THE GALOIS THEORY OF $\mathbb{Q}(\mu_{13})$

N. P. STRICKLAND

Consider the field $K = \mathbb{Q}(\mu_{13})$. This is generated by $\zeta = e^{2\pi i/13}$, which satisfies $\sum_{k=0}^{12} \zeta^k = 0$

The Galois group $G(K/\mathbb{Q})$ is identified with $\mathbb{F}_{13}^\times$ in the usual way. This is cyclic of order 12, generated by 2. The subgroups of the Galois group are as follows:

$$\begin{aligned}
C_1 &= \{1\} \\
C_2 &= \{1, -1\} \\
C_3 &= \{1, 3, -4\} \\
C_4 &= \{1, 5, -1, -5\} \\
C_6 &= \{1, 3, 4, -1, -3, -4\} \\
C_{12} &= \{1, 2, 3, 4, 5, 6, -1, -2, -3, -4, -5, -6\}.
\end{aligned}$$

We write $K_d = K^{C_{12/d}} \subseteq K$, so $[K_d : \mathbb{Q}] = d$ and $G(K_d/\mathbb{Q}) = C_{12}/C_{12/d} \simeq C_d$. Put

$$\alpha_d = \sum_{k \in C_{12/d}} \zeta^k \in K_d.$$

Explicitly, we have

$$\begin{aligned}
\alpha_1 &= -1 \\
\alpha_2 &= \zeta + \zeta^3 + \zeta^4 + \zeta^{-1} + \zeta^{-3} + \zeta^{-4} \\
\alpha_3 &= \zeta + \zeta^{-1} + \zeta^5 + \zeta^{-5} \\
\alpha_4 &= \zeta + \zeta^3 + \zeta^{-4} \\
\alpha_6 &= \zeta + \zeta^{-1} = 2\cos(2\pi/13) \\
\alpha_{12} &= \zeta.
\end{aligned}$$

We also use the elements

$$\begin{aligned}
r &= \left(-\tfrac{13}{2}(5 + 3\sqrt{-3})\right)^{1/3} \\
w &= \alpha_4 - \overline{\alpha_4} \\
y &= 2\alpha_2 + 1 = \sum_{k=0}^{11} (-1)^k \zeta^{2^k}
\end{aligned}$$

In the case of r, we use the unique cube root that lies in the first quadrant, so $r \simeq 0.91 + 3.49i$. It can be shown that

$$\begin{aligned}
x &= (r + \overline{r})/\sqrt{13} \\
y &= |r| = \sqrt{13} \\
w &= \sqrt{(3\sqrt{13} - 13)/2} \\
\alpha_2 &= (\sqrt{13} - 1)/2 \\
\alpha_3 &= (r + \overline{r} - 1)/3 \\
\alpha_4 &= \tfrac{1}{4}(\sqrt{13} - 1) + \tfrac{1}{2}\sqrt{(3\sqrt{13} - 13)/2} \\
K_2 &= \mathbb{Q}(\sqrt{13}) \\
K_3 &= \mathbb{Q}(r + \overline{r}) \\
K_4 &= \mathbb{Q}(\sqrt{(3\sqrt{13} - 13)/2}) \\
K_6 &= \mathbb{Q}(x).
\end{aligned}$$