 6 as one of it’s factors, and 6 is not a prime number since it can be factored into the prime numbers 2 x 3.

Suppose that y is a number whos prime factorization has exactly three different primes. What is the smallest possible value of y? Since each prime must be different, the solution is to find the first 3 prime numbers. 1 is not a prime number, so the 3 smallest prime numbers are 2, 3, and 5. The product of 2 x 3 x 5 is 30, so the smallest possible value of y is 30.

How many different prime factorization are there for 18? Note that rearranging the order of a prime factorization doesn’t make a new unique prime factorization. The answer is 1 because the only prime factorization for 18 is 2 x 2 x 3. As a larger take away from this lesson, note that every number has exactly one prime factorization. This is one of the reasons that our definition of prime numbers intentionally excludes 1. If we allowed 1 to be a prime, then the prime factorization of 18 could be 2 x 3 x 3, or 1 x 2 x 3 x 3, or 1 x 1 x 2 x 3 x 3, etc. For an infinite number prime factorizations (since we could just keep adding 1s).

Factorization:

It is important to remind ourselves that composite numbers can have factorizations which are not prime factorizations. Depending on the situation, different factorings can be more or less useful.

Zero as a Product:

To begin, we will look at a simple and powerful property that illustrates why it can be helpful to write something as a product. In the equation ab = 0, what must be true?

a) Both a = 0 and b = 0

b) Either a = 0 or b = 0

c) Neither a = 0 nor b = 0

The answer is b), either a = 0 or b = 0. Option a) might also be true, but it is not necesarily true, which was specific to the question, “what *must* be true”. This concept is known as the zero-product property: “If the product of two values is 0, then at least one of those values must be 0”.

In the equation (x + (-1)) \* (x + 5) = 0, how many different values of x are possible? The answer is 2: x = 1 or x = -5. If x = 1, then (x + (-1)) evaluates to 0, or if x = -5, then (x + 5) evaluates to 0. According to the zero-product property, in either scenario the result must be 0.

Finding Factorizations:

Previously, we learned that every number n has a trivial factorization of 1 x n. Additionally, we learned that any composite number has exactly one prime factorization (and potentially other factorizations). Our aim is to find those other factorizations.

Consider the number 33. It has a trivial factorization of 1 x 33 and a prime factorization of 3 x 11. Including an additional factor of 1 in any factorization doesn’t change it, because multiplication by 1 has no effect. So, for example, 1 x 3 x 11 doesn’t count as another factorization of 33. Is there another way to factor 33? The answer is no. But take the number 42, for example. 42 has a prime factorization of 2 x 3 x 7, but has other factorizations, such as 2 x 21, 3 x 14, and 7 x 6.

One way that we can find other factorizations given a numbers prime factorization is by simply multiplying two or more of the primes together. This only works for non-trivial prime factorizations. Looking back at 42, it’s prime factorization was 2 x 3 x 7. The factorization 2 x 21 can be obtained by multiplying the primes 3 x 7, the factorization 3 x 14 can be obtained by multiplying the primes 2 and 7, and the third factorization, 7 x 6, can be obtained by multiplying the primes 2 and 3.

Factoring can apply to unkown variables as well. For example, can be written as a \* a \* b \* c. We can factor as (a)(abc) or (ac)(ab), along with several other possibilities. This is the same concept that was demonstrated in the previous example. To find other factors of a \* a \* b \* c we are simply multiplying two or more of the variables together.

Which of the following is not a factorization of ?

a)

b) (ab)(bc)(ac)

c)

d) (b)(ac)(a)

The answer here is b) because multiplying the first and last terms (ab)(ac) gives us the solution , however, the term (bc) still remains, meaning that this cannot belong as one of the composites.

Matching Factorizations:

Knowing how to factor means that we can decide among several different ways of writing an expression, and sometimes this enables us to factor two quantities so they share a factor.

Which value of x is a solution to ?

a) x = 1

b) x = 2

c) x = 4

d) x = 15

The answer is x = 2 since 2 squared is 4 and 5 x 4 is 20. A different way of thinking about this would be writing the two expressions and 20 as factorizations with 5 being one of the factors. In other words, the equation could be written as:

5 \* x \* x = 5 \* 4

5 \* x \* x = 5 \* 2 \* 2

If the right side of the equation has been broken into it’s prime factors, then x must match the expression on the right for the two expressions to be equal ie. x must be 2.

Which number has more ways to be written as a product of two factors, 20 or 24? This can be solved by taking each number starting at 1 and going up sequentially and comparing how many factors of two numbers 20 and 24 have.

|  |  |  |
| --- | --- | --- |
|  | Factors of 20 | Factors of 24 |
| 1 | 1 x 20 | 1 x 24 |
| 2 | 2 x 10 | 2 x 12 |
| 3 | NA | NA |
| 4 | 4 x 5 | 4 x 6 |
| 5 | 5 x 4 | NA |
| 6 | NA | 6 x 4 |
| 7 | NA | NA |
| 8 | NA | 8 x 3 |
| 9 | NA | NA |
| 10 | 10 x 2 | NA |
| 11 | NA | NA |
| 12 | NA | 12 x 2 |

Besides their trivial factorizations 1 x 20 and 1 x 24, which have the factor 1 in common, in how many other ways can we write 20 and 24 as products of two factors that have one factor in common? The answer is 2. If we look at the chart above, we can see that 2 x 10 and 2 x 12 share the common factor 2, and 4 x 5 and 4 x 6 share the common factor, 4. This might seem irrelevant, but finding common factors between two expressions will be very useful for solving algebraic equations.

Consider two numbers: 66 and 24. One of the ways we can factor these two numbers and ensure that they share a common factor is 66 = 6 x 11 and 24 = 4 x 6. Pretend that these factors are the dimension of a rectangle. Both rectangles share the common side length 6. Can we could fit these rectangles together to make a larger rectangle? The answer is yes, we can create a rectangle that is 6 x 15. This is because the side length of 6 is shared among both rectangles, so we can add 11 and 4 to make 15. Let’s use this fact to factor an expression that contains variables.

We have a rectangle with dimensions 4 \* z. We also have 4 more rectangles with side lengths listed below:

A. 8x \*

B. 4xz \* 2z

C. 4 \*

D. 8xz \* z

How many of the 4 rectangles listed can adjoin with our original rectangle 4 \* z to make a larger rectangle? The answer is 2 – options C and D are able to make a larger rectangle with 4 \* z. This is because C shares the common side length/factor of 4, and D shares the common side length/factor z.

We can describe this property of sharing common side lengths/factors mathematically as having a rectangle a \* x and another rectangle a \* y which join to form a new rectangle, a \* (x + y). Note that the area of the resulting rectangle is the same as the sum of the areas of each individual rectangle. That is to say: a \* x + a \* y = a \* (x + y).

The Distributive Property:

Previously, we focused on how we could use factorization to represent a quantity. However, sometimes we need to work with expressions that involve more than just multiplication. For instance, we may see an equation containing multiplication and addition together. In this section, we’ll focus on equations that contain both multiplication and addition.

Let us say that we have an equation: x = 3 \* 2 + 2 \* 5. The answer for x in this case would be 16. Now let us say that x = (3 + 5) \* 2. The answer for x in this case once again is 16. The way that I got the second equation was by taking the common factor of both multiplications in the first equation (3 \* 2 and 2 \* 5) and applied the same rectangle rule that we seen before. Namely that a \* x + a \* y = a \* (x + y).

Continuing with this property (which is called the distributive property by the way), let’s look at another example. Which expression is equivalent to 6(2 + 5)?

a) (6)(2) + (6)(5)

b) (6)(2) + (2)(5)

c) (6)(6) + (2)(5)

d) (6)(7) + (6)(5)

The answer here is a). If you are unsure why, it’s probably because you’re not used to applying the distributive property. According to the logic that we’ve discussed before, we can *distribute* the 6 to each term in the brackets. So we distribute 6 to 2 and 5 which means that we multiply 2 and 5 by 6. In this case, option a) is what it looks like to distribute 6(2 + 5).

Looking at this from a different perspective, which expression is not equivalent to 6 x 13?

a) (6)(10) + (6)(3)

b) (6)(7) + (6)(8)

c) (6)(4) + (6)(9)

d) (6)(11) + (6)(2)

The answer is b). This is actually pretty easy question because if you noticed, each option has the common factor of 6 on both sides of the addition, so all that you need to do is apply the distributive property by adding the two factors that aren’t 6 together. In option b) 7 + 8 do not equal the side length of 13, so b) cannot be a factor of 6 x 13.

This property is also a useful tool for large multiplications. In America, students typically memorize the times table from 1 x 1 to 12 x 12. If we need to quickly calculate a multiplication that is greater than 12 x 12, we can split the multiplication into two smaller multiplications and add their areas. For example, take 6 x 13: This can be thought of as (6 x 10 + 6 x 3) or as (6 x 4 + 6 x 9) or as (6 x 11 + 6 x 2), etc.

Ellis memorizes the multiplication table 1 x 1 through 12 x 12. He also knows how to multiply by powers of 10 (that is, by 10, 100, 1000, ...). All of the expressions listed below have the same value as 8 x 612. With which expression would Ellis be able to calculate 8 x 612 without a calculator?

a) (8)(700 – 88)

b) (8)(611 + 1)

c) (8)(600 + 12)

d) (1 + 7)(612)

The answer is c) because Ellis can apply the distributive property to 600 and 12 easily. He knows powers of 10 which means that he knows he can do 8 \* 6 which is 48 \* 100 is 4800 and then do 8 \* 12 which is 96 and add them to get 4896. Here’s the mathematical way of expressing this:

8(600 + 12)

= 8 \* 600 + 8 \* 12

= 4800 + 96

= 4896

The Distributive Property (Part 2):

What we’ve noticed about sums and products in the last few problems applies to expressions with variables too. For example, which expression is 3(6 + x + y) equal to?

a) 18 + x + y

b) 9 + 3x + 3y

c) 18 + 3x + 3y

d) 18 + 6x + 6y

The answer is c). The distributive property multiplies each term inside the brackets by 3 so 3 \* 6 is 18, 3 \* x is 3x and 3 \* y is 3y.

We can use the distributive property as a verb. When we rewrite a(b + c) as ab + bc, we say that we’ve distributed the a. When we rewrite 3(6 + x + y) as 18 + 3x + 3y, we say we’ve distributed the 3.

Which expression gives the area of a rectangle with height 3x and width (2 + y)?

a) 6x + 3xy

b) 6xy

c) 6x + y

d) 2 + 3xy

The answer is a). We distribute the 3x to the 2: (3x \* 2 = 6x) and then to the y: (3x \* y = 3xy). Therefore, we get 6x + 3xy

The distributive property works for subtraction too. Which expression is equivelent to (-4p)(2r – 3p)?

a) 8pr -

b) -8pr -

c) -8pr +

d) 8pr +

The answer is c). -4p is distributed to 2r: and is then distributed to -3p: .

We can also use the distributive property when two or more sums are multiplied together. Consider a square with side length a + b. The area of the square is or (a + b)(a + b). Which other expression below represents the area of the square? That is, what expression is equivalent to ?

a)

b)

c) 2a + ab + 2b

d)

The answer is d). This might be more complicated to understand but the same logic of the distributive property is used here. We distribute the side length (a + b) to the a of the second side length. and then we distribute (a + b) to the b of the second side length. . Perhaps showing it step by step mathematically will be helpful:

= (a + b)(a + b)

= (a + b)a + (a + b)b

=

Common Factors:

The distributive property gives us a way to rewrite expressions. We are able to turn certain products into sums. In turn, we can break a tough arithmetic calculation into manageable pieces, or make an equation easier to solve. In this section, instead of turning products into sums, we’ll be turning sums into products in order to use factoring to simplify problems.

We will be revisiting factors in order to help us with this section. When we were doing factoring, we thought about it in terms of rectangles made up of blocks. Let’s try it once again. When we arrange 30 and 40 unit squares into rectangles, how many side lengths between 2 and 10 (inclusive) could the rectangles share?

|  |  |  |
| --- | --- | --- |
|  | Factors of 30 | Factors of 40 |
| 2 | 2 x 15 | 2 x 20 |
| 3 | 3 x 10 | NA |
| 4 | NA | 4 x 10 |
| 5 | 5 x 6 | 5 x 8 |
| 6 | 6 x 5 | NA |
| 7 | NA | NA |
| 8 | NA | 5 x 8 |
| 9 | NA | NA |
| 10 | 10 x 3 | 10 x 4 |

We can see that the two rectangles share the common side lengths/factors 2, 5 and 10. For the factors of 2, the rectangle of area 30 has 2 x 15, and the rectangle of area 40 has 2 x 20. For factors of 5, the rectangle of area 30 has 5 x 6, and the rectangle of area 40 has 5 x 8. Finally, for the factors of 10, the rectangle of area 30 has 10 x 3, and the rectangle of area 40 has 10 x 4.

In addition to the values we seen in this problem, 1 could also evenly divide both 30 and 40 (the trivial factorization), so we can then conclude that 1, 2, 5, and 10 are all common factors of 30 and 40.

Which of these numbers is not a factor of both 16 and 36?

a) 2

b) 4

c) 8

d) all three numbers are factors of both 16 and 36

The answer is c) because 8 is not a factor of 36.

How many common factors do 18, 24, and 25 have?

a) 0

b) 1

c) 2

d) 3 or more

The answer is b). 25 only has 1 prime factorization other than the trivial factorization being 5 x 5. Since 18 and 24 do not have 5 as a common factor, the only common factor that they share is 1.

Which pair of the numbers listed have the most common factors?

a) 20 and 32

b) 20 and 42

c) 32 and 42

d) All pairs have the same number of common factors

The easiest way to solve this problem is by breaking down each number into is prime factorization and then multiplying factors together to find alternative factorizations. Once we have all possible factors for 20, 32 and 42, we can compare them to see which ones share the most common factors.

The prime factorization of 20 = 2 x 2 x 5, so its factors are 1, 2, 4, 5, 10, 20

The prime factorization of 32 = 2 x 2 x 2 x 2 x 2, so its factors are 1, 2, 4, 8, 16, 32

The prime factorization of 42 = 2 x 3 x 7, so its factors are 1, 2, 3, 6, 7, 14, 21, 42

20 and 32 have 1, 2, and 4 as common factors

20 and 42 as well as 32 and 42 only have 1 and 2 as common factors

Common factors are the key to factoring expressions that involve sums. In the rest of this section, we’ll focus on factoring sums, first with numerical expressions, then with variables.

Factoring Sums:

Let’s ask ourselves why we even care about factoring expressions that involve sums. Let’s answer that question with a question. Do you think that you could solve the equation without factoring? Your answer is likely no, but if we are able to factor this equation, we can find that it is, in fact, equivalent to the equation x(2x – 8)(x - 5) = 0 which is much easier to solve. If you don’t believe me, try answering the following: Which value of x is not a solution to x(2x – 8)(x – 5) = 0?

a) x = 0

b) x = 2

c) x = 4

d) x = 5

The trick here is to recall the zero-product property. If ab = 0 then either a must = 0 or b must = 0. For the term (x – 5), it is as simple as finding a value for x such that the expression evaluates to 0. In this case 5 – 5 = 0. For the first term, x(2x – 8), we must find a value that, when multiplied by 2 gives us 8 to cancel with the 8 being subtracted. 2 \* 4 = 8 so x = 4 is also a solution to the problem. Additionally, a) is a solution as well since that would make everything 0. The outlier then is b), since if x = 2, neither term becomes 0 and the equation would not be true.

As a refresher question, two numbers having a common factor is like two rectangles having a common side length. Suppose we have two rectangles with dimensions 5 x 3 and 5 x 4. Since they have a dimension in common (5), we can combine them into one larger rectangle. The area of this new rectangle is the sum of the individual areas of both rectangles: 5 \* 3 + 5 \* 4. Which expression is equivalent to 5 \* 3 + 5 \* 4?

a) (5 + 3)(3 + 4)

b) 5(3 + 4)

c) 5 \* 3 + 4

d) 3 + 5 \* 4

The answer is b) since we do the opposite of the distributive property, and factor out the common factor 5 leaving us with 5(3 + 4). When we rewrite 5 \* 3 + 5 \* 4 as 5(3 + 4), we say that we’ve factored out the 5 and that 5(3 + 4) is factored.

In order to be able to factor an expression, all the terms need to share a factor other than 1 (you could technically factor out a 1, but it doesn’t change the expression at all since we can have infinite factors of 1).

Is it possible to factor 5 \* 7 + 3 \* 12? The answer is no, because neither side share a common factor.

How could we factor 4 \* 3 + 2 \* 7?

a) 2(3 + 7)

b) 2(6 + 7)

c) (4 + 2)(3 + 7)

d) 2(3 + 14)

The answer is b). The way that this can be obtained is by expanding the equation so that the 4 is written in terms of prime factors i.e. 2 \* 2 \* 3 + 2 \* 7. Then we can factor out an equal number of common factors from both sides: 2(2 \* 3 + 7). Be careful not to factor out both 2s on the left hand side of the plus sign – we can only factor out an equal number of common factors from both sides. Multiplying 2 \* 3 gives us 6 so the factored expression is 2(6 + 7). Here it is in steps:

4 \* 3 + 2 \* 7

= 2 \* 2 \* 3 + 2 \* 7

= 2(2 \* 3 + 7)

= 2(6 + 7)

Which value cannot be factored out of 54 + 60 + 72?

a) 2

b) 3

c) 6

d) 8

The answer is d) since both 54 and 60 are not divisible by 8. This means that all three numbers do not share the common factor 8.

Rewrite 54 + 60 + 72 with a 6 factored out. This is quite simple; we just divide each number by 6 and put the 6 outside of the addition.

54 + 60 + 72

= 6(9 + 10 + 12)

Factoring with Variables:

In order to factor expressions with variables, we need to be able to recognize common factors in terms with variables. One way we’ve found factors is to start with a number’s prime factorization. For example, in order to factor , we write each component in terms of its prime factorization.

=

=

Which of these is a common factor of 4ac and ?

a) 2a

b) 3a

c) 4b

d) 2c

The answer is a) because 2a is a possible factor of both expressions. To show this, we can write 4ac as (2a)(2c) and as .

Without counting 1 (which all expressions have as a common factor), how many common factors do and 5xy share?

a) 0

b) 1

c) 2

d) 3

The answer is d). To show this, we start with the prime factorization of both expressions.

5xy = 5 \* x \* y

For the first expression, we can multiply 2, 3, or all factors to get new factors.

- multiplying 2 factors we can get: 2x, 2y, , xy

- multiplying 3 factors we can get: , 2xy,

- multiplying all factors we get:

For the second expression, we can multiply 2 or all factors.

- multiplying 2 factors: 5x, 5y, xy

- multiplying all factors: 5xy

Therefore, we find a total of 3 common factors: x, y, and xy

Factoring Sums with Variables:

To summarize the process of factoring an expression, you can follow the following steps. Depending on the problem, some steps may be skipped:

1. Write a “prime factorization” of each term in the expression.

2. Identify the common factors of all terms, and pick one.

3. Rewrite each term in the expression as a product with the chosen common factor.

4. Use the distributive property in reverse to factor out the chosen common factor.

Which factored expression is equivalent to 6a + 2b?

a) 2(3a + b)

b) 4(3a + b)

c) 3a(2 + b)

d) 2(3 + 1)

The answer is a). 6a can be reduced to its prime factorization, 2 \* 3 \* a. Then we can factor out a 2 from the left and right side of the plus sign.

Which factored expression is not equivelant to ?

a) x(15x + 21)

b) 3x(5x + 7)

c)

d)

In this case, the answer is d). (15 + 21) are in brackets implying that they are one term, but we cannot factor out (15 + 21) from both sides of the plus sign. If that’s confusing, consider performing the addition within the brackets and you will notice that the sum is 36, which is not a common factor of either side of the addition.

Note that an expression is considered to be factored completely when no common factors remain between the added terms. Which of these three equivalent expressions is factored completely?

a) x(15x + 21)

b) 3x(5x + 7)

c)

d) none are factored completely

The answer here is b). We can factor a 3 out of a), and an x out of c), thus eliminating those two options.

Let’s get a bit more advanced: How can we rewrite the expression so that it is factored out completely?

a)

b)

c)

d)

The answer is b). This problem appears daunting, but it’s not actually that bad. We can immediately recognize that can be factored out of each term. If this wasn’t obvious to you, recall that is just b \* b \* b. If we expand the expression, it looks something like:

2 \* a \* b \* b \* b + 5 \* b \* b \* b + 7 \* b \* b \* b \* b

Remember that we can only factor out an equal number of variables. In this case, each term contains b \* b \* b, so we can factor that out, leaving the last b out in the final term. Perhaps an easier way to think about it would be to rewrite the expression as . Since 2, 5, and 7 are primes, and a is not a common factor, the expression is factored completely after factoring .

Which one of these equations does not represent a possible solution to the equation ?

a)

b)

c)

d)

The answer here is b). This is apparent when you look at the coefficients of each term (the number in front of each term). If the first coefficient is a 2, then 3 must have been factored out from the 6 in our original equation, however, if we factor a 3 from the second term with 18 as its coefficient, we’d get 6, but instead we get 3. Same for the last term: 24 / 3 does not equal 4. This is enough to inform us that the factoring was done incorrectly, because not every term factored out a common factor. Note also that this equation was a case of zero factorization since the expression on the left is equal to 0. When solving zero factorization equations, factoring completely is always preferred since it makes solving the solution easier the smaller the numbers are that we’re working with.

Factoring Numbers:

In this section, we’ll look at some more practical applications of factoring using word questions as opposed to purely mathematical equations.

Wayne bought 40 lollipops to give out to children on Halloween. However, he has a party to get to, so he plans to give out all 40 lollipops to the first few groups of children that knock on his door. That night, the sizes of the groups of trick-or-treaters ranged between 2 children and 10. If Wayne insists on every child getting the same number of lollipops, how many different sizes of groups of children could he accomodate?

a) 2

b) 3

c) 4

d) 5

e) 6

The answer is d). The way to solve this is to find the prime factorization of 40. The prime factorization of 40 is 2 x 2 x 2 x 5. In the case of this question, we can ignore the trivial factorization 1 x 40 since there are no groups of kids that are a size of 1. Using the rule where we multiply factors to find other factors, we can find that the possible factors between 2 and 10 (inclusive) are 2, 4, 5, 8, and 10. This means that 4 different sized groups can equally be accomodated i.e. no remainder in candy.

Annamarie is planning her garden. She has 45 tulips, 81 dahlias, and 63 cosmos to plant, and she insists on using every single flower. She has two goals:

- For every row in her garden to have just one type of flower ie. all tulips, all dahlias, or all cosmos

and

- For every row to have an equal number of flowers

What is the largest number of flowers Annamarie could have in each row?

a) 3

b) 7

c) 9

d) 13

The answer is c). This problem takes some critical thinking, so don’t worry if you didn’t get it. The solution is to find the prime factorizations of each type of flower, and then find the largest common factor among each kind.

- The prime factorization of 45 is 3 x 3 x 5

- The prime factorization of 81 is 3 x 3 x 3 x 3

- The prime factorization of 63 is 3 x 3 x 7

Using our multiplication rule, we can find all the possible factors of each kind of flower:

3 x 3 x 5 --> 3, 5, 9, 15, 45

3 x 3 x 3 x 3 --> 3, 9, 27, 81

3 x 3 x 7 --> 3, 7, 9, 21, 63

Here we can see that each type of flower shares the factors 3 and 9 in common. Since Annamarie wants the largest number of flowers in each row, 9 is the answer.

Using the Distributive Property:

We learned that the distributive property and factorization are closely related – they’re the same equation

a(b + c) = ab + ac

and

ab + ac = a(b + c)

just seen from different perspectives.

The distributive property turns a product into a sum, and it also happens to rid the expression of parentheses, as we wish to do in the next problem.

How can we write the expression 4(x(3(x(2(x + 1))))) without parentheses?

a)

b)

c)

d)

The answer is b). Here is the step by step on how to solve it:

4(x(3(x(2(x + 1)))))

= 4x(3(x(2(x + 1))))

= 12x(x(2(x + 1)))

=

= --> apply the distributive property!

=

Which of these values is the solution to 4(x + 3) = 12?

a) x = -3

b) x = -1

c) x = 0

d) x = 1

e) x = 3

The answer is c). Whereas we could probably guess and check for this question, the algebraic approach is to apply the distributive property:

4(x + 3) = 12

4x + 12 = 12

Since 12 is already on the right side of the plus sign, we want to eliminate 4x. The easiest way to do this is to make x = 0.

Which of these is not a solution to 4x(5 – x) – 12(5 – x) + 100 = 100?

a) x = 1

b) x = 3

c) x = 5

The answer here is a). The first step that will help us, is to remove unecesarry terms. In an equation, we are allowed to add, subtract, multiply, or divide any term on one side of the equals sign, so long as we do the same operation on the other side of the equals sign. In this case, 100 is on both sides of the equals sign, so we can subtract 100 from both sides, effectively removing 100 from the equation. This leaves us with only 4x(5 – x) – 12(5 – x) = 0 to deal with. Option c) is relatively simple – substituting 5 for x in both terms will apply the zero product property.

4 \* 5 \* (5 – 5) – 12 \* (5 – 5) = 0

20 \* 0 – 12 \* 0 = 0

0 – 0 = 0

0 = 0

Option b) is not as obvious. The easiest way to check in my opinion, is to substitute x with 3 and then apply the distributive property.

4 \* 3 \* (5 – 3) – 12(5 – 3) = 0

12 \* 2 – 12 \* 2 = 0

24 – 24 = 0

0 = 0

More Distribution:

To conclude this section, we’ll stretch our application of the distributive rule even further. First, we’ll take a quick detour back into arithmetic. Specifically, we’ll use the distributive property to make multiplications easier.

Which expression gives the area of a rectangle with side lengths 35 and 42?

a) (35)(40 + 2)

b) (30 + 5)(42)

c) (30 + 5)(40 + 2)

d) All of the above

The answer is d). This should be fairly straightforward hopefully; each expression represents the area of a rectangle that is 35 x 42. Adding the terms within the brackets reveals this.

Which expression is equivalent to (x + 3)(x + 4)?

a)

b)

c)

d)

The answer is c). Recall that (a + b)(a + c) is the same as This is because we apply the distributive property on each term in the second set of brackets. This looks like the following:

(x + 3)(x + 4)

= (x + 3)x + (x + 3)4

=

In math speak, the previous expression was an example of multiplying two binomials. A binomial is an expression with 2 terms. (x + 3) is a binomial and (x + 4) is a binomial, because they both have 2 terms. The general equation for multiplying two binomials is . This is acquired by simply applying the distributive property (so no need to memorize it).

Which expression is equivalent to (2x + y)(z + 7 + x)?

a)

b)

c)

d)

This one is more complex, but follows the same logic of using the distributive property. In order to solve this, we apply (2x + y) to each term in the second binomial. This goes as follows:

(2x + y)(z + 7 + x)

= (2x + y)z + (2x + y)7 + (2x + y)x

=

Therefore, the answer is b) (note that the order of additions doesn’t matter, only that both equations share all of the same terms).

Evaluating Expressions:

We’ve seen how using variables allows us to express quantities more compactly. Using tools from the previous sections, and adding some new ones, we’ll find ways to make expressions easier to understand and simpler to use.

Equivalent Expressions:

If two expressions always have the same value, they’re equivalent, and we can use them interchangeably. For example, if 3a + 5 always has the same value as 8a, we could just write 8a instead, which is more compact.

The truth is though, that 3a + 5 is not always equal to 8a. For instance, if a were 2, 3a + 5 would equal 11, but 8a would equal 16. In other cases, they may be equal, but because they are not always equal, we cannot replace 3a + 5 with 8a. An example of two expressions that are always equal is a + a + 7 + b + a + b + 3 and 3a + 2b + 10. But why are those expressions always equal, whereas 3a + 5 and 8a are not always equal? We can show that a + a + 7 + b + a + b + 3 and 3a + 2b + 10 are equal by simplifying the longer expression. By rearanging the terms we can come up with the following:

a + a + 7 + b + a + b + 3

= a + a + a + b + b + 7 + 3

= 3a + 2b + 10

We say that the expression has been simplified in this case. Later, we’ll get a complete definitions of a simplified expression. For now, let’s say an expression is simplified if it represents the same quantity as the original expression, but uses fewer terms.

Calculating by Grouping:

Grouping objects together can help us with calculations, whether it’s counting or finding areas. This is one method of simplification.

Let’s say that you have a rectangle which is made up of smaller rectangles (not all equally proportioned). The height of the rectangle can be modelled as (a + b) and the width of the rectangle can be modelled as (b + b + a). What is an easier way to model the area of the rectangle?

a)

b)

c)

d)

The answer is b). Unfortunately, this question is much easier to solve visually, but since I don’t have the luxury of spending all my time to create visuals, we will take the algebraic approach. The two side lengths must be multiplied, so the distributive property is applied as we’ve seen before. The solution would look like the following:

(a + b)(b + b + a)

= ab + ab + aa + bb + bb + ab

= We can now group like terms. For example ab + ab + ab = 3ab

=

When a term in an expression is added multiple times, we can simplify the expression by grouping similar terms together. For example:

=

=

Evaluating Simplified Expressions:

Now that we have an idea of how to simplify, we can put what we’ve learned to use. Grouping repeating terms together can be helpful when we want to evaluate an expression. The fewer times we need to plug a value in, the fewer opportunities there are to make an error.

You have three kinds of chocolate bars: coffee crisp, kitkat, and aero. You bought 3 kitkat, 3 aero and 1 coffee crisp. The kitkats have dimension of 3 \* y, aero have a dimension of 2 \* x, and the coffee crisp has a dimension of 2 \* 4. If x = 5 and y = 4, what is the total area of all the chocolate bars?

a) 52

b) 60

c) 72

d) 74

The answer is d). We can group each type of chocolate bar and then add up their areas.

(3)(3 \* y) + (3)(2 \* x) + (1)(2 \* 4)

= (3)(3 \* 4) + (3)(2 \* 5) + (2 \* 4)

= (3)(12) + (3)(10) + 8

= 36 + 30 + 8

= 74

What is the value of 1x + 2x + 3x + 4x + 5x when x = 13?

a) 169

b) 182

c) 195

d) 208

Think of each term in the equation as being a rectangle where the dimension is a \* x. Since all rectangles share the common factor x, each term is considered to be a like-term. In this case, we just add all of the coefficients (the number before the x) and simplify to have one large rectangle that will be of length (a + b + c + d + e) \* x. Then we substitute x with 13 and solve.

1x + 2x + 3x + 4x + 5x

= 3x + 3x + 4x + 5x

= 6x + 4x + 5x

= 10x + 5x

= 15x

= 15 \* 13

= 195

Therefore, the answer is c).

Like Terms:

In the previous section, we seen how simplified expression are easier to evaluate. But how can we know which terms we can group together? In this section, we’ll see exactly how to determine which terms can be groups and which cannot.

Previously, we simplified the expression 1x + 2x + 3x + 4x + 5x = 15x. We did so by thinking of each term as one or more rectangles with area 1x = x. This indicates that terms don’t have to be repeated *exactly*to be grouped, but something does need to be repeated. Perhaps this can help you solve the next problem.

What is a solution to the equation 3 + 8x – 8 + 3x + 5 = 22?

a) x = -1

b) x = 0

c) x = 1

d) x = 2

The answer is d). The easiest way to solve this is to group the like-terms 3x and 8x to form 11x. The terms that do not have xs in them can also be grouped i.e. 3, -8 and 5. Since 3 + 5 – 8 = 0, we’re left with 11x = 22. Knowing our multiplications, it is quickly apparent that x must be 2 if we want 11x to be = 22.

3 + 8x – 8 + 3x + 5 = 22

11x + 3 - 8 + 5 = 22

11x – 5 + 5 = 22

11x = 22

11 \* 2 = 22

x = 2

In general, rectangles with the same area, or a numerical multiple of the same area are like terms, and we can group them to calculate their total area. Consequently, terms that are repeated, or numerical multiples of terms that are repeated, can be combined to simplify and expression.

Simplifying Expressions:

We’ve already seen a few simplified expressions; now we’re ready for the definition of what a simplified expression is: “An expression with multiple summed terms is simplified if all the terms that can be combined have been combined”.

The total area of 4 rectangles is modelled as (2x)(x) + (2)(x) + (4)(x) + (x)(x) which is equal to . This is not completely simplified however. Which expression represents the same total area, simplified?

a)

b)

c)

d)

The answer is c). We can combine the terms, and the x terms.

=

=

In the previous problem, and are both multiples of , so they’re like-terms and we can combine them into . Similarily, 2x and 4x are both multiples of x, so we can combine them into 6x.

Which expression represents the same value as in simplified form?

a)

b)

c)

d)

The answer is d). and are both multiples of so they combine to make . Similarily, 2xy and xy are both multiples of xy, so they combine to make 3xy, and finaly, 5y and y are both multiples of y, so they combine to make 6y. Another way to think about this, is that each term is the area of a rectangle, and we are factoring out the unit area from each term. First, we group the terms that represent the same area, or a numerical multiple of the same area together. So becomes. Then, within each set of parentheses, we factor out the unit area from each term:

. Finally, adding inside all of the parentheses, we get or just .

We simplify an expression by combining like terms, or grouping together and factoring the terms that have all of the same variables with the same powers.

Which term is like 8x?

a) 8y

b) 8xy

c) 2x

d)

The answer is c) since both terms share the common factor, x.

Solve by Simplifying:

As we saw at the beginning of the lesson, equations with simplified expressions are easier to solve. We’ll conclude this lessson with two equations that wouldn’t be possible to solve without simplifying.

Which values of b and c are a solution to this equation?

a) b = 0, and c = 1

b) b = 1, and c = 0

c) All values of b and c are a solution

d) There is no solution

The answer is actually d). As was mentioned, you’d have to simplify to determine this fact. The process can be shown below:

As we will eventually learn, in algebra, any arithmetic operation that we perform on one side of the equation is valid so long as the exact same operation can be applied on the other side of the equation. In the second last step, I subtract from both sides of the equality operator so that we are left with 0 = 1 which is false, meaning that no possible solution for b or c exists that will avoid this equation from not having a solution. You could have still figured this out without knowing the subtraction trick by using intuition. Once we get to the second last step, , any value that we substitute for b will not make a difference because the +1 always makes it so that the right side will be 1 more than the left.

How many different solutions does this equation have?

a) 0

b) 1

c) 2

d) 3

This question has a bit of a tricky step, so don’t feel bad if you couldn’t figure out how to solve it. The first step that we should always take if possible is to simplify:

--> Factored out

By factoring out after simplifying, we can apply the zero-product rule. If x is 0, then becomes 0 which multiplies with (x-3). Their product will be 0, so we can conclude that 0 is one possible solution. Additionally, if x is 3, then the expression within the brackets (x – 3) becomes 0, which when multiplied with will also give us 0. Therefore, the answer to this question is c), because there are two possible values for x where we get 0.

Comparing Expressions:

Simplifying expressions is often seen as a rote task, assigned solely to torture students. However, we’ve seen some very practical reasons for simplifying: It makes expressions easier to evaluate and equations easier to solve. In this section, we’ll compare expressions (which will be easier when we simplify)!

Simplified Expressions:

We originally said that an expression is simplified if it represents the same quantity using fewer terms. Now we’re ready to say it more mathematically: An expression is simplified when all possible mathematical operations within it have been performed. For example, in the last section, we encountered the expression . It is not simplified because we can still perform some operations. Reordered, the expression is In this form, we see that we can perform three operations of adding together the like terms. When we do so, we get . At this point, there are two additions left in the expression, but we can’t complete them because none of the terms are like terms, and we do not know the values of the variables.

Which expression is the same as , simplified?

a)

b)

c)

d)

The answer is a). Even though and might appear as though they are like terms, because they have different exponents, they cannot be combined. Thus the equation can be solved as follows:

=

=

Which of these expressions is not simplified?

a)

b)

c)

d)

The answer is a) because can be simplified to .

Comparing Expressions:

One way to compare expressions is to examine them separately. It’s easier to see that expressions are equivalent or find what makes them not equivalent, when they’re simplified.

Which of these expressions has a larger value?

A.

B.

a) A

b) B

c) It depends on b and c

d) They’re equivalent

The answer here is b). If you simplify both expressions, you realize the expression A simplifies to and B simplifies to meaning that no matter what value of b or c is input, expression B will always be larger by 1.

Which expression is larger when x = 1000?

A.

B.

a) A.

b) B.

c) They have the same value.

The answer here is b). Expression A simplifies to and expression B simplifies to. If we substitute 1000 for A, we get 4,000,000, and for B, we get 4,000,000,000.

Sometimes, when comparing expressions, we don’t even have to evaluate them to determine their relative sizes. In the last problem, we saw that expression A was equivalent to and that expression B was equivalent to . Instead of finding their exact values with x = 1000, we could just look at the expressions with 1000 plugged in, before evaluating:

-would look like 4 x 1000 x 1000.

-would look like 4 x 1000 x 1000 x 1000

Without calculating, we can see that when x = 1000, the value of is 1000 times greater than . This is because the value of is x times the value of .

Finding Differences:

Another way to compare expressions is to find their difference; that is, to subtract one expression from the other.

Rewrite the expressionso that it has no parentheses and is simplified.

a) 0

b) 1

c)

d)

The answer for this question is a). In it’s factored form, this equation cannot be simplified any further. However, by applying the distributive property, we can expand this equation, and then simplify it.

=

= 0

When n is a positive integer, what is ?

a) 1

b) even

c) odd

d) it depends

The answer is c). To find the solution, we expand both terms like we did previously. Let’s just focus on the first term, :

= (n + 1)(n + 1)

= (n + 1)(n) + (n + 1)(1)

=

Now, the original expression is which simplifies to 2n + 1. Because we always add 1 at the end, which is odd, the expression will always evaluate to be odd granted that n is a positive number.

When n and m are integers, which expression is not a multiple of 10?

a) (2m + 8n)(5n + 10m)

b) 5(1 + 2(2m + n))

c) 2n(5mn + 1)-2(10 + n)

b) They’re all multiples of 10

The answer is b). I won’t go over how to solve each option, but the idea is that you need to use the distributive property, and then simplify for each of them. Options a) and b) can both have a 10 factored out which makes them multiples of 10, however, b) can only have 15 factored out, which makes it a multiple of 15.

Problem Solving:

In this section, we’ll put everything we’ve learned to use in order to solve a variety of problems. To start, we’ll think about simplifying and use it to find equivalent expressions.

When the expression is simplified, how many terms does it have?

a) 3

b) 4

c) 5

d) 6

The answer is b). The terms and cancel which removes two terms. Then, and make , and make and cannot combine with anything since there are no like terms. At the end, we end up with the formula which has 4 terms.

How many of these expressions are equivalent to 4n – 1?

A. 5(1 + 2n) – 6(1 + n)

B. 3n – 7 - (-n – 6)

C. 3(2(n-2))+2(n + 5)

D. 1/2(6n + 4) – (1 – 1)

a) 0

b) 1

c) 2

d) 3

e) 4

The answer is c). Equations A and B simplify to 4n-1. Here are the steps for both:

A. 5(1 + 2n) – 6(1 + n)

= 5 + 10n – 6 – 6n

= 4n – 1

B. 3n – 7 - (-n – 6)

= 3n – 7+ n + 6

= 4n – 1

In case you wanted to know, equation C Simplifies to 8n – 2, and D simplifies to 4n + 1.

Which expression is not equivalent to the others?

a) (x – 7)(x + 7)

b)

c)

d)

The answer is b). The quickest way to figure out this problem would be to realize that -7x and 7x cancel in option c), which just leaves which is the same as option d) and by logical deduction, we can conclude that b) is the outlier since cannot equal. If we wanted to double check our answer and solve a), however, we can use the distributive property like so:

(x – 7)(x + 7)

=

=

Also notice that whenever we have the pattern (x – a)(x + a), the negative sign always overpowers the positive sign. We can conclude that .

Distributing and Simplifying:

Which expression is equivalent to (4 + x)(2x – 3)?

a)

b)

c)

d)

The solution here is just to use the distributive property on the expression with the two binomials like so:

(4 + x)(2x – 3)

=

=

Therefore, the answer is a).

In this question, we will need to find the difference between the square of a sum, and the sum of two squares. What is the simplified form of ?

a) 0

b)

c)

d)

The answer is c). This can be solved from a practical point of view, or an algebraic one. The practical way is much more difficult, but perhaps more rewarding than the algebraic one. Let’s start with the practical standpoint. The first term represents a square of a sum. Imagine a rectangle that has a dimension of a x b.

a b

Now imagine that we add the two side lengths, and then square them:

a b

b

a

This essentially creates a big square that has side the dimension or which is our first binomial. Now if we picture what the second binomial looks like, we will see that it’s actually the bottom left and top right squares of our first binomial:

b

a b

a

Because we are subtracting the second binomial, those squares cancel, and we are left with:

a

b

b

a

Which is 2ab.

The algebraic way to solve this is probably simpler:

=

=

=

=

= 2ab

This is a commonly used formula, known as the square of a binomial. Can you find the cube of a binomial? Which expression is equivalent to ?

a)

b)

c)

d)

The answer here is b). I am not artistic, nor patient enough to draw the 3D cube for this one like I did for the last question, but we can solve this algebraically:

=

=

=

=

=

When we rewrite an expression like as a sum, it’s called a binomial expansion. For example, we just did the binomial expansions for N = 2 and N = 3:

-

and

-

There are some tricks to make this even faster for larger exponents, but we’ll cover those in another course.

Solving Equations:

We’ve breifly covered certain rules that allow us to go backwards when solving for a particular variable. Given that we have an equation with one unknown, we can always solve for the unknown by working our way backwards.

What value of x solves the equation 2x + 5 = 13?

a) 2

b) 3

c) 4

d) 5

The answer is c). To solve the equation 2x + 5 = 13, we begin with the right-hand side of the equation, 13. We start here because the right side of the equation is simplified, but most importantly, because there are no unknowns. We know that the left-hand side must be equivalent to 13, so we work our way backwards. We ask ourselves, what number, multiplied by 2, and then added to 5 gives us 13. The number must be at least 5 less than 13 since adding 5 to the number gives us 13. So we can safely subtract 5 from both sides of the equation, leaving us with 8 on the right-hand side. What multiplied by 2 gives us 8? Well this is as simple as performing the reverse operation of multiplication by 2 on the right-hand side of our equation. 8 / 2 is 4, which is our answer. Here are the steps:

2x + 5 = 13

2x = 8 --> Subtract 5 from both sides

x = 8/2 --> Divide by 2 on both sides

x = 4

Keeping Equations Balanced:

When we’re working backwards to solve an equation, we have to make sure that the equation remains true. To understand how, we can use a balance as an analogy.

Pedro is using an old-fashioned scale to weight his school backpack. His backpack is placed on the left of the scale, along with a weight that weighs 2lbs. On the right side of the balance are placed 5 weights that each weigh 2lbs. The scale is perfectly balanced. What does he need to do to find its weight?

a) Remove 2 lbs from the left side

b) Remove 2 lbs from the right side

c) Remove 2 lbs from both sides

d) Add 2 lbs to both sides

The answer is c). We must remove 2lbs from both sides to keep the scale balanced. This will reveal that his backpack weighs 8lbs since 4 weights (each weighing 2lbs) will remain on the right side of the scale.

By undoing operations which are performed on a variable, we find the equation’s solution. This is known as an **inverse operation**. For example, addition can be reversed by subtraction, multiplication by division, and vice versa. Using the above problem as an example, we had to remove the weight from both sides (subtraction) because 2lbs had been added to the backpack.

Solve the equation for x:

x – 5 = -3

a) -15

b) -8

c) 2

d) 8

Applying the rule of inverse operations, we can *add* 5 to both sides. This cancels out the -5 on the left-hand side of the equation so that we are left with x = 2, therefore, the answer is c).

What value of x is the solution to 4x – 10 = -6?

a) -1

b) 0

c) 1

d) 4

In order to solve this, we do something similar to the previous problem i.e. adding 10 to both sides of the equation to cancel the -10 on the left-hand side. Thus we are left with 4x = 4. We have not solved for x yet, however. To do this, we perform the inverse operation being applied to x. In this case, since x is being multiplied by 4, we should divide both sides by 4: 4x / 4 = 4 / 4 which leaves us with x = 1.

Reversing Procedures:

When we’re given the steps of a procedure, we don’t need to also be given the steps for reversing that procedure. We can figure them out using inverse operations.

For example, to convert a given temperature from Fahrenheit to Celsius, the procedure is to take the Fahrenheit temperature, subtract 32, and then multiply by 5/9. Say that we want to covert the boiling point of water (212F) to Celcius. We do 212 – 32 = 180. Then 180 \* 5/9 = 100. Therefore, 212F is equal to 100C.

What is the procedure to convert Celsius to Fahrenheit?  
a) Take the Celsius temperature, subtract 32, and then divide by 5/9

b) Take the Celsius temperature, divide by 5/9, and then subtract 32

c) Take the Celsius temperature, divide by 5/9, and then add 32

The answer to this question is c). Since the procedure to convert Fahrenheit to Celsius requires that we subtract, then multiply, the inverse operation is to divide then add.

Following the same line of questioning: Jack wants to bake a surprise birthday cake for his sister. The recipe requires an oven temperature of 180C, but his oven operates in Fahrenheit. What temperature should Jack set the oven to?

a) 82F

b) 132F

c) 266F

d) 356F

The answer is d). Following the same logic as the previous question, we simply divide 180 by 5/9 and then add 32.

At Gianni’s pizza parlor, customers can order a custom pizza by choosing any number of toppings, like pepperoni and cheese. This is the procedure to calculate the price in dollars of a custom pizza: Take the number of toppings, multiply by 1.25, then add 12. A customer walks in and realizes she only has cash to pay for her pizza. What procedures does she need to follow to determine how many toppings she can afford to choose?

a) Take the amount of money, multiply by 1.25, then subtract 12

b) Take the amount of money, subtract 12, then multiply by 1.25

c) Take the amount of moeny, subtract 12, then divide by 1.25

d) Take the amount of money, divide by 1.25, then subtract 12

The answer here is c) since it is the inverse of operations.

Alina has $16 in her wallet. What is the maximum number of toppings she can afford?

a) 2

b) 3

c) 4

d) 5

Performing the inverse operation, we take 16, subtract 12 which yields 4, then divide by 1.25 which is 3.2. Since we cannot have 0.2% of a topping, the maximum number of toppings she can get is 3 (b)).

Different Formatting:

When equations look different than we’re used to, we may be able to rewrite them into a format we’re comfortable with.

Solve the equation for x:

13 = 5 – 2x

a) -4

b) 3

c) 4

d) 9

The answer is a). We subtract 5 from both sides to get 8 = -2x. Then we isolate x by dividing both sides by -2 which gives us -4 = x.

More Complicated Equations:

The next four problems look slightly more complicated than the ones we’ve seen so far, but the method of inverse operations still applies.

Solve the equation for x:

10(10 – 3x) = -20

a) -4

b) 0

c) 4

d) 10

The answer here is c). Since 10 is a factor of both sides of the equation, it is best to remove that first so that both sides are simplified:

10(10 + -3x) = -20

10(10 + -3x) / 10 = -20 / 10

10 + -3x = -2

Then we can complete the operation as normal:

10 – 10 + -3x = -2 – 10 --> Subtract 10 from both sides

-3x = -12

x = 4 --> Remove the common factor -3

Solve the equation for x:

a) -20

b) -19

c) -8

d) 43

The answer is c). It will be easiest if I show you the visual:

In this example, we learned that it is crucial to follow the correct order of operations. In particular, we need to remember that the numerators and denominators of fractions count as groups, which means that they get priority over other operations.

Solve the equation for x:

4x + 13(x + 3) + 7(3x) = 115

a) 1

b) 2

c) 5

d) 6

Here is how we solve this:

4x + 13x + 39 + 21x = 115

38x + 39 = 115 --> Gather like terms

38x + 39 – 39 = 115 – 39

38x = 76

38x / 38 = 76 / 38

x = 2

If 10 / x = 2, what is the value of x?

a) 2

b) 5

c) 12

d) 20

We can use the aforementioned triangle rule to solve this equation. Recall that the numerator goes on the top of the triangle (10 in this case) and then the denominator goes in the bottom left (x), and finaly, the value on the right of the equals sign goes on the bottom right corner. Thus to solve for x, we should divide 10 by 2 to get 5. The triangle rule is actually just a simplified method for remembering inverse operations.

Both Sides:

We’re able to solve equations by undoing the operations applied to a variable in reverse order. What happens, however, if the unknown value appears on both sides of the equation? In this lesson, we’ll learn how to deal with such situations by rearranging values and simplifying expressions.

Balancing Equations:

When a variable appears on both sides of an equation, we can use the balance analogy from the last lesson to figure out how to solve it. As long as we add or remove the same value from both sides, the equation will stay balanced and the value of the solution will remain unchanged.

Solve the equation for x:

5x = 3x + 2

a) -2

b) -1

c) 1

d) 2

The answer here is c). Since 5x and 3x are like terms, we are allowed to subtract 3x from both sides of the equation to give us 2x = 2. Dividing both sides by the common factor 2 reveals that x = 1.

Solve the equation for x:

-21 + 9x = 2x

a) -3

b) -2

c) 2

d) 3

Using the same logic as the previous example, we subtract 9x from both sides to get -21 = -7x. Dividing by -7 on both sides gives us 3 = x.

If a yellow rectangle with dimensions 9 \* x has the same area as the sum of two blue rectangles with areas 3 \* x and 3 \* 6, what is x? Here, we can recall that (a \* b)(a \* c) = a(b + c). Let us first add the two blue rectangles: (3 \* x)(3 \* 6) = 3(x + 6). Now we say that

3(x + 6) = 9x

3(x + 6) / 3 = 9x / 3 --> Remove the common factor, 3

x + 6 = 3x

x – x + 6 = 3x – x --> Subtract 1x from both sides

6 = 2x

3 = x --> Divide by the common factor, 2

Therefore, the answer is c).