CUBICS

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Consider a cubic $f(x) = x^3 + bx + c$ with $b \neq 0$. Put $\Delta = -4b^3 - 27c^2$.

Proposition 0.1. *We have $\Delta = 0$ iff $f(x)$ has a repeated root. In particular, if $f(x)$ is irreducible then $\Delta \neq 0$.*

Proof. First, suppose that $\Delta = 0$, so $27c^2 = -4b^3$. Put $\alpha = -(3c)/(2b)$, and note that

$$\begin{aligned}
\alpha^2 &= \frac{9c^2}{4b^2} = \frac{27c^2/3}{4b^2} = \frac{-4b^3/3}{4b^2} = -b/3 \\
\alpha^3 &= \frac{-27c^3}{8b^3} = \frac{4b^3c}{8b^3} = c/2 \\
(x-\alpha)^2(x+2\alpha) &= (x^2 - 2\alpha x + \alpha^2)(x+2\alpha) \\
&= x^3 - 3\alpha^2 x + 2\alpha^3 = x^3 + bx + c = f(x),
\end{aligned}$$

so $f(x)$ has a repeated root at α.

For the converse, one can just expand everything out to check that

$$(18bx - 27c)f(x) + (9xc - 4b^2 - 6bx^2)f'(x) = \Delta.$$

In particular, the left hand side is actually a constant, independent of x. If $f(x)$ has a repeated root at α then $f(\alpha) = f'(\alpha) = 0$ so we can substitute $x = \alpha$ in the above to get $0 = \Delta$. $\square$

From now on we assume that $f(x)$ has no repeated roots, so $\Delta \neq 0$. We put $\delta = \sqrt{\Delta}$. For definiteness, if $\Delta \geq 0$ we take δ to be the nonnegative square root of Δ, and if $\Delta < 0$ then we take δ to be the square root of Δ lying in the upper half-plane. Next, we put

$$\begin{aligned}
m &= (-27c + \sqrt{-27}\delta)/2 \\
n &= (-27c - \sqrt{-27}\delta)/2,
\end{aligned}$$

so

$$\begin{aligned}
m + n &= -27c \\
m - n &= \sqrt{-27}\delta \\
mn &= \tfrac{1}{4}((-27c)^2 - (\sqrt{-27}\delta)^2) = \tfrac{1}{4}(27^2c^2 + 27(-4b^3 - 27c^2)) \\
&= -27b^3 = (-3b)^3.
\end{aligned}$$

As $mn = -b^3$ and $b \neq 0$ we have $m, n \neq 0$.

Now let μ_0 be one of the cube roots of m. For definiteness, we can take μ_0 be the unique cube root of m that can be written as $\mu_0 = re^{i\theta}$ with $0 \leq \theta < 2\pi/3$. As $m \neq 0$ we also have $\mu_0 \neq 0$.

Now put

$$\omega = e^{2\pi i/3} = \cos(2\pi/3) + i\sin(2\pi/3) = \frac{\sqrt{3}i - 1}{2},$$

and note that $\omega^3 = 1$ and

$$\omega^2 = \overline{\omega} = 1/\omega = -1 - \omega = e^{-2\pi i/3} = \frac{-\sqrt{3}i - 1}{2}.$$

The other two cube roots of m are then $\mu_1 = \omega\mu_0$ and $\mu_2 = \overline{\omega}\mu_0$. We put

$$\begin{aligned}\nu_0 &= -3b/\mu_0 \\ \nu_1 &= -3b/\mu_1 = \overline{\omega}\nu_0 \\ \nu_2 &= -3b/\mu_0 = \omega\nu_0.\end{aligned}$$

As $\mu_k^3 = m$ and $(-3b)^3 = mn$ and $b \neq 0$ we find that $\nu_k^3 = n$ and $\nu_k \neq 0$. We now put $\alpha_k = (\mu_k + \nu_k)/3$. The binomial expansion gives

$$\alpha_k^3 = \tfrac{1}{27}(\mu_k^3 + 3\mu_k(\mu_k\nu_k) + 3\nu_k(\mu_k\nu_k) + \nu_k^3).$$

Here $\mu_k^3 = m$ and $\nu_k^3 = n$, so $\mu_k^3 + \nu_k^3 = m + n = -27c$. On the other hand, we have $\mu_k\nu_k = -3b$ by the definition of ν_k. This means that

$$\begin{aligned}\alpha_k^3 &= \tfrac{1}{27}(m - 9b\mu_k - 9b\nu_k + n) \\ &= \tfrac{1}{27}(-27c - 27b\frac{\mu_k + \nu_k}{3}) \\ &= -c - b\alpha_k,\end{aligned}$$

so $f(\alpha_k) = \alpha_k^3 + b\alpha_k + c = 0$. We thus have three roots of $f(x)$ which appear to be different, but we need to check that there is not some hidden reason why they could be the same. This will follow from the formula below.

Proposition 0.2. *With δ and α_k defined as above, we have*

$$\delta = (\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)(\alpha_2 - \alpha_1).$$

Proof. First, from the definition of the numbers α_k and some easy manipulation we have

(A) $$3(\alpha_0 - \alpha_1) = \mu_0 + \nu_0 - \omega\mu_0 - \overline{\omega}\nu_0 = (1 - \omega)(\mu_0 - \overline{\omega}\nu_0)$$

(B) $$3(\alpha_0 - \alpha_2) = \mu_0 + \nu_0 - \overline{\omega}\mu_0 - \omega\nu_0 = (1 - \overline{\omega})(\mu_0 - \omega\nu_0)$$

(D) $$3(\alpha_2 - \alpha_1) = \overline{\omega}\mu_0 + \omega\nu_0 - \omega\mu_0 - \omega\nu_0 = (\overline{\omega} - \omega)(\mu_0 - \nu_0)$$

We now want to multiply these together. On the one hand, by fairly direct expansion we see that

(E) $$(1 - \omega)(1 - \overline{\omega})(\overline{\omega} - \omega) = -3\sqrt{3}i = -\sqrt{-27}.$$

On the other hand, as the roots of $x^3 - 1$ are 1, ω and $\overline{\omega}$ we have

$$(x - \overline{\omega})(x - \omega)(x - 1) = x^3 - 1.$$

We saw earlier that $\nu_0 \neq 0$, so it is legitimate to substitute $x = \mu_0/\nu_0$ and then multiply through by ν_0^3 to get

(F) $$(\mu_0 - \overline{\omega}\nu_0)(\mu_0 - \omega\nu_0)(\mu_0 - \nu_0) = \mu_0^3 - \nu_0^3 = m - n = \sqrt{-27}\delta$$

If we multiply (A), (B) and (C), then use (D) and (E) to simplify the right hand side, we obtain

$$27(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)(\alpha_2 - \alpha_1) = -\sqrt{-27}\sqrt{-27}\delta = 27\delta.$$

After dividing through by 27, we get the claimed identity. □

References