Density Fourier transform

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What is Fourier Transform?

There are many definitions:

- on section $x \in (-L/2, +L/2)$ with periodic boundary condition $f(x) = f(x+L)$

$$\begin{aligned} f(x) &= \sum_{i=-\infty}^{+\infty} F_i e^{+ik_i x}; \\ F_i &= \frac{1}{L}\int_{-L/2}^{+L/2} f(x) e^{-ik_i x} dx; \end{aligned} \qquad k_i = \frac{2\pi}{L} i, i \in \mathbb{Z} \tag{1}$$

- on finite or spectral-limited functions

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(k) e^{+ikx} dk; \\ F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x) e^{-ikx} dx; \end{aligned} \qquad k \in \mathbb{R} \tag{2}$$

- etc...

Fourier transform is **linear integral transform**.

Properties - Accuracy

Decomposition coefficients asymptotics are closely tied to the class of function:

$$\begin{aligned} f(x) \in C^n(-L/2; L/2) \rightarrow F_i \propto \frac{1}{i^n}; \\ f(x) \in C^\infty(-L/2; L/2) \rightarrow F_i \propto \frac{1}{i!}; \end{aligned} \tag{3}$$

We can estimate residue of finite series approx:

$$\begin{aligned} f(x) \in C_n = \sum_{i=-k}^{+k} F_i e^{+ik_i x} + O\left(\frac{1}{k^{n-1}}\right); \\ f(x) \in C_\infty = \sum_{i=-k}^{+k} F_i e^{+ik_i x} + O\left(\frac{1}{(k-1)!}\right); \end{aligned} \tag{4}$$

Approximation **accuracy is growing** fast!

Properties - Other important features

- Inversion symmetry:

$$f(x) \in \mathbb{R} \rightarrow F_i = \tilde{F}_{-i} \tag{5}$$

- Spectral density preservation:

$$\sum_{i=-\infty}^{+\infty} |F_i|^2 = \int_{-L/2}^{L/2} |f(x)|^2 dx; \tag{6}$$

- Convolution image:

$$(f*g)(x) = \int_{-L/2}^{L/2} \tilde{f}(y) g(x-y) dy \rightarrow (FG)_i = \tilde{F}_i G_i \tag{7}$$

Density Fourier - Definition

The random value X is distributed on section $(-L/2, +L/2)$, then it has dustribution $\rho(x)$. If we know many points x_j then we know subsample of distribution:

$$\hat{\rho}(x) = \frac{1}{n} \sum_{j=1}^{n} \delta(x - x_j) \sim \rho(x). \tag{8}$$

Fourier transform can be applied to such approximation:

$$\hat{F}_i = \sum_{j=1}^{n} \exp(-ik_i x_j) \sim F_i \tag{9}$$