# THE FIELD $\mathbb{Q}(\mu_{20})$

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Put

$$\begin{aligned}
\zeta &= e^{i\pi/5} \\
\alpha &= \sqrt{(5-\sqrt{5})/2} \\
\beta &= \sqrt{(5+\sqrt{5})/2} \\
\tau &= (\sqrt{5}+1)/2.
\end{aligned}$$

Put $K = \mathbb{Q}(\mu_{20})$, which is the same as $\mathbb{Q}(i, \zeta)$. One can check that

$$\begin{aligned}
\alpha\beta &= \sqrt{5} & \beta/\alpha &= \tau \\
\beta &= 3\alpha - \alpha^3 & \alpha &= \beta^3 - 3\alpha \\
\zeta &= \frac{5+i\alpha^3}{5-i\alpha^3} = \frac{3-\alpha^2+i\alpha}{2} & \zeta^3 &= \frac{5+i\beta^3}{5-i\beta^3} = \frac{3-\beta^2+i\beta}{2} \\
\alpha &= 2\sin(\pi/5) = i(\zeta^{-1}-\zeta) & \beta &= 2\sin(3\pi/5) = i(\zeta^{-3}-\zeta^3).
\end{aligned}$$

We also have the factorization

$$t^4 - 5t^2 + 5 = (t-\alpha)(t+\alpha)(t-\beta)(t+\beta).$$

We now see that the field $L = \mathbb{Q}(\alpha, \beta)$ is the same as $\mathbb{Q}(\alpha)$ or $\mathbb{Q}(\beta)$ and is a Galois extension of $\mathbb{Q}$ with Galois group $C_4$. One can also check that $\alpha^{-2} + \beta^{-2} = 1$, so the cyclic permutations of $(0, \pm\alpha^{-1}, \pm\beta^{-1})$ are all unit vectors. One can check that they form the vertices of an icosahedron.