UNIFORMISATION OF EMBEDDED SURFACES

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Contents

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1. Introduction

[sec-intro]

Let $X \subset S^3$ be a smoothly embedded closed surface of genus $g > 1$. As we will explain, X automatically has a very rich and rigid geometric structure. Indeed, X inherits a Riemannian metric from S^3, and after specifying some conventions we also obtain a well-defined orientation. Now consider a point $x \in X$, and let T_xX denote the corresponding tangent space. Let $J_x \colon T_xX \to T_xX$ be the anticlockwise rotation through $\pi/2$ (which is meaningful given the metric and orientation). This depends smoothly on x and satisfies $J_x^2 = -1$, so it gives an almost complex structure on X. It has been known since the early twentieth century that any almost complex structure on a manifold of real dimension two integrates to give a genuine complex structure. Thus, X can be regarded as a compact Riemann surface. It is known that any compact Riemann surface can be regarded as a projective algebraic variety over $\mathbb{C}$, and also as a branched cover of the Riemann sphere. Alternatively, as we have assumed that the genus is larger than one, the universal cover of X is conformally equivalent to the open unit disc Δ. This means that X is conformally equivalent to the quotient Δ/Π for some Fuchsian group Π.

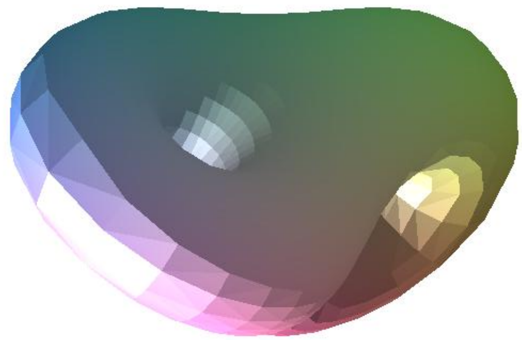
To the best of our knowledge, the literature contains no examples where a significant fraction of this structure can be made explicit. This monograph is a partially successful attempt to provide such an example, involving the surface

$$EX^* = \{x \in S^3 \mid (3x_3^2 - 2)x_4 + \sqrt{2}(x_1^2 - x_2^2)x_3 = 0\},$$

with weaker results for a one-parameter family of surfaces in which EX^* appears. To display EX^* visually, we apply the stereographic projection map $s \colon S^3 \to \mathbb{R}^3 \cup \{\infty\}$ defined as follows:

$$s(x) = \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right)$$
$$s^{-1}(u) = \left(\frac{2u_1}{\|u\|^2+1}, \frac{2u_2}{\|u\|^2+1}, \frac{2u_3}{\|u\|^2+1}, \frac{\|u\|^2-1}{\|u\|^2+1}\right).$$

The image $s(EX^*)$ looks like this:



Our work is organised around the following definitions:

Definition 1.0.1. [defn-G]
Let G be the group of order 16 generated by λ, μ and ν subject to relations

$$\lambda^4 = \mu^2 = \nu^2 = (\mu\nu)^2 = (\lambda\mu)^2 = (\lambda\nu)^2 = 1,$$

so

$$G = \{\lambda^i \mu^j \nu^k \mid 0 \leq i < 4,\ 0 \leq j, k < 2\}.$$

We use the following notation for subgroups:

$$D_8 = \langle \lambda, \mu \rangle \qquad C_4 = \langle \lambda \rangle$$
$$C_2 = \langle \lambda^2 \rangle \qquad C_2' = \langle \mu\nu \rangle.$$

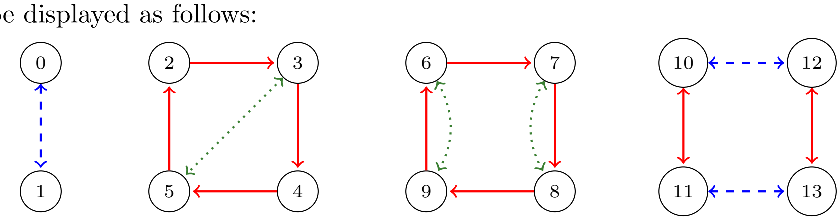
Definition 1.0.2. [defn-V-star]
We write V^* for the set $\{0, \dots, 13\}$ equipped with the action of G by the following permutations:

$$\lambda \mapsto (2\ 3\ 4\ 5)\ (6\ 7\ 8\ 9)\ (10\ 11)\ (12\ 13)$$
$$\mu \mapsto (0\ 1)\ (3\ 5)\ (6\ 9)\ (7\ 8)\ (10\ 12)\ (11\ 13)$$
$$\nu \mapsto (3\ 5)\ (6\ 9)\ (7\ 8).$$

Remark 1.0.3. [rem-V-star]
The orbits in V^* are

$$\{0, 1\} \simeq G/\langle \lambda, \nu \rangle$$
$$\{2, 3, 4, 5\} \simeq G/\langle \mu, \nu \rangle$$
$$\{6, 7, 8, 9\} \simeq G/\langle \lambda\mu, \lambda\nu \rangle$$
$$\{10, 11, 12, 13\} \simeq G/\langle \lambda^2, \nu \rangle.$$

The action can be displayed as follows:



The solid red arrows show the action of λ, the dotted green arrows show the action of ν, and the dashed blue arrows show the action of $\mu\nu$.

Definition 1.0.4. [defn-precromulent]
A *precromulent surface* is a compact Riemann surface X of genus 2 with an action of G such that

(a) The elements of D_8 act conformally, and the elements of $G \setminus D_8$ act anticonformally.
(b) The set $V = \{v \in X \mid \mathrm{stab}_{D_8}(v) \neq 1\}$ is isomorphic to V^* as a G-set.

A *precromulent labelling* of X is a specific choice of isomorphism $V^* \simeq V$, or equivalently, a listing of the points in V as $v_0, \dots, v_{13}$ such that G permutes these points in accordance with the permutations listed in Definition 1.0.2. A *cromulent labelling* is a precromulent labelling such that

(c) λ acts on the tangent space $T_{v_0}X$ as multiplication by i.
(d) In the set $X' = \{x \in X \mid \mathrm{stab}_G(x) = 1\}$, there is a connected component F' whose closure contains $\{v_0, v_3, v_6, v_{11}\}$.

We will show in Proposition 3.7.12 that every precromulent surface has precisely two cromulent labellings, which are exchanged by the action of λ^2. A *cromulent surface* is a precromulent surface with a choice of cromulent labelling.

A *(precromulent) isomorphism* between precromulent surfaces will mean a G-equivariant conformal isomorphism. A *(cromulent) isomorphism* between cromulent surfaces will mean a G-equivariant conformal isomorphism that is compatible with the specified labellings.

Now fix a parameter $a \in (0, 1)$. We put

$$EX(a) = \{x \in \mathbb{R}^4 \mid \|x\| = 1,\ ((a^{-2} + 1)x_3^2 - 2)x_4 + a^{-1}(x_1^2 - x_2^2)x_3 = 0\},$$

and observe that $EX^* = EX(1/\sqrt{2})$. In Section 6 we will give $EX(a)$ a G-action and labelling making it a cromulent surface. We call these surfaces the *embedded family.* Although our central problem is to study uniformisations of the surfaces $EX(a)$, we will also discuss many other features of their geometry and topology. In particular, we will give an alternative definition which is much more geometric but takes longer to state. Two special features of the case $a = 1/\sqrt{2}$ are as follows:

(a) By a *great circle* we mean the intersection of S^3 with a two-dimensional vector subspace of $\mathbb{R}^4$. For all a, the fixed set of the element $\nu \in G$ is the disjoint union of three curves, each of which is diffeomorphic to S^1. If $a = 1/\sqrt{2}$ (but for no other value) then one of those components is a great circle.
(b) One can show that the complexified variety

$$CEX(a) = \{x \in \mathbb{C}^4 \mid \sum_i x_i^2 = 1,\ ((a^{-2} + 1)x_3^2 - 2)x_4 + a^{-1}(x_1^2 - x_2^2)x_3 = 0\}$$

is smooth for all $a \neq 1/\sqrt{2}$, but $CEX(1/\sqrt{2})$ is singular at the eight points in the G-orbit of $(i\sqrt{2}, 0, \sqrt{2}, 1)$.

Next, put

$$PX_0(a) = \{(w, z) \in \mathbb{C}^2 \mid w^2 = z^5 - (a^2 + a^{-2})z^3 + z\}.$$

This is an affine hyperelliptic curve. By well-known methods we can construct a compact Riemann surface $PX(a)$ which is the union of $PX_0(a)$ with a single extra point. In Section 3 we will give this a G-action and labelling making it a cromulent surface. We call these surfaces the *projective family.*

Finally, let Π be the abstract group generated by symbols β_k (for $k \in \mathbb{Z}/8$) subject to the following relations:

$$\beta_{k+4} = \beta_k^{-1}$$
$$\beta_0\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7 = 1.$$

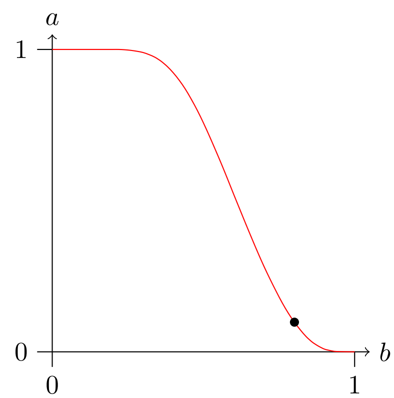
In Section 4 we will give a free action of Π on the unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$, depending on a parameter $a \in (0, 1)$. We will show that the orbit space $HX(a)$ for this action is a compact Riemann surface of genus two, and give it a cromulent structure. We call these surfaces the *hyperbolic family.*

In Theorem 3.7.2, Corollary 3.7.11 and Theorem 4.5.1 we will show that

- For any two cromulent surfaces, there is at most one isomorphism between them.
- For any cromulent surface X there is a unique $a \in (0, 1)$ such that $X \simeq PX(a)$, and there is a unique $b \in (0, 1)$ such that $X \simeq HX(b)$.

In other words, the projective family and the hyperbolic family are both universal. We conjecture that the embedded family is also universal, but we have not proved this.

As a consequence of universality, for every $a \in (0,1)$ there is a unique $b \in (0,1)$ such that $PX(a) \simeq HX(b)$. In Section 5 we will develop two different methods for computing a as a function of b or *vice versa*, and for computing the corresponding cromulent isomorphism. One method involves a rich theory based on Fuchsian differential equations and the Schwarzian derivative; the other is less illuminating, but in some respects more efficient and direct. The graph of a against b is as follows.



We have conducted a fairly extensive heuristic search for closed-form relationships between the above graph and various other functions that we know to be relevant, but without success.

Finally, we want to find a and b such that $EX^* \simeq PX(a) \simeq HX(b)$. Our best estimates are $a \simeq 0.0983562$ and $b \simeq 0.8005319$, corresponding to the marked point on the above graph. We have some reason to hope that all the quoted digits are accurate, but we have not performed a rigorous error analysis. In Section 8 we will explain the methods used to calculate b (and then a is calculated from b as described previously). The first step is to find the unique smooth function f on EX^* such that e^{2f} times the standard metric has curvature equal to -1. We can then find the lengths of certain curves with respect to this rescaled metric, and the value of b can be determined from these lengths.

1.1. **Maple code.** To carry out the work described above, we need to check a very large number of reasonably complex formulae and combinatorial facts, and we also need to perform extensive numerical calculations. Most of the formulae could individually be checked by hand with sufficient effort. However, the number and size of the formulae are so large that computer assistance is required for the project as a whole. We have used Maple for this. The code and documentation are distributed alongside this monograph, and there is an overview of the structure in Section 9. This monograph contains many lines like this:

```
group_check.mpl: check_group_properties(), check_character_table()
```

This indicates that some set of claims that have recently been made in the text can be checked by executing the functions `check_group_properties()` and `check_character_table()`, which are defined in the file `group_check.mpl`. These functions are set up so that they will print their own names, then they will run silently unless they detect any errors. One can set the global variable `assert_verbosely` to `true`, and then the checking functions will print additional information about the individual claims being checked. One can check the complete set of claims for the whole monograph by reading the file `check_all.mpl`. While this does not quite reach the level of rigour provided by formal proof assistants such as Isabelle, it is a major step in that direction.

The worksheet `text_check.mw` also provides another means to check the consistency of the text with the Maple code. (Some fragments of LaTeXcode were generated automatically by Maple to ensure correctness, but technical problems with precise control of formatting dissuaded us from using this approach more extensively.)

One can also repeat all the numerical calculations by following the instructions in Section 9.4.

2. General theory of precromulent surfaces

[sec-general]

2.1. **Representations of G.** [sec-representations]

We first discuss the representation theory of G, which will be useful for organising various algebraic calculations later. We assume that the reader is familiar with the basic ideas of representation theory, which are discussed in [17], for example.

Proposition 2.1.1. [prop-characters]

The centre of G is $\{1, \lambda^2, \mu\nu, \lambda^2\mu\nu\}$, and the commutator subgroup is $\{1, \lambda^2\}$. The character table is as follows:

	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
1	1	1	1	1	1	1	1	1	2	2
λ^2	1	1	1	1	1	1	1	1	-2	-2
$\mu\nu$	1	1	-1	-1	1	1	-1	-1	2	-2
$\lambda^2\mu\nu$	1	1	-1	-1	1	1	-1	-1	-2	2
$\lambda^{\pm 1}$	1	-1	1	-1	-1	1	-1	1	0	0
$\mu, \lambda^2\mu$	1	1	-1	-1	-1	-1	1	1	0	0
$\lambda^{\pm 1}\mu$	1	-1	-1	1	1	-1	-1	1	0	0
$\nu, \lambda^2\nu$	1	1	1	1	-1	-1	-1	-1	0	0
$\lambda^{\pm 1}\nu$	1	-1	1	-1	1	-1	1	-1	0	0
$\lambda^{\pm 1}\mu\nu$	1	-1	-1	1	-1	1	1	-1	0	0

Proof. The commutator of λ and μ is λ^2, and it is clear from the form of the defining relations that $\{1, \lambda^2\}$ is normal and that $G/\{1, \lambda^2\}$ is elementary abelian. It follows that the commutator subgroup is precisely $\{1, \lambda^2\}$. It is a straightforward calculation that the elements 1, λ^2, $\mu\nu$ and $\lambda^2\mu\nu$ are central, but that no other element commutes with λ. It follows that the centre is as claimed. We now see that if α is a non-central element then the corresponding conjugacy class is just $\{\alpha, \lambda^2\alpha\}$. This means that there are ten conjugacy classes, as listed in the left hand column. The characters of degree one are the same as the homomorphisms from the abelianization $G/\{1, \lambda^2\}$ to S^1. As $G/\{1, \lambda^2\}$ is elementary abelian of order 8, it is easy to check that $\chi_0, \dots, \chi_7$ is a complete list of such characters. There are two different retractions of G onto D_8, one sending $\mu\nu$ to the identity, and the other sending $\mu\nu$ to λ^2. There is a standard action of D_8 as the isometries of a square in $\mathbb{R}^2$, and by pulling this back along the two projections we get two two-dimensional representations of G, with characters χ_8 and χ_9. These are irreducible, because in each case the sum of the squares of the character values is equal to the group order. We now have ten different irreducible representations, which matches the number of conjugacy classes, so the list is complete.

```
group_check.mpl: check_group_properties(), check_character_table()
```

□

Remark 2.1.2. Maple notation for the elements of G is as follows:

$$\begin{array}{cccc} 1 = 1 & \lambda = L & \lambda^2 = LL & \lambda^3 = LLL \\ \mu = M & \lambda\mu = LM & \lambda^2\mu = LLM & \lambda^3\mu = LLLM \\ \nu = N & \lambda\nu = LN & \lambda^2\nu = LLN & \lambda^3\nu = LLLN \\ \mu\nu = MN & \lambda\mu\nu = LMN & \lambda^2\mu\nu = LLMN & \lambda^3\mu\nu = LLLMN \end{array}$$

To make this work reliably, the code in the file `group.mpl` protects the symbols `L`, `N`, `LLMN` and so on, so they cannot be assigned values. The function `G_mult` computes the group operation, so `G_mult(M,L)` returns `LLLM`, for example. The functions `G_inv` and `G_conj` compute inverses and conjugates. To retrieve $\chi_8(\lambda^2)$ (for example), one can enter `character[8][LL]`. The variable `G16` contains the list of all elements of G. All of this is set up by the file `group.mpl`.

Note that this discussion of the contents of `group.mpl` is incomplete, as will be the case with similar comments throughout this monograph. For full information, the reader should consult the code itself, and the comments therein. The full set of files for this project contains a `doc` directory. The file `defs.html` in that directory is an index of all defined symbols, with links to the relevant lines in in the files of Maple code.

2.2. Automorphisms of V^*. [sec-aut-V]

As we stated in the introduction, every precromulent surface has precisely two cromulent labellings. In order to prove this, we will need to understand the automorphisms of the G-set V^*, and it is convenient to treat that question now.

Proposition 2.2.1. [prop-aut-V]

$\mathrm{Aut}(V^*)$ is isomorphic to C_2^5, with the following generators:

$$\begin{aligned} \phi_0 &= (0\ 1) \\ \phi_1 &= (2\ 4)(3\ 5) \\ \phi_2 &= (6\ 8)(7\ 9) \\ \phi_3 &= (10\ 11)(12\ 13) \\ \phi_4 &= (10\ 12)(11\ 13). \end{aligned}$$

Readers may find it helpful to consider the picture in Remark 1.0.3 when reading the argument below. The permutation ϕ_i is represented in Maple as `aut_V_phi[i]`.

Proof. First, it is straightforward to check directly that the above permutations commute with λ, μ and ν (so they define automorphisms of V^*). It is also easy to see that they are commuting involutions and that they generate a group A isomorphic to C_2^5.

Now consider an arbitrary permutation ϕ that commutes with λ, μ and ν; we must show that $\phi \in A$. As ϕ commutes with G, we must have $\mathrm{stab}_G(\phi(i)) = \mathrm{stab}_G(i)$ for all i. The stabilisers are as follows:

$$\begin{aligned} \mathrm{stab}_G(0) = \mathrm{stab}_G(1) &= \langle \lambda, \nu \rangle \\ \mathrm{stab}_G(2) = \mathrm{stab}_G(4) &= \langle \mu, \nu \rangle \\ \mathrm{stab}_G(3) = \mathrm{stab}_G(5) &= \langle \lambda^2\mu, \lambda^2\nu \rangle \\ \mathrm{stab}_G(6) = \mathrm{stab}_G(8) &= \langle \lambda\mu, \lambda\nu \rangle \\ \mathrm{stab}_G(7) = \mathrm{stab}_G(9) &= \langle \lambda^{-1}\mu, \lambda^{-1}\nu \rangle \\ \mathrm{stab}_G(10) = \cdots = \mathrm{stab}_G(v_{13}) &= \langle \lambda^2, \nu \rangle. \end{aligned}$$

It follows that ϕ must preserve each of the following sets:

$$\{0,1\},\ \{2,4\},\ \{3,5\},\ \{6,8\},\ \{7,9\},\ \{10,11,12,13\}.$$

The restriction of ϕ to $\{2,3,4,5\}$ must commute with the restrictions of λ and μ, which are $(2\ 3\ 4\ 5)$ and $(3\ 5)$. It follows easily that the restriction of ϕ is $\phi_1 = (2\ 4)(3\ 5)$ or the identity. A similar argument shows that the restriction of ϕ to $\{6,7,8,9\}$ is $\phi_2 = (6\ 8)(7\ 9)$ or the identity, and the restriction to $\{10,11,12,13\}$ must be a transposition pair or the identity. Here the possible transposition pairs are $\phi_3 = (10\ 11)(12\ 13)$ and $\phi_4 = (10\ 12)(11\ 13)$ and $\phi_3\phi_4 = (10\ 13)(11\ 12)$. The claim follows easily.

`group_check.mpl: check_aut_V()`

$\square$

2.3. Quotients. [sec-quotients]

Let X be a precromulent surface. It is standard that for any finite group H of conformal automorphisms, the quotient X/H always has a canonical structure as a compact connected Riemann surface such that the projection $X \to X/H$ is a branched cover. We will need to understand the genus of X/H, which is determined by its Euler characteristic, which is given by the following result:

Lemma 2.3.1. [lem-chi-quotient]

For any subgroup $H \leq D_8$ we have $\chi(X/H) = |V/H| - 16/|H|$.

Proof. We can write $X = A \cup B$, where A is a union of small discs around the points of V, and B is the closure of the complement of A. This means that the set $C = A \cap B$ is a disjoint union of circles, so $\chi(C) = 0$, so $\chi(A) + \chi(B) = \chi(X)$. Similarly, C/H is again a union of circles, so $\chi(C/H) = 0$, so $\chi(A/H) + \chi(B/H) = \chi(X/H)$. Now A and A/H are homotopy equivalent to V and V/H, so $\chi(A) = 14$ and $\chi(A/H) = |V/H|$. As X has genus $g = 2$ we have $\chi(X) = 2 - 2g = -2$, so $\chi(B) = -2 - 14 = -16$. Next, note that the action of H on B is free. Thus, if we choose a finite regular cell structure on B/H, then the preimage in B of each cell in B/H will be a disjoint union of $|H|$ cells. Using this we see that $\chi(B/H) = \chi(B)/|H| = -16/|H|$, so $\chi(X/H) = |V/H| - 16/|H|$. □

Recall that we use the following notation for subgroups of D_8

$$D_8 = \langle \lambda, \mu \rangle \qquad C_4 = \langle \lambda \rangle \qquad C_2 = \langle \lambda^2 \rangle.$$

Corollary 2.3.2. [cor-quotient-types]
The surfaces X/C_2, X/C_4 and X/D_8 are all conformally equivalent to $\mathbb{C}_\infty$. However, $X/\langle \mu \rangle$ and $X/\langle \lambda\mu \rangle$ are elliptic curves.

Proof. By the classification of compact connected Riemann surfaces, it will suffice to show that $\chi(X/C_2) = \chi(X/C_4) = \chi(X/D_8) = 2$ and $\chi(X/\langle \mu \rangle) = \chi(X/\langle \lambda\mu \rangle) = 0$. If $H \leq D_8$ is generated by a single element σ, then $|V/H|$ is just the number of cycles (including 1-cycles) in the permutation corresponding to σ. This gives everything in the following table except for the case $H = D_8$, which is easily handled in an *ad-hoc* way.

H	$\lvert H \rvert$	σ	$\lvert V/H \rvert$	$\chi(X/H)$
C_2	2	λ^2	10	2
C_4	4	λ	6	2
$\langle \mu \rangle$	2	μ	8	0
$\langle \lambda\mu \rangle$	2	$\lambda\mu$	8	0
D_8	8		4	2

□

Remark 2.3.3. [rem-conj-quot]
Recall that the action of an element $g \in G$ gives an isomorphism $X/H \to X/gHg^{-1}$. In particular, the action of λ gives isomorphisms $X/\langle \mu \rangle \to X/\langle \lambda^2\mu \rangle$ and $X/\langle \lambda\mu \rangle \to X/\langle \lambda^3\mu \rangle$. Because of this, we will mostly restrict attention to $X/\langle \mu \rangle$ and $X/\langle \lambda\mu \rangle$, and ignore $X/\langle \lambda^2\mu \rangle$ and $X/\langle \lambda^3\mu \rangle$.

Remark 2.3.4. [rem-smooth-branch]
All of the above relies on the standard fact that if Z is a Riemann surface and H is a finite group of holomorphic automorphisms, then Z/H has a natural structure as a Riemann surface, and in particular has a smooth structure. We offer some remarks about this, some of which will be needed later.

More precisely, the claim is that this structure makes Z/H into a coequaliser for the action in the analytic category: if U is an H-invariant open subset of Z, and $f\colon U \to W$ is an H-invariant analytic function to another Riemann surface W, then U/H is open in Z/H, and there is a unique analytic function $g\colon U/H \to W$ such that the composite $U \to U/H \xrightarrow{g} W$ is f. Note here that coequalisers are automatically unique up to unique isomorphism. Thus, it does not matter if we make some arbitrary choices in the process of constructing a coequaliser; the result will be independent of those choices.

The proof of the claim is local on Z. Given $z \in Z$, put

$$\begin{aligned} C_0 &= \{\alpha \in H \mid \alpha(z) = z\} \\ C_1 &= \{\alpha \in H \mid \alpha = 1 \text{ on some neighbourhood of } z\} \\ C &= C_0/C_1. \end{aligned}$$

Then each element $\alpha \in C$ must act on $T_z^* Z$ as multiplication by some scalar $\chi(\alpha) \in \mathbb{C}^\times$; this defines a homomorphism $\chi\colon C \to \mathbb{C}^\times$. By power series methods, one can check that χ must be injective, and thus that C must be cyclic, of order n say. Now choose a local parameter f_0 with $f_0(z) = 0$, and put

$$f(w) = |C|^{-1} \sum_{\alpha \in C} \chi(\alpha)^{-1} f_0(\alpha(w)).$$

This is the same as f_0 to first order, so it is again a local parameter, and it satisfies $f(\alpha(w)) = \chi(\alpha)\, f(w)$. Using this, we reduce to the case where the group μ_n of n'th roots of unity acts on $\mathbb{C}$ by multiplication. Here, the map $\sigma_n\colon z \mapsto z^n$ is easily seen to be a coequaliser.

Note, however, that the map $\sigma_n\colon \mathbb{C} \to \mathbb{C}$ is not a coequaliser in the smooth category (provided that $n > 1$). Indeed, the function $f(z) = |z|^2$ is smooth and μ_n-invariant. There is a unique map $g\colon \mathbb{C} \to \mathbb{R}$ with $f = g \circ \sigma_n$, namely $g(w) = |w|^{2/n}$. However, g is not smooth. Because of this, if we start with a smooth surface Z and an orientation-preserving action of a finite group H, there is no obvious way to obtain a smooth structure on Z/H. Given z and C as above, we can choose a chart ϕ at z on which C acts by rotation, and using this we obtain a chart $\overline{\phi}$ on the quotient. However, if ψ is another local chart at z on which C acts by rotation, then $\overline{\psi}^{-1} \circ \overline{\phi}$ need not be smooth.

We can always obtain a smooth structure on Z/H by choosing a smooth invariant Riemannian metric, using this to give Z a conformal structure, and then taking a major detour through the analytic category as above. However, the result will depend on the choice of metric, and we do not know any way to shortcut the detour.

2.4. **Curve systems.** [sec-curve-systems]

In this section, we define what we mean by a *curve system* on a precromulent surface. Later we will exhibit curve systems for the projective family, the hyperbolic family and the embedded family. We will also show that the projective family is universal, so in fact every precromulent surface has a curve system.

Definition 2.4.1. [defn-precromulent-C]

Let X be a labelled precromulent surface. For any $\gamma \in G$ we put $X^\gamma = \{x \in X \mid \gamma(x) = x\}$. We then put

$$\begin{aligned}
C_0 &= \text{ the component of } v_2 \text{ in } X^{\mu\nu} \\
C_1 &= \text{ the component of } v_0 \text{ in } X^{\lambda\nu} \\
C_2 &= \text{ the component of } v_0 \text{ in } X^{\lambda^3\nu} \\
C_3 &= \text{ the component of } v_{11} \text{ in } X^{\lambda^2\nu} \\
C_4 &= \text{ the component of } v_{10} \text{ in } X^{\nu} \\
C_5 &= \text{ the component of } v_0 \text{ in } X^{\nu} \\
C_6 &= \text{ the component of } v_0 \text{ in } X^{\lambda^2\nu} \\
C_7 &= \text{ the component of } v_1 \text{ in } X^{\nu} \\
C_8 &= \text{ the component of } v_1 \text{ in } X^{\lambda^2\nu}.
\end{aligned}$$

Remark 2.4.2. [rem-Ck-circle]

The elements $\mu\nu$ and $\lambda^k\nu$ act on X as antiholomorphic involutions. A standard result, which we will recall as Corollary 3.6.9, shows that the fixed set of an antiholomorphic involution on a compact Riemann surface is always diffeomorphic to a finite disjoint union of circles. Thus, each of the sets C_k above is a circle. If α and β are distinct antiholomorphic involutions in G then $X^\alpha \cap X^\beta$ is fixed by the holomorphic element $\alpha\beta$ and so is contained in the finite set V. Thus, for example, we have $C_0 \cap C_1 \subseteq V$. On the other hand, C_4 and C_5 are two components in X^ν, so they are either equal or disjoint. In fact, we will see later that they are always disjoint, but this will require some further theory. More generally, C_4, C_5 and C_7 are disjoint, and C_3, C_6 and C_8 are disjoint.

Remark 2.4.3. The antiholomorphic involution that fixes C_k is represented in Maple as `c_involution[k]`. For example, `c_involution[6]` evaluates to `LLN`, which is our Maple notation for $\lambda^2\nu$.

Definition 2.4.4. [defn-curve-system]

Let X be a labelled precromulent surface. A *curve system* on X is a system of maps $c_k\colon \mathbb{R} \to X$ (for $0 \le k \le 8$) such that:

(a) Each c_k is real-analytic and 2π-periodic and induces an embedding $\mathbb{R}/2\pi\mathbb{Z} \to X$.

(b) The vertices $v_0, \dots, v_{13}$ occur as values of the maps $c_0, \dots, c_8$, as follows:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0			0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{4}$				
1	0	π					$\frac{\pi}{2}$		$-\frac{\pi}{2}$					
2	0	π						$\frac{\pi}{2}$		$-\frac{\pi}{2}$				
3				$\frac{\pi}{2}$		$-\frac{\pi}{2}$						0		π
4			$-\frac{\pi}{2}$		$\frac{\pi}{2}$						0		π	
5	0											π		
6	0										π			
7		0												π
8		0											π	

In more detail, if the above table has an angle θ in column j of row i, then $c_i(\theta) = v_j$, but if column j of row i is empty, then $v_j \notin c_i(\mathbb{R})$.

(c) The group G acts on the curves c_k as follows:

$$\begin{array}{lll}
\lambda(c_0(t)) = c_0(t + \pi/2) & \mu(c_0(t)) = c_0(-t) & \nu(c_0(t)) = c_0(-t) \\
\lambda(c_1(t)) = c_2(t) & \mu(c_1(t)) = c_2(t + \pi) & \nu(c_1(t)) = c_2(-t) \\
\lambda(c_2(t)) = c_1(-t) & \mu(c_2(t)) = c_1(t + \pi) & \nu(c_2(t)) = c_1(-t) \\
\lambda(c_3(t)) = c_4(t) & \mu(c_3(t)) = c_3(t + \pi) & \nu(c_3(t)) = c_3(-t) \\
\lambda(c_4(t)) = c_3(-t) & \mu(c_4(t)) = c_4(-t - \pi) & \nu(c_4(t)) = c_4(t) \\
\lambda(c_5(t)) = c_6(t) & \mu(c_5(t)) = c_7(t) & \nu(c_5(t)) = c_5(t) \\
\lambda(c_6(t)) = c_5(-t) & \mu(c_6(t)) = c_8(-t) & \nu(c_6(t)) = c_6(-t) \\
\lambda(c_7(t)) = c_8(t) & \mu(c_7(t)) = c_5(t) & \nu(c_7(t)) = c_7(t) \\
\lambda(c_8(t)) = c_7(-t) & \mu(c_8(t)) = c_6(-t) & \nu(c_8(t)) = c_8(-t)
\end{array}$$

Remark 2.4.5. The details of axiom (b) are represented in Maple in several different ways, which are useful for different purposes. Consider, for example, the fact that $c_2(\pi/2) = v_7$ but v_7 does not lie on C_3.

- `v_on_c[7,2]` is `Pi/2`, but `v_on_c[7,3]` is `NULL`.
- `c_gen[2](Pi/2)` evaluates to `v_gen[7]`. Here `v_gen[7]` is just a symbol, with no assigned value. On the other hand, `c_gen[2](Pi/4)` just evaluates to itself, corresponding to the fact that we have no axiom about the value of $c_2(\pi/4)$.
- `v_track[7]` is a list of equations, one of which is the equation `2=Pi/2`. There is no equation in the list with 3 on the left hand side.
- `c_track[2]` is a list of equations, one of which is the equation `7=Pi/2`. On the other hand, `c_track[3]` has no equation with 7 on the left hand side.

The details of axiom (c) are encoded in the table `act_c_data`, which is indexed by pairs `[g,i]` with `g` in G and `i` in $\{0, \dots, 8\}$. If `act_c_data[g,i]` evaluates to `[j,m,a]` then the corresponding axiom is $g.c_i(t) = c_j(mt + a)$.

Remark 2.4.6. Suppose we have a curve system $(c_k)_{k=0}^{8}$ and a strictly increasing analytic diffeomorphism $u \colon \mathbb{R} \to \mathbb{R}$ with $u(-t) = -u(t)$ and $u(t + \pi/4) = u(t) + \pi/4$; then the maps $c_k \circ u$ give another curve system. Thus, curve systems are not unique. However, they are unique up to a kind of reparametrisation slightly more general than that described above; we will not spell out the details.

Proposition 2.4.7. [`prop-curve-system`]
Let $(c_k)_{k=0}^{8}$ be a curve system on a labelled precromulent surface X. Then:

(1) *For each k the map c_k gives a diffeomorphism $\mathbb{R}/2\pi\mathbb{Z} \to C_k$.*
(2) *The sets C_4, C_5 and C_7 are disjoint.*
(3) *The sets C_3, C_6 and C_8 are disjoint.*

(4) *For all $i \neq j$ we have $C_i \cap C_j \subseteq V$ (so a precise list of elements of $C_i \cap C_j$ can be read off from axiom (b)).*

Proof. Axiom (c) gives $\mu\nu(c_0(t)) = \mu(c_0(-t)) = c_0(t)$, so $c_0(\mathbb{R}) \subseteq X^{\mu\nu}$. Moreover, $c_0(\mathbb{R})$ is connected and contains $c_0(0)$, which is v_2 by axiom (b). This proves that $c_0(\mathbb{R}) \subseteq C_0$. Axiom (a) tells us that c_0 gives a smooth embedding $\mathbb{R}/2\pi\mathbb{Z} \to C_0$, but C_0 is diffeomorphic to a circle by Remark 2.4.2, and any smooth embedding of a circle in a circle is necessarily a diffeomorphism. The same line of argument shows that c_k induces a diffeomorphism $\mathbb{R}/2\pi\mathbb{Z} \to C_k$ for all k. Next, axiom (b) tells us that $v_0 \notin c_4(\mathbb{R}) = C_4$, so C_5 is a component of X^ν which is different from C_4 and therefore disjoint from C_4. The same line of argument shows that C_4, C_5 and C_7 are disjoint, and also that C_3, C_6 and C_8 are disjoint. Now consider an intersection $C_i \cap C_j$ that is not covered by (b) or (c). We then find that $C_i \subseteq X^\gamma$ and $C_j \subseteq X^\delta$ for some antiholomorphic involutions $\gamma, \delta \in G$ with $\gamma \neq \delta$, so $\gamma\delta$ is a nontrivial element of D_8. Any element of $C_i \cap C_j$ is fixed by $\gamma\delta$, and so lies in V by the definition of V. □

Proposition 2.4.8. [prop-empty-boxes]

Suppose we have a system of maps $c_k \colon \mathbb{R} \to X$ such that axioms (a) and (c) are satisfied. Suppose also that

(p) *The part of axiom (b) corresponding to the nonempty boxes in the table is satisfied.*

(q) *The sets $c_3(\mathbb{R})$, $c_6(\mathbb{R})$ and $c_8(\mathbb{R})$ are disjoint.*

Then the maps c_k give a curve system.

Proof. First note that part (a) of Proposition 2.4.7 used only axioms that we are still assuming here, so we again have $C_k = c_k(\mathbb{R})$ for all k. We also see from axiom (c) that $\lambda(C_3) = C_4$ and $\lambda(C_6) = C_5$ and $\lambda(C_8) = C_7$, so C_4, C_5 and C_7 are also disjoint.

Next, we can redraw the table in axiom (b) as follows:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	A	A	0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{4}$	A	A	A	A
1	0	π	A	A	A	A	$\frac{\pi}{2}$	A	$-\frac{\pi}{2}$	A	A	A	A	A
2	0	π	A	A	A	A	A	$\frac{\pi}{2}$	A	$-\frac{\pi}{2}$	A	A	A	A
3	B	B	A	$\frac{\pi}{2}$	A	$-\frac{\pi}{2}$	A	A	A	A	B	0	B	π
4	B	B	$\frac{\pi}{2}$	A	$\frac{\pi}{2}$	A	A	A	A	A	0	B	π	B
5	0	B	B	A	B	A	A	A	A	A	B	π	B	B
6	0	B	A	B	A	B	A	A	A	A	π	B	B	B
7	B	0	B	A	B	A	A	A	A	A	B	B	B	π
8	B	0	A	B	A	B	A	A	A	A	B	B	π	B

All the boxes that were blank in the original table have been marked A or B. Consider for example column 5, corresponding to v_5. It follows from the definition of a precromulent labelling that the stabiliser group of v_5 is $\{1, \mu\nu, \lambda^2\mu, \lambda^2\nu\}$, so in particular v_5 is not fixed by ν, $\lambda\nu$ or $\lambda^3\nu$, so it cannot lie in $c_i(\mathbb{R})$ for $i \in \{1, 2, 4, 5, 7\}$. This accounts for all the boxes in column 5 marked A. We also have $-\pi/2$ in row 3, indicating that $v_5 = c_3(-\pi/2) \in C_3$. As C_3, C_6 and C_8 are disjoint, we see that $v_5 \notin c_6(\mathbb{R})$ and $v_5 \notin c_8(\mathbb{R})$, which accounts for the remaining two boxes in column 5 marked B. The same line of argument works for all the other columns. □

Definition 2.4.9. [defn-std-isotropy]

We say that X has *standard isotropy* if

$$\begin{aligned} X^{\mu\nu} &= C_0 \\ X^{\lambda\nu} &= C_1 \\ X^{\lambda^3\nu} &= C_2 \\ X^{\nu} &= C_4 \amalg C_5 \amalg C_7 \\ X^{\lambda^2\nu} &= C_3 \amalg C_6 \amalg C_8 \\ X^{\lambda^2\mu\nu} &= \emptyset. \end{aligned}$$

We will show later that every cromulent surface has standard isotropy.

2.5. **Holomorphic curve systems.** [sec-holomorphic-curves]

For any curve system, it turns out that each map $c_k\colon \mathbb{R} \to X$ can be extended to give a holomorphic map defined on a suitable neighbourhood of $\mathbb{R}$ in $\mathbb{C}$. We will start by developing the relevant theory in a slightly more abstract setting.

Proposition 2.5.1. [prop-chart]

Let X be a Riemann surface, and let $c\colon \mathbb{R} \to X$ be a real analytic map with $c'(0) \neq 0$. Then there is a unique germ of an analytic map $\phi\colon \mathbb{C} \to X$ with $\phi(t) = c(t)$ for small real values of t. Similarly, there is a unique local conformal parameter z at $c(0)$ such that $z(c(t)) = t$ for small $t \in \mathbb{R}$. Moreover, if τ is an antiholomorphic involution on X with $\tau(c(t)) = c(t)$ for all t, then $z(\tau(u)) = \overline{z(u)}$.

Proof. The claim is local on X, so we may assume that $X = \mathbb{C}$ and $c(0) = 0$. As c is real analytic, there are coefficients $a_k \in \mathbb{C}$ such that $c(t) = \sum_k a_k t^k$, with the sum being absolutely convergent for small real values of t. It follows in a standard way that the sum is still absolutely convergent for small complex values of t, and this gives us a germ of a complex analytic map $\phi\colon \mathbb{C} \to X$ extending c. This is unique, by the Identity Principle. We also have $\phi'(0) = c'(0) \neq 0$, so ϕ is locally invertible near 0, and the inverse is the unique local parameter z such that $z(c(t)) = t$.

Now suppose that τ is an antiholomorphic involution on X with $\tau(c(t)) = c(t)$ for all t. Then the map $u \mapsto \overline{z(\tau(u))}$ has the defining property of z, and so is the same as z, as claimed. □

Given $c\colon \mathbb{R} \to X$, we can apply the above proposition to $c(t_0 + t)$ for various different values of t_0, and then patch the results together. To organise this construction, we introduce the following definitions:

Definition 2.5.2. [defn-band]

Consider a point $z = x + iy \in \mathbb{C}$. We let $\mathcal{Q}(z)$ denote the set of pairs $(U_0, \widetilde{c}_0)$, where U_0 is a convex open subset of $\mathbb{C}$ containing x and z, and $\widetilde{c}_0\colon U_0 \to X$ is a holomorphic map with $\widetilde{c}_0|_{U_0\cap\mathbb{R}} = c|_{U_0\cap\mathbb{R}}$. We then put $V = \{z \mid \mathcal{Q}(z) \neq \emptyset\}$. If $z \in V$ then we choose any $(U_0, \widetilde{c}_0) \in \mathcal{Q}(z)$ and put $\widetilde{c}(z) = \widetilde{c}_0(z)$; a straightforward argument with the identity principle shows that this is independent of the choice of $(U_0, \widetilde{c}_0)$.

Proposition 2.5.3. [prop-band-chart]

The set V is open in $\mathbb{C}$ and contains $\mathbb{R}$, and it is closed under conjugation. The map $\widetilde{c}\colon V \to X$ is holomorphic, and satisfies $\widetilde{c}|_{V\cap\mathbb{R}} = c$ and $\widetilde{c}(\overline{z}) = \tau(\widetilde{c}(z))$. Moreover, if $c(t + 2\pi) = c(t)$ for all t, then we also have $V + 2\pi = V$ and $\widetilde{c}(t + 2\pi) = \widetilde{c}(t)$.

Proof. Straightforward. □

Corollary 2.5.4. [cor-band-charts]

Suppose that X is a cromulent surface, and $(c_k)_{k=0}^{8}$ is a curve system. Then each map $c_k\colon \mathbb{R} \to X$ has a canonical holomorphic extension $\widetilde{c}_k\colon V_k \to X$, where V_k is a 2π-periodic open neighbourhood of $\mathbb{R}$ in $\mathbb{C}$. □

2.6. **Fundamental domains.** [sec-fundamental]

Definition 2.6.1. [defn-fundamental]

Let X be a compact Hausdorff space, and let H be a finite group acting continuously on X. Let F be a closed subset of X.

- (a) We say that F is a *fundamental domain* for H if $X = \bigcup_{\gamma\in H} \gamma(F)$, and $\mathrm{int}(F) \cap \gamma(F) = \emptyset$ for all $\gamma \in H \setminus \{1\}$.
- (b) We say that F is a *retractive fundamental domain* if, in addition, there is a continuous map $r\colon X \to F$ such that
  - (i) $r(x) = x$ for all $x \in F$ (so r is a retraction).
  - (ii) $r(\gamma(x)) = r(x)$ for all $x \in X$ and $\gamma \in H$.

Proposition 2.6.2. [prop-fundamental]

Let F be a retractive fundamental domain for H, with retraction r. Then

- (a) *For all $x \in X$, the point $r(x)$ lies in the same H-orbit as x.*
- (b) *For all $\gamma \in H$ we have $F \cap \gamma(F) = \{x \in F \mid \gamma(x) = x\}$.*

(c) *There is a canonical homeomorphism* $X/H \simeq F$.

(d) *There is a canonical homeomorphism* $X \simeq (G \times F)/\sim$, *where* $(\gamma_0, x_0) \sim (\gamma_1, x_1)$ *iff* $x_0 = x_1$ *and* $\gamma_1^{-1}\gamma_0 \in \mathrm{stab}_G(x_0)$.

Proof. (a) As F is a fundamental domain, we have $x = \gamma(y)$ for some $y \in F$ and $\gamma \in H$. This gives $r(x) = r(\gamma(y))$, but we can use axioms (ii) and (i) to see that $r(\gamma(y)) = r(y) = y$, so $r(x) = y$. Thus, x and $r(x)$ lie in the same H-orbit.

(b) Now suppose that $x \in F \cap \gamma(F)$, so $x = \gamma(y)$ for some $y \in F$. We now have $x = r(x) = r(\gamma(y)) = r(y) = y$, so $x = \gamma(x)$ as required.

(c) We have an inclusion $j \colon F \to X$ and a projection $p \colon X \to X/H$. We will show that pj is a homeomorphism.

As r is continuous with $r(\gamma(x)) = r(x)$ for all x and γ, we see that there is a unique map $\overline{r} \colon X/H \to F$ with $\overline{r}p = r$, and that this is continuous. As r is a retraction we have $\overline{r}pj = rj = 1$. Next, as x is in the same orbit as $r(x)$, we have $p(x) = pjr(x) = pj\overline{r}p(x)$. As p is surjective it follows that $pj\overline{r} = 1$. This proves that $\overline{r}$ is an inverse for pj, as required.

(d) We have a continuous map $m \colon G \times F \to X$ given by $m(\gamma, x) = \gamma(x)$. As $X = \bigcup_\gamma \gamma(F)$ we see that m is surjective. The source and target are compact Hausdorff spaces, so m is automatically a quotient map. If $m(\gamma_0, x_0) = m(\gamma_1, x_1)$ then the element $\gamma = \gamma_1^{-1}\gamma_0$ has $\gamma(x_0) = x_1$. Applying r to this gives $x_0 = x_1$, and it follows that $\gamma \in \mathrm{stab}_G(x_0)$. It follows that m induces a homeomorphism $(G \times F)/\sim \to X$, as claimed.

□

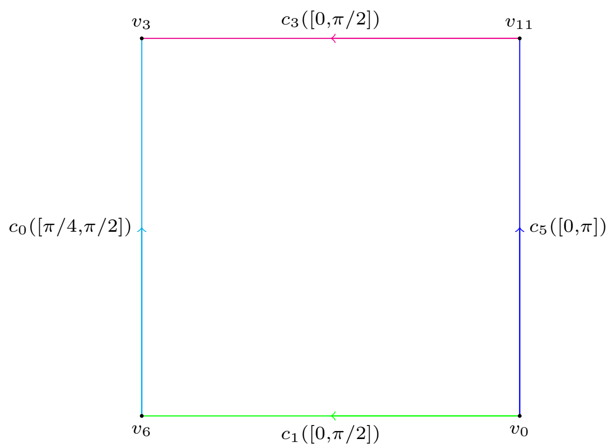
Now let X be a labelled precromulent surface with a curve system. By the axioms for a curve system, we have

$$v_0 = c_1(0) = c_5(0) \qquad v_3 = c_0(\pi/2) = c_3(\pi/2)$$
$$v_6 = c_0(\pi/4) = c_1(\pi/2) \qquad v_{11} = c_3(0) = c_5(\pi).$$

This means that the set

$$DF_{16} = c_0([\tfrac{\pi}{4}, \tfrac{\pi}{2}]) \cup c_1([0, \tfrac{\pi}{2}]) \cup c_3([0, \tfrac{\pi}{2}]) \cup c_5([0, \pi])$$

fits together as follows:



(Using Proposition 2.4.7, we see that the four boundary arcs cannot have any additional intersection points.)

Remark 2.6.3. Information about the above picture is stored as a table in Maple in the global variable `F16_curve_limits`, which is defined in `cromulent.mpl`. For example, `F16_curve_limits[1]` is the range `0..Pi/2`, whereas `F16_curve_limits[2]` is `NULL` (because none of the sides of DF_{16} lies along C_2). Note that Maple does not display the full structure of tables by default; to see all entries in `F16_curve_limits`, one needs to enter `eval(F16_curve_limits)`, not just `F16_curve_limits`.

Remark 2.6.4. The colours in the above diagram will be used systematically throughout this monograph: the curve c_0 is cyan, the curves c_1 and c_2 are green, the curves c_3 and c_4 are magenta, and the curves

c_5 to c_8 are blue. The colour of c_k is represented in Maple as `c_colour[k]`. Readers who have trouble distinguishing these colours can try changing the definitions of `c_colour[k]` in the file `cromulent.mpl` and regenerating the diagrams using the functions in various files called `plots.mpl` appearing in several different directories. However, colour should not be strictly necessary for any of the diagrams.

Lemma 2.6.5. [lem-F-stabilisers]
The stabilisers of points in DF_{16} are as follows:

$$\begin{aligned} \mathrm{stab}_G(v_0) &= \langle \lambda, \nu \rangle & \mathrm{stab}_G(v_3) &= \langle \lambda^2\mu, \lambda^2\nu \rangle \\ \mathrm{stab}_G(v_6) &= \langle \lambda\mu, \lambda\nu \rangle & \mathrm{stab}_G(v_{11}) &= \langle \lambda^2, \nu \rangle \end{aligned}$$

$$\begin{aligned} \mathrm{stab}_G(c_0(t)) &= \{1, \mu\nu\} && \text{for } \pi/4 < t < \pi/2 \\ \mathrm{stab}_G(c_1(t)) &= \{1, \lambda\nu\} && \text{for } 0 < t < \pi/2 \\ \mathrm{stab}_G(c_3(t)) &= \{1, \lambda^2\nu\} && \text{for } 0 < t < \pi/2 \\ \mathrm{stab}_G(c_5(t)) &= \{1, \nu\} && \text{for } 0 < t < \pi. \end{aligned}$$

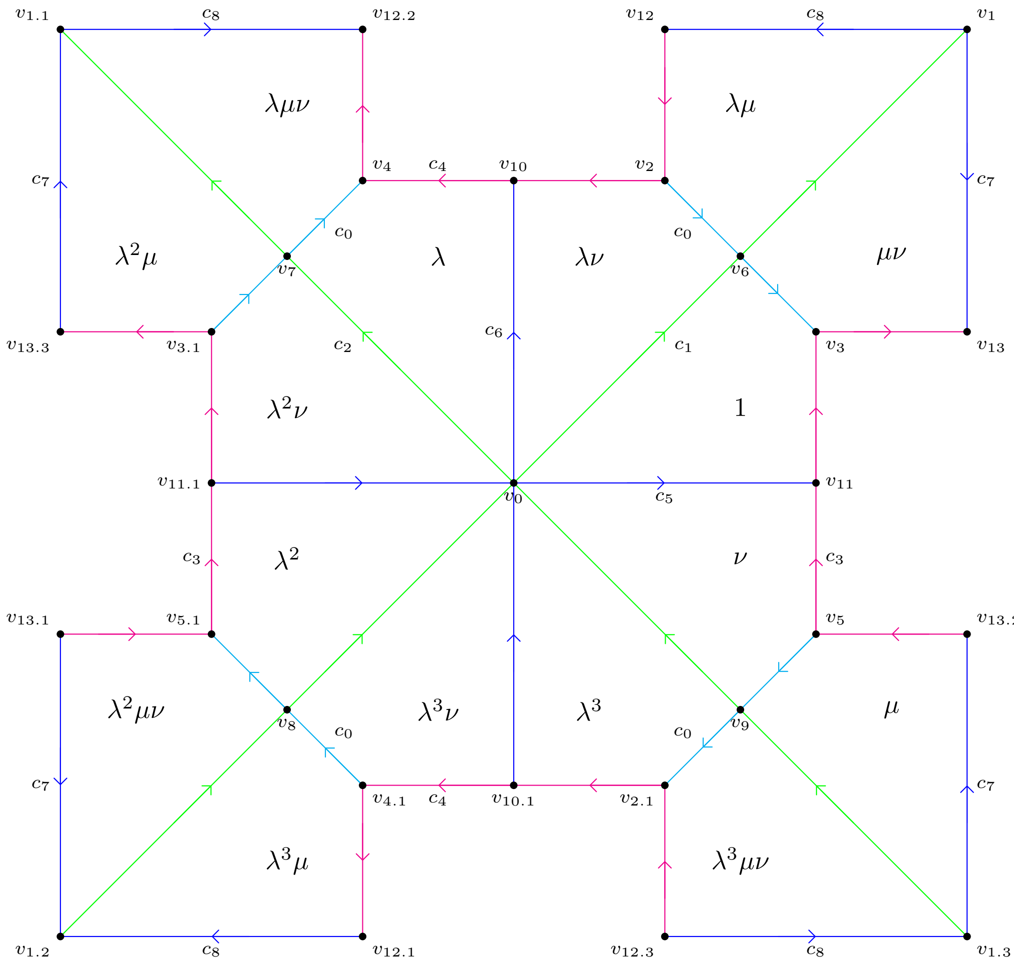
Proof. The stabilisers for the points v_i are determined by Definition 1.0.2. Next, axiom (c) in Definition 2.4.4 tells us that $\mathrm{stab}_G(c_0(t)) \supseteq \{1, \mu\nu\}$ for all t. Moreover, if $\pi/4 < t < \pi/2$ then axiom (b) tells us that $c_0(t) \neq v_i$ for all i, so $\mathrm{stab}_G(c_0(t)) \cap D_8 = \{1\}$. It is easy to see that any subgroup strictly larger than $\{1, \mu\nu\}$ has nontrivial intersection with D_8, so we must have $\mathrm{stab}_G(c_0(t)) = \{1, \mu\nu\}$ as claimed. The same line of argument works for c_1, c_3 and c_5. □

Definition 2.6.6. [defn-standard-F]
A *standard fundamental domain* is a subset $F_{16} \subseteq X$ that is a retractive fundamental domain for G and is homeomorphic to a square and has boundary DF_{16}.

Remark 2.6.7. [rem-standard-F]
We will see later that every cromulent surface has a unique standard fundamental domain. Conversely, suppose that X is a labelled precromulent surface with a given curve system and a standard fundamental domain, and that $\lambda_* = i \colon T_{v_0}X \to T_{v_0}X$. Then the interior of the standard fundamental domain has the property specified in Definition 1.0.4(d), which proves that X is actually cromulent.

Remark 2.6.8. If F_{16} is a standard fundamental domain, the Proposition 2.6.2(d) allows us to identify X with $(G \times F_{16})/\sim$ for a certain equivalence relation $\sim$. This relation depends only on the stabilisers of points in DF_{16}, which are given by Lemma 2.6.5. We can also identify F_{16} with $[0, 1]^2$ and thus identify X with a quotient of $G \times [0, 1]^2$.

If we perform only some of the identifications given by the above equivalence relation, we obtain the following space, which we call Net_0. It is clearly homeomorphic to a square.



`nets_check.mpl: check_nets()`

Remark 2.6.9. There is code for dealing with nets in the file `nets.mpl`. This uses the object oriented programming framework described in Section 9.3. Information about Net_0 is stored in the variable `net_0`, as an instance of the class `net`. This means that

- One can enter `net_0["v"][12.1]` to retrieve the coordinates of the point $v_{12.1}$ in the above picture.
- One can enter `net_0["squares"][M]` to retrieve the list `[9,1,13.2,5]` corresponding to the vertices of the region marked μ (recall that μ is represented as `M` in Maple).
- One can enter `net_0["plot"]` to generate a picture of the net as a Maple plot structure. Note that this is an example of a method rather than a property: it performs an operation rather than simply returning information that was previously stored.
- One can enter `net_0["check"]` to perform various consistency checks on the combinatorial structure of the net.

There are also various other properties and methods.

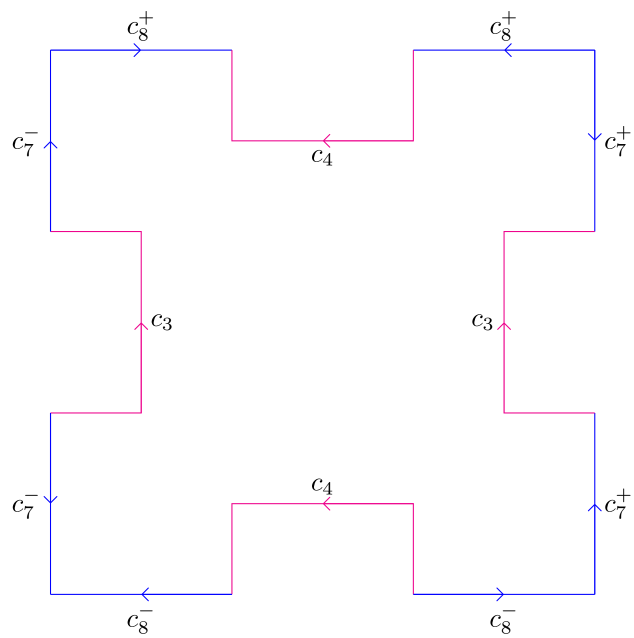
Remark 2.6.10. [rem-match-net]

Elsewhere we will consider a number of other constructions which give partial or global maps between cromulent surfaces and $\mathbb{R}^2$ or $\mathbb{C}$ or $\mathbb{C}_\infty \simeq S^2$ or $\mathbb{R}^2/\mathbb{Z}^2$. We will usually arrange the details of such maps so that they match up with Net_0 as far as possible: v_0 will go to the origin, $c_5(t)$ will go to the positive x-axis for small $t > 0$, $c_6(t)$ will go to the positive y-axis for small $t > 0$, and so on.

We can obtain the space X by performing some additional identifications on the boundary. The points marked v_{12}, $v_{12.1}$, $v_{12.2}$ and $v_{12.3}$ all map to v_{12}, and similarly for the other points with fractional labels.

The above net inherits an orientation from $\mathbb{R}^2$, but it also inherits an orientation from X, so we can ask whether these orientations are the same. To see that they are, recall that λ acts on the tangent space $T_{v_0}X$ as multiplication by i. (This was part of the definition of a cromulent labelling.) On the other hand, we have $\lambda(c_1(t)) = c_2(t)$, and from this we see that λ acts on the net near v_0 as an anticlockwise turn through $\pi/2$. This implies that the orientations are compatible as claimed.

To explain the gluing conditions on the boundary in more detail, we use the following, less cluttered version of the above diagram:

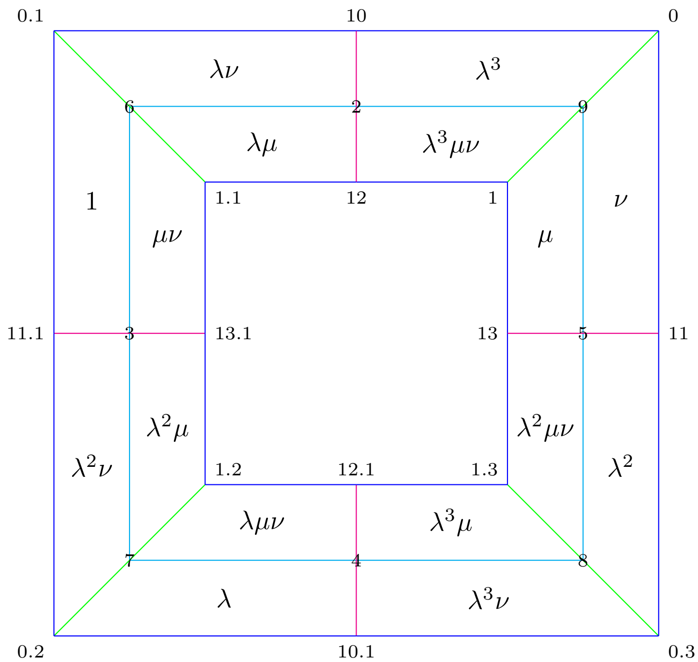


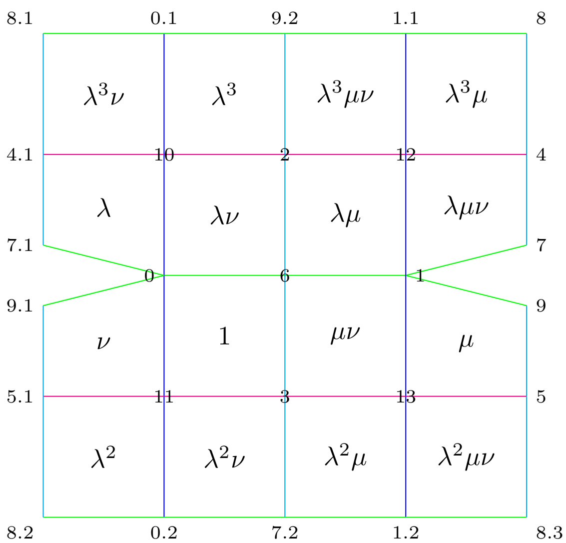
The gluing rules are as follows:

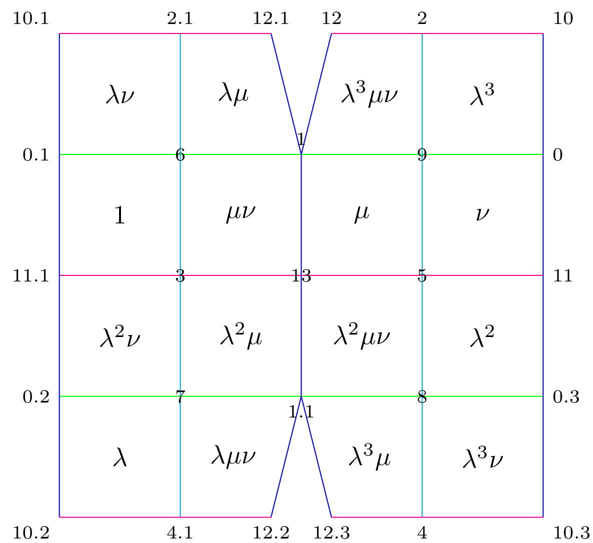
- The two edges marked c_7^+ are identified together.
- The two edges marked c_7^- are identified together.
- The two edges marked c_8^+ are identified together.
- The two edges marked c_8^- are identified together.
- The curve marked c_3 consisting of three edges at the left of the diagram is identified with the corresponding curve at the right of the diagram.
- The curve marked c_4 consisting of three edges at the top of the diagram is identified with the corresponding curve at the bottom of the diagram.

The edges c_k^+ (for $k \in \{7, 8\}$) become the arcs $c_k([0, \pi]) \subseteq X$, and the edges c_k^- become the arcs $c_k([-\pi, 0])$.

Here are three more ways we can perform partial gluing to get a net for X; we will call them Net_1, Net_2 and Net_3.



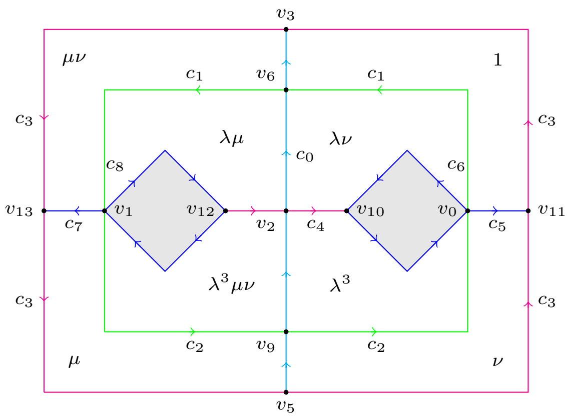


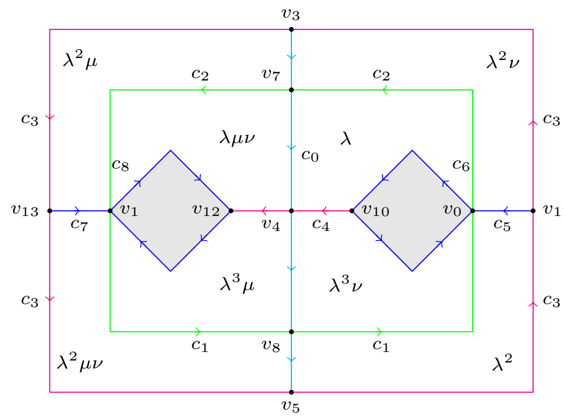


```
nets_check.mpl: check_nets()
```

In all of $\mathrm{Net}_0, \ldots, \mathrm{Net}_3$, the labels 0, 11, 3 and 6 occur in anticlockwise order around the region marked 1. This shows that all the nets have orientation compatible with each other and thus also compatible with the orientation of X.

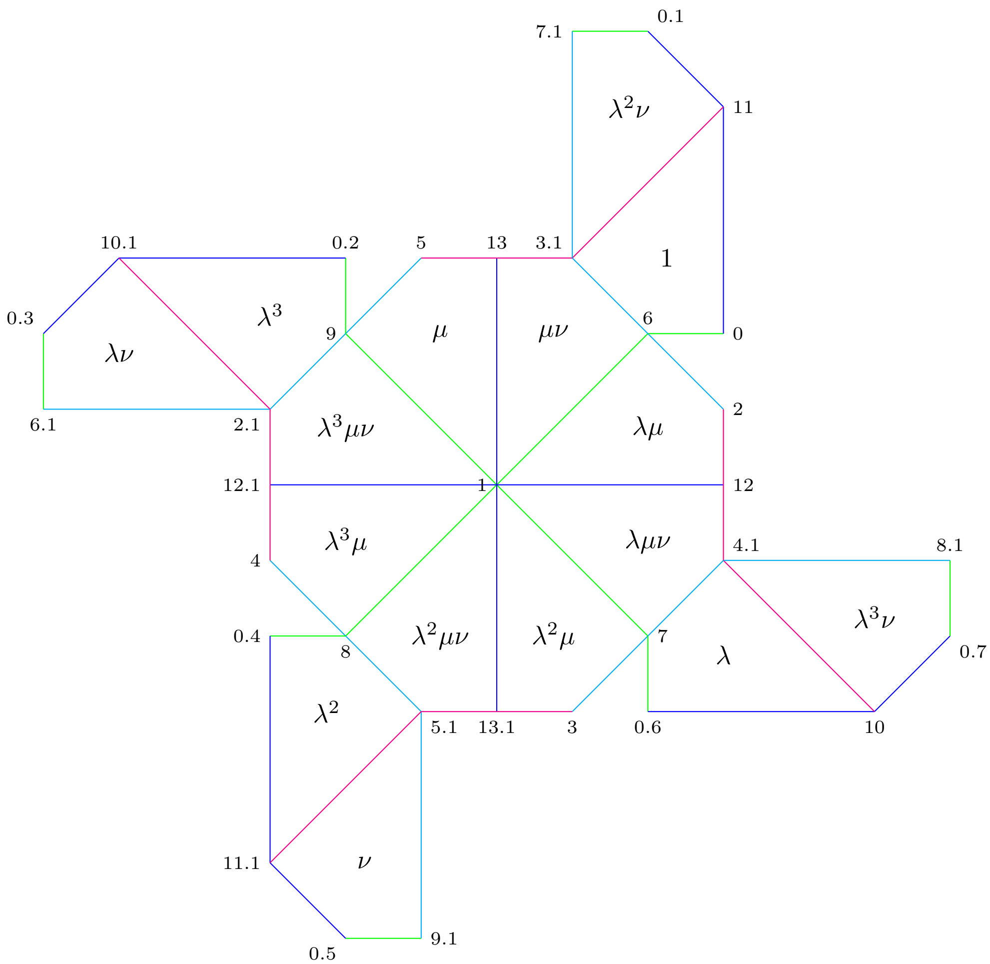
Here is another way to assemble the pieces. The left hand picture (which we call Net_4^+) consists of eight distorted copies of F_{16}, and is homeomorphic to a disc with two holes, or a "pair of pants". The right hand picture (Net_4^-) consists of the other eight translates of F_{16}. The surface can be obtained by gluing the two pictures together along $C_3 \amalg C_6 \amalg C_8$: this is a "pair of pants decomposition".





In this case the orientation of Net_4^+ is compatible with the orientation of X, but the orientation of Net_4^- is reversed.

We now give another net which we call Net_5. Note that the central octagon is the same as for Net_0, but the outer pieces have been rearranged.



The point about Net_5 is that it allows us to read off a convenient presentation of the fundamental group $\pi_1(X, v_0)$.

Definition 2.6.11. We define $\beta_i \colon [0,1] \to X$ for $i \in \mathbb{Z}/8$ by

$$\beta_0(t) = c_5(2\pi t)$$
$$\beta_1(t) = \begin{cases} c_1(-3t\pi) & 0 \leq t \leq 1/6 \\ c_0((-1-3t)\pi/2) & 1/6 \leq t \leq 2/6 \\ c_4((6t-1)\pi/2) & 2/6 \leq t \leq 4/6 \\ c_0((3t-2)\pi/2) & 4/6 \leq t \leq 5/6 \\ c_1((3-3t)\pi) & 5/6 \leq t \leq 1, \end{cases}$$

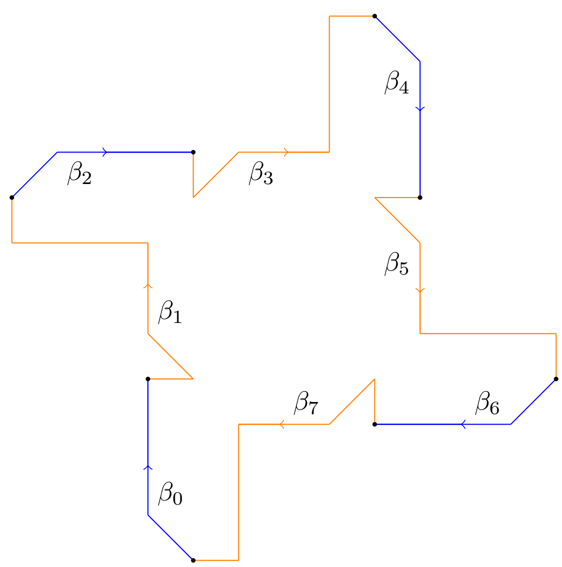
then $\beta_{i+2j}(t) = \lambda^j \beta_i(t)$ for $i \in \{0,1\}$ and $j \in \{1,2,3\}$.

It is straightforward to check that $\beta_k(0) = \beta_k(1) = v_0$ for all k, so β_k represents an element of $\pi_1(X, v_0)$.

Proposition 2.6.12. [prop-pi-one]
$\pi_1(X, v_0)$ is generated by the elements β_i, subject only to the relations $\beta_i\beta_{i+4} = 1$ and

$$\beta_0\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7 = 1.$$

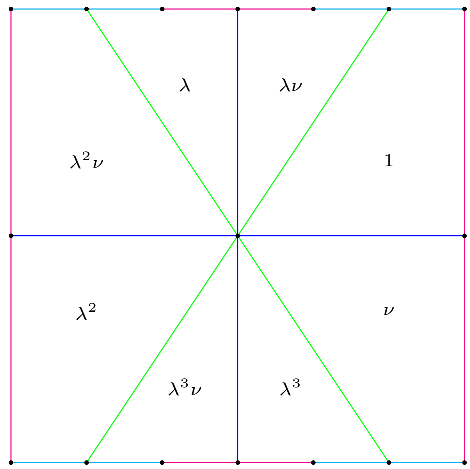
Proof. Inspection of the definitions, together with part (c) of Definition 2.4.4, shows that $\beta_{i+4}(t) = \beta_i(1-t)$ for all i, so β_{i+4} is inverse to β_i in $\pi_1(X, v_0)$. The paths β_i appear in Net_5 as follows:



The surface X can be recovered by gluing β_i to the reverse of β_{i+4} for all i, so we have a presentation of X of the type used in the standard approach to the classification of surfaces, which gives the claimed presentation of the fundamental group. ☐

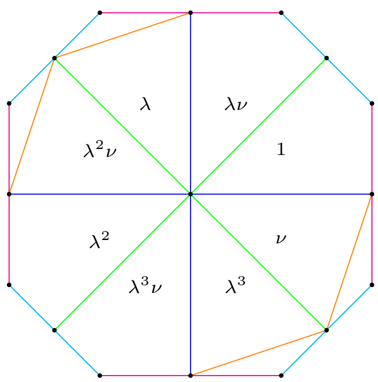
Remark 2.6.13. [rem-groupoid]
For some purposes it is more convenient to work with the fundamental groupoid $\Pi_1(X)$, or the full subgroupoid $\Gamma \subset \Pi_1(X)$ with objects $\{v_i \mid 0 \leq i < 14\}$. This has an action of G by groupoid automorphisms. Each side of F_{16} gives a generator, and the interior of F_{16} gives a relation. One can check that the G-orbits of these generators and relations give an equivariant presentation for Γ, fom which one can recover our earlier presentation of the group $\pi_1(X, v_0) = \Gamma(v_0, v_0)$. Details are in the file `groupoid.mpl`.

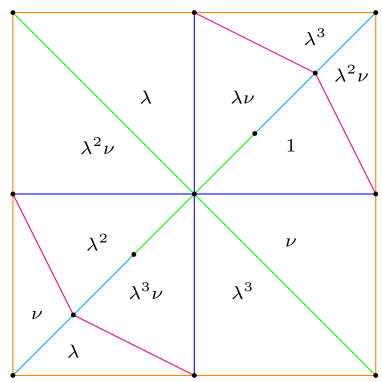
Given a subgroup $H \leq G$, it is usually straightforward to find a subset of Net_0 that gives a fundamental domain for the action of H, and thus to understand the topology of X/H. In the case $H \leq D_8$, this will be consistent with Corollary 2.3.2. We will do this explicitly for the cases $H = \langle\lambda\rangle$ and $H = \langle\lambda\mu\rangle$, where X/H is an elliptic curve. First, the inner octagon in Net_0 is a fundamental domain for $\langle\mu\rangle$. We can redraw this octagon in a slightly distorted form as follows:



One can now check that $X/\langle\mu\rangle$ is obtained by gluing the left edge of the square to the right edge, and the top to the bottom, which gives a torus as expected.

For $X/\langle\lambda\mu\rangle$, it is best to cut some corners off the inner octagon as shown on the left below, and then rearrange the pieces as shown on the right.





One can again check that $X/\langle\lambda\mu\rangle$ is obtained by gluing the left edge of the square to the right edge, and the top to the bottom, which gives a torus as expected.

2.7. **Homology.** [sec-homology]

We next consider the homology groups of a cromulent surface. For any compact Riemann surface of genus 2, it is standard that $H_0(X) \simeq H_2(X) \simeq \mathbb{Z}$ and $H_1(X) \simeq \mathbb{Z}^4$, and that all other homology groups are zero. Our main task is to give specific generators for $H_2(X)$, and understand the action of G in terms of those generators. One approach is to recall that $H_1(X)$ is just the abelianization of $\pi_1(X, v_0)$; we see from Proposition 2.6.12 that this is the free abelian group generated by $\{\beta_0, \beta_1, \beta_2, \beta_3\}$. However, we will use different generators that interact with the group action in a more convenient way.

Proposition 2.7.1. [prop-homology]

Let X be a cromulent surface with a curve system. Then there is an isomorphism $\psi\colon H_1(X) \to \mathbb{Z}^4$, with the following effect on the homology classes of the curves c_k:

$$
\begin{array}{ll}
\psi(c_0) = (\ \ 0,\ \ 0,\ \ 0,\ \ 0) & \\
\psi(c_1) = (\ \ 1,\ \ 1,-1,-1) & \psi(c_2) = (-1,\ \ 1,\ \ 1,-1) \\
\psi(c_3) = (\ \ 0,\ \ 1,\ \ 0,-1) & \psi(c_4) = (-1,\ \ 0,\ \ 1,\ \ 0) \\
\psi(c_5) = (\ \ 1,\ \ 0,\ \ 0,\ \ 0) & \psi(c_6) = (\ \ 0,\ \ 1,\ \ 0,\ \ 0) \\
\psi(c_7) = (\ \ 0,\ \ 0,\ \ 1,\ \ 0) & \psi(c_8) = (\ \ 0,\ \ 0,\ \ 0,\ \ 1).
\end{array}
$$

This is equivariant with respect to the following action of G on $\mathbb{Z}^4$:

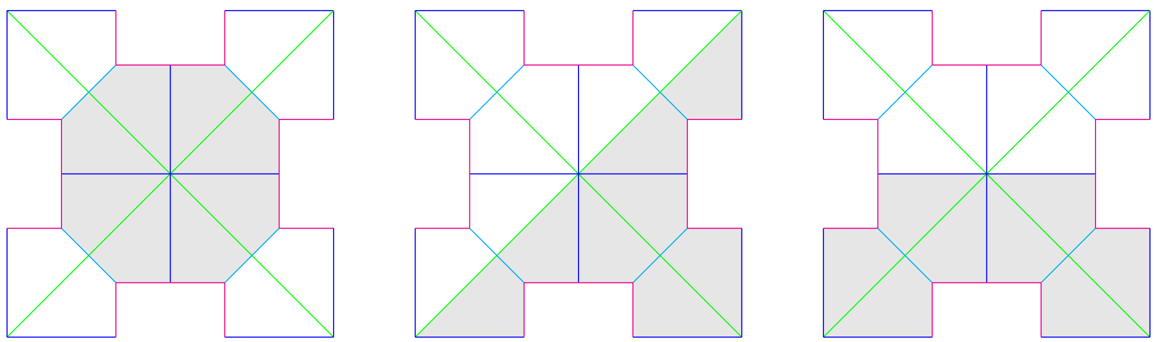
$$\begin{aligned}
\lambda(n) &= (-n_2, \quad n_1, -n_4, \quad n_3) \\
\mu(n) &= (\quad n_3, -n_4, \quad n_1, -n_2) \\
\nu(n) &= (\quad n_1, -n_2, \quad n_3, -n_4).
\end{aligned}$$

Moreover, the intersection product on $H_1(X)$ corresponds to the following bilinear form on $\mathbb{Z}^4$:

$$(n, m) = n_1m_2 - n_2m_1 - n_3m_4 + n_4m_3.$$

Proof. We use the net Net_0 for X discussed in Section 2.6. This gives a CW structure on X with a single 0-cell, a single 2-cell, and four 1-cells corresponding to c_3, c_4, c_7 and c_8. The attaching map for the 2-cell is a product of commutators and so is homologically trivial. It follows that the homology classes $[c_k]$ for $k \in \{3, 4, 7, 8\}$ give a basis for $H_1(X)$.

We next derive some relations. Consider the following subsets of Net_0:



r_0 $\qquad$ r_1 $\qquad$ r_2

The boundary of r_0 consists of four fragments of c_0 (which together make up the whole of c_0) together with a fragment of c_3 repeated twice with opposite orientations, and a fragment of c_4 repeated twice with opposite orientations. From this we conclude that $[c_0] = 0$ in $H_1(X)$. Similarly, the boundary of r_1 consists of c_1, c_3 and c_4 together with mutually cancelling fragments of c_7 and c_8. Here c_1 and c_4 run clockwise but c_3 runs anticlockwise. We therefore have $[c_1] - [c_3] + [c_4] = 0$. Applying the same method to r_2 gives $[c_4]+[c_5]-[c_7] = 0$. Next, part of the definition of a curve system is that $\lambda(c_1(t)) = c_2(t)$ and $\lambda(c_3(t)) = c_4(t)$ and $\lambda(c_4(t)) = c_3(-t)$, which gives $\lambda_*[c_1] = [c_2]$ and $\lambda_*[c_3] = [c_4]$ and $\lambda_*[c_4] = -[c_3]$. We can therefore apply λ_* to the relation $[c_1] - [c_3] + [c_4] = 0$ to get $[c_2] - [c_3] - [c_4] = 0$. Similarly, we can apply λ_* to the relation $[c_4] + [c_5] - [c_7] = 0$ to get $-[c_3] + [c_6] - [c_8] = 0$. This is enough to show that $[c_5]$, $[c_6]$, $[c_7]$ and $[c_8]$ form an alternative basis for $H_1(X)$. By writing everything in terms of this basis we get an isomorphism $\psi\colon H_1(X) \to \mathbb{Z}^4$, and by inspecting the above relations we see that this is given by the claimed formulae.

Next, it is part of the definition of cromulence that

$$\begin{aligned}
&\lambda(c_5(t)) = c_6(t) && \mu(c_5(t)) = c_7(t) && \nu(c_5(t)) = c_5(t) \\
&\lambda(c_6(t)) = c_5(-t) && \mu(c_6(t)) = c_8(-t) && \nu(c_6(t)) = c_6(-t) \\
&\lambda(c_7(t)) = c_8(t) && \mu(c_7(t)) = c_5(t) && \nu(c_7(t)) = c_7(t) \\
&\lambda(c_8(t)) = c_7(-t) && \mu(c_8(t)) = c_6(-t) && \nu(c_8(t)) = c_8(-t).
\end{aligned}$$

The action in homology can be read off from this in an obvious way, and we find that it works as described in the statement of the Proposition.

Finally, we need to analyse the intersection pairing. From the definition of a curve system and associated discussion, we see that $C_5 \cap C_6 = \{v_0\}$ and $C_7 \cap C_8 = \{v_1\}$ and

$$C_5 \cap C_7 = C_5 \cap C_8 = C_6 \cap C_7 = C_6 \cap C_8 = \emptyset.$$

This means that the corresponding products in homology are $[c_5] \cdot [c_6] = \pm 1$ and $[c_7] \cdot [c_8] = \pm 1$ and

$$[c_5] \cdot [c_7] = [c_5] \cdot [c_8] = [c_6] \cdot [c_7] = [c_6] \cdot [c_8] = 0.$$

In the net, v_0 is the origin, c_5 runs to the right along the x-axis and c_6 runs upwards along the y-axis, so $[c_5] \cdot [c_6] = +1$. Moreover, the map μ preserves orientation and has $\mu_*[c_5] = [c_7]$ and $\mu_*[c_6] = -[c_8]$; this means that $[c_7] \cdot [c_8] = -1$. The claimed description of the intersection product follows easily.

```
homology_check.mpl: check_homology()
```

∎

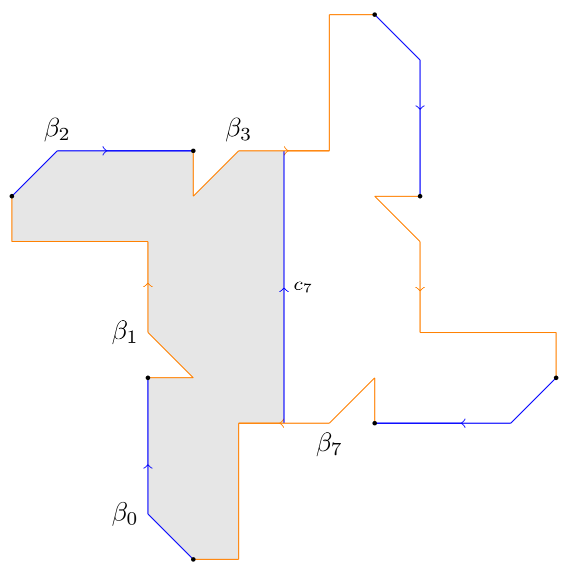
Remark 2.7.2. The vector $\psi(c_4) = (-1, 0, 1, 0)$ (for example) is represented in Maple as `c_homology[4]`. The action of G on $\mathbb{Z}^4$ is represented by `act_Z4`, which is a table indexed by the elements of G, whose entries are functions. For example, the expression `act_Z4[LMN]([1,2,3,4])` evaluates to `[-4,3,-2,1]`, corresponding to the fact that $\lambda\mu\nu(1, 2, 3, 4) = (-4, 3, -2, 1)$. Most other actions of G on other sets are also represented in this way.

We now relate the above description to the fundamental group.

Lemma 2.7.3. *The homology classes of the generators $\beta_i \in \pi_1(X, v_0)$ are as follows:*

$$
\begin{aligned}
[\beta_0] &= (1, 0, 0, 0) & [\beta_1] &= (-1, -1, 1, 0) \\
[\beta_2] &= (0, 1, 0, 0) & [\beta_3] &= (1, -1, 0, 1) \\
[\beta_4] &= (-1, 0, 0, 0) & [\beta_5] &= (1, 1, -1, 0) \\
[\beta_6] &= (0, -1, 0, 0) & [\beta_7] &= (-1, 1, 0, -1)
\end{aligned}
$$

Proof. The claim for β_0 is immediate from the definitions, and the claims for β_2, β_4 and β_6 follow using the group action. Now consider the left hand half of Net_5:



Define $u(t) = \beta_3(t/2) = \beta_7(1 - t/2)$ for $0 \leq t \leq 1$. The boundary of the above region gives a relation

$$[\beta_0] + [\beta_1] + [\beta_2] + [u] - [c_7] - [u] = 0,$$

so

$$[\beta_1] = [c_7] - [\beta_0] - [\beta_2] = (-1, -1, 1, 0)$$

as claimed. The remaining claims for β_3, β_5 and β_7 now follow using the group action. ∎

For some purposes it is convenient to use a different basis. We put

$$
\begin{aligned}
u_{11} &= (1, 0, 1, 0) & u_{12} &= (0, 1, 0, 1) \\
u_{21} &= (1, 0, -1, 0) & u_{22} &= (0, 1, 0, -1).
\end{aligned}
$$

These elements do not generate all of $\mathbb{Z}^4$, but only a subgroup of index 4. However, they give a basis for $\mathbb{Q}^4$. One can check that

$$\begin{aligned}
\lambda(u_{11}) &= u_{12} & \mu(u_{11}) &= u_{11} & \nu(u_{11}) &= u_{11} \\
\lambda(u_{12}) &= -u_{11} & \mu(u_{12}) &= -u_{12} & \nu(u_{12}) &= -u_{12} \\
\lambda(u_{21}) &= u_{22} & \mu(u_{21}) &= -u_{21} & \nu(u_{21}) &= u_{21} \\
\lambda(u_{22}) &= -u_{21} & \mu(u_{22}) &= u_{22} & \nu(u_{22}) &= -u_{22}.
\end{aligned}$$

This shows that $\{u_{11}, u_{12}\}$ is a basis for a subrepresentation U_1 with character χ_8, whereas $\{u_{21}, u_{22}\}$ is a basis for a subrepresentation U_2 with character χ_9.

We now consider the homology of certain quotient spaces X/H. For most subgroups $H \leq G$, the quotient X/H is either a disc or a sphere, and so the first homology group is trivial. The interesting cases are the elliptic curves $X/\langle\mu\rangle$ and $X/\langle\lambda\mu\rangle$. We write $q_+ \colon X \to X/\langle\mu\rangle$ and $q_- \colon X \to X/\langle\lambda\mu\rangle$ for the quotient maps.

Proposition 2.7.4. [prop-elliptic-homology]
There is a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[d, "\psi_+"', "\simeq"] & & \mathbb{Z}^4 \arrow[ll, two heads, "\theta_+"'] \arrow[rr, two heads, "\theta_-"] \arrow[d, "\psi"', "\simeq"] & & \mathbb{Z}^2 \arrow[d, "\simeq"', "\psi_-"] \\
H_1(X/\langle\mu\rangle) & & H_1(X) \arrow[ll, two heads, "(q_+)_*"] \arrow[rr, two heads, "(q_-)_*"'] & & H_1(X/\langle\lambda\mu\rangle)
\end{tikzcd}$$

where

$$\begin{aligned}
\theta_+(n) &= (n_1 + n_3,\ n_2 - n_4) \\
\theta_-(n) &= (n_2 + n_3,\ n_1 + n_4).
\end{aligned}$$

Proof. First, we can regard $H_1(X)$ as the abelianization of $\pi_1(X, v_2)$, and $H_1(X/\langle\mu\rangle)$ as the abelianization of $\pi_1(X/\langle\mu\rangle, q_+(v_2))$. Because q_+ is a branched covering, any loop u based at $q_+(v_2)$ can be lifted to give a path $\widetilde{u}$ in X starting at v_2. The endpoint $\widetilde{u}(1)$ will lie over $q_+(v_2)$, but $\mu(v_2) = v_2$ so this ensures that $\widetilde{u}(1) = v_2$. This means that $\widetilde{u}$ is again a loop, and we deduce that the map

$$(q_+)_* \colon \pi_1(X, v_2) \to \pi_1(X/\langle\mu\rangle, q_+(v_2))$$

is surjective. It follows that the map

$$(q_+)_* \colon H_1(X) \to H_1(X/\langle\mu\rangle)$$

is also surjective. A similar argument (using the basepoint $v_6 = \lambda\mu(v_6)$) shows that $(q_-)_*$ is also surjective on H_1. Now recall that ψ is equivariant for the following action:

$$\begin{aligned}
\mu(n) &= (n_3, -n_4, n_1, -n_2) \\
\lambda\mu(n) &= (n_4, n_3, n_2, n_1).
\end{aligned}$$

It follows easily that θ_+ is a coequaliser for the action of μ, and θ_- is a coequaliser for the action of $\lambda\mu$. This implies that there are unique maps ψ_+ and ψ_- making the diagram commute. As ψ is an isomorphism and $(q_\pm)_*$ is surjective, we see that $\psi_\pm$ is also surjective. However, $X/\langle\mu\rangle$ and $X/\langle\lambda\mu\rangle$ are both homeomorphic to the torus, so in each case $H_1 \simeq \mathbb{Z}^2$, and this implies that any surjective homomorphism $\mathbb{Z}^2 \to H_1$ is automatically an isomorphism. $\square$

We next discuss the Jacobian variety JX. This is usually constructed as an abelian variety using methods of algebraic geometry, as we will recall in Section 3.5. However, in our cromulent setting it is possible to construct JX as a space using only topological methods.

We first give some definitions related to coverings and subgroups of the fundamental group. These are standard, but we just want to pin down some issues of naturality.

Definition 2.7.5. Let Y be a (locally tame) path-connected space with a basepoint y_0. We define $\widetilde{Y}$ to be the space of paths in Y starting at y_0, modulo the equivalence relation of homotopy relative to endpoints. This has a natural basepoint $\widetilde{y}_0$, which is the equivalence class of the constant path at y_0. All this is clearly functorial for based maps. Moreover, path join gives a free left action of $\pi_1(Y, y_0)$ on $\widetilde{Y}$. Evaluation at 1 gives a projection $\pi\colon \widetilde{Y} \to Y$, which induces a homeomorphism $\widetilde{Y}/\pi_1(Y, y_0) \to Y$. Given a subgroup $H \leq \pi_1(Y, y_0)$, we have a based covering map $\widetilde{Y}/H \to Y$, whose effect in π_1 is the inclusion $H \to \pi_1(Y, y_0)$. Given a based homeomorphism $f\colon Y_0 \to Y_1$ with $f_*(H_0) = H_1$, we get an induced based homeomorphism $\widetilde{Y}_0/H_0 \to \widetilde{Y}_1/H_1$.

Definition 2.7.6. [defn-JX]
We put $J'X = (X/\langle\mu\rangle) \times (X/\langle\lambda\mu\rangle)$, and define $q'\colon X \to J'X$ by $q'(x) = (q_+(x), q_-(x))$. The resulting map

$$\theta' = (\mathbb{Z}^4 \xrightarrow{\psi} H_1(X) \xrightarrow{q'_+} H_1(X/\langle\mu\rangle) \oplus H_1(X/\langle\lambda\mu\rangle) \xrightarrow{(\psi_+,\psi_-)^{-1}} \mathbb{Z}^4)$$

is then given by

$$\theta'(n) = (\theta_+(n), \theta_-(n)) = (n_1 + n_3, n_2 - n_4, n_1 + n_4, n_2 + n_3).$$

One can check that the image of θ' is the subgroup

$$\Theta' = \{m \in \mathbb{Z}^4 \mid \sum_i m_i = 0 \pmod 2\},$$

which has index two. Note also that $J'X \simeq (S^1)^4$, so the natural map $\pi_1(J'X, q'(v_0)) \to H_1(J'X)$ is an isomorphism. Thus, Θ' corresponds to a subgroup of index two in $\pi_1(J'X, q'(v_0))$, and so gives rise to a double cover of $J'X$, which we call JX. Because $\pi_1(JX) = \Theta'$, we see that q' lifts to give a map $q\colon X \to JX$. By construction, this induces an isomorphism $H_1(X) \to H_1(JX)$.

Remark 2.7.7. If Y and Z are homotopy equivalent to $(S^1)^n$ and $(S^1)^m$, then standard methods of homotopy theory show that the natural map

$$H_1\colon [Y, Z] \to \mathrm{Hom}(H_1(Y), H_1(Z))$$

is bijective. (It makes no difference here whether we work in a based context or an unbased context.) Using this, we can produce a map $JX \times JX \to JX$ which makes JX into an abelian group up to homotopy, and we can produce a map $G \to [JX, JX]$ giving an action of G on JX up to homotopy. Using analytic methods, we can improve this: JX becomes an abelian variety, with an action of G by automorphisms of abelian varieties. However, we cannot build a topological group structure or a G-action by homeomorphisms without using analysis. The element $\lambda^2 \in G$ normalises the subgroups $\langle\mu\rangle$ and $\langle\lambda\mu\rangle$, and preserves the basepoint v_0, so this induces an involution on $J'X$ and on JX. However, no other nontrivial element of G shares these properties.

3. The projective family

[sec-P]
In this section we construct a family of precromulent surfaces $PX(a)$ (for $a \in (0, 1)$) as branched covers of the Riemann sphere $\mathbb{C}_\infty$. Later we will consider an arbitrary precromulent surface X, and attempt to find an isomorphism $X \simeq PX(a)$ for some a. The notion of a cromulent labelling will emerge naturally from this analysis.

3.1. Definitions. [sec-P-defs]

Definition 3.1.1. [defn-P]
For any $a \in (0, 1)$ we put $A = a^2 + 1/a^2 \in (2, \infty)$ and define $r_a\colon \mathbb{C} \to \mathbb{C}$ by

$$r_a(z) = z(z - a)(z + a)(z - 1/a)(z + 1/a) = z^5 - Az^3 + z.$$

Next, we put

$$R_0(a) = \mathbb{C}[z, w]/(w^2 - r_a(z))$$
$$PX_0(a) = \mathrm{spec}(R_0(a)) = \{(w, z) \in \mathbb{C}^2 \mid w^2 = r_a(z)\},$$

so $PX_0(a)$ is a smooth affine hyperelliptic curve. Unfortunately, if we just take the closure in $\mathbb{C}P^2$, the resulting curve is singular at infinity. To get a nonsingular completion, we put

$$PX(a) = \{[z] \in \mathbb{C}P^4 \mid z_1^2 - z_2 z_3 - z_4 z_5 + Az_3 z_4 = 0,\ z_2 z_4 = z_3^2,\ z_2 z_5 = z_3 z_4,\ z_3 z_5 = z_4^2\}.$$

There is a map $j\colon PX_0(a) \to PX(a)$ given by

$$j(w,z) = [w:1:z:z^2:z^3].$$

We define points $v_0, \dots, v_{13} \in PX(a)$ by

$$\begin{aligned}
&v_0 = [0:1:0:0:0] = j(0,0) \\
&v_1 = [0:0:0:0:1] \\
&v_2 = j(-(a^{-1}-a), -1) && v_6 = j\left(\ \ \frac{1+i}{\sqrt{2}}(a^{-1}+a),\ \ i\right) && v_{10} = j(0,-a) \\
&v_3 = j(-i(a^{-1}-a),\ \ 1) && v_7 = j\left(-\frac{1-i}{\sqrt{2}}(a^{-1}+a), -i\right) && v_{11} = j(0,\ \ a) \\
&v_4 = j(\ \ (a^{-1}-a), -1) && v_8 = j\left(-\frac{1+i}{\sqrt{2}}(a^{-1}+a),\ \ i\right) && v_{12} = j(0,-a^{-1}) \\
&v_5 = j(\ \ i(a^{-1}-a),\ \ 1) && v_9 = j\left(\ \ \frac{1-i}{\sqrt{2}}(a^{-1}+a), -i\right) && v_{13} = j(0,\ \ a^{-1}).
\end{aligned}$$

We let G act on $\mathbb{C}P^4$ and $PX(a)$ by

$$\begin{aligned}
\lambda[z] &= [iz_1 : z_2 : -z_3 : z_4 : -z_5] \\
\mu[z] &= [z_1 : z_5 : z_4 : z_3 : z_2] \\
\nu[z] &= [\overline{z}_1 : \overline{z}_2 : \overline{z}_3 : \overline{z}_4 : \overline{z}_5].
\end{aligned}$$

```
cromulent.mpl: check_precromulent("P")
projective/PX_check.mpl: check_P_action()
projective/PX_check.mpl: check_j_P()
```

We will prove as Proposition 3.1.12 that this gives a precromulent surface; then we will see in Proposition 3.3.1 that it is actually cromulent.

Remark 3.1.2. The parameters a and A are `a_P` and `A_P` in Maple. The polynomial $r_a(z)$ is `r_P(z)`. Points in $\mathbb{C}^2$ are represented as lists of length two. (Maple distinguishes between lists and vectors, and we have generally preferred to use lists, for technical reasons that we will not explore here.) One can check whether a list lies in $PX_0(a)$ using the function `is_member_P_0(z)`. Points in $\mathbb{C}P^4$ are represented by lists of length 5, and one can check whether two lists are projectively equivalent using the function `is_equal_P(w,z)`. One can also check whether a point lies in $PX(a)$ using the function `is_member_P(z)`. The map $j\colon PX_0(a) \to PX(a)$ is `j_P(z)`. The points $v_i \in PX(a)$ are `v_P[i]`. The action of $g \in G$ on $PX(a)$ is given by `act_P[g](z)`, and the corresponding action on $PX_0(a)$ is `act_P_0[g](z)`. All this comes from the file `projective/PX.mpl`.

Remark 3.1.3. [rem-a-Po]

All the above functions treat `a_P` as a symbol. There is also a global variable `a_P0` which holds a numerical value for `a_P`. In fact, there are two such variables, called `a_P0` and `a_P1`. This is intended to cover the case where `a_P0` is an exact expression (such as a rational number) and `a_P1` is a floating point approximation to the same number. However, we have usually taken `a_P0` to be a floating point number, so that there is no distinction between `a_P0` and `a_P1`. These variables should be set using the function `set_a_P0`, defined in the file `projective/PX0.mpl`. This will then set a large number of other variables by substituting `a_P0` or `a_P1` for `a_P`. For example, `v_P0[i]` and `v_P1[i]` are obtained by applying these substitutions to `v_P[i]`. When the file `projective/PX0.mpl` is loaded, it calls the function `set_a_P0` to set `a_P0` to a particular value (approximately 0.0984), which is close to the value for which EX^* is isomorphic to $PX(a)$. However, one can call `set_a_P0` again to change the value if desired.

Remark 3.1.4. [rem-simplify-P]

When working with an expression m involving `a_P`, it is sometimes convenient to use the function `simplify_P(m)` (defined in `projective/PX.mpl`). This will try some substitutions like $\sqrt{1-a^2} = \sqrt{1-a}\sqrt{1+a}$ that are not always used by the default simplification functions.

Remark 3.1.5. [rem-r-coprime]
As $a \in (0, 1)$ we see that $r_a(z)$ has no repeated roots, so $r_a(z)$ and $r_a'(z)$ are coprime in $\mathbb{R}[z]$ or $\mathbb{C}[z]$. More specifically, one can check by direct expansion that $m_0(z)r_a(z) + m_1(z)r_a'(z) = 1$, where

$$\begin{aligned} m_0(z) &= \frac{(100 - 30A^2)z^3 + (18A^3 - 70A)z}{4(A^2 - 4)} \\ m_1(z) &= \frac{(6A^2 - 20)z^4 - (6A^3 - 22A)z^2 + (4A^2 - 16)}{4(A^2 - 4)}. \end{aligned}$$

(These are `r_P_cofactor0(z)` and `r_P_cofactor1(z)` in the Maple code.)

`projective/PX_check.mpl: check_r_P_cofactors()`

Lemma 3.1.6. [lem-R-domain]
The ring $R_0(a)$ is an integral domain, and the set

$$B = \{z^i w^j \mid i \in \mathbb{N},\ j \in \{0, 1\}\}$$

is a basis for $R_0(a)$ over $\mathbb{C}$.

Proof. From the description $R_0(a) = \mathbb{C}[z, w]/(w^2 - r_a(z))$ it is clear that $\{1, w\}$ is a basis for $R_0(a)$ as a module over $\mathbb{C}[z]$. It follows that B is a basis for $R_0(a)$ over $\mathbb{C}$. Next, for any element $f = p(z) + q(z)w \in R_0(a)$, we put

$$N(f) = (p(z) + q(z)w)(p(z) - q(z)w) = p(z)^2 - q(z)^2 r_a(z) \in \mathbb{C}[z].$$

It is easy to check that $N(fg) = N(f)N(g)$. Moreover, by considering the highest power of z that divides the various terms, we see that $N(f) \neq 0$ whenever $f \neq 0$. Thus, if g is also nonzero we have $N(fg) = N(f)N(g)$, which is nonzero because $\mathbb{C}[z]$ is a domain, so $fg \neq 0$ as required. □

Lemma 3.1.7. [lem-P-differentials]
The module $\Omega^1(PX_0(a))$ of Kähler differentials is freely generated over $R_0(a)$ by the element

$$\omega_0 = m_0(z)w\,dz + 2m_1(z)\,dw$$

(where m_0 and m_1 are as in Remark 3.1.5). In particular, we have

$$\begin{aligned} dz &= w\,\omega_0 \\ dw &= \tfrac{1}{2}r_a'(z)\omega_0. \end{aligned}$$

Proof. We will put $\Omega^1 = \Omega^1(PX_0(a))$ for brevity. Differentiating the equation $w^2 = r_a(z)$ gives $2w\,dw = r_a'(z)\,dz$, and (essentially by definition) the module Ω^1 is generated by dw and dz subject only to this relation. Now

$$\begin{aligned} w\,\omega_0 &= m_0(z)w^2\,dz + 2m_1(z)w\,dw \\ &= m_0(z)r_a(z)\,dz + m_1(z)r_a'(z)\,dz \\ &= (m_0(z)r_a(z) + m_1(z)r_a'(z))\,dz = dz. \end{aligned}$$

A similar argument shows that $\frac{1}{2}r_a'(z)\omega_0 = dw$, so both dz and dw lie in the submodule of Ω^1 generated by ω_0. This implies that ω_0 generates all of Ω^1. Now note that our original presentation of Ω^1 implies that $\Omega^1[w^{-1}]$ is freely generated over $R_0(a)[w^{-1}]$ by dw. Thus, if $f \in R_0(a)$ satisfies $f\omega_0 = 0$ then $f\Omega^1 = 0$ so $f\Omega^1[w^{-1}] = 0$ so f must map to zero in $R_0(a)[w^{-1}]$. However, as $R_0(a)$ is an integral domain we see that the map $R_0(a) \to R_0(a)[w^{-1}]$ is injective, so f must be zero. It follows that Ω^1 is *freely* generated by ω_0, as claimed. □

Lemma 3.1.8. [lem-nonzero]
Consider a point $[z] \in PX(a)$.

(a) *If any of the coordinates $z_2, \dots, z_5$ is zero, then $[z] \in \{v_0, v_1\}$.*
(b) *Either $z_2 \neq 0$ or $z_5 \neq 0$.*

Proof. We put

$$\begin{aligned}
r_0(z) &= z_1^2 - z_2z_3 - z_4z_5 + Az_3z_4 \\
r_1(z) &= z_2z_4 - z_3^2 \\
r_2(z) &= z_2z_5 - z_3z_4 \\
r_3(z) &= z_3z_5 - z_4^2,
\end{aligned}$$

so that $PX(a) = \{[z] \mid r_0(z) = \dots = r_3(z) = 0\}$.

(a) Suppose that $[z] \in PX(a)$ and $z_2z_3z_4z_5 = 0$. Using r_2 we see that $(z_2z_5)^2$ and $(z_3z_4)^2$ are both equal to $z_2z_3z_4z_5$ and thus to zero, so $z_2z_5 = z_3z_4 = 0$. Thus, either $z_2 = 0$ or $z_5 = 0$; and either $z_3 = 0$ or $z_4 = 0$. If $z_3 = 0$ then r_3 gives $z_4 = 0$; conversely, if $z_4 = 0$ then r_1 gives $z_3 = 0$. We must therefore have $z_3 = z_4 = 0$. Substituting this in r_0 gives $z_1 = 0$. In summary, at most one of the coordinates z_i can be nonzero, and the nonzero coordinate must be z_2 or z_5. It follows that $[z] \in \{v_0, v_1\}$ as required.

(b) The claim is clear if $z_2z_3z_4z_5 \neq 0$, and follows from (a) if $z_2z_3z_4z_5 = 0$.

□

Proposition 3.1.9. [prop-P-j]
The map j gives an isomorphism $PX_0(a) \simeq PX(a) \setminus \{v_1\}$.

Proof. The lemma shows that for $[z] \in PX(a) \setminus \{v_1\}$ we have $z_2 \neq 0$. We can thus define $k\colon PX(a) \setminus \{v_1\} \to \mathbb{C}^2$ by $k[z] = (z_1/z_2, z_3/z_2)$. It will be harmless to rescale z so that $z_2 = 1$, and then the last three defining relations for $PX(a)$ become $z_4 = z_3^2$ and $z_5 = z_4z_4$ and $z_3z_5 = z_4^2$, so $(z_2, z_3, z_4, z_5) = (1, z_3, z_3^2, z_3^3)$. If we use these to rewrite the first relation we get

$$z_1^2 - z_3 - z_3^5 + (a^2 + a^{-2})z_3^3 = 0,$$

so $k[z] \in PX_0(a)$. The equations $jk = 1$ and $kj = 1$ are now clear.

`projective/PX_check.mpl: check_j_P()`

□

Definition 3.1.10. [defn-P-cover]
We put

$$\begin{aligned}
U_0 &= PX(a) \setminus \{v_1\} \\
U_1 &= PX(a) \setminus \{v_0\} = \mu(U_0) \simeq U_0 \\
U_{01} &= U_0 \cap U_1.
\end{aligned}$$

Remark 3.1.11. [rem-P-cover]
We use j to silently identify U_0 with $PX_0(a)$, which is the spectrum of the ring $R_0(a) = \mathbb{C}[z, w]/(w^2 - r_a(z))$. This in turn identifies U_{01} with the spectrum of the ring

$$R'(a) = R_0(a)[z^{-1}] = \mathbb{C}[z^{\pm 1}, w]/(w^2 - r_a(z)).$$

The set

$$B_0 = \{z^i \mid i \geq 0\} \amalg \{z^i w \mid i \geq 0\}$$

is a basis for the subring $R_0(a) \subset R'(a)$ over $\mathbb{C}$. The set $R_1(a) = \mu(R_0(a))$ is also a subring of $R'(a)$, with basis

$$B_1 = \{z^i \mid i \leq 0\} \amalg \{z^i w \mid i \leq -3\}.$$

Proposition 3.1.12. [prop-P-precromulent]
The above definitions make $PX(a)$ into a precromulent surface.

Proof. First, the standard Jacobian condition shows that $PX_0(a)$ is smooth, so Proposition 3.1.9 shows that $PX(a)$ is smooth except possibly at v_1. Next, straightforward calculation shows that the action of G on $\mathbb{C}P^4$ preserves $PX(a)$, with D_8 acting conformally and $G \setminus D_8$ acting anticonformally, and

$$\begin{aligned} \lambda(j(w,z)) &= j(iw,-z) \\ \mu(j(w,z)) &= j(-w/z^3, 1/z) \\ \nu(j(w,z)) &= j(\overline{w}, \overline{z}). \end{aligned}$$

In particular, we see that μ gives an isomorphism between $PX(a) \setminus \{v_1\}$ and $PX(a) \setminus \{v_0\}$, showing that $PX(a)$ is smooth everywhere. It is clearly closed in $\mathbb{C}P^4$ and therefore compact. It is standard that for any polynomial $r(z)$ of degree $2g+1$, the hyperelliptic curve $w^2 = r(z)$ has genus g; in particular, $PX(a)$ has genus 2. A straightforward but lengthy check shows that G permutes the points v_i in accordance with Definition 1.0.4(b). The tangent space to $PX_0(a)$ at $(0,0)$ is $\mathbb{C} \oplus 0$, and λ acts on this is multiplication by i. All that is left is to check that D_8 acts freely on $PX(a) \setminus V$, where $V = \{v_0, \dots, v_{13}\}$. The fixed points of λ^2 on $PX_0(a)$ are pairs (w,z) with $(-w,z) = (w,z)$ and $w^2 = z(z^2-a^2)(z^2-a^{-2})$, which means that $w=0$ and $z \in \{0, a, -a, a^{-1}, -a^{-1}\}$. It follows that

$$PX(a)^{\langle \lambda^2 \rangle} = \{v_0, v_1, v_{10}, v_{11}, v_{12}, v_{13}\} \subseteq V.$$

All fixed points of λ or λ^3 are also fixed by λ^2 and so lie in V. A similar analysis shows that the fixed points of μ, $\lambda\mu$, $\lambda^2\mu$ and $\lambda^3\mu$ also lie in V, as required. $\square$

Remark 3.1.13. [rem-P-quotient]
We can define $p\colon PX(a) \to \mathbb{C}_\infty$ by $p([z]) = z_3/z_2$, so $pj(w,z) = z$. Using $\lambda^2 j(w,z) = (-w,z)$, it is not hard to check that p gives an isomorphism $PX(a)/\langle \lambda^2 \rangle \to \mathbb{C}_\infty$. Moreover, p is equivariant if we use the following action on $\mathbb{C}_\infty$:

$$\lambda(z) = -z \qquad \mu(z) = 1/z \qquad \nu(z) = \overline{z}.$$

This action is represented by `act_C` in Maple. For example, `act_C[L](3)` evaluates to `-3`.

3.2. **The curve system.** [sec-P-curves]

Definition 3.2.1. [defn-P-curves]
We define maps $j'\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P^4$ by

$$j'(w,x,y) = [w : x^3 : x^2 y : xy^2 : y^3].$$

Note that $j(w,z) = j'(w,1,z)$, and when $x \neq 0$ we have $j'(w,x,y) = j(w/x^3, y/x)$. We then define $c_k\colon \mathbb{R} \to PX(a)$ as follows:

$$\begin{aligned}
c_0(t) &= j'(-\sqrt{(a^{-1}-a)^2 + 4\sin^2(2t)},\ e^{it},\ -e^{-it}) \\
c_1(t) &= j'\left(\frac{1+i}{8\sqrt{2}}\sin(t)\sqrt{16\cos(t)^2 + (a+a^{-1})^2\sin(t)^4},\ \frac{1+\cos(t)}{2},\ \frac{1-\cos(t)}{2}i\right) \\
c_2(t) &= \lambda(c_1(t)) \\
c_3(t) &= j'\left(-i\frac{a^{-1}-a}{8}\sin(t)\sqrt{(1+a)^4 - (1-a)^4\cos(t)^2}\sqrt{(1+a)^2-(1-a)^2\cos(t)^2},\right. \\
&\qquad\qquad \left.\frac{(1+a)+(1-a)\cos(t)}{2},\ \frac{(1+a)-(1-a)\cos(t)}{2}\right) \\
c_4(t) &= \lambda(c_3(t)) \\
c_5(t) &= j\left(\frac{\sin(t)}{8}\sqrt{2a(3-\cos(t))(4-a^4(1-\cos(t))^2)},\ a\frac{1-\cos(t)}{2}\right) \\
c_6(t) &= \lambda(c_5(t)) \\
c_7(t) &= \mu(c_5(t)) \\
c_8(t) &= \lambda\mu(c_5(t)).
\end{aligned}$$

Maple notation for $c_k(t)$ is `c_P[k](t)`. The versions with a numerical values for a are `c_P0[k](t)` and `c_P1[k](t)`. The map j' is `jj_P`.

Proposition 3.2.2. [prop-P-curves]
The above maps give a curve system on $PX(a)$ (in the sense of Definition 2.4.4).

Proof. Combine Lemmas 3.2.3 and 3.2.6.

```
cromulent.mpl: check_precromulent("P")
```

□

Lemma 3.2.3. [lem-P-curves-bc]
For $0 \leq k \leq 8$, the map $c_k \colon \mathbb{R} \to \mathbb{C}P^4$ is smooth, with $c_k(t + 2\pi) = c_k(t)$, and the image is contained in $PX(a)$. Moreover, parts (b) and (c) of Definition 2.4.4 are satisfied.

Proof. Direct calculation. □

Lemma 3.2.4. [lem-P-pc]
The composites $\mathbb{R} \xrightarrow{c_k} PX(a) \xrightarrow{p} \mathbb{C}_\infty$ and their images are as follows:

$$
\begin{aligned}
pc_0(t) &= -e^{-2it} & pc_0(\mathbb{R}) &= S^1 \\
pc_1(t) &= \frac{1-\cos(t)}{1+\cos(t)} i & pc_1(\mathbb{R}) &= [0,\infty]i \\
pc_2(t) &= -\frac{1-\cos(t)}{1+\cos(t)} i & pc_2(\mathbb{R}) &= [-\infty,0]i \\
pc_3(t) &= \frac{(1+a)-(1-a)\cos(t)}{(1+a)+(1-a)\cos(t)} & pc_3(\mathbb{R}) &= [a,a^{-1}] \\
pc_4(t) &= -\frac{(1+a)-(1-a)\cos(t)}{(1+a)+(1-a)\cos(t)} & pc_4(\mathbb{R}) &= [-a^{-1},-a] \\
pc_5(t) &= \frac{1-\cos(t)}{2} a & pc_5(\mathbb{R}) &= [0,a] \\
pc_6(t) &= -\frac{1-\cos(t)}{2} a & pc_6(\mathbb{R}) &= [-a,0] \\
pc_7(t) &= \frac{2}{1-\cos(t)} a^{-1} & pc_7(\mathbb{R}) &= [a^{-1},\infty] \\
pc_8(t) &= -\frac{2}{1-\cos(t)} a^{-1} & pc_8(\mathbb{R}) &= [-\infty,-a^{-1}]
\end{aligned}
$$

Proof. Direct calculation.

```
projective/PX_check.mpl: check_pc_P()
```

□

Lemma 3.2.5. [lem-jj-equal]
Suppose that $j'(w_0,x,y) = j'(w_1,x,y)$ and that this point lies in $PX(a)$. Then $w_0 = w_1$.

Proof. By assumption we have

$$[w_0 : x^3 : x^2y : xy^2 : y^3] = [w_1 : x^3 : x^2y : xy^2 : y^3].$$

It follows easily that $w_0 = w_1$ unless $x = y = 0$. However, as this point lies in $PX(a)$, Lemma 3.1.8 tells us that x and y cannot both vanish. □

Lemma 3.2.6. [lem-P-curves-a]
For $0 \leq k \leq 8$, the induced map $c_k \colon \mathbb{R}/2\pi\mathbb{Z} \to PX(a)$ is a smooth embedding.

Proof. Because of the group action, it will suffice to treat the cases $k = 0, 1, 3, 5$.

Using Lemma 3.2.4 we see that $(pc_k)'(t) \neq 0$ except when $k > 0$ and $t \in \pi\mathbb{Z}$. It follows that $c_k'(t) \neq 0$ except possibly when $k > 0$ and $t \in \pi\mathbb{Z}$. Moreover, one can check that to first order in ϵ we have

$$c_1(\epsilon) \simeq \left[\frac{1+i}{2\sqrt{2}}\epsilon : 1 : 0 : 0 : 0\right]$$
$$c_1(\epsilon + \pi) \simeq \left[\frac{1-i}{2\sqrt{2}}\epsilon : 0 : 0 : 0 : 1\right].$$

It follows easily that $c_1'(t) \neq 0$ for all t, so c_1 is at least an immersion. Similar calculations show that $c_0, \ldots, c_8$ are all immersions.

We now just need to show that $c_k \colon \mathbb{R}/2\pi\mathbb{Z} \to PX(a)$ is injective.

(a) Consider the case $k = 0$. If $u = c_0(t)$ we have $p(u) = -e^{-2it}$, and it follows that the quantity $a^{-2} + a^2 - p(u)^2 - p(u)^{-2}$ is equal to $a^{-2} + a^2 - 2\cos(4t)$ and so is strictly positive. It follows in turn that
$$-q(u)\, p(u)^{-2}\, (a^{-2} + a^2 - p(u)^2 - p(u)^{-2})^{-1/2} = e^{it}.$$
From this it is clear that when $c_0(s) = c_0(t)$ we have $e^{is} = e^{it}$ and so $s - t \in 2\pi\mathbb{Z}$ as required.

(b) Now suppose instead that $k \in \{1, 3, 5\}$ and $c_k(s) = c_k(t)$. We then have $pc_k(s) = pc_k(t)$, and using Lemma 3.2.4 we can deduce that $\cos(s) = \cos(t)$. Now, we can use the identities $\sin(2t) = \sin(t)\cos(t)$ and $\sin^2(t) = 1 - \cos^2(t)$ to rewrite $c_k(t)$ in the form
$$c_k(t) = j'(u(\cos(t))\sin(t),\ v(\cos(t)),\ w(\cos(t))),$$
for some functions u, v and w. As $\cos(s) = \cos(t)$ and $c_k(s) = c_k(t)$ we can use Lemma 3.2.5 to see that
$$u(\cos(t))\sin(s) = u(\cos(t))\sin(t).$$
Moreover, in each case one can check that $u(\cos(t))$ is never zero, so $\sin(s) = \sin(t)$. We thus have $s - t \in 2\pi\mathbb{Z}$ again.

□

Proposition 3.2.7. [prop-P-std-isotropy]
$PX(a)$ has standard isotropy (as in Definition 2.4.9).

Proof. First, we put $D_k = p(C_k) \subseteq \mathbb{C}_\infty$; these sets are described by Lemma 3.2.4. Note that we have $\lambda^2(c_0(t)) = c_0(t + \pi)$, and $\lambda^2(c_k(t)) = c_k(-t)$ for $1 \leq k \leq 8$. It follows that for all k we have $\lambda^2(C_k) = C_k$. In view of Remark 3.1.13, it follows that $C_k = p^{-1}(D_k)$.

It is clear that the sets $D_4 = [-a^{-1}, -a]$, $D_5 = [0, a]$ and $D_7 = [a^{-1}, \infty]$ are disjoint, and it follows that C_4, C_5 and C_7 are disjoint. Similarly, C_3, C_6 and C_8 are disjoint.

Next, recall that the map $p \colon PX(a) \to \mathbb{C}_\infty$ is equivariant with respect to the action described in Remark 3.1.13 (given by $\lambda(z) = -z$ and $\mu(z) = z^{-1}$ and $\nu(z) = \overline{z}$). In particular, if $x \in PX(a)$ is fixed by an element $\alpha \in G$, then $p(x)$ is also fixed by α.

(a) Consider a point $x \in PX(a)$ with $\mu\nu(x) = x$. Then $p(x)$ is also fixed by $\mu\nu$, which means that $p(x) = 1/\overline{p(x)}$, so $p(x) \in S^1 = D_0$, so $x \in p^{-1}(D_0) = C_0$. We thus have $PX(a)^{\langle\mu\nu\rangle} = C_0$ as claimed.

(b) Consider a point $x \in PX(a)$ with $\lambda\nu(x) = x$. If $x = v_0 = c_1(0)$ or $x = v_1 = c_1(\pi)$ then it is clear that $x \in C_1$. Suppose instead that $x \notin \{v_0, v_1\}$, so can be written as $j(w, z)$ with $z \neq 0$. In general we have $\lambda\nu(j(w, z)) = j(i\overline{w}, -\overline{z})$. As $\lambda\nu(x) = x$ we see that $w = i\overline{w}$ and $z = -\overline{z}$. Put $\omega = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$, so $\omega^2 = i$ and $\omega = i\overline{\omega}$. We find that $w = \omega w_1$ and $z = iz_1$ for some $w_1, z_1 \in \mathbb{R}$. The equation $w^2 = r_a(z)$ becomes $w_1^2 = z_1(z_1^2 + a^2)(z_1^2 + a^{-2})$. From this it is clear that $z_1 > 0$. By elementary calculus, there is a unique $t \in (0, \pi)$ with $z_1 = (1 - \cos(t))/(1 + \cos(t))$. For this t we find that $p(c_1(t)) = z$, and thus that x is either $c_1(t)$ or $\lambda^2(c_1(t)) = c_1(-t)$. Either way we have $x \in c_1(\mathbb{R}) = C_1$, so $PX(a)^{\langle\lambda\nu\rangle} = C_1$ as claimed.

(c) As $\lambda^3\nu$ is conjugate to $\lambda\nu$, we can use the group action to deduce that $PX(a)^{\langle\lambda^3\nu\rangle} = C_2$.

(d) Now consider a point $x \in PX(a)$ with $\nu(x) = x$. If $x = v_1$ then $x \in C_7$. Otherwise, we have $x = j(w, z)$ for some $(w, z) \in PX_0(a)$. As $\nu(x) = x$ we see that w and z are real. As $r_a(z) = w^2$ we have $r_a(z) \geq 0$. Recall that the roots of $r_a(z)$, listed in increasing order, are $-a^{-1}, -a, 0, a$ and a^{-1}. It follows that

$$p(x) = z \in [-a^{-1}, -a] \amalg [0, a] \amalg [a^{-1}, \infty] = D_4 \amalg D_5 \amalg D_7,$$

and thus that $x \in C_4 \amalg C_5 \amalg C_7$.

(e) Consider instead a point $x \in PX(a)$ with $\lambda^2\nu(x) = x$. Then the point $y = \lambda(x)$ satisfies $\nu(x) = x$ and so lies in $C_4 \amalg C_5 \amalg C_7$. However, one can check from Definition 2.4.1 that

$$\begin{aligned} \lambda(C_3) &= C_4 & \qquad \lambda(C_4) &= C_3 \\ \lambda(C_5) &= C_6 & \qquad \lambda(C_6) &= C_5 \\ \lambda(C_7) &= C_8 & \qquad \lambda(C_8) &= C_7, \end{aligned}$$

so $x \in C_3 \amalg C_6 \amalg C_8$.

(f) Finally, consider a point $x \in PX(a)$ with $x = \lambda^2\mu\nu(x)$. Then x cannot be equal to v_1, so $x = j(w, z)$ for some w and z. The equation $x = \lambda^2\mu\nu(x)$ gives $z = -1/\overline{z}$ and so $|z|^2 = -1$, which is impossible. Thus, there are no such points x.

□

This is a convenient place to record the following result, which will be needed later.

Lemma 3.2.8. [lem-right-angle]

(a) *The curves C_0 and C_3 cross at right angles at v_3*
(b) *The curves C_0 and C_1 cross at right angles at v_6*
(c) *The curves C_3 and C_5 cross at right angles at v_{11}.*

If we were willing to wait until we had proved Lemma 3.6.11, we could give a non-computational argument based on that. However, we will just calculate the relevant derivatives instead.

Proof. For (a), recall that $v_3 = c_0(\pi/2) = c_3(\pi/2)$, so we need to compare $c_0'(\pi/2)$ with $c_2'(\pi/2)$. Because the map $p\colon PX(a) \to \mathbb{C}_\infty$ is conformal, it will suffice to show that $(pc_0)'(\pi/2)$ and $(pc_3)'(\pi/2)$ are nonzero and that the ratio between them is purely imaginary. This is easy to do using the formulae in Lemma 3.2.4. Specifically, we have $(pc_0)'(\pi/2) = -2i$ and $(pc_3)'(\pi/2) = 2(1-a)/(1+a)$. We can prove (b) in the same way using $(pc_0)'(\pi/4) = 2$ and $(pc_1)'(\pi/2) = 2i$. We need a slightly different method for v_{11} because p has derivative zero there. We instead define a rational map $q\colon PX(a) \to \mathbb{C}_\infty$ by $q(z) = z_1/z_2$, so $qj(w, z) = w$. This is conformal and satisfies $q(v_{11}) = 0$, so it will suffice to show that $(qc_3)'(0)$ and $(qc_5)'(\pi)$ are nonzero, and that the ratio between them is purely imaginary. A standard calculation from the definitions gives

$$\begin{aligned} (qc_3)'(0) &= -i(1-a^2)(1+a^2)^{1/2}/\sqrt{2} \\ (qc_5)'(\pi) &= -(1-a^2)^{1/2}(1+a^2)^{1/2}a^{1/2}/\sqrt{2} \end{aligned}$$

as required. □

3.3. **Fundamental domains.** [sec-P-fundamental]

Proposition 3.3.1. [prop-P-fundamental]
If we put

$$\begin{aligned} PF_{16}'(a) &= \{(w, z) \in PX_0(a) \mid Re(z), Im(z), Re(w) \geq 0,\ Re(w) \geq Im(w),\ |z| \leq 1\} \\ PF_{16}(a) &= j(PF_{16}'(a)) \subset PX(a), \end{aligned}$$

then $PF_{16}(a)$ is a standard fundamental domain for $PX(a)$ (as in Definition 2.6.6). Thus, $PX(a)$ is cromulent (by Remark 2.6.7).

Proof. For brevity, we will write F' and F for $PF_{16}'(a)$ and $PF_{16}(a)$. Put

$$\begin{aligned} Z &= \{x + iy \in \mathbb{C} \mid x, y \geq 0,\ x^2 + y^2 \leq 1\} & &= \{r\,e^{i\theta} \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/2\} \\ W &= \{x + iy \in \mathbb{C} \mid x \geq 0,\ x \geq y\} & &= \{r\,e^{i\theta} \mid 0 \leq r,\ -\pi/2 \leq \theta \leq \pi/4\} \\ W^2 &= \{x + iy \in \mathbb{C} \mid x \geq 0 \text{ or } y \leq 0\} & &= \{r\,e^{i\theta} \mid 0 \leq r,\ -\pi \leq \theta \leq \pi/2\}. \end{aligned}$$

We then have $F' = (W \times Z) \cap PX_0(a)$.

We now claim that $r_a(Z) \subseteq W^2$. Indeed, it is clear that ∂Z is a simple closed curve. The image $r_a(\partial Z)$ consists of the points $r_a(t) \in \mathbb{R}$ (for $0 \leq t \leq 1$) and $r_a(it) \in i\mathbb{R}$ (for $0 \leq t \leq 1$) and $r_a(e^{it})$ (for $0 \leq t \leq \pi/2$). Here

$$r_a(e^{it}) = -(4\sin(t)^2 + (a^{-1} - a)^2)e^{3it},$$

so $\arg(r_a(e^{it})) = 3t - \pi \in [-\pi, \pi/2]$, so $r_a(e^{it}) \in W^2$. We now see that $r_a(\partial Z)$ is a simple closed curve in W^2. The argument principle shows that $r_a(Z)$ is the interior of $r_a(\partial Z)$, and this is contained in W^2 as claimed.

Now consider a point $v \in PX(a)$. If $p(v) = x + iy \in \mathbb{C}$ then we put

$$s_0(v) = \frac{|x| + i|y|}{\max(1, x^2 + y^2)} \in Z$$
$$s(v) = j(\sqrt{r_a(s_0(v))}, s_0(v)) \in F'.$$

(Here $r_a(s_0(v))$ lies in W^2, and $\sqrt{r_a(s_0(v))}$ refers to the unique choice of square root that lies in W.) For the exceptional case $v = v_1$, we put $s(v_1) = (0, 0)$. It is easy to see that s is a retraction. Using Remark 3.1.13, we see that $s_0(v) = s_0(v')$ iff $Gv = Gv'$, and also that $s_0(v) \in G.p(v)$. After recalling that $\lambda^2 j(w, z) = j(-w, z)$, we deduce that $s(v) = s(v')$ iff $Gv = Gv'$, and also that $s(v) \in G.v$. It follows that $PX(a) = \bigcup_{\gamma \in G} \gamma.F'$, with

$$F' \cap \gamma F' = \{v \in F' \mid \gamma(v) = v\}.$$

If v lies in the interior of F' then it is easy to see that $p(v)$ lies in the interior of Z, and thus that $\mathrm{stab}_G(v) \subseteq \mathrm{stab}_G(p(v)) = \{1, \lambda^2\}$. On the other hand, for v in the interior of F' we also have $r_a(p(v)) \neq 0$, so v is not fixed by λ^2, so $\mathrm{stab}_G(v) = 1$. We now see that $\mathrm{int}(F') \cap \gamma(F') = \emptyset$ for $\gamma \neq 1$, so F' is a retractive fundamental domain for $PX(a)$.

Next, the formulae in Lemma 3.2.4 show that

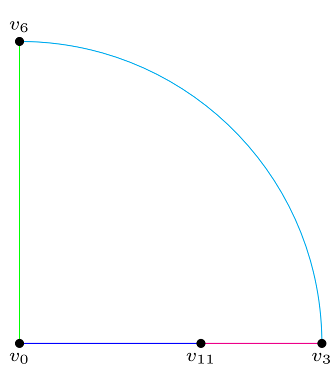
$$\begin{aligned}\partial Z &= [0, a] \cup [a, 1] \cup e^{[0,\pi/2]i} \cup [i, 0] \\ &= pc_5([0, \pi]) \cup pc_3([0, \pi/2]) \cup pc_0([\pi/4, \pi/2]) \cup pc_1([0, \pi/2]).\end{aligned}$$

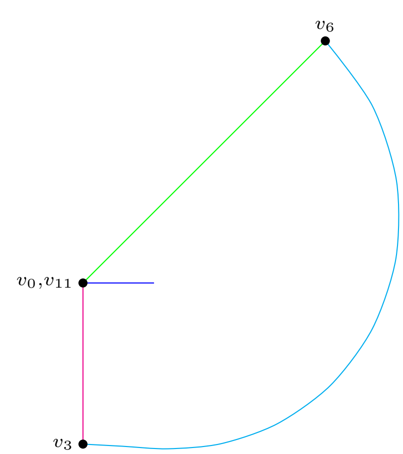
From this we deduce that

$$\partial F' = c_5([0, \pi]) \cup c_3([0, \pi/2]) \cup c_0([\pi/4, \pi/2]) \cup c_1([0, \pi/2]) = DF_{16},$$

so F' is a standard fundamental domain. □

We can illustrate the surface $PX(a)$ as follows. The picture on the left shows the image under $p\colon PX(a) \to \mathbb{C}_\infty$ of the fundamental domain F, and the picture on the right shows the image under q. (In both cases the origin is at v_0.)





We next consider differential forms on $PX_0(a)$ and $PX(a)$.

Remark 3.3.2. [rem-anticonformal-forms]
Holomorphic differential forms are clearly functorial for conformal isomorphisms. In fact, they are also functorial for anticonformal isomorphisms. Indeed, given an anticonformal map $\phi\colon Z_0 \to Z_1$ of Riemann surfaces and a holomorphic function $f \in \mathcal{O}(Z_1)$, we can define $\phi^{\#}(f) \in \mathcal{O}(Z_0)$ by $\phi^{\#}(f)(z) = \overline{f(\phi(z))}$. It is not hard to see that there is a unique locally determined map $\phi^{\#}\colon \Omega^1(Z_1) \to \Omega^1(Z_0)$ satisfying $\phi^{\#}(f\,dg) = \phi^{\#}(f)\,d\phi^{\#}(g)$ for all $f, g \in \mathcal{O}(Z_1)$. We therefore have an action of G on $\Omega^1(PX(a))$.

Proposition 3.3.3. [prop-holomorphic-forms]

(a) *The differential form $\omega_0 \in \Omega^1(U_0)$ (from Lemma 3.1.7) extends to give a holomorphic differential form on all of $PX(a)$ (which we also call ω_0).*
(b) *The form $\omega_1 = \mu^*(\omega_0)$ satisfies $\omega_1 = z\,\omega_0$ when restricted to U_0.*
(c) *The set $\{\omega_0, \omega_1\}$ is a basis for $\Omega^1(PX(a))$ over $\mathbb{C}$.*
(d) *The group G acts on this space by*

$$\begin{aligned} \lambda^*(\omega_0) &= i\omega_0 & \mu^*(\omega_0) &= \omega_1 & \nu^{\#}(\omega_0) &= \omega_0 \\ \lambda^*(\omega_1) &= -i\omega_1 & \mu^*(\omega_1) &= \omega_0 & \nu^{\#}(\omega_1) &= \omega_1. \end{aligned}$$

Proof. As $z\omega_0 \in \Omega^1(U_0)$ and $\mu(U_1) = U_0$ we have a holomorphic form $\omega_0' = \mu^*(z\,\omega_0) \in \Omega^1(U_1)$. Recall that $w\omega_0 = dz$. After restricting to U_{01} we can apply μ^* to this equation, giving

$$-wz^{-3}\mu^*(\omega_0) = d(z^{-1}) = -z^{-2}\,dz = -z^{-2}w\omega_0,$$

which implies that $\mu^*(\omega_0) = z\omega_0$, and thus that $\mu^*(z\omega_0) = \omega_0$. This implies that ω_0 and ω_0' have the same restriction to U_{01}, so we can patch them together to give a holomorphic form on all of $U_0 \cup U_1 = PX(a)$. Claims (a) and (b) are now clear, and (d) is a straightforward calculation. This just leaves (c). Consider a holomorphic form $\alpha \in \Omega^1(PX(a))$. Lemma 3.1.7 tells us that there is a unique function $f_0 \in R_0(a)$ such that $\alpha = f_0\omega_0$ on U_0. By applying the same logic to $\mu^*(\alpha)$, and applying μ^* again, we see that there is also a unique function $f_1 \in R_1(a)$ such that $\alpha = f_1\omega_1$ on U_1. On U_{01} we now see that $f_0\omega_0 = \alpha = f_1\omega_1 = f_1 z\omega_0$, so $f_0 = f_1 z$. Using the bases described in Remark 3.1.11 we see that $f_0 \in R_0(a) \cap R_1(a)z = \mathbb{C}\{1, z\}$, and it follows that $\alpha \in \mathbb{C}\{\omega_0, \omega_1\}$. Moreover, it is easy to see that ω_1 vanishes at v_0 but ω_0 does not, and the other way around at v_1. This means that ω_0 and ω_1 are linearly independent, so they form a basis for $\Omega^1(PX(a))$. □

Remark 3.3.4. [rem-P-parameter]
The coordinate w is a local parameter on $PX_0(a)$ at the point $v_0 = (0, 0)$. In terms of this parameter we have $\omega_0 = 2\,dw + O(w^4)$ and $\omega_1 = 2w^2\,dw + O(w^6)$.

Definition 3.3.5. [defn-period-p]
The *periods* for $PX(a)$ are the numbers $p_{jk}(a) = \int_{c_j} \omega_k \in \mathbb{C}$ (for $0 \leq j \leq 8$ and $k \in \{0, 1\}$).

3.4. **Galois theory.** [sec-galois]
Let $PK(a)$ denote the field of rational functions on $PX_0(a)$ (which is the same as the field of rational functions on $PX(a)$). This can be described as

$$PK(a) = \mathbb{C}(z)[w]/(w^2 - r_a(z)).$$

This field has an action of the group D_8, and for any subgroup $H \leq D_8$, we can identify the fixed field $PK(a)^H$ with the field of rational functions on the quotient $PX(a)/H$, with its standard structure as a Riemann surface. The subgroups of D_8 can be enumerated as follows:

$$
\begin{tikzcd}
 & & H_1 = 1 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
H_6 = \langle \lambda^2\mu \rangle \arrow[dr] & H_4 = \langle \mu \rangle \arrow[d] & H_2 = \langle \lambda^2 \rangle \arrow[dl] \arrow[d] \arrow[dr] & H_5 = \langle \lambda\mu \rangle \arrow[d] & H_7 = \langle \lambda^3\mu \rangle \arrow[dl] \\
 & H_8 = \langle \lambda^2, \mu \rangle \arrow[dr] & H_3 = \langle \lambda \rangle \arrow[d] & H_9 = \langle \lambda^2, \lambda\mu \rangle \arrow[dl] & \\
 & & H_{10} = D_8 & &
\end{tikzcd}
$$

We will describe the fixed fields $L_i = PK(a)^{H_i}$ in terms of the following elements:

$$
\begin{aligned}
t_0 &= z & u_0 &= w & & \\
t_1 &= z^2 & u_1 &= \frac{2w(1-z)}{(1+z^2)^2} & v_1 &= w(1-1/z^3)/2 \\
t_2 &= \frac{2z}{1+z^2} & u_2 &= -\frac{1+i}{\sqrt{2}}\,\frac{2w(i+z)}{(1-z^2)^2} & v_2 &= w(1-i/z^3)/2 \\
t_3 &= \frac{2iz}{1-z^2} & u_3 &= \frac{-2iw(1+z)}{(1+z^2)^2} & v_3 &= w(1+1/z^3)/2 \\
t_4 &= \frac{2z^2}{1+z^4} & u_4 &= -\frac{1-i}{\sqrt{2}}\,\frac{2w(i-z)}{(1-z^2)^2} & v_4 &= w(1+i/z^3)/2.
\end{aligned}
$$

One can check that

$$
\begin{aligned}
L_1 &= \mathbb{C}(t_0)\{1, u_0\} \\
L_2 &= \mathbb{C}(t_0) \\
L_3 &= \mathbb{C}(t_1) \\
L_4 &= \mathbb{C}(t_2)\{1, u_1\} = \mathbb{C}(t_2)\{1, v_1\} \\
L_5 &= \mathbb{C}(t_3)\{1, u_2\} = \mathbb{C}(t_3)\{1, v_2\} \\
L_6 &= \mathbb{C}(t_2)\{1, u_3\} = \mathbb{C}(t_2)\{1, v_3\} \\
L_7 &= \mathbb{C}(t_3)\{1, u_4\} = \mathbb{C}(t_3)\{1, v_4\} \\
L_8 &= \mathbb{C}(t_2) \\
L_9 &= \mathbb{C}(t_3) \\
L_{10} &= \mathbb{C}(t_4).
\end{aligned}
$$

Details for L_4 and L_5 will be given in Section 3.5. The cases L_6 and L_7 can be recovered from this, because H_6 and H_7 are conjugate to H_4 and H_5 respectively. The other cases are relatively easy (and are easily seen to be consistent with Corollary 2.3.2, which gives the genera of the quotients $PX(a)/H_i$). One can find further information in the files `parabolic/galois.mpl` and `parabolic/PK_subfields.mpl`.

`parabolic/galois_check.mpl: check_PK(), check_PK_subfields()`

3.5. **Elliptic quotients.** [sec-ellquot]

We next study the quotients $PX(a)/\langle\mu\rangle$ and $PX(a)/\langle\lambda\mu\rangle$.

First note that there is no natural action of the full group G on $PX(a)/\langle\mu\rangle$. Instead, there is an action of the centraliser of μ, which is $\langle\lambda^2,\mu,\nu\rangle \simeq C_2^3$. This action factors through the quotient group $\langle\lambda^2,\mu,\nu\rangle/\langle\mu\rangle \simeq C_2^2$. Similarly, we have a natural action of the group $\langle\lambda^2,\lambda\mu,\mu\nu\rangle/\langle\lambda\mu\rangle \simeq C_2^2$ on $PX(a)/\langle\lambda\mu\rangle$.

Definition 3.5.1. [defn-ellquot]
We put $b_\pm = (a^{-1} \pm a)/2$, and define affine curves $E_0^\pm(a)$ as follows.

$$\begin{aligned}
q_a^+(x) &= 2x(x-1)\left(b_+^2x^2-1\right) \\
q_a^-(x) &= 2x(x-1)\left(b_-^2x^2+1\right) \\
E_0^+(a) &= \{(y,x)\in\mathbb{C}^2 \mid y^2 = q_+(x)\} \\
E_0^-(a) &= \{(y,x)\in\mathbb{C}^2 \mid y^2 = q_-(x)\}.
\end{aligned}$$

We can obtain smooth projective completions of these curves by taking the closures of their images under the map $j\colon \mathbb{C}^2 \to \mathbb{C}P^3$ given by

$$j(y,x) = [y:1:x:x^2].$$

The results are

$$\begin{aligned}
E^+(a) &= \{[z] \mid z_2z_4 - z_3^2 = z_1^2 - 2(z_4-z_3)(b_+^2z_4 - z_2) = 0\} \\
E^-(a) &= \{[z] \mid z_2z_4 - z_3^2 = z_1^2 - 2(z_4-z_3)(b_-^2z_4 + z_2) = 0\}.
\end{aligned}$$

We define an action of the group $\langle\lambda^2,\mu,\nu\rangle$ on $E^+(a)$ as follows:

$$\begin{aligned}
\lambda^2[z] &= [-z_1:z_2:z_3:z_4] & \lambda^2 j(y,x) &= j(-y,x) \\
\mu[z] &= [\ \ z_1:z_2:z_3:z_4] & \mu j(y,x) &= j(\ \ y,x) \\
\nu[z] &= [\ \ \overline{z_1}:\overline{z_2}:\overline{z_3}:\overline{z_4}] & \nu j(y,x) &= j(\ \ \overline{y},\overline{x}).
\end{aligned}$$

Similarly, we define an action of the group $\langle\lambda^2,\lambda\mu,\mu\nu\rangle$ on $E^-(a)$ as follows:

$$\begin{aligned}
\lambda^2[z] &= [-z_1:z_2:z_3:z_4] & \lambda^2 j(y,x) &= j(-y,x) \\
\lambda\mu[z] &= [\ \ z_1:z_2:z_3:z_4] & \lambda\mu j(y,x) &= j(\ \ y,x) \\
\mu\nu[z] &= [\ \ \overline{z_1}:\overline{z_2}:\overline{z_3}:\overline{z_4}] & \mu\nu j(y,x) &= j(\ \ \overline{y},\overline{x}).
\end{aligned}$$

projective/ellquot_check.mpl: check_ellquot()

Remark 3.5.2. Code for all this is in ellquot.mpl. The polynomials q_a^+ and q_a^- are q_Ep and q_Em. Elements of $E_0^+(a)$ and $E_0^-(a)$ are represented as lists of length two, whereas elements of $E^+(a)$ and $E^-(a)$ are represented as lists of length four. The functions is_equal_Ep and is_equal_Em (which are actually the same) can be used to test projective equality. The function is_member_Ep_0 can be used to test whether a point lies in $E_0^+(a)$, and similarly for is_member_Em_0, is_member_Ep and is_member_Em. The inclusion $E_0^+(a) \to E^+(a)$ and its inverse are j_Ep and j_inv_Ep, and similarly for j_Em and j_inv_Em. The function NF_Ep can be used to reduce a polynomial in $z_1,\dots,z_5$ to normal form modulo the Gröbner basis for the ideal that defines $E^+(a)$. There is a similar function NF_Em for $E^-(a)$. Actions of G are given by act_Ep_0, act_Em_0, act_Ep and act_Em.

Proposition 3.5.3. [prop-P-Ep]
There is a unique morphism $\phi^+\colon PX(a) \to E^+(a)$ satisfying

$$\phi^+(j(w,z)) = j\left(\frac{2w(1-z)}{(1+z^2)^2}, \frac{2z}{1+z^2}\right)$$

for all $(w,z) \in PX_0(a)$ with $z \neq \pm i$. Moreover, this is equivariant with respect to $\langle\lambda^2,\mu,\nu\rangle$, and it induces an isomorphism $PX(a)/\langle\mu\rangle \to E^+(a)$.

Two variants of this map are represented by P_to_Ep and P_to_Ep_0.

Proof. First define $\psi\colon \mathbb{C}^5 \to \mathbb{C}^4$ by

$$\psi(z) = (2(z_2-z_3)z_1,\ z_2^2 + 2z_2z_4 + z_3z_5,\ 2z_2(z_3+z_5),\ 4z_2z_4).$$

This is homogeneous of degree two, so it induces a map $\overline{\psi}\colon U \to \mathbb{C}P^3$, where $U = \{[z] \in \mathbb{C}P^5 \mid \psi(z) \neq 0\}$. Now put

$$V = \{j(w,z) \in j(PX_0(a)) \mid z \notin \{0, i, -i\}\},$$

and note that this is open and dense in $PX(a)$ and is preserved by G. It is straightforward to check that $V \subseteq j(PX_0(a)) \subseteq U$, so we can define ϕ_0^+ to be the restriction of $\overline{\psi}$ to $PX_0(a)$. It follows easily from the definitions that for $j(w,z) \in V$ we have

$$\phi_0^+ j(w,z) = j\left(\frac{2w(1-z)}{(1+z^2)^2}, \frac{2z}{1+z^2}\right),$$

and that this lies in $E^+(a)$. From this we also see that the restriction of ϕ_0^+ to V is equivariant, and in particular that $\phi_0^+ = \phi_0^+\mu$ on V. By continuity, we must have $\phi_0^+ = \phi_0^+\mu$ on all of $PX_0(a)$. We can thus patch together ϕ_0^+ and $\phi_0^+\mu$ to get a morphism from $PX(a) = j(PX_0(a)) \cup \mu j(PX_0(a))$ to $E^+(a)$. The equivariance conditions are satisfied on the open dense subset V, so they are satisfied everywhere.

Now consider a point $(y,x) \in E_0^+(a)$ with $x \notin \{0,1\}$. Let u be a square root of $1-x^2$, and put

$$v_\pm = \left(\frac{\pm(2-x)u - (x+2)(x-1)}{x^3(x-1)} y,\ \frac{1 \pm u}{x}\right).$$

Straightforward algebra shows that $v_+, v_- \in PX_0(a)$ with $\mu(v_+) = v_-$, and that $(\phi^+)^{-1}\{(y,x)\} = \{v_+, v_-\}$. It follows that the induced map $PX(a)/\langle\mu\rangle \to E^+(a)$ is generically bijective. As the source and target are both smooth and complete algebraic curves, it follows that the map is an isomorphism, as claimed.

`projective/ellquot_check.mpl: check_ellquot()`

□

Proposition 3.5.4. [prop-P-Em]
There is a unique map $\phi^-\colon PX(a) \to E^-(a)$ satisfying

$$\phi^-(j(w,z)) = \left(-\frac{\sqrt{2}(1+i)w(i+z)}{(1-z^2)^2}, \frac{2iz}{1-z^2}\right) \in E_+^0(a)$$

for all $(w,z) \in PX_0(a)$ with $z \neq \pm 1$. Moreover, this is equivariant with respect to $\langle\lambda^2, \lambda\mu, \mu\nu\rangle$, and it induces an isomorphism $PX(a)/\langle\lambda\mu\rangle \to E^-(a)$.

Two variants of this map are represented by `P_to_Em` and `P_to_Em_0`.

Proof. Similar to the previous proposition, using the formulae

$$\psi(z) = (\sqrt{2}(1-i)z_1(z_2 - iz_3),\ z_2^2 - 2z_2z_4 + z_3z_5,\ 2iz_2(z_3 - z_5),\ -4z_2z_4)$$

and

$$v_\pm = \left(\frac{1+i}{\sqrt{2}}\ \frac{\pm(x-2)u - (x+2)(x-1)}{x^3(x-1)} y,\ \frac{\pm u - 1}{x} i\right).$$

`projective/ellquot_check.mpl: check_ellquot()`

□

Definition 3.5.5. We will write $v_i^+ = \phi^+(v_i) \in E^+(a)$, and similarly for v_i^-.

Remark 3.5.6. [rem-Ep-infinity]
One can check that

$$\begin{aligned} v_6^+ = v_9^+ &= [\ \ a + a^{-1} : 0 : 0 : -\sqrt{2}] \\ v_7^+ = v_8^+ &= [-a - a^{-1} : 0 : 0 : -\sqrt{2}], \end{aligned}$$

and these are the only points in $E^+(a) \setminus j(E_0^+(a))$. Similarly, we have

$$\begin{aligned} v_2^- = v_3^- &= [\ \ a - a^{-1} : 0 : 0 : -\sqrt{2}] \\ v_4^- = v_5^- &= [-a + a^{-1} : 0 : 0 : -\sqrt{2}], \end{aligned}$$

and these are the only points in $E^-(a) \setminus j(E_0^-(a))$.

Remark 3.5.7. [rem-elliptic-group]
One can also check that

$$\begin{aligned} v_0^+ = v_1^+ &= j(0,0) = [0:1:0:0] \in E^+(a) \\ v_0^- = v_1^- &= j(0,0) = [0:1:0:0] \in E^-(a). \end{aligned}$$

We use these points as the basepoints in $E^+(a)$ and $E^-(a)$. As these are elliptic curves, each of them has a unique group structure for which the specified basepoint is the zero element. We also find that

$$\begin{aligned} j^{-1}\phi^+ c_5(t) &= (\sqrt{a}t, 0) + O(t^2) \\ j^{-1}\phi^+ c_1(t) &= (e^{i\pi/4}t, 0) + O(t^2) \\ j^{-1}\phi^- c_5(t) &= (e^{-i\pi/4}\sqrt{a}t, 0) + O(t^2) \\ j^{-1}\phi^- c_1(t) &= (t, 0) + O(t^2). \end{aligned}$$

Thus, if we use the map $z \mapsto j^{-1}(z)_1$ as a local coordinate at the basepoint $v_0^\pm$, then $E^+(a)$ looks like our standard picture Net_0, but $E^-(a)$ is rotated clockwise by $\pi/4$.

projective/ellquot_check.mpl: check_ellquot_origin()

Definition 3.5.8. [defn-T-matrices]
We define matrices $T_i^\pm$ as follows:

$$T_1^+ = \begin{bmatrix} b_+^2 - 1 & 0 & 0 & 0 \\ 0 & 1 & -2b_+^2 & b_+^4 \\ 0 & 1 & -b_+^2 - 1 & b_+^2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \qquad T_1^- = \begin{bmatrix} b_-^2 + 1 & 0 & 0 & 0 \\ 0 & -1 & -2b_-^2 & -b_-^4 \\ 0 & -1 & -b_-^2 + 1 & b_-^2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$$T_2^+ = \begin{bmatrix} 2(1-b_+) & 0 & 0 & 0 \\ 0 & b_+ & 2b_+(b_+ - 2) & b_+(b_+ - 2)^2 \\ 0 & 1 & -2 & -b_+(b_+ - 2) \\ 0 & b_+^{-1} & -2 & b_+ \end{bmatrix} \qquad T_2^- = \begin{bmatrix} 2(1-b_-) & 0 & 0 & 0 \\ 0 & b_- & 2ib_-(2i - b_-) & -b_-(2i - b_-)^2 \\ 0 & i & -2i & -ib_-(2i - b_-) \\ 0 & -b_-^{-1} & -2i & b_- \end{bmatrix}$$

$$T_3^+ = \begin{bmatrix} -2(1+b_+) & 0 & 0 & 0 \\ 0 & b_+ & -2b_+(b_+ + 2) & b_+(b_+ + 2)^2 \\ 0 & -1 & 2 & b_+(b_+ + 2) \\ 0 & b_+^{-1} & 2 & b_+ \end{bmatrix} \qquad T_3^- = \begin{bmatrix} 2(i+b_-) & 0 & 0 & 0 \\ 0 & -b_- & -2ib_-(b_- + 2i) & b_-(b_- + 2i)^2 \\ 0 & i & -2i & ib_-(b_- + 2i) \\ 0 & b_-^{-1} & -2i & -b_- \end{bmatrix}$$

One can check that in $PGL_4(\mathbb{C})$ we have

$$(T_1^+)^2 = (T_2^+)^2 = (T_3^+)^2 = T_1^+T_2^+T_3^+ = (T_1^-)^2 = (T_2^-)^2 = (T_3^-)^2 = T_1^-T_2^-T_3^- = 1.$$

One can also check that these matrices preserve the defining equations for $E^+(a)$ or $E^-(a)$ as appropriate, so we have holomorphic involutions $\tau_i^+ \colon E^+(a) \to E^+(a)$ and $\tau_i^- \colon E^-(a) \to E^-(a)$ for $i = 1,2,3$. We also define $\tau_0^\pm$ to be the identity.

projective/ellquot_check.mpl: check_translations()

The maps τ_i^+ are `Ep_trans[i]` (on $E_0^+(a)$) or `Ep_0_trans[i]` (on $E^+(a)$), and the maps τ_i^- are `Em_trans[i]` or `Em_0_trans[i]`.

Because $E^\pm(a)$ is an elliptic curve, it is standard that the line bundle Ω^1 is trivial. For an elliptic curve in Weierstrass form $y^2 = x^3 + ax + b$, it is also standard that dx/y is a generator for Ω^1. As our conventions are slightly different, it is not quite so standard that the same formula remains valid, but we will now prove that it is.

Proposition 3.5.9. [prop-elliptic-forms]
There is a unique differential form $\omega^\pm$ on $E^\pm(a)$ such that $j^*(\omega^\pm) = dx/y$ on $E_0^\pm(a)$. Moreover, this is everywhere finite and nonzero, so it generates the module $\Omega^1_{E^\pm(a)}$. Near the origin we have $j^*(\omega^\pm) = (1 + O(y^2))dy$.

Proof. Differentiating the relation $y^2 = q_a^+(x)$ gives $2y\,dy = (q_a^+)'(x)\,dx$ and thus $dx/y = y\,dx/(q_a^+)(x) = 2dy/(q_a^+)'(x)$. Here $q_a^+(x)$ has no repeated roots, so there are no points where $q_a^+(x)$ and $(q_a^+)'(x)$ both vanish. It follows that dx/y is finite and nonzero everywhere in $E_0^+(a)$. Next, recall that $E^+(a) = j(E_0^+(a)) \cup$

$\{v_6^+, v_7^+\}$. Calculation shows that $\tau_1^+(v_6^+)$ and $\tau_1^+(v_7^+)$ lie in $j(E_0^+(a))$, so $E^+(a) = j(E_0^+(a)) \cup \tau_1^+ j(E_0^+(a))$. One can check from the definitions that

$$\tau_1^+ j(y,x) = j\left(\frac{b_-^2 y}{(b_+^2 x - 1)^2}, \frac{x-1}{b_+^2 x - 1}\right),$$

and thus that $(\tau_1^+)^*(x/dy)$ agrees with x/dy on their common domain. We can thus patch together dx/y with $(\tau_1^+)^*(x/dy)$ to get a form ω^+ which is finite and nonzero everywhere on $E^+(a)$, as required. One can check that $(q_a^+)'(x) = 2 + O(x)$, and the relation $y^2 = q_a^+(x)$ gives $x = O(y^2)$. We have seen that $j^*(\omega^+) = 2dy/(q_a^+)'(x)$, so $j^*(\omega^+) = (1 + O(y^2))dy$ as claimed.

The same method works for $E^-(a)$.

`projective/ellquot_check.mpl: check_translations()`

□

Remark 3.5.10. [`rem-translate`]

For any i, the form $(\tau_i^+)^*(\omega^+)$ must have the form $u\,\omega^+$ for some function u which is holomorphic everywhere on $E^+(a)$, and so is constant. As $(\tau_i^+)^2 = 1$ we see that $u^2 = 1$, so $u = \pm 1$. In the case $i = 1$, we saw in the proof of the above proposition that $u = 1$. By the same method one can check that $u = 1$ for $i = 2, 3$ as well. This implies that all the maps τ_i^+ are actually translations with respect to the standard group structure on $E^+(a)$. More specifically, the zero element is $o = v_0^+ = v_1^+$, and one can check that

$$\begin{aligned}
\tau_1^+(o) &= v_3^+ = v_5^+ \\
\tau_2^+(o) &= v_{11}^+ = v_{13}^+ \\
\tau_3^+(o) &= v_{10}^+ = v_{12}^+.
\end{aligned}$$

Thus, we have $\tau_1^+(p) = p + v_3^+$ and so on.

The situation for $E^-(a)$ is similar, but with

$$\begin{aligned}
\tau_1^-(o) &= v_7^- = v_9^- \\
\tau_2^-(o) &= v_{11}^- = v_{13}^- \\
\tau_3^-(o) &= v_{10}^- = v_{12}^-.
\end{aligned}$$

`projective/ellquot_check.mpl: check_translations()`

Proposition 3.5.11. [`prop-Ep-Em`]

There are (unbranched) double covering maps

$$E^+(a) \xrightarrow{\pi^+} E^-(a) \xrightarrow{\pi^-} E^+(a)$$

given generically by

$$\begin{aligned}
\pi^+(j(y,x)) &= j\left(\frac{\sqrt{2}y((1-x)^2 + b_-^2 x^2)}{((1-x)^2 - b_-^2 x^2)^2}, \frac{2x(x-1)}{((1-x)^2 - b_-^2 x^2)}\right) \\
\pi^-(j(y,x)) &= j\left(\frac{\sqrt{2}y((1-x)^2 - b_+^2 x^2)}{((1-x)^2 + b_+^2 x^2)^2}, \frac{2x(x-1)}{((1-x)^2 + b_+^2 x^2)}\right).
\end{aligned}$$

(More precisely, the above formulae are valid for all points (y,x) where the denominators are nonzero.) These are in fact surjective group homomorphisms, with

$$\begin{aligned}
\ker(\pi^+) &= \{j(0,0), j(0,1)\} = \{v_0^+, v_3^+\} \\
\ker(\pi^-) &= \{j(0,0), j(0,1)\} = \{v_0^-, v_7^-\}.
\end{aligned}$$

These maps are `Ep_to_Em` and `Em_to_Ep` (or `Ep_0_to_Em_0` and `Em_0_to_Ep_0`).

Proof. We define $\widetilde{\pi}^+, \widetilde{\pi}^- \colon \mathbb{C}^4 \to \mathbb{C}^4$ by

$$\begin{aligned}
\widetilde{\pi}^+(z)_1 &= \sqrt{2} z_1(z_2 - 2z_3 + b_+^2 z_4) \\
\widetilde{\pi}^+(z)_2 &= (2 - b_+^2)^2 z_4^2 + (z_2 - 2z_3 + 2(2 - b_+^2)z_4)(z_2 - 2z_3) \\
\widetilde{\pi}^+(z)_3 &= 2(z_2 - b_+^2 z_4)(z_4 - z_3) + 4(z_4^2 - 2z_3 z_4 + z_2 z_4) \\
\widetilde{\pi}^+(z)_4 &= 4(z_4 - 2z_3 + z_2)z_4 \\
\widetilde{\pi}^-(z)_1 &= \sqrt{2} z_1(z_2 - 2z_3 - b_-^2 z_4) \\
\widetilde{\pi}^-(z)_2 &= (2 + b_-^2)^2 z_4^2 + (z_2 - 2z_3 + 2(2 + b_-^2)z_4)(z_2 - 2z_3) \\
\widetilde{\pi}^-(z)_3 &= 2(z_2 + b_-^2 z_4)(z_4 - z_3) + 4(z_4^2 - 2z_3 z_4 + z_2 z_4) \\
\widetilde{\pi}^-(z)_4 &= 4(z_4 - 2z_3 + z_2)z_4.
\end{aligned}$$

Recall that

$$\begin{aligned}
E^+(a) &= \{[z] \mid \rho_0(z) = \rho_1(z) = 0\} \\
E^-(a) &= \{[z] \mid \rho_0(z) = \rho_2(z) = 0\}
\end{aligned}$$

where

$$\begin{aligned}
\rho_0(z) &= z_2 z_4 - z_3^2 \\
\rho_1(z) &= z_1^2 - 2(z_4 - z_3)(b_+^2 z_4 - z_2) \\
\rho_2(z) &= z_1^2 - 2(z_4 - z_3)(b_-^2 z_4 + z_2).
\end{aligned}$$

We claim that there are no nonzero points in $\mathbb{C}^4$ where $\rho_0(z) = \rho_1(z) = 0$ and $\widetilde{\pi}^+(z) = 0$. This can be proved by using Gröbner basis methods to prove that the ideals

$$I_k = (z_k - 1, \rho_0(z), \rho_1(z), \widetilde{\pi}^+(z)_1, \dots, \widetilde{\pi}^+(z)_4)$$

all contain 1, or just by solving the equations in a more elementary way. One can also use Gröbner bases to check that $\rho_i(\widetilde{\pi}^+(z)) \in (\rho_0(z), \rho_1(z))$ for $i \in \{0, 2\}$. It follows that the rule $\pi^+([z]) = [\widetilde{\pi}^+(z)]$ gives a well-defined morphism $\pi^+ \colon E^+(a) \to E^-(a)$. Straightforward algebra shows that $\pi^+(j(y,x))$ is given by the stated formula whenever $(1-x)^2 - b_-^2 x^2 \neq 0$. In particular, we have $\pi^+(j(0,0)) = j(0,0)$, so π^+ preserves basepoints. It is a standard fact that any basepoint preserving morphism of elliptic curves is a group homomorphism, and in this context, any non-constant group homomorphism is a covering map, so we just need to identify $\ker(\pi^+)$. If $(1-x)^2 - b_-^2 x^2 \neq 0$ then the stated formula for $\pi^+(j(x,y))$ is valid, and we see that $\pi^+(j(x,y)) = j(0,0)$ iff $(x,y) \in \{(0,0),(0,1)\}$. The exceptional points where $(1-x)^2 - b_-^2 x^2 = 0$ are as follows:

$$\begin{aligned}
w_1 &= j(\ \ 2b_-/(1+b_-)^2,\ 1/(1+b_-)) & w_2 &= j(\ \ 2b_-/(1-b_-)^2,\ 1/(1-b_-)) \\
w_3 &= j(-2b_-/(1+b_-)^2,\ 1/(1+b_-)) & w_4 &= j(-2b_-/(1-b_-)^2,\ 1/(1-b_-)).
\end{aligned}$$

These satisfy $\pi^+(w_1) = \pi^+(w_2) = v_2^+ \neq v_0^+$ and $\pi^+(w_3) = \pi^+(w_4) = v_4^+ \neq v_0^+$, so they do not contribute to the kernel. This just leaves the points in $E^+(a)$ that do not lie in the image of j, which are v_6^+ and v_7^+; direct calculation shows again that these are not in the kernel of π^+. This completes the proof that $\ker(\pi^+) = \{v_0^+, v_3^+\}$. (As an alternative, we could reach the same conclusion by calculating Gröbner bases for each of the ideals

$$(z_i - 1, \rho_0(z), \rho_1(z), \widetilde{\pi}^+(z)_1, \widetilde{\pi}^+(z)_3, \widetilde{\pi}^+(z)_4),$$

or by showing that the degree of the relevant field extension is two and appealing to some more abstract arguments.)

The proof for π^- is essentially the same.

`projective/ellquot_check.mpl: check_isogenies()`

□

Definition 3.5.12. [defn-PJ]

We note that Proposition 3.5.11 implies that the elements v_3^+ and v_7^- have order two, so (v_7^-, v_3^+) generates a subgroup Z of order two in $(E^-(a) \times E^+(a))$. We define

$$PJ(a) = (E^-(a) \times E^+(a))/Z.$$

We also define $\theta^+ : PJ(a) \to E^+(a)$ and $\theta^- : PJ(a) \to E^-(a)$ by

$$\begin{aligned} \theta^+((w^-, w^+) + Z) &= \pi^-(w^-) \\ \theta^-((w^-, w^+) + Z) &= \pi^+(w^+). \end{aligned}$$

Proposition 3.5.13. [prop-P-to-PJ]

There is a unique morphism $\phi \colon PX(a) \to PJ(a)$ such that $\phi(v_0) = o$ and $\theta^+\phi = \phi^+$ and $\theta^-\phi = \phi^-$.

Proof. Suppose we have $(w, z) \in PX_0(a)$ and $u \in \mathbb{C}$ with $u^2 = z$. When the relevant denominators are nonzero, we then put

$$\begin{aligned} x_- &= \frac{w/u - (1-z)^2}{2(b_-^2 z - (1-z)^2)} \\ x_+ &= i\frac{w/u + (i+z)^2}{2(b_+^2 z + i(i+z)^2)} \\ y_- &= \frac{1}{\sqrt{2}}\,\frac{u(1-z)(2b_+^2(1-z)^2 - b_-^2(w/u + 1 + z^2))}{(b_-^2 z - (1-z)^2)^2} \\ y_+ &= \frac{1+i}{2}\,\frac{u(i+z)(2b_-^2(i+z)^2 + b_+^2(w/u + 1 - z^2))}{(b_+^2 z + i(i+z)^2)^2}, \end{aligned}$$

and $\phi_0(w, z, u) = (y_-, x_-, y_+, x_+)$. One can check that

(a) $y_-^2 = q_a^-(x_-)$ and $y_+^2 = q_a^+(x_+)$, so $\phi_0(w, z, u) \in E_0^-(a) \times E_0^+(a)$.
(b) $\phi_0(w, z, -u) = (\tau_1^- \times \tau_1^+)\phi_0(w, z, u)$.
(c) $\pi^-(j(y^-, x^-)) = \phi_0^+(w, z)$ and $\pi^+(j(y^+, x^+)) = \phi_0^-(w, z)$.

It follows that the image of $\phi_0(w, z, u)$ in $PJ(a)$ is independent of the choice of u, so we can call it $\phi_0(w, z)$. This defines a rational map from $EX_0(a)$ to $PJ(a)$, but $EX_0(a)$ is dense subset of the smooth curve $EX(a)$, and $PJ(a)$ is complete, so this extends uniquely to give a morphism $\phi \colon PX(a) \to PJ(a)$. Point (c) above shows that $\theta^\pm\phi = \phi^\pm$.

```
projective/ellquot_check.mpl: check_isogenies()
```

□

The map ϕ is represented in Maple as `P_0_to_J_0`.

Corollary 3.5.14. [cor-jacobian]

$PJ(a)$ can be regarded as the Jacobian variety of $PX(a)$.

Proof. The Jacobian variety J can be characterised by the fact that it is an abelian variety equipped with a map $\delta \colon PX(a) \to J$ of varieties such that the induced map $H_1PX(a) \to H_1J$ is an isomorphism. We saw in Definition 2.7.6 that the map

$$(\phi^+, \phi^-)_* \colon H_1PX(a) \to H_1(E^+(a) \times E^-(a))$$

is injective, and has image of index two. As the map

$$\theta = (\theta^+, \theta^-) \colon PJ(a) \to E^+(a) \times E^-(a)$$

is a connected double covering, it also gives an index two subgroup of $\pi_1 = H_1$. As $(\phi^+, \phi^-) = \theta\phi$, we deduce that $\phi_* \colon H_1PX(a) \to H_1PJ(a)$ is an isomorphism, as required. □

It is standard that any elliptic curve has an analytic parametrisation via the Weierstrass $\wp$-function. Details for the present case are as follows.

Definition 3.5.15. [defn-weierstrass-xi]
We put

$$g_2^+ = 4(\tfrac{1}{3} + b_+^2) \qquad g_2^- = 4(\tfrac{1}{3} - b_-^2)$$
$$g_3^+ = \tfrac{8}{3}(\tfrac{1}{9} - b_+^2) \qquad g_3^- = \tfrac{8}{3}(\tfrac{1}{9} + b_-^2)$$
$$p_0^+(z) = \wp(z/\sqrt{2}; g_2^+, g_3^+) \qquad p_0^-(z) = \wp(iz/\sqrt{2}; g_2^-, g_3^-)$$
$$p_1^+(z) = \wp'(z/\sqrt{2}; g_2^+, g_3^+) \qquad p_1^-(z) = \wp'(iz/\sqrt{2}; g_2^-, g_3^-)$$

and then

$$\xi^+(z) = j\left(-\frac{p_1^+(z)}{\sqrt{2}(p_0^+(z)+1/3)^2}, \frac{1}{(p_0^+(z)+1/3)}\right) \in \mathbb{C}P^3$$
$$\xi^-(z) = j\left(i\frac{p_1^-(z)}{\sqrt{2}(p_0^-(z)+1/3)^2}, \frac{1}{(p_0^-(z)+1/3)}\right) \in \mathbb{C}P^3.$$

In Maple, the parameters $g_i^\pm$ are `Wg2p`, `Wg3p`, `Wg2m` and `Wg3m`. The maps $\xi^\pm$ are `C_to_Ep_0` and `C_to_Em_0`.

Remark 3.5.16. [rem-xi-differential]
It is a standard fact that $\wp(z) = z^{-2} + O(z^2)$, so $\wp'(z) = -2z^{-3} + O(z)$. Using this, we find that $j^{-1}\xi^+(z) = (z,0) + O(z^2)$ and $j^{-1}\xi^-(z) = (z,0) + O(z^2)$.

Proposition 3.5.17. [prop-xi-functions]
There are lattices $\Lambda^+, \Lambda^- \subset \mathbb{C}$ such that $\xi^\pm$ induces an isomorphism $\mathbb{C}/\Lambda^\pm \to E^\pm(a)$. Moreover, the forms $\omega^\pm$ on $E^\pm(a)$ (from Proposition 3.5.9) satisfy $(\xi^\pm)^*(\omega^\pm) = dz$.

Proof. Given any g_2, g_3 we can define $f(x) = 4x^3 - g_2x - g_3$ and $F_0 = \{(y,x) \in \mathbb{C}^2 \mid y^2 = f(x)\}$, and we can then define F to be the normalisation of F_0. It is standard that

$$\wp'(z; g_2, g_3)^2 = f(\wp(z; g_2, g_3)),$$

so we have a meromorphic function $z \mapsto (\wp'(z), \wp(z))$ from $\mathbb{C}$ to F. It is also standard that this induces an isomorphism $\mathbb{C}/\Lambda \to F$, for a suitable lattice $\Lambda \subset \mathbb{C}$. The first claim follows from this by a change of coordinates.

Next, we have $(\xi^\pm)^*(\omega^\pm) = u(z)\,dz$ for some function $u(z)$ which is holomorphic, nowhere zero, and periodic with respect to $\Lambda^\pm$. This forces $u(z)$ to be constant. Remark 3.5.16, together with the last part of Proposition 3.5.9, shows that $u = 1$.

`projective/ellquot_check.mpl: check_weierstrass()`

∎

We now want to understand the lattices Λ^+ and Λ^- in more detail, which is essentially the same as calculating the periods $p_{jk}(a) = \int_{c_j} \omega_k$ as in Definition 3.3.5.

Definition 3.5.18. We define the complete elliptic integral $K(k)$ (for $0 < k < 1$) by

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}.$$

In this range the square roots are real and positive and there is no need for branch cuts.

Our definition is the same as Maple's `EllipticK(k)`, but slightly different conventions are used in some other sources.

Definition 3.5.19. [defn-period-rs]
We put

$$m_+ = \frac{1+a}{\sqrt{2(1+a^2)}} \qquad m_- = \frac{1-a}{\sqrt{2(1+a^2)}}$$
$$\alpha_+ = 2b_+^{-1/2}K(m_+) \qquad \alpha_- = 2b_+^{-1/2}K(m_-).$$

(Note here that the first factor in α_- involves b_+, not b_-.)

In Maple, these are mp_period, mm_period, ap_period and am_period. These are defined in the file projective/picard_fuchs.mpl.

Remark 3.5.20. [rem-legendre]

A mixture of theoretical arguments and numerical calculations makes it clear that we also have

$$\alpha_+ = \pi P_{-1/4}(A/2) + 2Q_{-1/4}(A/2)$$
$$\alpha_- = \pi P_{-1/4}(A/2)$$

where $A = a^{-2} + a^2$ as before, and P and Q are Legendre functions. We do not have a complete proof, but we will mention some ingredients. Consider the differential operator

$$\mathcal{L} = 198\frac{\partial^2}{\partial A^2} + 192A\frac{\partial^3}{\partial A^3} + (32A^2 - 128)\frac{\partial^4}{\partial A^4}.$$

One can check by direct calculation that

$$\mathcal{L}((z^5 - Az^3 + z)^{-1/2})\, dz = \frac{d}{dz}\left(\frac{33z^8 - 3Az^{10} - 27z^{12}}{(z^5 - Az^3 + z)^{7/2}}\right) dz.$$

The terms on the left can be interpreted as meromorphic differential forms on $PX(a)$, all of whose residues are zero. This means that their integrals round a loop depend only on the homology class of the loop. On the other hand, the integral of the right hand side around any loop is zero. Using this, we see that the periods (when expressed as a function of A) are annihilated by $\mathcal{L}$. This is the Picard-Fuchs equation for the family $\{PX(a) \mid a \in (0,1)\}$. Maple asserts that the annihilator of $\mathcal{L}$ is spanned by 1, A, $P_{-1/4}(A/2)$ and $Q_{-1/4}(A/2)$. This could presumably be checked using hypergeometric series, but we have not attempted that. The specific coefficients in the stated expressions for α_+ and α_- were obtained by graphical and numerical experimentation.

projective/picard_fuchs_check.mpl: check_picard-fuchs()

Proposition 3.5.21. [prop-periods]

The periods are

$$\begin{aligned}
p_{0,0} &= 0 & p_{0,1} &= 0\\
p_{1,0} &= (i+1)\alpha_- & p_{1,1} &= (i-1)\alpha_-\\
p_{2,0} &= (i-1)\alpha_- & p_{2,1} &= (i+1)\alpha_-\\
p_{3,0} &= i\alpha_- & p_{3,1} &= i\alpha_-\\
p_{4,0} &= -\alpha_- & p_{4,1} &= \alpha_-\\
p_{5,0} &= (\alpha_+ + \alpha_-)/2 & p_{5,1} &= (\alpha_+ - \alpha_-)/2\\
p_{6,0} &= i(\alpha_+ + \alpha_-)/2 & p_{6,1} &= -i(\alpha_+ - \alpha_-)/2\\
p_{7,0} &= (\alpha_+ - \alpha_-)/2 & p_{7,1} &= (\alpha_+ + \alpha_-)/2\\
p_{8,0} &= i(\alpha_+ - \alpha_-)/2 & p_{8,1} &= -i(\alpha_+ + \alpha_-)/2.
\end{aligned}$$

In Maple, p_{ij} is p_period[i,j].

The proof will be given after some preparatory lemmas.

Lemma 3.5.22. [lem-int-q]

$$\int_0^{1/b_+} \frac{dx}{\sqrt{q_a^+(x)}} = \alpha_+/2$$
$$\int_{-1/b_+}^0 \frac{dx}{\sqrt{-q_a^+(x)}} = \alpha_-/2.$$

Proof. For the first integral we use the substitution $x = (1-t^2)/(b_+ - t^2)$, and for the second we use the substitution $x = b_+^{-1}(1 - 2/t^2)^{-1}$. In both cases we get an extra factor of -1 in the integrand, which is cancelled by the fact that the limits are reversed, because x is a decreasing function of t.

`projective/picard_fuchs_check.mpl: check_period_integrals()`

□

Lemma 3.5.23. [`lem-elliptic-periods`]

$$\int_{\phi^+\circ c_5} \frac{dx}{y} = \alpha_+$$
$$\int_{\phi^+\circ c_6} \frac{dx}{y} = i\alpha_-.$$

Proof. From the definitions we have $j^{-1}(c_5(t)) = (w(t), z(t))$, where $z(t) = a\sin(t/2)^2 \in [0, a]$ and $w(t)$ is a positive multiple of $\sin(t)$. It follows that $j^{-1}\phi^+(c_5(t)) = (y(t), x(t))$, where $x(t) = 2/(z(t)+z(t)^{-1})$, and $y(t)$ is again a positive multiple of $\sin(t)$. On the other hand, we have $(y(t), x(t)) \in E_0^+(a)$ so $y(t)^2 = q_a^+(x(t))$, so for $0 \leq t \leq \pi$ we must have $y(t) = \sqrt{q_a^+(x(t))}$, and for $\pi \leq t \leq 2\pi$ we must have $y(t) = -\sqrt{q_a^+(x(t))}$. Moreover, as t runs from 0 to π we see that $z(t)$ increases from 0 to a, and so $x(t)$ increases from 0 to $1/b_+$. On the other hand, as t increases from π to 2π we see that $x(t)$ decreases from $1/b_+$ to 0. It follows that

$$\int_{\phi^+\circ c_5} \frac{dx}{y} = \int_{x=0}^{1/b_+} \frac{dx}{\sqrt{q_a^+(x)}} + \int_{x=1/b_+}^{0} \frac{dx}{-\sqrt{q_a^+(x)}} = 2\int_0^{1/b_+} \frac{dx}{\sqrt{q_a^+(x)}} = \alpha_+.$$

The second integral is similar. We have $j^{-1}(c_6(t)) = (i\,w(t), -z(t))$, where $z(t) = a\sin(t/2)^2 \in [0, a]$ and $w(t)$ is a positive multiple of $\sin(t)$. It follows that $j^{-1}\phi^+(c_5(t)) = (y(t), x(t))$, where $x(t) = -2/(z(t) + z(t)^{-1})$, and $y(t)$ is again a positive multiple of $i\sin(t)$. In this range $q_a^+(x(t)) \leq 0$ so it is natural to consider $\sqrt{-q_a^+(x(t))}$. As $(y(t), x(t)) \in E_0^+(a)$, we must have $(y(t)/i)^2 = -q_a^+(x(t))$, so for $0 \leq t \leq \pi$ we must have $y(t) = i\sqrt{-q_a^+(x(t))}$, and for $\pi \leq t \leq 2\pi$ we must have $y(t) = -i\sqrt{-q_a^+(x(t))}$. Moreover, as t runs from 0 to π and then to 2π, we see that $x(t)$ decreases from 0 to $-1/b_+$, and then increases back to 0 again. It follows that

$$\int_{\phi^+\circ c_6} \frac{dx}{y} = \int_0^{-1/b_+} i\frac{dx}{\sqrt{-q_a^+(x)}} + \int_{-1/b_+}^{0} -i\frac{dx}{\sqrt{-q_a^+(x)}} = 2i\int_0^{1/b_+} \frac{dx}{\sqrt{-q_a^+(x)}} = i\alpha_-.$$

`projective/picard_fuchs_check.mpl: check_period_integrals()`

□

Lemma 3.5.24. [`lem-form-pullback`]

For the forms $\omega^\pm$ on $E^\pm(a)$ and ω_i on $PX(a)$ (as in Propositions 3.3.3 and 3.5.9), we have

$$(\phi^+)^*(\omega^+) = \omega_0 + \omega_1$$
$$(\phi^-)^*(\omega^+) = \frac{1+i}{\sqrt{2}}(i\omega_0 - \omega_1).$$

Proof. On the affine pieces $PX_0(a)$ and $E_0^+(a)$ we have $\omega^+ = dx/y$ and

$$\phi^+(w, z) = \left(\frac{2w(1-z)}{(1+z^2)^2}, \frac{2z}{1+z^2}\right),$$

so

$$\begin{aligned}(\phi^+)^*(\omega^+) &= \frac{(1+z^2)^2}{2w(1-z)} d\left(\frac{2z}{1+z^2}\right) \\ &= \frac{(1+z^2)^2}{2w(1-z)} \frac{2-2z^2}{(1+z^2)^2} dz = \frac{1+z}{w} dz.\end{aligned}$$

On the other hand, we have $\omega_0 = dz/w$ and $\omega_1 = z\,dz/w$, so $(\phi^+)^*(\omega^+) = \omega_0 + \omega_1$. The argument for $(\phi^-)^*(\omega^-)$ is similar.

`projective/picard_fuchs_check.mpl: check_period_integrals()`

□

Corollary 3.5.25. [cor-basic-periods]
We have $p_{5,0} = (\alpha_+ + \alpha_-)/2$ *and* $p_{5,1} = (\alpha_+ - \alpha_-)/2$.

Proof. Lemma 3.5.23 says that

$$\int_{c_5} (\phi^+)^* \left(\frac{dx}{y}\right) = \alpha_+.$$

On the left hand side, Lemma 3.5.24 says that the integrand is $\omega_0 + \omega_1$, so the integral is $p_{5,0} + p_{5,1}$. Similarly, the second equation in Lemma 3.5.23 becomes $p_{6,0} + p_{6,1} = i\alpha_-$. On the other hand, we have seen that $\lambda \circ c_5 = c_6$ and $\lambda^*\omega_0 = i\omega_0$ and $\lambda^*\omega_1 = -i\omega_1$; it follows that $p_{6,0} + p_{6,1} = ip_{5,0} - ip_{5,1}$. Linear algebra now gives $p_{5,0} = (\alpha_+ + \alpha_-)/2$ and $p_{5,1} = (\alpha_+ - \alpha_-)/2$ as claimed. □

Proof of Proposition 3.5.21. It is standard that there is a well-defined pairing $H_1(PX(a)) \otimes \Omega^1(PX(a)) \to \mathbb{C}$ such that $([\gamma], \alpha) = \int_\gamma \alpha$ for all closed curves γ in $PX(a)$ and all $\omega \in \Omega^1(PX(a))$. In particular, we have $p_{i,j} = ([c_i], \omega_j)$. Thus, relations in $H_1(PX(a))$ will give relations between periods. Note also that for $g \in D_8$ we have $(g_*[c_i], \omega_j) = ([g \circ c_i], \omega_j) = ([c_i], g^*\omega_j)$. Proposition 3.3.3 gives the action of D_8 on ω_0 and ω_1, whereas Definition 2.4.4(c) and Proposition 3.2.2 give the action on the curves c_i. In particular, we have

$$[c_6] = \lambda_*[c_5] \qquad [c_7] = \mu_*[c_5] \qquad [c_8] = (\mu\lambda)_*[c_5],$$

whereas

$$\lambda^*\omega_0 = i\omega_0 \qquad \lambda^*\omega_1 = -i\omega_1 \qquad \mu^*\omega_0 = \omega_1 \qquad \mu^*\omega_1 = \omega_0.$$

It therefore follows from Corollary 3.5.25 that

$$\begin{aligned} p_{5,0} &= (\alpha_+ + \alpha_-)/2 & p_{5,1} &= (\alpha_+ - \alpha_-)/2 \\ p_{6,0} &= i(\alpha_+ + \alpha_-)/2 & p_{6,1} &= -i(\alpha_+ - \alpha_-)/2 \\ p_{7,0} &= (\alpha_+ - \alpha_-)/2 & p_{7,1} &= (\alpha_+ + \alpha_-)/2 \\ p_{8,0} &= i(\alpha_+ - \alpha_-)/2 & p_{8,1} &= -i(\alpha_+ + \alpha_-)/2. \end{aligned}$$

Next, we see from Proposition 2.7.1 that

$$\begin{aligned} [c_0] &= 0 \\ [c_1] &= [c_5] + [c_6] - [c_7] - [c_8] \\ [c_2] &= -[c_5] + [c_6] + [c_7] - [c_8] \\ [c_3] &= [c_6] - [c_8] \\ [c_4] &= -[c_5] + [c_7]. \end{aligned}$$

We can now apply the maps $(-, \omega_0)$ and $(-, \omega_1)$ to deduce that

$$\begin{aligned} p_{0,0} &= 0 & p_{0,1} &= 0 \\ p_{1,0} &= (i+1)\alpha_- & p_{1,1} &= (i-1)\alpha_- \\ p_{2,0} &= (i-1)\alpha_- & p_{2,1} &= (i+1)\alpha_- \\ p_{3,0} &= i\alpha_- & p_{3,1} &= i\alpha_- \\ p_{4,0} &= -\alpha_- & p_{4,1} &= \alpha_-. \end{aligned}$$

projective/picard_fuchs_check.mpl: check_periods()

□

Proposition 3.5.26. *The lattices* Λ^+ *and* Λ^- *(from Proposition 3.5.17) are*

$$\begin{aligned} \Lambda^+ &= \{n\alpha_+ + m\alpha_- i \mid n, m \in \mathbb{Z}\} \\ \Lambda^- &= \{(n\alpha_+ + m\alpha_- i)/\sqrt{2} \mid n, m \in \mathbb{Z},\ n = m \pmod 2\}. \end{aligned}$$

Proof. We can identify Λ^+ with $\{\int_\gamma dz \mid \gamma \in \pi_1(\mathbb{C}/\Lambda^+)\}$. Now ξ^+ induces an isomorphism $\mathbb{C}/\Lambda^+ \to E^+(a)$, under which dz corresponds to ω^+ (by the last part of Proposition 3.5.17). This means that $\Lambda^+ =$

$\{\int_\gamma \omega^+ \mid \gamma \in \pi_1(E^+(a))\}$. On the other hand, we know from Proposition 2.7.4 that the group $\pi_1(E^+(a)) = H_1(E^+(a))$ is a quotient of $H_1(PX(a))$, so

$$\Lambda^+ = \{\int_\gamma (\phi^+)^*(\omega^+) \mid \gamma \in \pi_1(PX(a))\}.$$

Using Lemma 3.5.24 we now see that Λ^+ is spanned by the numbers $p_{k0} + p_{k1}$, and by inspecting Proposition 3.5.21 we conclude that $\Lambda^+ = \{n\alpha_+ + m\alpha_- i \mid n, m \in \mathbb{Z}\}$ as claimed.

In the same way, using the relation $(\phi^-)^*(\omega^-) = \frac{1+i}{\sqrt{2}}(i\omega_0 - \omega_1)$ we see that Λ^- is generated by the numbers $\frac{1+i}{\sqrt{2}}(ip_{k0} - p_{k1})$. These numbers can again be read off from Proposition 3.5.21, giving

$$\Lambda^- = \{(n\alpha_+ + m\alpha_- i)/\sqrt{2} \mid n, m \in \mathbb{Z},\ n = m \pmod 2\}$$

as claimed.

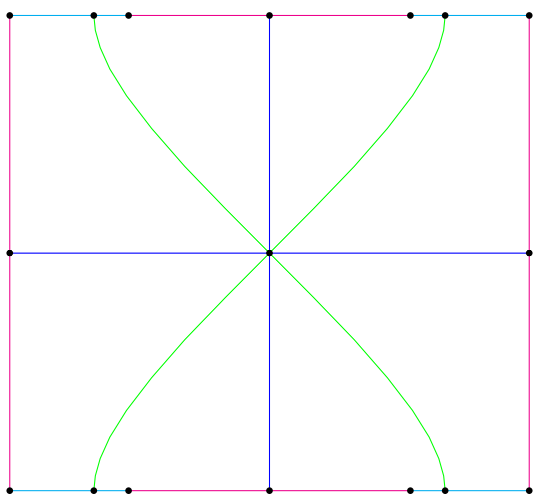
projective/ellquot_check.mpl: check_weierstrass()

□

The closed curves $c_k(t) \in PX(a)$ can be mapped to $E^+(a)$ using ϕ^+ and then lifted via ξ^+ to give curves in $\mathbb{C}$ which are usually not closed. There are no closed formulae for these curves, but the functions c_TEp_approx[k] and c_TEm_approx[k] give good approximations. The set

$$\{x + iy \mid |x| \leq \alpha_+/2, |y| \leq \alpha_-/2\}$$

is a fundamental domain for the action of Λ^+ on $\mathbb{C}$, and the parts of the lifted curves lying in that domain can be illustrated as follows:

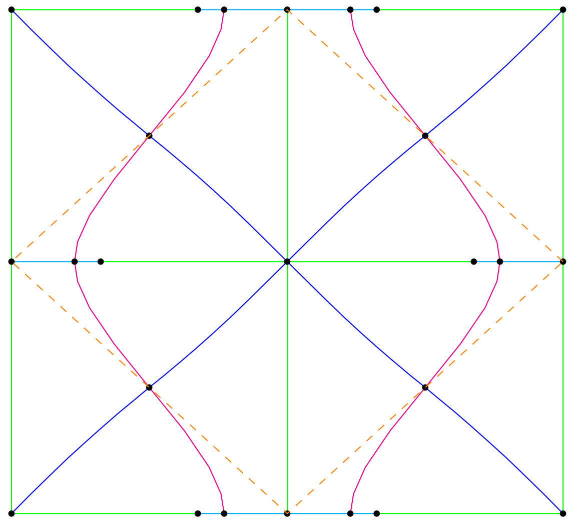


This is combinatorially equivalent to the net for $X/\langle\mu\rangle$ which we exhibited in Section 2.6. We have used the value $a = 0.1$, which is roughly right for the embedded surface EX^*. The above domain is not square; the ratio width/height $= \alpha_+/\alpha_-$ is approximately 1.1.

The situation for $E^-(a)$ is a little more complicated. The picture below shows the domain

$$\{x + iy \mid |x| \leq \alpha_+/\sqrt{2}, |y| \leq \alpha_-/\sqrt{2}\},$$

which covers $\mathbb{C}/\Lambda^-$ twice. The diagonal blue curves, which represent c_5 and c_6, are close to being straight, but they are not exactly straight. The dashed orange lines enclose a fundamental domain for Λ^-. This is a rhombus, but the angles are not $\pi/2$. It is combinatorially equivalent to the net for $X/\langle\lambda\mu\rangle$ which we exhibited in Section 2.6, but is rotated through $\pi/4$ as well as being slightly distorted.



3.6. **Some general theory of Riemann surfaces.** [sec-riemann]

In the next section, we will give a classification of (pre)cromulent surfaces. In the present section, we develop some more general theory of Riemann surfaces, which will feed into that classification. All of it is essentially standard; we discuss it here in order to have a convenient reference with a uniform approach.

3.6.1. *Involutions.* Throughout this section, Z will be a compact connected Riemann surface, with a conformal involution $\alpha\colon Z \to Z$, and an anticonformal involution $\beta\colon Z \to Z$ that commutes with α. We also assume that α has only isolated fixed points (which means that the total number of fixed points is finite). We will write Δ for the open unit disc in $\mathbb{C}$.

Lemma 3.6.1. [lem-invariant-metric]

Z admits a smooth Riemannian metric that is compatible with the conformal structure and is invariant under the action of α and β.

Proof. Any coordinate patch clearly admits a smooth conformal Riemannian metric, and one can use a partition of unity to combine such local metrics to give a local metric, say μ_0. Any smooth automorphism of Z acts in an evident way on the set of metrics, and that set is convex, so we can define

$$\mu = (\mu_0 + \alpha^*\mu_0 + \beta^*\mu_0 + \alpha^*\beta^*\mu_0)/4.$$

This is the required invariant metric. □

For the rest of this section we will assume that an invariant metric has been chosen.

Remark 3.6.2. [rem-std-param]

Let C be a closed connected one-dimensional smooth submanifold of Z. Then C is necessarily diffeomorphic to the circle. Now fix a point $a \in C$, and a unit vector $v \in T_aC$. It is then standard that there is a unique smooth map $c_1\colon \mathbb{R} \to C$ with $c_1(0) = a$ and $c_1'(0) = v$ and $\|c_1'(t)\| = 1$ for all t. One can check that c_1 induces a diffeomorphism $\mathbb{R}/\mathbb{Z}d \to C$ for some $d > 0$. We put $c(t) = c_1(td/2\pi)$, so c induces a diffeomorphism $\mathbb{R}/2\pi\mathbb{Z} \to C$, which we call a *standard parametrisation* of C. It depends on the choice of a and v, but if we make different choices then the new standard parametrisation will be of the form $t \mapsto c(p+t)$ or $t \mapsto c(p-t)$ for some constant p.

Remark 3.6.3. [rem-circle-involution]

Let $c\colon \mathbb{R}/2\pi\mathbb{Z} \to C$ be as in the previous remark, let $\gamma\colon Z \to Z$ be an involution that preserves the metric, and suppose that $\gamma(C) = C$. Then $\gamma(c(t))$ must have the form $c(p+t)$ or $c(p-t)$ for some constant p (which is well-defined modulo 2π).

(a) If $\gamma(c(t)) = c(p+t)$ then the equation $\gamma^2 = 1$ gives $2p = 0 \pmod{2\pi}$, so we can take $p = 0$ or $p = \pi$. If $p = 0$ then of course $\gamma|_C = 1$. If $p = \pi$ then γ acts freely on C, so $C/\langle\gamma\rangle$ is again a circle.

(b) If $\gamma(c(t)) = c(p-t)$ then we can define a new standard parametrisation by $c^*(t) = c(p/2+t)$, and this satisfies $\gamma(c^*(t)) = c^*(-t)$. It follows that the points $a = c^*(0)$ and $b = c^*(\pi)$ are fixed by γ, but that γ acts freely on $C \setminus \{a,b\}$. If we put $P = c^*([0,\pi])$ and $Q = c^*([-\pi,0])$ then the evident maps

$$[0,\pi] \xrightarrow{c^*} P \to C/\langle\gamma\rangle \leftarrow Q \xleftarrow{c^*} [-\pi,0]$$

are homeomorphisms. Moreover, we have $P \cup Q = C$ and $P \cap Q = \{a,b\}$.

Definition 3.6.4. [defn-local-parameter]
Suppose that $a \in Z$, that U_0 is an open neighbourhood of a, and that $f_0 \colon U_0 \to \mathbb{C}$ is a holomorphic map. We say that f_0 is a *centred local parameter* at a if $f_0(a) = 0$, and that df_0 generates the cotangent space to Z at a. We say that the pair (U_0, f_0) is *normalised* if f_0 gives a conformal isomorphism from U_0 to the open unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

Remark 3.6.5. [rem-shrinking]
Let $f_0 \colon U_0 \to \mathbb{C}$ be a centred local parameter that need not be normalised. The holomorphic inverse function theorem then guarantees that there is a smaller open neighbourhood U with $a \in U \subseteq U_0$ and a number $\epsilon > 0$ such that f_0 restricts to give a conformal isomorphism $U \to \{z \in \mathbb{C} \mid |z| < \epsilon\}$. This means that the map $f = \epsilon^{-1} f_0|_U \colon U \to \Delta$ is a conformal isomorphism, so (U, f) is normalised. The operation that converts (U_0, f_0) to (U, f) will be called *shrinking*.

Lemma 3.6.6. [lem-al-fixed]
Suppose that $a \in Z$ with $\alpha(a) = a$. Then there is a normalised local parameter $f \colon U \to \Delta$ at a such that $\alpha(U) = U$ and $f(\alpha(u)) = -f(u)$ for all $u \in U$.

Proof. Choose any centred local parameter $f_0 \colon U_0 \to \mathbb{C}$. Put $U_1 = U_0 \cap \alpha(U_0)$ and $f_1 = f_0|_{U_1}$; this still gives a centred local parameter. We can expand $f_1(\alpha(u))$ as a power series $\sum_{k=1}^{\infty} a_k f_1(u)^k$. Because $\alpha^2 = 1$, we have $a_1 = \pm 1$. We claim that a_1 cannot be equal to 1. Indeed, as α has isolated fixed points, we cannot have $f_1(\alpha(u)) = f_1(u)$ as a power series. Thus, if $a_1 = 1$ then there must exist $k > 1$ such that $a_i = 0$ for $1 < i < k$ and $a_k \neq 0$, so

$$f_1(\alpha(u)) = f_1(u) + a_k f_1(u)^k + O(f_1(u)^{k+1}).$$

If we substitute this into itself and use $\alpha^2 = 1$ we get $2a_k = 0$, which is a contradiction. We must therefore have $a_1 = -1$.

Now put $f_2(u) = (f_1(u) - f_1(\alpha(u)))/2$, so $f_2 \colon U_1 \to \mathbb{C}$ is holomorphic with $f_2(a) = 0$ and $f_2(\alpha(u)) = -f_2(u)$. We also have $f_2(u) = f_1(u) + O(f_1(u)^2)$, and thus that $f_2(u)$ is again a centred local parameter. We can therefore produce the required pair (U, f) by shrinking. □

Lemma 3.6.7. [lem-al-not-fixed]
Suppose that $a \in Z$ with $\alpha(a) \neq a$. Then there is a normalised local parameter $f \colon U \to \Delta$ at a such that $\alpha(\overline{U}) \cap \overline{U} = \emptyset$.

Proof. By standard arguments with compact Hausdorff spaces, we can choose open neighbourhoods V of a and W of $\alpha(a)$ such that $\overline{V} \cap \overline{W} = \emptyset$. Put $U_0 = V \cap \alpha(W)$, so U_0 is an open neighbourhood of a with $\overline{U_0} \cap \alpha(\overline{U_0}) = \emptyset$. Now let $f \colon U \to \Delta$ be any normalised local parameter at a with $U \subseteq U_0$. □

Lemma 3.6.8. [lem-bt-fixed]
Suppose that $a \in Z$ with $\beta(a) = a$. Then there is a normalised local parameter $f \colon U \to \Delta$ at a with $\beta(U) = U$ and $f(\beta(u)) = \overline{f(u)}$ for all $u \in U$.

Proof. Choose any centred local parameter $f_0 \colon U_0 \to \mathbb{C}$. Put $U_1 = U_0 \cap \beta(U_0)$ and $f_1 = f_0|_{U_1}$; this still gives a centred local parameter. The map $u \mapsto \overline{f_1(\beta(u))}$ is another centred local parameter on U_1, so we have $\overline{f_1(\beta(u))} = c\, f_1(u) + O(f_1(u)^2)$ for some $c \neq 0$. Using $\beta^2 = 1$ we find that $\overline{c}\, c = 1$, so $c = e^{2i\theta}$ for some $\theta \in \mathbb{R}$. Now put

$$f_2(u) = (e^{i\theta} f_1(u) + e^{-i\theta} \overline{f_1(\beta(u))})/2 = e^{i\theta} f_1(u) + O(f_1(u)^2).$$

This is again a centred local parameter at a, and it satisfies $f_2(\beta(u)) = \overline{f_2(u)}$. Shrinking now gives the required pair (U, f). □

Corollary 3.6.9. [cor-fixed-circles]

The fixed set $Z^{\langle\beta\rangle}$ is a closed submanifold of Z, and so is diffeomorphic to a finite disjoint union of circles. The same applies to $Z^{\langle\alpha\beta\rangle}$.

Proof. Any normalised local parameter $f\colon U \to \Delta$ as in the lemma gives a diffeomorphism

$$U \cap Z^{\langle\beta\rangle} \to \Delta \cap \mathbb{R} = (-1, 1),$$

and it follows easily from this that $Z^{\langle\beta\rangle}$ is a closed submanifold of Z. As $\alpha\beta$ is an equally good example of an anticonformal involution, we see that $Z^{\langle\alpha\beta\rangle}$ is also a closed submanifold. $\square$

Remark 3.6.10. [rem-preserved-circle]

As α commutes with β, it preserves the set $Z^{\langle\beta\rangle}$. However, if $Z^{\langle\beta\rangle}$ has several components, then they need not be preserved individually. If a certain component is preserved, then Remark 3.6.3 will apply.

Lemma 3.6.11. [lem-bt-al-fixed]

Suppose that $a \in Z$ satisfies $\alpha(a) = \beta(a) = a$. Then there is a normalised local parameter $f\colon U \to \Delta$ at a such that $f(\alpha(u)) = -f(u)$ and $f(\beta(u)) = \overline{f(u)}$ for all $u \in U$. Moreover:

(a) *f induces a conformal isomorphism $g\colon U/\langle\alpha\rangle \to \Delta$ with $g([u]) = f(u)^2$.*
(b) *f restricts to give a diffeomorphism from $U^{\langle\beta\rangle}$ to the real axis in Δ.*
(c) *Similarly, f restricts to give a diffeomorphism from $U^{\langle\alpha\beta\rangle}$ to the imaginary axis in Δ.*
(d) *g restricts to give homeomorphisms $U^{\langle\beta\rangle}/\langle\alpha\rangle \to [0, 1)$ and $U^{\langle\alpha\beta\rangle}/\langle\alpha\rangle \to (-1, 0]$.*

Proof. We choose a local parameter f_0 with $f_0(\alpha(u)) = -f(u)$ as in Lemma 3.6.6. We then take this as the initial choice in the proof of Lemma 3.6.8. This gives a normalised local parameter f with $f(\beta(u)) = \overline{f(u)}$, and by inspecting the construction we see that the property $f(\alpha(u)) = -f(u)$ is retained as well. The additional properties (a) to (d) follow easily. $\square$

3.6.2. *Branched coverings.* First, we give a formal definition:

Definition 3.6.12. We put $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$, and $\Delta' = \Delta \setminus \{0\}$. We let $\pi\colon \Delta \times \{0, 1\} \to \Delta$ denote the projection, and we let $\sigma\colon \Delta \to \Delta$ denote the squaring map.

Let $f\colon X \to Y$ be a holomorphic map between Riemann surfaces. We say that f is a *branched double covering* if for each $y \in Y$ there is a diagram of one of the following types:

$$\begin{tikzcd} \Delta \times \{0,1\} \arrow[r, "q"] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ \Delta \arrow[r, "p"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \Delta \arrow[r, "q"] \arrow[d, "\sigma"'] & X \arrow[d, "f"] \\ \Delta \arrow[r, "p"'] & Y \end{tikzcd}$$

where

(a) p is a holomorphic chart with $p(0) = y$.
(b) The square is a pullback, so q gives a holomorphic isomorphism from $\Delta \times \{0, 1\}$ or Δ to $f^{-1}(p(\Delta))$.

Remark 3.6.13. We note that in the left hand case we have $|p^{-1}\{y\}| = 2$, whereas in the right hand case we have $|p^{-1}\{y\}| = 1$, so the two cases are disjoint. In the right hand case we say that y is a *branch point*, and we write $B(f)$ for the set of branch points. We note that all points in $p(\Delta) \setminus \{y\}$ have two preimages and so are not branch points; thus, the set $B(f)$ is discrete.

Lemma 3.6.14. [lem-removable]

Let X and Y be Riemann surfaces such that Y is isomorphic to Δ or $\Delta \amalg \Delta$, and let a be a point in X. Then any holomorphic map $f\colon X \setminus \{a\} \to Y$ has a unique holomorphic extension $f\colon X \to Y$.

Proof. By choosing a chart around a we can reduce to the case where $X = \Delta$ and $a = 0$. Now, even if $Y \simeq \Delta \amalg \Delta$ the image $f(\Delta')$ will be connected and therefore contained in one of the two copies of Δ. We can thus assume that $Y = \Delta$. This case is just the standard theorem on removable singularities in complex analysis. $\square$

Proposition 3.6.15. [prop-un-branched]

Let Y be a Riemann surface, let V be a discrete subset of Y, and put $Y' = Y \setminus V$. Let $\mathcal{X}$ be the category of branched double coverings $f\colon X \to Y$ with $B(f) \subseteq V$, and let $\mathcal{X}'$ be the category of unbranched double coverings of Y'. (In both cases, the morphisms are holomorphic isomorphisms covering the identity on Y or Y'.) Let $R\colon \mathcal{X} \to \mathcal{X}'$ be the evident restriction functor, given by $R(X \xrightarrow{f} Y) = (X' \xrightarrow{f'} Y')$ where $X' = f^{-1}(Y')$ and $f' = f|_{X'}$. Then R is an equivalence of categories.

Proof. Suppose we have objects $(X_i \xrightarrow{f_i} Y) \in \mathcal{X}$ for $i = 0, 1$ and an isomorphism $g'\colon X_0' \to X_1'$ with $f_1' g_1 = f_0'$. We claim that there is a unique holomorphic extension $g\colon X_0 \to X_1$, and that this satisfies $f_1 g = f_0$. This can be checked locally on Y, so we can restrict attention to a small neighbourhood of a point in V and therefore assume that $(X_1 \xrightarrow{f_1} Y)$ is either $(\Delta \times \{0,1\} \xrightarrow{\pi} \Delta)$ or $(\Delta \xrightarrow{\sigma} \Delta)$. In either case it is clear from Lemma 3.6.14 that g' has a unique holomorphic extension $g\colon X_0 \to X_1$. We can also apply the uniqueness clause in the same lemma to the map $f_1' g' = f_0'$; this gives $f_1 g = f_0$. We now see that R is full and faithful.

We now need to show that R is essentially surjective. Consider an unbranched covering $f'\colon X' \to Y'$. For each $v \in V$ we can choose a chart $p_v\colon \Delta \to Y$ with $p_v(0) = v$. As V is discrete we may assume, after shrinking the charts if necessary, that the sets $p_v(\Delta)$ are disjoint. In particular, this means that $p_v(\Delta) \cap V = \{v\}$, so $p_v(\Delta') \subseteq Y'$. The pullback $p_v^*(X')$ is an unbranched double cover of Δ', so it is isomorphic to $(\Delta' \times \{0,1\} \xrightarrow{\pi'} \Delta')$ or to $(\Delta' \xrightarrow{\sigma'} \Delta')$. In either case there is an evident way to extend $p_v^*(X')$ to give a branched cover of all of Δ, and this in turn extends X' to give a branched cover of $p_v(\Delta)$. These extensions can be patched together to give a branched cover $(X \xrightarrow{f} Y)$ extending the original unbranched cover, as required. □

Definition 3.6.16. [defn-monodromy]

Let $(X \xrightarrow{f} Y)$ be a branched double cover with $B(f) \subseteq V$, giving an unbranched cover $(X' \xrightarrow{f'} Y')$. For any closed loop $u\colon [0,1] \to Y'$, we define the *monodromy* $\mu_X(u) \in \mathbb{Z}/2$ as follows. The fibre $F = (f')^{-1}\{u(0)\}$ will have precisely two elements. For any element a, there is a unique continuous lift $\widetilde{u}\colon [0,1] \to X'$ with $f\widetilde{u} = u$ and $u(0) = a$. We put $\sigma(a) = \widetilde{u}(1) \in F$. This defines a permutation $\sigma\colon F \to F$; we put $\mu_X(u) = 0$ if σ is the identity, and $\mu_X(u) = 1$ if σ is the transposition. This depends only on the homotopy class of the loop u.

Next, for $v \in V$ we let ω_v denote a small loop in Y' that winds once around v and does not wind around any of the other points in V. It is clear that v is a branch point for X iff $\mu_X(\omega_v) = 1$.

Lemma 3.6.17. [lem-homology-monodromy]

There is a unique homomorphism $\overline{\mu}_X\colon H_1(Y') \to \mathbb{Z}/2$ such that $\mu_X(u) = \overline{\mu}_X([u])$ for all loops u, where $[u]$ denotes the homology class represented by u.

Proof. We can reduce to the case where Y' is connected, and choose a basepoint $b \in Y'$. The Hurewicz map then gives an isomorphism $h_b\colon \pi_1(Y', b)_{\mathrm{ab}} \to H_1(Y')$, and it follows that there is a unique homomorphism $\overline{\mu}\colon H_1(Y') \to \mathbb{Z}/2$ with $\mu(u) = \overline{\mu}(\langle u \rangle)$ for all loops u based at b. If u is a loop that is not based at b, we can choose a path w from b to $u(0)$, and let u' be the loop given by w followed by u followed by the reverse of w, so u' is based at b. We then have $[u'] = [u]$ and $\mu(u) = \mu(u')$, so we still have $\mu(u) = \overline{\mu}([u])$. □

Lemma 3.6.18. [lem-covering-classification]

The map

$$[X \xrightarrow{f} Y] \mapsto \overline{\mu}_X$$

gives a bijection from the set of isomorphism classes in $\mathcal{X}$ (or $\mathcal{X}'$) to $\mathrm{Hom}(H_1(Y'), \mathbb{Z}/2)$.

Proof. We can again reduce to the case where Y' is connected, choose a basepoint $b \in Y'$, and use the Hurewicz isomorphism $H_1(Y') = \pi_1(Y', b)_{\mathrm{ab}}$. The claim is then that unbranched double covers of Y' are classified by homomorphisms $\pi_1(Y', b) \to \mathbb{Z}/2$, which is standard covering theory. □

Definition 3.6.19. Let $f\colon X \to Y$ be a branched double covering. We define $\chi\colon X \to X$ by

$$\chi(x) = \begin{cases} x' & \text{if } f^{-1}\{f(x)\} = \{x, x'\} \text{ with } x' \neq x \\ x & \text{if } f^{-1}\{f(x)\} = \{x\}. \end{cases}$$

This is easily seen to be holomorphic.

Proposition 3.6.20. [prop-branched-unique]
Suppose that Y is isomorphic to $\mathbb{C}_\infty$, and that $V \subset Y$ is a finite subset of even size. Then:

(a) *There is a branched covering $f: X \to Y$ for which $B(f) = V$.*
(b) *If $f_0: X_0 \to Y$ and $f_1: X_1 \to Y$ are as in (a), then there is an isomorphism $g: X_0 \to X_1$ with $f_1 g = f_0$, and we have $\chi_1 g = g\chi_0$.*
(c) *If $g, h: X_0 \to X_1$ are as in (b), then either $h = g$ or $h = \chi_1 g$.*

Proof. Recall that for each $v \in V$ we have a loop ω_v and a homology class $[\omega_v] \in H_1(Y')$. A standard calculation using the Mayer-Vietoris sequence shows that $H_1(Y')$ is generated by these classes subject only to the relation $\sum_{v\in V}[\omega_v] = 0$. As $|V|$ is even it follows that there is a unique homomorphism $\nu: H_1(Y') \to \mathbb{Z}/2$ with $\nu([\omega_v]) = 1$ for all $v \in V$. By Lemma 3.6.18, there exist unbranched coverings of Y' with monodromy ν, and any two such are isomorphic. By Proposition 3.6.15, it follows that there exist branched coverings of Y whose branch set is precisely V, and any two such are isomorphic. This proves (a) and (b) except for the fact that $\chi_1 g = g\chi_0$. This fact and claim (c) are standard covering theory and are left to the reader. □

3.7. **The projective family is universal.** [sec-P-universal]
We will prove the following two theorems.

Theorem 3.7.1. [thm-classify-precromulent]
Let X be a precromulent surface. Then there is a unique number $a \in (0, 1)$ such that $X \simeq PX(a)$ as G-equivariant Riemann surfaces. Moreover, there are precisely two isomorphisms $X \to PX(a)$, which are related by the action of λ^2.

Theorem 3.7.2. [thm-classify-cromulent]
Let X be a cromulent surface. Then there is a unique number $a \in (0, 1)$ such that $X \simeq PX(a)$ as G-equivariant Riemann surfaces. Moreover, there is precisely one cromulent isomorphism $X \to PX(a)$.

The proofs will be given after some preliminary results. First, however, we record a consequence of Theorem 3.7.2:

Corollary 3.7.3. *Let X be a cromulent surface. Then X admits a curve system (as in Definition 2.4.4) and has standard isotropy (as in Definition 2.4.9).*

Proof. Definition 3.2.1 and Proposition 3.2.7 show that this holds for $PX(a)$, which is sufficient by Theorem 3.7.2. □

Proposition 3.7.4. [prop-v-zero]
There is a unique point $v_0 \in X$ such that $\lambda(v_0) = v_0$ and $\lambda_* = i: T_{v_0}X \to T_{v_0}X$. Similarly, there is a unique point $v_1 = \mu(v_0) \in X$ such that $\lambda(v_1) = v_1$ and $\lambda_* = -i: T_{v_1}X \to T_{v_1}X$.

Proof. In V^* there are precisely two points that are fixed by λ, and they are exchanged by μ. The same must therefore be true in X. Let a be one of these points, so the other one is $b = \mu(a)$. Note that the holomorphic involution λ^2 fixes a and b, so it acts as -1 on T_aX and T_bX by Lemma 3.6.6, so λ acts as $\pm i$. Now consider the commutative diagram on the left below, and the resulting commutative diagram on the right:

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\lambda"'] & X \arrow[d, "\lambda^{-1}"] \\ X \arrow[r, "\mu"'] & X \end{tikzcd} \qquad \begin{tikzcd} T_aX \arrow[r, "\mu_*"] \arrow[d, "\lambda_*"] & T_bX \arrow[d, "\lambda_*^{-1}"] \\ T_aX \arrow[r, "\mu_*"'] & T_bX. \end{tikzcd}$$

From this we see that the eigenvalue of λ on T_aX is the same as the eigenvalue of λ^{-1} on T_bX. The claim follows easily from this. □

Definition 3.7.5. [defn-e-points]
We define points $e_0, e_1, e_\infty \in X/\langle\lambda^2\rangle$ as follows. First, we let v_0 and v_1 be as in Proposition 3.7.4, and we put $e_0 = [v_0]$ and $e_\infty = [v_1]$. Next, we note that in V^* there are precisely two points with stabiliser

$\langle \lambda^2\mu, \lambda^2\nu\rangle$ (namely 3 and 5), and that these are exchanged by λ^2. It follows that the same is true in V. These points therefore form an equivalence class in $X/\langle\lambda^2\rangle$, which we call e_1.

We saw in Corollary 2.3.2 that $X/\langle\lambda^2\rangle$ is isomorphic to $\mathbb{C}_\infty$. It is well-known that the conformal automorphisms of $\mathbb{C}_\infty$ are the Möbius transformations, and that these act freely and transitively on the triples of distinct points in $\mathbb{C}_\infty$. This validates the following definition:

Definition 3.7.6. [`defn-p`]
We let $p\colon X/\langle\lambda^2\rangle \to \mathbb{C}_\infty$ denote the unique conformal isomorphism such that $p(e_i) = i$ for $i \in \{0, 1, \infty\}$. We will also use the symbol p for the composite $X \to X/\langle\lambda^2\rangle \xrightarrow{p} \mathbb{C}_\infty$.

Lemma 3.7.7. [`lem-p-equivariance`]
If we let G act on $\mathbb{C}_\infty$ by

$$\lambda(z) = -z \qquad\qquad \mu(z) = 1/z \qquad\qquad \nu(z) = \overline{z}$$

then the map p is equivariant.

Proof. The maps $x \mapsto -p(\lambda(x))$ and $x \mapsto 1/p(\mu(x))$ and $x \mapsto \overline{p(\nu(x))}$ all have the defining property of p. $\square$

Lemma 3.7.8. [`lem-v-eleven`]
There is a unique point $v_{11} \in V$ such that the number $a = p(v_{11})$ lies in $(0, 1)$. Moreover, if we define $v_{10} = \lambda(v_{11})$ and $v_{12} = \lambda\mu(v_{11})$ and $v_{13} = \mu(v_{11})$, then we have

$$p(v_{10}) = -a \qquad\qquad p(v_{11}) = a \qquad\qquad p(v_{12}) = -1/a \qquad\qquad p(v_{13}) = 1/a.$$

Proof. Put $W = \{x \in X \mid \mathrm{stab}_G(x) = \langle\lambda^2, \nu\rangle\}$. Because the group $\langle\lambda^2, \nu\rangle$ is normal in G, this is a G-set. As X is precromulent, it is equivariantly isomorphic to the G-set

$$W^* = \{i \in V^* \mid \mathrm{stab}_G(i) = \langle\lambda^2, \nu\rangle\} = \{10, 11, 12, 13\} \simeq G/\langle\lambda^2, \nu\rangle.$$

As p gives an equivariant isomorphism $X/\langle\lambda^2\rangle \to \mathbb{C}_\infty$, it must restrict to give an equivariant injection

$$W/\langle\lambda^2\rangle \to \{z \in \mathbb{C}_\infty \mid \mathrm{stab}_G(z) = \langle\lambda^2, \nu\rangle\}.$$

The domain here is just W, and the codomain is the set

$$U = \mathbb{R}_\infty \setminus \{0, 1, -1, \infty\} = (-\infty, -1) \amalg (-1, 0) \amalg (0, 1) \amalg (1, \infty).$$

The action of G on $\mathbb{C}_\infty$ permutes the four components of U transitively, so the preimage under p of each component must contain precisely one point of W. The claim is clear from this. $\square$

Proof of Theorem 3.7.1. The map $p\colon X \to \mathbb{C}_\infty$ is a branched covering, with branch set $p(U)$, where $U = \{x \in X \mid \lambda^2(x) = x\}$. This is equivariantly isomorphic to the G-set

$$U^* = \{i \in V^* \mid \lambda^2(i) = i\} = \{0, 1, 10, 11, 12, 13\},$$

and using this we see that $p(U) = \{0, \infty, \pm a, \pm 1/a\}$. This is the same as the branch set for the map $p\colon PX(a) \to \mathbb{C}_\infty$ defined in Remark 3.1.13. Our claim now follows from Proposition 3.6.20. $\square$

Corollary 3.7.9. [`cor-precromulent-aut`]

(a) *If X is a precromulent surface, then the group of precromulent automorphisms of X is $C_2 = \{1, \lambda^2\}$.*
(b) *If X and Y are isomorphic precromulent surfaces, then there are precisely two precromulent isomorphisms between them. If one of them is ϕ, then the other is $\phi\lambda_X^2 = \lambda_Y^2\phi$.*

Proof. Clear from the theorem. $\square$

Corollary 3.7.10. [`cor-cromulent-aut`]
If X is a cromulent surface, then the only cromulent automorphism is the identity.

Proof. Any cromulent automorphism is also a precromulent automorphism, and so is 1 or λ^2; but λ^2 does not preserve the labelling. $\square$

Corollary 3.7.11. [cor-cromulent-iso]
Let X and Y be isomorphic cromulent surfaces; then there is a unique cromulent isomorphism between them. □

Proposition 3.7.12. [prop-labellings]
Let X be a precromulent surface. Then X has precisely two cromulent labellings, which are related by the action of λ^2.

Proof. By Theorem 3.7.1 we can reduce to the case $X = PX(a)$. In particular, this means that we have a curve system, and nets as in Section 2.6.

The labelling given in Definition 3.1.1 is cromulent, by Proposition 3.3.1. It follows easily that if we change the labelling by λ^2, then it remains cromulent.

Now suppose we have another cromulent labelling, say $(v_i')_{i=0}^{13}$. This must have the form $v_i' = v_{\phi(i)}$ for some ϕ in the group $\mathrm{Aut}(V^*)$, which is described by Proposition 2.2.1. Proposition 3.7.4 shows that we must have $\phi(0) = 0$ and $\phi(1) = 1$. Proposition 2.2.1 shows that $\phi(2) \in \{2, 4\}$, and after replacing ϕ by $\phi\lambda^2$ if necessary, we may assume that $\phi(2) = 2$. Assuming this, we see from Proposition 2.2.1 that $\phi(i) = i$ for $i \in \{2, 3, 4, 5\}$, and that $\phi(6) \in \{6, 8\}$.

Next, as the new labelling is cromulent, there must be a connected component $F' \subseteq \{x \in X \mid \mathrm{stab}_G(x) = 1\}$ whose closure contains the set $U = \{v_{\phi(0)}, v_{\phi(3)}, v_{\phi(6)}, v_{\phi(11)}\}$. From the discussion in Section 2.6 we see that F' must be $\gamma(PF_{16}'(a))$ for some $\gamma \in G$. We have seen that U contains v_0, v_3 and either v_6 or v_8. By inspecting the nets in Section 2.6, we see that this can only be consistent if $\gamma = 1$ and $\phi(6) = 6$ and $\phi(11) = 11$. By consulting Proposition 2.2.1 again, we conclude that $\phi = 1$, as required. □

Proof of Theorem 3.7.2. Let X be a cromulent surface. By Theorem 3.7.1, there is a unique a such that $X \simeq PX(a)$ as precromulent surfaces. Choose a precromulent isomorphism $\phi\colon X \to PX(a)$. The points $\phi^{-1}(v_i)$ give a cromulent labelling of X. By Proposition 3.7.12, this must either be the given labelling of X, or its twist by λ^2. Thus, after replacing ϕ by $\phi\lambda^2$ if necessary, we may assume that ϕ is a cromulent isomorphism. It is unique by Corollary 3.7.11. □

Remark 3.7.13. [rem-p-extra]
We now see that the map $p\colon X/\langle\lambda^2\rangle \to \mathbb{C}_\infty$ from Definition 3.7.6 must factor as a cromulent isomorphism $X \to PX(a)$, followed by the map $p\colon PX(a) \to \mathbb{C}_\infty$ from Remark 3.1.13. This implies that we have the following additional properties:

$$\begin{aligned} p(v_0) &= 0 \\ p(v_1) &= \infty \\ p(v_2) = p(v_4) &= -1 \\ p(v_3) = p(v_5) &= 1 \\ p(v_6) = p(v_8) &= i \\ p(v_7) = p(v_9) &= -i \\ p(v_{10}) &= -a \\ p(v_{11}) &= a \\ p(v_{12}) &= -1/a \\ p(v_{13}) &= 1/a. \end{aligned}$$

The number $p(v_i)$ is recorded in Maples as `v_C[i]`.

Remark 3.7.14. [rem-p-hat]
For some purposes it is more convenient to work with the round sphere $S^2 \subset \mathbb{R}^3$ rather than the Riemann sphere $\mathbb{C}_\infty$. We identify them using the stereographic projection map

$$\xi(x + iy) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

This has been normalised so that the unit circle in $\mathbb{C}$ is sent to the equator. It is a standard fact that ξ is conformal. The resulting complex structure on the tangent spaces $T_y S^2 = \{t \in \mathbb{R}^3 \mid t.y = 0\}$ can be described in terms of the cross product of vectors: we have $(a+ib)t = at + bt\times y$. This means that an ordered basis (u,v) for T_yS^2 is oriented iff $\det(y,v,u)>0$. Note also that if $\xi(z)=y$ then

$$\begin{aligned}\xi(-z) &= (-y_1,\ -y_2,\ \ \ y_3)\\ \xi(1/z) &= (\ \ y_1,\ -y_2,\ -y_3)\\ \xi(\overline{z}) &= (\ \ y_1,\ -y_2,\ \ \ y_3).\end{aligned}$$

It follows that the composite $\widehat{p}=\xi p\colon X\to S^2$ has properties as follows:

$$\begin{aligned}\widehat{p}(v_0) &= (\ \ 0,\ \ 0,-1)\\ \widehat{p}(v_1) &= (\ \ 0,\ \ 0,\ \ 1)\\ \widehat{p}(v_2) = \widehat{p}(v_4) &= (-1,\ \ 0,\ \ 0)\\ \widehat{p}(v_3) = \widehat{p}(v_5) &= (\ \ 1,\ \ 0,\ \ 0)\\ \widehat{p}(v_6) = \widehat{p}(v_8) &= (\ \ 0,\ \ 1,\ \ 0)\\ \widehat{p}(v_7) = \widehat{p}(v_9) &= (\ \ 0,-1,\ \ 0)\\ \widehat{p}(v_{10}) &= (-2a,0,a^2-1)/(a^2+1)\\ \widehat{p}(v_{11}) &= (\ \ 2a,0,a^2-1)/(a^2+1)\\ \widehat{p}(v_{12}) &= (-2a,0,1-a^2)/(a^2+1)\\ \widehat{p}(v_{13}) &= (\ \ 2a,0,1-a^2)/(a^2+1)\end{aligned}$$

$$\begin{array}{lll}\widehat{p}_1(\lambda(x)) = -\widehat{p}_1(x) & \widehat{p}_2(\lambda(x)) = -\widehat{p}_2(x) & \widehat{p}_3(\lambda(x)) = \ \ \widehat{p}_3(x)\\ \widehat{p}_1(\mu(x)) = \ \ \widehat{p}_1(x) & \widehat{p}_2(\mu(x)) = -\widehat{p}_2(x) & \widehat{p}_3(\mu(x)) = -\widehat{p}_3(x)\\ \widehat{p}_1(\nu(x)) = \ \ \widehat{p}_1(x) & \widehat{p}_2(\nu(x)) = -\widehat{p}_2(x) & \widehat{p}_3(\nu(x)) = \ \ \widehat{p}_3(x).\end{array}$$

The points $\widehat{p}(v_i)$ are recorded in Maple as `v_S2[i]`, and the induced action of $g\in G$ on $u\in S^2$ is `act_S2[g](u)`.

4. The hyperbolic family

[sec-H]

4.1. The groups Π and $\widetilde{\Pi}$. [sec-Pi]

Later we will construct a family of cromulent surfaces as quotients of the unit disc by different actions of a certain group Π. In this section we define and study Π, together with a larger group $\widetilde{\Pi}$ such that $\widetilde{\Pi}/\Pi = G$. Actions of $\widetilde{\Pi}$ and Π will be given in the following section.

Definition 4.1.1. [defn-Pi]
Let Π be the abstract group generated by symbols β_k (for $k\in\mathbb{Z}/8$) subject to the following relations:

$$\begin{aligned}\beta_{k+4} &= \beta_k^{-1}\\ \beta_0\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7 &= 1.\end{aligned}$$

We saw in Proposition 2.6.12 that any cromulent surface has fundamental group isomorphic to Π. Note also that the abelianization Π_{ab} is freely generated (as an abelian group) by $\beta_0,\dots,\beta_3$, and so is isomorphic to $\mathbb{Z}^4$.

We can introduce an alternative set of generators as follows:

$$\begin{array}{llll}\alpha_0 = \beta_3^{-1}\beta_2^{-1}\beta_0 & \alpha_1 = \beta_1\beta_2\beta_3 & \alpha_2 = \beta_2^{-1} & \alpha_3 = \beta_3^{-1}\\ \beta_0 = \alpha_2^{-1}\alpha_3^{-1}\alpha_0 & \beta_1 = \alpha_1\alpha_3\alpha_2 & \beta_2 = \alpha_2^{-1} & \beta_3 = \alpha_3^{-1}.\end{array}$$

In terms of these, the relation $\beta_0\cdots\beta_7=1$ becomes the standard relation

$$[\alpha_0,\alpha_1][\alpha_2,\alpha_3] = \alpha_0\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_2\alpha_3\alpha_2^{-1}\alpha_3^{-1} = 1$$

for the fundamental group of a surface of genus 2. (However, we prefer to use the generators β_k, for reasons of symmetry.)

hyperbolic/Pi_check.mpl: check_Pi_alpha()

Remark 4.1.2. [rem-runs]

Note that the relation $\beta_0 \cdots \beta_7 = 1$ can be conjugated to give

$$\beta_k \beta_{k+1} \cdots \beta_{k+7} = 1$$

for any $k \in \mathbb{Z}/8$. This in turn gives

$$\begin{aligned}
\beta_k\beta_{k+1}\beta_{k+2}\beta_{k+3}\beta_{k+4}\beta_{k+5}\beta_{k+6} &= \beta_{k+7}^{-1} &&= \beta_{k+3} \\
\beta_k\beta_{k+1}\beta_{k+2}\beta_{k+3}\beta_{k+4}\beta_{k+5} &= (\beta_{k+6}\beta_{k+7})^{-1} &&= \beta_{k+3}\beta_{k+2} \\
\beta_k\beta_{k+1}\beta_{k+2}\beta_{k+3}\beta_{k+4} &= (\beta_{k+5}\beta_{k+6}\beta_{k+7})^{-1} &&= \beta_{k+3}\beta_{k+2}\beta_{k+1} \\
\beta_k\beta_{k+1}\beta_{k+2}\beta_{k+3} &= (\beta_{k+4}\beta_{k+5}\beta_{k+6}\beta_{k+7})^{-1} &&= \beta_{k+3}\beta_{k+2}\beta_{k+1}\beta_k \\
\beta_k\beta_{k+1}\beta_{k+2} &= (\beta_{k+3}\beta_{k+4}\beta_{k+5}\beta_{k+6}\beta_{k+7})^{-1} &&= \beta_{k+3}\beta_{k+2}\beta_{k+1}\beta_k\beta_{k-1} \\
\beta_k\beta_{k+1} &= (\beta_{k+2}\beta_{k+3}\beta_{k+4}\beta_{k+5}\beta_{k+6}\beta_{k+7})^{-1} &&= \beta_{k+3}\beta_{k+2}\beta_{k+1}\beta_k\beta_{k-1}\beta_{k-2} \\
\beta_k &= (\beta_{k+1}\beta_{k+2}\beta_{k+3}\beta_{k+4}\beta_{k+5}\beta_{k+6}\beta_{k+7})^{-1} &&= \beta_{k+3}\beta_{k+2}\beta_{k+1}\beta_k\beta_{k-1}\beta_{k-2}\beta_{k-3}.
\end{aligned}$$

Thus:

- Any increasing or decreasing run of length at least 5 can be replaced by a strictly shorter run in the opposite direction.
- Any run of length 4 can be reversed.

Definition 4.1.3. [defn-beta-reduced]

A word in the letters $\{\beta_i \mid i \in \mathbb{Z}/8\}$ is *reduced* if

(a) There are no subwords of the form $\beta_i\beta_{i+4}$
(b) There are no subwords of the form $\beta_i\beta_{i+1}\beta_{i+2}\beta_{i+3}\beta_{i+4}$.
(c) There are no subwords of the form $\beta_i\beta_{i-1}\beta_{i-2}\beta_{i-3}$.

Proposition 4.1.4. [prop-beta-reduced]

Every element $\pi \in \Pi$ can be expressed in a unique way as a reduced word in the letters β_i. Moreover, if π is represented by a word w, then the corresponding reduced word w' can be obtained from w by repeatedly cancelling pairs of the form $\beta_i\beta_{i+4}$, and shortening or reversing runs as in Remark 4.1.2.

Proof. A straightforward argument by induction on the length shows that for every word there is a reduced word (of the same length or less) that represents the same element of Π. Uniqueness is less obvious, but follows from Dehn's algorithm and small cancellation theory. In more detail, put $X = \{\beta_0, \beta_1, \beta_2, \beta_3\}$, then put

$$\begin{aligned}
\sigma_i^+ &= \beta_i\beta_{i+1} \cdots \beta_{i+7} \\
\sigma_i^- &= \beta_i\beta_{i-1} \cdots \beta_{i-7},
\end{aligned}$$

where each β_j is replaced by an element of $X \amalg X^{-1}$ in the obvious way. Then the set

$$S = \{\sigma_i^+ \mid i \in \mathbb{Z}/8\} \amalg \{\sigma_i^- \mid i \in \mathbb{Z}/8\}$$

consists of reduced words in the free group FX, and is closed under taking inverses and cyclic permutations. If α and β are distinct elements of S, then they have length 8, but they share at most one initial letter. The shared fraction of 1/8 is less than 1/6, so the main theorem of [9] is applicable, and the conclusion follows easily. (For a textbook treatment, see Chapter V of [15].) □

Remark 4.1.5. Elements of Π are represented in Maple as lists of integers. For example, the list [5,3,6] represents $\beta_5\beta_3\beta_6$. The function is_Pi_reduced(L) decides whether a list L corresponds to a reduced word. The function Pi_reduce(L) finds the unique reduced word that represents the same element as L. Here L is allowed to have arbitrary integer entries, but the first step in the reduction process is to reduce them modulo 8 so that they lie in $\{0, 1, \ldots, 7\}$. Multiplication and inversion are implemented by the functions Pi_mult and Pi_inv. All of this (together with various other things) is in the file hyperbolic/Pi.mpl.

Corollary 4.1.6. [cor-trivial-centre]
The centre of Π *is trivial, so the conjugation map* $\Pi \to \mathrm{Aut}(\Pi)$ *is injective.*

Proof. Any nontrivial element $\pi \in \Pi \setminus \{1\}$ can be represented by a nonempty reduced word w. Let i and j be the indices of the first and last letters in w, and put

$$L = \{i-1, i, i+1, i+4, j-1, j+1, j+4\}$$

(so $|L| \leq 7$). Choose $k \in (\mathbb{Z}/8) \setminus L$, so that $\beta_k w$ and $w\beta_k$ are reduced words. Their first letters are β_k and β_i, so they are different. It follows that π is not central. □

Proposition 4.1.7. [prop-Aut-Pi]
The group Π *has automorphisms* λ_*, μ_* *and* ν_*, *which act on the generators* β_i *as follows:*

	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
λ_*	β_2	β_3	β_4	β_5	β_6	β_7	β_0	β_1
μ_*	$\beta_2\beta_0\beta_1$	$\beta_5\beta_4\beta_3$	$\beta_0\beta_7\beta_6$	$\beta_2\beta_3\beta_1$	$\beta_5\beta_4\beta_6$	$\beta_7\beta_0\beta_1$	$\beta_2\beta_3\beta_4$	$\beta_5\beta_7\beta_6$
ν_*	β_0	$\beta_2\beta_1\beta_2$	β_6	$\beta_0\beta_7\beta_0$	β_4	$\beta_6\beta_5\beta_6$	β_2	$\beta_4\beta_3\beta_4$

Moreover, we have $\lambda_*^4 = \mu_*^2 = \nu_*^2 = (\lambda_*\nu_*)^2 = 1$, *and the automorphisms* $(\lambda_*\mu_*)^2$ *and* $(\nu_*\mu_*)^2$ *are inner. More precisely, for any* $\pi \in \Pi$ *we have*

$$(\lambda_*\mu_*)^2(\pi) = (\beta_7\beta_6)\ \pi\ (\beta_7\beta_6)^{-1}$$
$$(\nu_*\mu_*)^2(\pi) = (\beta_6\beta_0\beta_7\beta_6)\ \pi\ (\beta_6\beta_0\beta_7\beta_6)^{-1}.$$

Proof. We will only discuss μ_*; similar arguments cover the other two cases. Put $B = \{\beta_k : k \in \mathbb{Z}/8\} \subset \Pi$ and define $\mu_* : B \to \Pi$ by the above table. To show that this extends to an endomorphism of Π, we must check that it respects the relations, or in other words that

$$\mu_*(\beta_k)\mu_*(\beta_{k+4}) = 1$$
$$\mu_*(\beta_0)\mu_*(\beta_1)\mu_*(\beta_2)\mu_*(\beta_3)\mu_*(\beta_4)\mu_*(\beta_5)\mu_*(\beta_6)\mu_*(\beta_7) = 1.$$

The first of these follows easily by inspecting the definition of $\mu_*(\beta_k)$ and using the relation $\beta_j\beta_{j+4} = 1$ three times. For the second, the left hand side can be grouped as

$$\beta_2(\beta_0\beta_1\beta_5\beta_4)\beta_3(\beta_0(\beta_7\beta_6\beta_2\beta_3)(\beta_1\beta_5)\beta_4)(\beta_6\beta_7\beta_0\beta_1\beta_2\beta_3\beta_4\beta_5)\beta_7\beta_6.$$

Working from the inside out, we see that the content of each matched pair of parentheses cancels down to 1. This leaves $\beta_2\beta_3\beta_7\beta_6$, which cancels further down to 1. We thus have an endomorphism μ_* as claimed. It satisfies

$$\mu_*^2(\beta_0) = \mu_*(\beta_2\beta_0\beta_1) = \mu_*(\beta_2)\mu_*(\beta_0)\mu_*(\beta_1) = \beta_0(\beta_7(\beta_6\ \beta_2)(\beta_0(\beta_1\ \beta_5)\beta_4)\beta_3) = \beta_0.$$

By similar arguments, it also satisfies $\mu_*^2(\beta_i) = \beta_i$ for all other indices i, so $\mu_*^2 = 1$. In particular, this means that μ_* is an automorphism. The identities $\lambda_*^4 = \nu_*^2 = (\lambda_*\nu_*)^2 = 1$ can be verified in a similar way. Moreover, it will also suffice to check the identities for $(\lambda_*\mu_*)^2(\pi)$ and $(\nu_*\mu_*)^2(\pi)$ when $\pi = \beta_k$ for some k, and this is again straightforward (but long).

hyperbolic/Pi_check.mpl: check_Pi_relations()

□

Definition 4.1.8. [defn-tPi]

We let $\widetilde{\Pi}$ be the abstract group generated by symbols $\lambda, \mu, \nu, \beta_k$ (for $k \in \mathbb{Z}/8$) subject to the following relations:

$$\begin{aligned}
\beta_{k+4} &= \beta_k^{-1} \\
\beta_0\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7 &= 1 \\
\lambda^4 = \mu^2 = \nu^2 &= (\lambda\nu)^2 = 1 \\
(\lambda\mu)^2 &= \beta_7\beta_6 \\
(\nu\mu)^2 &= \beta_6\beta_0\beta_7\beta_6 \\
\lambda\beta_k\lambda^{-1} &= \lambda_*(\beta_k) \\
\mu\beta_k\mu &= \mu_*(\beta_k) \\
\nu\beta_k\nu &= \nu_*(\beta_k).
\end{aligned}$$

(Here $\lambda_*(\beta_k)$, $\mu_*(\beta_k)$ and $\nu_*(\beta_k)$ refer to the words given in the table in Proposition 4.1.7.)

Proposition 4.1.9. [prop-tPi]
Π can be identified with the subgroup of $\widetilde{\Pi}$ generated by $\beta_0, \dots, \beta_7$. Moreover, this subgroup is normal, and there is a canonical isomorphism $\widetilde{\Pi}/\Pi \simeq G$.

Proof. Let Π' be the subgroup of $\widetilde{\Pi}$ generated by $\beta_0, \dots, \beta_7$. Using the defining relations for $\lambda\beta_k\lambda^{-1}$, $\mu\beta_k\mu$ and $\nu\beta_k\nu$ we see that Π' is normal. If we just set all the elements β_k to the identity in our presentation for $\widetilde{\Pi}$, we obtain a presentation for $\widetilde{\Pi}/\Pi'$; from this it is clear that $\widetilde{\Pi}/\Pi' = G$. There is an evident surjective homomorphism $\phi\colon \Pi \to \Pi'$, sending β_k to β_k for all k. All that is left is to show that this is injective. For this, we let $\gamma\colon \Pi \to \mathrm{Inn}(\Pi)$ be the usual conjugation map, given by $\gamma(\alpha)(\pi) = \alpha\pi\alpha^{-1}$. This is surjective by the definition of $\mathrm{Inn}(\Pi)$, and injective by Corollary 4.1.6, so it is an isomorphism. Next, we define

$$\begin{aligned}
\delta(\lambda) &= \lambda_* \in \mathrm{Aut}(\Pi) \\
\delta(\mu) &= \mu_* \in \mathrm{Aut}(\Pi) \\
\delta(\nu) &= \nu_* \in \mathrm{Aut}(\Pi) \\
\delta(\beta_k) &= \gamma(\beta_k) \in \mathrm{Inn}(\Pi) \lhd \mathrm{Aut}(\Pi).
\end{aligned}$$

Using Proposition 4.1.7 we see that this is compatible with the defining relations for $\widetilde{\Pi}$, so it extends to give a homomorphism $\widetilde{\Pi} \to \mathrm{Aut}(\Pi)$. We can now chase the diagram

$$\begin{tikzcd}
\Pi \arrow[r, "\phi"] \arrow[d, "\gamma"', "\simeq"] & \widetilde{\Pi} \arrow[r, two heads] \arrow[d, "\delta"'] & G \arrow[d] \\
\mathrm{Inn}(\Pi) \arrow[r, tail, "\mathrm{inc}"'] & \mathrm{Aut}(\Pi) \arrow[r, two heads] & \mathrm{Out}(\Pi)
\end{tikzcd}$$

to see that ϕ is injective as required. ∎

Remark 4.1.10. Elements of $\widetilde{\Pi}$ are represented as pairs `[T,L]`, with `T` in G and `L` in Π. The multiplication rule is not obvious; it is implemented by the function `Pi_tilde_mult`, which uses data stored in the table `G_Pi_cocycle`. Inversion is implemented by the function `Pi_tilde_inv`. All this is in `hyperbolic/Pi.mpl`.

We next give an alternate presentation of Π, which will be helpful when we want to analyse actions on the unit disc.

Proposition 4.1.11. [prop-Pi-Sg]
Π is generated by the elements

$$\begin{aligned}
\sigma_a &= \beta_5\beta_6 & \sigma_b &= \beta_2 & \sigma_c &= \beta_7\beta_0 \\
\sigma_d &= \beta_4 & \sigma_e &= \beta_1\beta_2 & \sigma_f &= \beta_3\beta_4
\end{aligned}$$

subject only to the relations

$$\sigma_a\sigma_c\sigma_e\sigma_f = \sigma_b\sigma_e^{-1}\sigma_b^{-1}\sigma_a^{-1} = \sigma_d\sigma_f^{-1}\sigma_d^{-1}\sigma_c^{-1} = 1.$$

Proof. First, it is straightforward to check that the stated relations hold in Π. Thus, if we let Σ denote the abstract group with the indicated generators and relations, we have a homomorphism $\phi\colon \Sigma \to \Pi$ given by $\phi(\sigma_t) = \sigma_t$ for all $t \in \{a,b,c,d,e,f\}$. Next, we define elements $\beta_0,\dots,\beta_7 \in \Sigma$ by

$$\beta_0 = \sigma_d^{-1} \qquad \beta_1 = \sigma_e\sigma_b^{-1} \qquad \beta_2 = \sigma_b \qquad \beta_3 = \sigma_f\sigma_d^{-1} \qquad \beta_{i+4} = \beta_i^{-1}.$$

We claim that $\beta_0\beta_1\cdots\beta_7 = 1$ in Σ. It will suffice to prove the conjugate relation $\beta_5\beta_6\beta_7\beta_0\beta_1\beta_2\beta_3\beta_4 = 1$. We can write out the left hand side and group the terms as

$$(\sigma_b\sigma_e^{-1}\sigma_b^{-1})(\sigma_d\sigma_f^{-1}\sigma_d^{-1})\sigma_e(\sigma_b^{-1}\sigma_b)\sigma_f(\sigma_d^{-1}\sigma_d).$$

The relation $\sigma_b\sigma_e^{-1}\sigma_b^{-1}\sigma_a^{-1} = 1$ converts the first parenthesised term to σ_a, and similarly the second becomes σ_c. The third parenthesised term is clearly the identity, as is the fourth. This leaves $\sigma_a\sigma_c\sigma_e\sigma_f$, which is the identity by the first defining relation for Σ. This means that we have a homomorphism $\psi\colon \Pi \to \Sigma$ given by $\psi(\beta_i) = \beta_i$ for all i. Straightforward calculations now show that $\phi\psi(\beta_i) = \beta_i$ for all i, and $\psi\phi(\sigma_t) = \sigma_t$ for all t, so $\phi\psi = 1_\Pi$ and $\psi\phi = 1_\Sigma$.

`hyperbolic/Pi_check.mpl: check_Pi_sigma`

□

4.2. **Cromulent actions.** [sec-H-action]

We next describe a family of actions of $\widetilde{\Pi}$ on the unit disc

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}.$$

This has a standard Riemannian metric as follows:

$$ds^2 = 4|dz|^2/(1-|z|^2)^2.$$

We recall some standard facts about this, most of which can be found in [1, Section 4.1], for example. The Gaussian curvature of the metric is equal to -1. The conformal automorphisms of Δ have the form

$$z \mapsto \lambda\frac{z-\alpha}{1-\overline{\alpha}z},$$

with $|\alpha| < 1$ and $|\lambda| = 1$, and that these all preserve the metric. Similarly, the anticonformal automorphisms have the form

$$z \mapsto \lambda\frac{\overline{z}-\alpha}{1-\overline{\alpha}\,\overline{z}}.$$

The geodesic distance function is

$$d_{\text{hyp}}(z,w) = 2\operatorname{arctanh}\left|\frac{z-w}{1-\overline{z}w}\right|.$$

`hyperbolic/HX_check.mpl: check_hyperbolic_metric()`

Now fix a number b with $0 < b < 1$, and put $b_+ = \sqrt{1+b^2}$ and $b_- = \sqrt{1-b^2}$. We define an action of $\widetilde{\Pi}$ on the unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ as follows:

$$\begin{aligned}
\lambda(z) &= iz & \beta_0(z) &= \frac{b_+z+1}{z+b_+} \\
\mu(z) &= \frac{b_+z-b^2-i}{(b^2-i)z-b_+} & \beta_1(z) &= \frac{b_+^3z-(2+i)b^2-i}{((i-2)b^2+i)z+b_+^3} \\
\nu(z) &= \overline{z} & \beta_{2n}(z) &= i^n\beta_0(z/i^n) \\
& & \beta_{2n+1}(z) &= i^n\beta_1(z/i^n).
\end{aligned}$$

One can check by direct calculation that the defining relations for $\widetilde{\Pi}$ are satisfied.

`hyperbolic/HX_check.mpl: check_Pi_action()`

Remark 4.2.1. The parameters b, b_+ and b_- are a_H, ap_H and am_H in the Maple code. The action of $g \in \widetilde{\Pi}$ on $z \in \Delta$ is given by act_Pi_tilde(g,z). The group G can be considered as a subset of $\widetilde{\Pi}$ which is not a subgroup. For $g \in G$, the alternative notation act_H[g](z) also works for act_Pi_tilde(g,z). This notation is potentially misleading, because we do not actually have an action of G.

Remark 4.2.2. [rem-a-Ho]
The evident analogue of Remark 3.1.3 applies to a_H. Most parts of the code treat a_H as a symbol, but there are also global variables a_H0 and a_H1 which holds numerical values for a_H. These should be set using the function set_a_H0, which is defined in hyperbolic/HX0.mpl. There is also a function simplify_H analogous to the function simplify_P mentioned in Remark 3.1.4: it applies some simplification rules like $\sqrt{1-b^4} = b_+b_-$ which are not always used by Maple.

We now put $HX(b) = \Delta/\Pi$. Later we will give this the structure of a cromulent surface. As a first step, we note that Π is normal in $\widetilde{\Pi}$, so there is an induced action of $G = \widetilde{\Pi}/\Pi$ on $HX(b)$.

Definition 4.2.3. [defn-v-H]
We define points $v_0, \dots, v_{13} \in \Delta$ as follows:

$$
\begin{aligned}
v_0 &= 0 & v_6 &= \frac{1+i}{\sqrt{2}}\,\frac{\sqrt{2}-b_-}{b_+} & v_{10} &= i(b_+ - b) \\
v_1 &= \frac{1+i}{2}b_+ & v_7 &= i\,v_6 & v_{11} &= b_+ - b \\
v_2 &= \frac{b\,b_- - b_+}{i - b^2} & v_8 &= -v_6 & v_{12} &= (b+b_+)\frac{i+(i+2)b^2}{(b+b_+)^2+b^2} \\
v_3 &= \frac{b\,b_- - b_+}{ib^2 - 1} & v_9 &= -i\,v_6 & v_{13} &= i\,\overline{v_{12}} \\
v_4 &= i\,v_3 \\
v_5 &= -i\,v_2.
\end{aligned}
$$

Definition 4.2.4. [defn-v-H-extra]
We also consider additional points in the same Π-orbits:

$$
\begin{aligned}
v_{1.1} &= i\,v_1 &&= \beta_2\beta_1(v_1) \\
v_{1.2} &= -v_1 &&= \beta_1\beta_2\beta_3\beta_4(v_1) \\
v_{1.3} &= -i\,v_1 &&= \beta_3\beta_4(v_1) \\
v_{2.1} &= \overline{v_2} &&= \beta_6(v_2) \\
v_{3.1} &= -\overline{v_3} &&= \beta_4(v_3) \\
v_{4.1} &= \overline{v_4} &&= \beta_6(v_4) \\
v_{5.1} &= -\overline{v_5} &&= \beta_4(v_5) \\
v_{10.1} &= \overline{v_{10}} &&= \beta_6(v_{10}) \\
v_{11.1} &= -\overline{v_{11}} &&= \beta_4(v_{11}) \\
v_{12.1} &= -v_{12} &&= \beta_1(v_{12}) \\
v_{12.2} &= -\overline{v_{12}} &&= \beta_2\beta_1(v_{12}) \\
v_{12.3} &= \overline{v_{12}} &&= \beta_6(v_{12}) \\
v_{13.1} &= -v_{13} &&= \beta_4\beta_3\beta_4(v_{13}) \\
v_{13.2} &= \overline{v_{13}} &&= \beta_3\beta_4(v_{13}) \\
v_{13.3} &= -\overline{v_{13}} &&= \beta_4(v_{13}).
\end{aligned}
$$

Remark 4.2.5. The points v_i are represented as v_H[i], and there are also points v_H0[i] and v_H1[i] that are obtained by substituting the numerical values a_H0 or a_H1 for the symbol a_H. The entries in the above table have the form $v_i = \gamma_i(v_j)$ where i is not an integer, j is the integer part of i, and γ_i is an element of Π. The element γ_i is represented in maple as v_H_fraction_offset[i].

One can check that λ, μ and ν act as follows:

$$\begin{aligned}
&\lambda(v_0) = v_0 && \mu(v_0) = \beta_2(\beta_3(\beta_4(v_1))) && \nu(v_0) = v_0 \\
&\lambda(v_1) = \beta_2(\beta_1(v_1)) && \mu(v_1) = \beta_2(v_0) && \nu(v_1) = \beta_3(\beta_4(v_1)) \\
&\lambda(v_2) = \beta_4(v_3) && \mu(v_2) = v_2 && \nu(v_2) = \beta_6(v_2) \\
&\lambda(v_3) = v_4 && \mu(v_3) = \beta_2(v_5) && \nu(v_3) = v_5 \\
&\lambda(v_4) = \beta_4(v_5) && \mu(v_4) = \beta_5(\beta_6(v_4)) && \nu(v_4) = \beta_6(v_4) \\
&\lambda(v_5) = v_2 && \mu(v_5) = \beta_2(\beta_3(\beta_4(v_3))) && \nu(v_5) = v_3 \\
&\lambda(v_6) = v_7 && \mu(v_6) = \beta_2(v_9) && \nu(v_6) = v_9 \\
&\lambda(v_7) = v_8 && \mu(v_7) = \beta_5(v_8) && \nu(v_7) = v_8 \\
&\lambda(v_8) = v_9 && \mu(v_8) = \beta_5(\beta_4(\beta_3(v_7))) && \nu(v_8) = v_7 \\
&\lambda(v_9) = v_6 && \mu(v_9) = \beta_2(\beta_3(\beta_4(v_6))) && \nu(v_9) = v_6 \\
&\lambda(v_{10}) = \beta_4(v_{11}) && \mu(v_{10}) = v_{12} && \nu(v_{10}) = \beta_6(v_{10}) \\
&\lambda(v_{11}) = v_{10} && \mu(v_{11}) = \beta_2(\beta_3(\beta_4(v_{13}))) && \nu(v_{11}) = v_{11} \\
&\lambda(v_{12}) = \beta_4(v_{13}) && \mu(v_{12}) = v_{10} && \nu(v_{12}) = \beta_6(v_{12}) \\
&\lambda(v_{13}) = \beta_2(\beta_1(v_{12})) && \mu(v_{13}) = \beta_2(v_{11}) && \nu(v_{13}) = \beta_3(\beta_4(v_{13}))
\end{aligned}$$

This shows that G permutes the corresponding points in $HX(b)$ in accordance with Definition 1.0.4. The elements of Π appearing here are recorded in the table `v_action_witness_H`, which is defined in the file `hyperbolic/HX.mpl`.

```
cromulent.mpl: check_precromulent("H")
hyperbolic/HX_check.mpl: check_v_H()
```

4.3. **The curve system.** [sec-H-curves]

We would now like to construct curves $C_0, \dots, C_8$ in Δ or $HX(b)$. These will be the fixed sets of certain anticonformal involutions of Δ, or the images in $HX(b)$ of those fixed sets. Such involutions can be classified as follows:

(a) Suppose that $m \in \mathbb{C}$ with $|m| > 1$. Put
$$\xi_m(z) = \frac{m\,\overline{z} - 1}{\overline{z} - \overline{m}}.$$
This is an anticonformal involution on $\mathbb{C}_\infty$ that preserves Δ. Maple notation (defined in the file `hyperbolic/HX.mpl`) is `xi(m,z)`. We put
$$\Xi_m = \{z \in \Delta \mid \xi_m(z) = z\}.$$
If we put $d = \sqrt{|m|^2 - 1}$, then the fixed set of ξ_m in $\mathbb{C}$ is the circle of radius d centred at m, and Ξ_m is the intersection of this circle with Δ. This is a geodesic in Δ, and the following formula gives an isometric parametrisation $\omega_m : \mathbb{R} \to \Xi_m$:
$$\omega_m(s) = \left(\frac{id - 1}{\overline{m}}\right) \frac{(id + 1) - i|m|e^s}{i|m|e^s + (id - 1)}.$$
Maple notation is `xi_curve(m,s)`. The endpoints of Ξ_m on the unit circle are $(1 \pm id)/\overline{m}$. If we have another involution of the same type with parameter m', then $\xi_m \xi_{m'} = \xi_{m'} \xi_m$ iff Ξ_m and $\Xi_{m'}$ cross at right angles, iff $\mathrm{Re}(\overline{m} m') = 1$.

(b) Suppose instead that $u \in \mathbb{C}$ with $|u| = 1$. We then have an anticonformal involution $z \mapsto u\overline{z}$. The fixed set in Δ is the straight line joining the two square roots of u. This is isometrically parameterised by the map
$$s \mapsto \sqrt{u}\frac{e^s - 1}{e^s + 1} = \sqrt{u}\tanh(s/2).$$

(c) Every anticonformal involution on Δ arises in one of the above two ways.

```
hyperbolic/HX_check.mpl: check_xi()
```

Definition 4.3.1. [defn-H-curves]
We define constants $s_0, \dots, s_4$ as follows:

$$s_0 = 2\log\left(\frac{\sqrt{2}b}{b_+ - b_-}\right) \qquad s_1 = \frac{1}{2}\log\left(\frac{\sqrt{2}+b_+}{\sqrt{2}-b_+}\right) \qquad s_2 = \log\left(\frac{1+b}{b_-}\right)$$
$$s_3 = \frac{1}{2}\log\left(\frac{b+b_++1}{b+b_+-1}\right) \qquad s_4 = \frac{1}{4}\log\left(\frac{b_+^2+2b_++2}{b_+^2-2b_++2}\right).$$

We then define maps $\widetilde{c}_k \colon \mathbb{R} \to \Delta$ for $0 \le k \le 8$ as follows:

$$\begin{aligned}
&\widetilde{c}_0(t) = \omega_{(1+i)/b_+}((t/\pi - 1/4)s_0) && \\
&\widetilde{c}_1(t) = e^{i\pi/4}\tanh(t\,s_1/\pi) && \widetilde{c}_2(t) = e^{3i\pi/4}\tanh(t\,s_1/\pi) \\
&\widetilde{c}_3(t) = \omega_{b_+}(-t\,s_2/\pi) && \widetilde{c}_4(t) = \omega_{ib_+}(-t\,s_2/\pi) \\
&\widetilde{c}_5(t) = \tanh(t\,s_3/\pi) && \widetilde{c}_6(t) = i\,\tanh(t\,s_3/\pi) \\
&\widetilde{c}_7(t) = \omega_{ib_+/2+1/b_+}(t\,s_3/\pi - s_4) && \widetilde{c}_8(t) = \omega_{b_+/2+i/b_+}(-t\,s_3/\pi + s_4).
\end{aligned}$$

We write c_k for the composite

$$\mathbb{R} \xrightarrow{\widetilde{c}_k} \Delta \to \Delta/\Pi = HX(b).$$

We also put $\widetilde{C}_k = \widetilde{c}_k(\mathbb{R}) \subset \Delta$ and $C_k = c_k(\mathbb{R}) \subset \Delta/\Pi$.

Remark 4.3.2. The function $\widetilde{c}_k(t)$ is `c_H[k](t)`, and the constant s_j is `s_H[j]`. For $k \in \{0, 3, 4, 7, 8\}$ we see that $\widetilde{c}_k(\mathbb{R})$ is a circular arc. The centre is recorded as `c_H_p[k]`, and the radius as `c_H_r[k]`.

The factors s_k are chosen to make the maps c_k periodic with period 2π. In more detail:

Proposition 4.3.3. [prop-c-H-cycle]
For all $t \in \mathbb{R}$ we have

$$\begin{aligned}
&\widetilde{c}_0(t+2\pi) = \beta_0\beta_2\beta_4\beta_6(\widetilde{c}_0(t)) && \\
&\widetilde{c}_1(t+2\pi) = \beta_0\beta_7\beta_6\beta_5(\widetilde{c}_1(t)) && \widetilde{c}_2(t+2\pi) = \beta_2\beta_1\beta_0\beta_7(\widetilde{c}_2(t)) \\
&\widetilde{c}_3(t+2\pi) = \beta_0\beta_7(\widetilde{c}_3(t)) && \widetilde{c}_4(t+2\pi) = \beta_2\beta_1(\widetilde{c}_4(t)) \\
&\widetilde{c}_5(t+2\pi) = \beta_0(\widetilde{c}_5(t)) && \widetilde{c}_6(t+2\pi) = \beta_2(\widetilde{c}_6(t)) \\
&\widetilde{c}_7(t+2\pi) = \beta_0\beta_2\beta_1(\widetilde{c}_7(t)) && \widetilde{c}_8(t+2\pi) = \beta_2\beta_3\beta_4(\widetilde{c}_8(t)).
\end{aligned}$$

Thus, for $0 \le k \le 8$ we have $c_k(t+2\pi) = c_k(t)$.

(The group element for $\widetilde{c}_k(t)$ is recorded as `c_H_cycle[k]`.)

Proof. Computer calculation.

```
hyperbolic/HX_check.mpl: check_c_H_monodromy()
```

∎

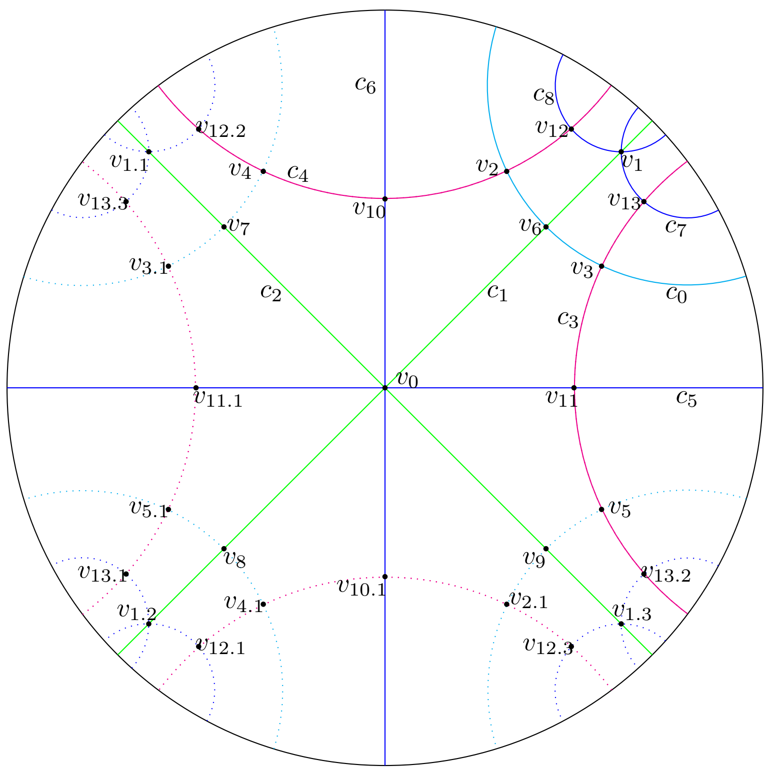
For the equivariance properties of a curve system, it will suffice to check that the equations below hold in Δ, and this can be done by direct calculation.

$$\begin{array}{lll}
\lambda(\widetilde{c}_0(t)) = \beta_4(\widetilde{c}_0(t+\pi/2)) & \mu(\widetilde{c}_0(t)) = \widetilde{c}_0(-t) & \nu(\widetilde{c}_0(t)) = \beta_6(\widetilde{c}_0(-t)) \\
\lambda(\widetilde{c}_1(t)) = \widetilde{c}_2(t) & \mu(\widetilde{c}_1(t)) = \beta_5\beta_4\beta_3(\widetilde{c}_2(\pi+t)) & \nu(\widetilde{c}_1(t)) = \widetilde{c}_2(-t) \\
\lambda(\widetilde{c}_2(t)) = \widetilde{c}_1(-t) & \mu(\widetilde{c}_2(t)) = \beta_2\beta_3\beta_4(\widetilde{c}_1(\pi+t)) & \nu(\widetilde{c}_2(t)) = \widetilde{c}_1(-t) \\
\lambda(\widetilde{c}_3(t)) = \widetilde{c}_4(t) & \mu(\widetilde{c}_3(t)) = \beta_2\beta_3\beta_4(\widetilde{c}_3(\pi+t)) & \nu(\widetilde{c}_3(t)) = \widetilde{c}_3(-t) \\
\lambda(\widetilde{c}_4(t)) = \beta_4(\widetilde{c}_3(-t)) & \mu(\widetilde{c}_4(t)) = \widetilde{c}_4(-t-\pi) & \nu(\widetilde{c}_4(t)) = \beta_6(\widetilde{c}_4(t)) \\
\lambda(\widetilde{c}_5(t)) = \widetilde{c}_6(t) & \mu(\widetilde{c}_5(t)) = \beta_2\beta_3\beta_4(\widetilde{c}_7(t)) & \nu(\widetilde{c}_5(t)) = \widetilde{c}_5(t) \\
\lambda(\widetilde{c}_6(t)) = \widetilde{c}_5(-t) & \mu(\widetilde{c}_6(t)) = \beta_2\beta_3\beta_4(\widetilde{c}_8(-t)) & \nu(\widetilde{c}_6(t)) = \widetilde{c}_6(-t) \\
\lambda(\widetilde{c}_7(t)) = \beta_2\beta_1(\widetilde{c}_8(t)) & \mu(\widetilde{c}_7(t)) = \beta_2(\widetilde{c}_5(t)) & \nu(\widetilde{c}_7(t)) = \beta_3\beta_4(\widetilde{c}_7(t)) \\
\lambda(\widetilde{c}_8(t)) = \beta_2\beta_1(\widetilde{c}_7(-t)) & \mu(\widetilde{c}_8(t)) = \beta_2(\widetilde{c}_6(-t)) & \nu(\widetilde{c}_8(t)) = \beta_3\beta_4(\widetilde{c}_8(-t))
\end{array}$$

(The elements of Π appearing here are recorded in the table `c_action_witness_H`.)

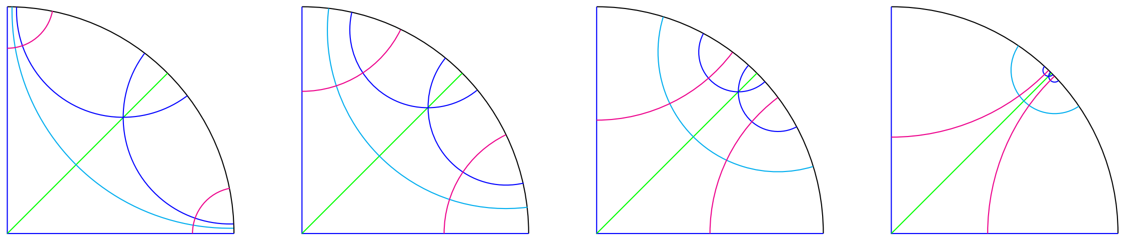
```
cromulent.mpl: check_precromulent("H")
```

In the case $b = 0.75$, these curves and vertices can be illustrated as follows:



The dotted curves are images of the undotted curves under the action of various elements of Π. Combinatorially, the whole picture is essentially the same as the first net described in Section 2.6.

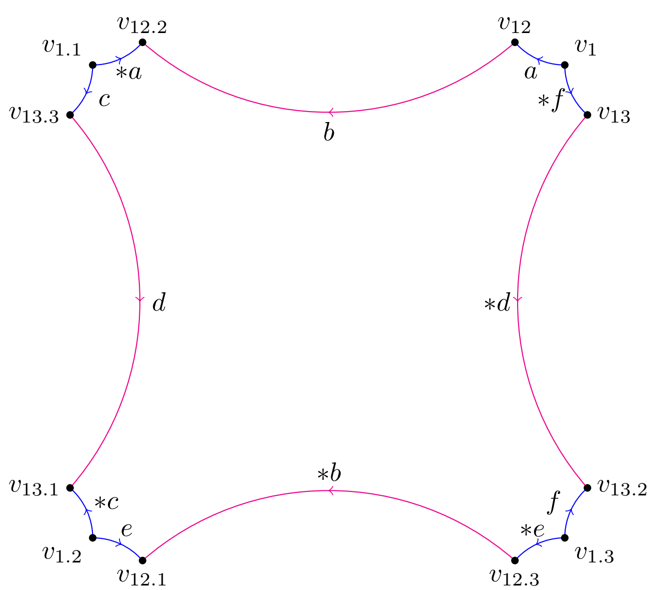
The following diagrams show how the picture varies as b varies from 0 to 1.



$b = 0.20$ $b = 0.50$ $b = 0.75$ $b = 0.95$

In particular, we see that there is no change in the combinatorial structure.

We write $HF_1(b)$ for the following region.



Note that we have marked each edge with a direction and a label. These edges are portions of the curves $\widetilde{C}_k$, or images of those curves under the action of elements of Π, so in particular they are geodesics. One can check that when $b = \sqrt{2/\sqrt{3}-1} \simeq 0.3933$, the fundamental domain is actually a right angled regular dodecagon.

`hyperbolic/HX_check.mpl: check_H_F1()`

Proposition 4.3.4. [`prop-Pi-free`]

The group Π acts freely on Δ, and $HF_1(b)$ is a fundamental domain for this action.

Proof. We will use the standard theorem of Poincaré (which is presented as Theorem VII.1.7 in the textbook [11], for example). Recall that in Proposition 4.1.11 we introduced elements $\sigma_t \in \Pi$ for $t \in \{a, b, c, d, e, f\}$. We claim that σ_t carries the edge labelled $*t$ to the edge labelled t, and carries the interior of $HF_1(b)$ to the exterior. As the edges are geodesic, this claim can be checked by calculating the effect of σ_t on the ends of the relevant edge, which is straightforward. Thus, we have a system of edge pairings as in the Poincaré Theorem.

For the next ingredient, we need to know the internal angles of the hyperbolic polygon $HF_1(b)$. We claim that they are all $\pi/2$. For example, side a is part of the curve $\widetilde{C}_8$, and by inspecting the formula for $\widetilde{c}_8(t)$ we see that $\widetilde{C}_8 = \Xi_{m_8}$, where $m_8 = b_+/2 + i/b_+$. Similarly, side b is part of $\widetilde{C}_4 = \Xi_{m_4}$, where $m_4 = ib_+$. As we mentioned previously, geodesics Ξ_m and $\Xi_{m'}$ are orthogonal if and only if $\mathrm{Re}(\overline{m}m') = 1$. It is clear that $\mathrm{Re}(\overline{m_4}m_8) = 1$, so edges a and b meet at right angles. Similarly, $*f$ is part of $\widetilde{C}_7 = \Xi_{m_7}$, where $m_7 = ib_+/2 + 1/b_+$, and this also meets a at right angles, by the same test. It follows by symmetry that the remaining internal angles are $\pi/2$.

We next need to understand the edge cycle map. We write $\overline{a}$ for the edge a considered in the reverse direction, and similarly for the other edges. We write E for the set of directed edges, so $|E| = 24$. We define $\alpha\colon E \to E$ by

$$\alpha(t) = *t \qquad \alpha(*t) = t \qquad \alpha(\overline{t}) = \overline{*t} \qquad \alpha(\overline{*t}) = \overline{t}.$$

We also define $\beta(u)$ to be the directed edge different from u that has the same initial point as u. For example, we have $\beta(a) = *f$, $\beta(b) = \overline{a}$ and so on. The edge cycle map γ is the composite $\beta\alpha\colon E \to E$. It can be written in disjoint cycle notation as

$$\gamma = (a\; c\; e\; f)(*f\;\; *e\;\; *c\;\; *a)(b\; \overline{*e}\; \overline{*b}\; \overline{*a})(\overline{a}\; \overline{b}\; \overline{e}\;\; *b)(d\; \overline{*f}\; \overline{*d}\; \overline{*c})(\overline{c}\; \overline{d}\; \overline{f}\;\; *d).$$

For each cycle in γ, we can consider the sum of the internal angles at the initial points of the corresponding edges. The key hypothesis in the Poincaré Theorem is that this sum must be a multiple of 2π. This is clearly satisfied here, as each cycle has length 4 and all internal angles are $\pi/2$.

The theorem now tells us that the side pairing maps σ_t generate a group Σ that acts freely on Δ, with $HF_1(b)$ as a fundamental domain. Moreover, Σ is generated freely by the σ_t subject only to a small family of relations, one for each cycle in γ. The relations are constructed in an obvious way from the cycles, with a factor σ_t for each entry t or $*t$, and a factor σ_t^{-1} for each entry $\overline{t}$ or $\overline{*t}$. Specifically, the relations are as follows:

$$\begin{aligned}
\sigma_a\sigma_c\sigma_e\sigma_f &= 1 & \qquad \sigma_f^{-1}\sigma_e^{-1}\sigma_c^{-1}\sigma_a^{-1} &= 1\\
\sigma_b\sigma_e^{-1}\sigma_b^{-1}\sigma_a^{-1} &= 1 & \sigma_a\sigma_b\sigma_e\sigma_b^{-1} &= 1\\
\sigma_d\sigma_f^{-1}\sigma_d^{-1}\sigma_c^{-1} &= 1 & \sigma_c\sigma_d\sigma_f\sigma_d^{-1} &= 1.
\end{aligned}$$

The relations on the right are equivalent to those on the left and so can be ignored. Proposition 4.1.11 now allows us to identify Σ with Π.

`hyperbolic/HX_check.mpl: check_H_F1()`

□

Remark 4.3.5. [rem-move-inwards]

We can make a more constructive statement as follows. Any conformal automorphism of Δ has the form $\gamma(z) = \lambda(z - \alpha)/(1 - \overline{\alpha}z)$ for some λ, α with $|\lambda| = 1$ and $|\alpha| < 1$. Suppose that $\alpha \neq 0$. One can check that

$$|\gamma(z)|^2 - |z|^2 = \left|\frac{\alpha}{1 - \overline{\alpha}z}\right|^2 (1 - |z|^2)\left(|z - \overline{\alpha}^{-1}|^2 - (|\alpha|^{-2} - 1)\right).$$

`hyperbolic/HX_check.mpl: check_move_inwards()`

This means that $|\gamma(z)| = |z|$ if and only if $|z - \overline{\alpha}^{-1}| = \sqrt{|\alpha|^{-2} - 1}$, and this locus describes a circle that cuts $\partial\Delta$ at right angles, or in other words, a hyperbolic geodesic. Now put

$$B = \{\beta_0, \beta_2, \beta_4, \beta_6, \beta_0\beta_7, \beta_1\beta_2, \beta_2\beta_1, \beta_3\beta_4, \beta_4\beta_3, \beta_5\beta_6, \beta_6\beta_5, \beta_7\beta_0\}.$$

Using the above analysis one can check that

$$HF_1(b) = \{z \in \Delta \mid |z| \leq |\gamma(z)| \text{ for all } \gamma \in B\}.$$

(Note that there is one element of B for each of the twelve sides of $HF_1(b)$.) Now suppose we have a point $z_0 \in \Delta$ that lies outside $HF_1(b)$. We can choose $\gamma_0 \in B$ such that the point $z_1 = \gamma_0(z_0)$ has $|z_1| < |z_0|$. If z_1 still does not lie in $HF_1(b)$, we can choose $\gamma_1 \in B$ such that the point $z_2 = \gamma_1(z_1)$ has $|z_2| < |z_1|$, and so on. As the orbit Πz_0 is discrete, it can only contain finitely many points of absolute value less than or equal to $|z_0|$, so this process will terminate after a finite number of steps. This gives us a point $z_n \in HF_1(b) \cap \Pi z_0$, which will be unique unless it lies on the boundary of $HF_1(b)$. This algorithm is implemented by the function `retract_F1_H0_aux`, defined in `hyperbolic/HX0.mpl`.

Corollary 4.3.6. [cor-bound-i]
For $\gamma \in \Pi \setminus \{1\}$ we have $1/\sqrt{2} \leq |\gamma(0)| < 1$.

Proof. Because 0 lies in the interior of F we see that the orbit $\Pi.0$ is discrete. We can thus choose $\gamma \in \Pi \setminus \{1\}$ such that $|\gamma(0)|$ is minimal. Because F is a fundamental domain, it is clear that $\gamma(0) \notin F$. Thus, by Remark 4.3.5, there is an element $\delta \in B$ such that $|\delta\gamma(0)| < |\gamma(0)|$. By our choice of γ, we must have $\delta\gamma = 1$, so $\gamma = \delta^{-1}$. As B is closed under inversion, we must have $\gamma \in B$. From the definitions we see that when $\gamma \in B$ we have

$$|\gamma(0)|^2 \in \left\{ \frac{1}{2}\left(1 + \frac{1-b^2}{1+b^2}\right), \frac{4}{5}\left(1 + \frac{b^2(3-b^2)}{5+2b^2+b^4}\right) \right\}.$$

From these expressions it is clear that $|\gamma(0)|^2 \geq 1/2$, as required.

```
hyperbolic/HX_check.mpl: check_Pi_bound()
```

□

Corollary 4.3.7. [cor-bound-ii]
Consider an element $\gamma \in \Pi \setminus \{1\}$, given by $\gamma(z) = (az+b)/(cz+d)$ with $ad - bc = 1$. Then

$$\begin{aligned} |c| &\geq 1 \\ 1 < |d/c| &\leq \sqrt{2} \\ |cz+d| &\leq (1+\sqrt{2})|c|. \end{aligned}$$

Proof. As γ preserves Δ, it can be written in the form $\gamma(z) = \mu^2(z+\alpha)/(\overline{\alpha}z+1)$ with $|\alpha| < 1$ and $|\mu| = 1$. This can be rewritten as $\gamma(z) = (az+b)/(cz+d)$, where

$$\begin{aligned} a &= \mu/\sqrt{1-|\alpha|^2} & b &= \mu\alpha/\sqrt{1-|\alpha|^2} \\ c &= \overline{\mu\alpha}/\sqrt{1-|\alpha|^2} & d &= \overline{\mu}/\sqrt{1-|\alpha|^2}, \end{aligned}$$

and then $ad - bc = 1$.

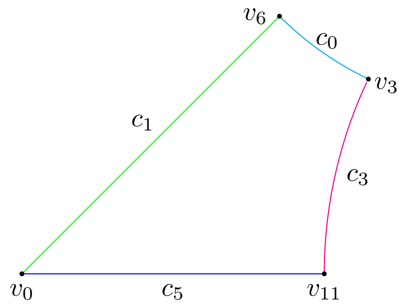
As $\gamma(0) = \mu^2\alpha$, Corollary 4.3.6 tells us that $1/\sqrt{2} \leq |\alpha| < 1$. Now $|c| = |\alpha|/\sqrt{1-|\alpha|^2}$, which is a strictly increasing function of $|\alpha|$, and is equal to 1 when $|\alpha| = 1/\sqrt{2}$. We deduce that $|c| \geq 1$ as claimed. Similarly, we have $|d/c| = 1/|\alpha| \in (1, \sqrt{3}]$. Finally, we have

$$|cz+d|/|c| = |z + d/c| \leq |z| + |d/c| < 1 + \sqrt{2},$$

so $|cz+d| \leq (1+\sqrt{2})c$ as claimed. □

4.4. **Fundamental domains.** [sec-H-fundamental]

Definition 4.4.1. [defn-HF-sixteen]
We define $HF_{16}(b)$ to be the region indicated below:



Proposition 4.4.2. [prop-HF-sixteen]

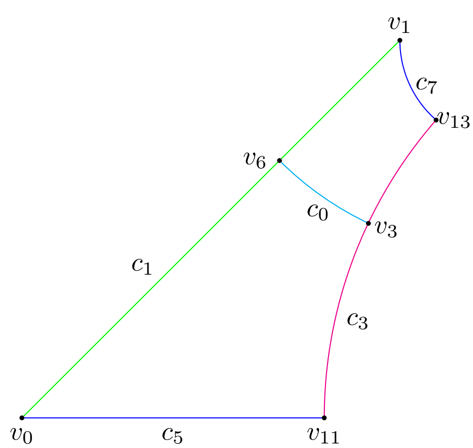
$HF_{16}(b)$ is a fundamental domain for the action of $\widetilde{\Pi}$ on Δ. Similarly, the image in $HX(b)$ is a fundamental domain for the action of G on $HX(b)$.

Proof. First put

$$HF_8(b) = \{z \in HF_1(b) \mid 0 \leq \arg(z) \leq \pi/4\}.$$

By inspecting the following picture, we see that

$$\begin{aligned} HF_{16}(b) \cup \lambda\nu\mu(HF_{16}(b)) &= HF_8(b) \\ HF_{16}(b) \cap \lambda\nu\mu(HF_{16}(b)) &= c_0([\pi/4, \pi/2]). \end{aligned}$$



Now put

$$\begin{aligned} T_8 &= \{1, \lambda, \lambda^2, \lambda^3, \nu, \lambda\nu, \lambda^2\nu, \lambda^3\nu\} \\ T_{16} &= T_8 \amalg \{\tau\lambda\mu\nu \mid \tau \in T_8\}. \end{aligned}$$

It is easy to check that

$$\bigcup_{\tau \in T_{16}} HF_{16}(b) = \bigcup_{\tau \in T_8} HF_8(b) = HF_1(b).$$

It is also easy to check that the homomorphism $\pi\colon \widetilde{\Pi} \to G$ restricts to give a bijection $T_{16} \to G$, so that

$$\widetilde{\Pi} = \coprod_{\tau \in T_{16}} \Pi\tau.$$

We also know that $HF_1(b)$ is a fundamental domain for Π, so $\Delta = \bigcup_{\phi\in\Pi} \phi(HF_1(b))$. Putting this together, we deduce that $\Delta = \bigcup_{\phi\in\widetilde{\Pi}} \phi(HF_{16}(b))$.

Now consider an element $\phi \in \widetilde{\Pi} \setminus \{1\}$ and a point $z \in HF_{16}(b)$ such that $\phi(z)$ also lies in $HF_{16}(b)$. We can write $\phi = \psi\tau$ with $\psi \in \Pi$ and $\tau \in T_{16}$. Now the point $z' = \tau(z)$ lies in $HF_1(b)$, so the point $z'' = \phi(z) = \psi(z')$ lies in $\psi(HF_1(b))$, but it also lies in $HF_{16}(b)$ by assumption, and one can see directly that $HF_{16}(b)$ is in the interior of $HF_1(b)$. As $HF_1(b)$ is a fundamental domain for Π, this can only be consistent if $\psi = 1$. This means that $\tau(z) \in HF_{16}(b) \cap \tau(HF_{16}(b))$, and the action of T_{16} is simple enough that we can just do a check of cases to show that $\tau(z) \in \partial(HF_{16}(b))$. This proves that $HF_{16}(b)$ is a fundamental domain as claimed. □

Proposition 4.4.3. [prop-square-diffeo-H]
$HF_{16}(b)$ is homeomorphic to the unit square.

Proof. This is visually obvious, but we will outline a construction of an explicit homeomorphism. Suppose we have two disjoint geodesics in Δ, the first with endpoints $a_1, a_2 \in S^1$, and the second with endpoints $a_3, a_4 \in S^1$. It is easy to produce a Möbius transformation m that sends Δ to the upper half-plane and $\{a_1, a_2\}$ to $\{1, -1\}$. This means that m sends our first geodesic to the upper half of the unit circle, and our second geodesic to another semicircle that crosses the real line orthogonally at $m(a_3)$ and $m(a_4)$. As the two geodesics are disjoint, the second circle must either be wholly inside or wholly outside the unit circle. By composing m with $z \mapsto -1/z$ if necessary, we may assume that it lies outside. There is in fact a one-parameter family of choices of m that have the properties mentioned so far, and one can check that there is precisely one choice for which $m(a_3) + m(a_4) = 0$. With this condition, we see that m sends the second geodesic to a semicircle centred at the origin, of radius $r > 1$ say. Now the function $p(z) = \log|m(z)|/\log(r)$ takes the values 0 and 1 on our two geodesics. In the cases of interest it is convenient to adjust this procedure slightly. Rather than explaining the intermediate steps, we just describe the outcome. We put

$$\zeta_1 = \sqrt{\frac{i-b}{i+b}} = \frac{1+ib}{b_+} \in S^1 \qquad \zeta_2 = \frac{b_+ + ib_-}{1+i} \in S^1$$

$$r_1 = \sqrt{\frac{1-b}{1+b}} = \frac{b_-}{1+b} \in \mathbb{R}^+ \qquad r_2 = \frac{b_+ + b_-}{\sqrt{2}b} \in \mathbb{R}^+$$

$$m_1(z) = \frac{\zeta_1 - z}{1 - \zeta_1 z} \qquad m_2(z) = \frac{i\zeta_2 - z}{1 - \zeta_2 z}$$

$$p_1(z) = \log|m_1(z)|/\log(r_1) \qquad p_2(z) = \log|m_2(z)|/\log(r_2),$$

then $p(z) = (1 - p_1(z), p_2(z))$. This defines a map $p\colon HF_{16}(b) \to [0,1]^2$, with boundary behaviour as discussed in Section 2.6. Recall that Möbius transformations send circles to circles (provided that we interpret straight lines as circles of infinite radius). It follows that for any point $t \in [0,1]^2$, the fibres $p_1^{-1}\{1 - t_1\}$ and $p_2^{-1}\{t_2\}$ are circles, with centres c_1 and c_2 say. It is not too hard to obtain formulae for c_1 and c_2; the observation that $m_1 = m_1^{-1}$ and $m_2 = m_2^{-1}$ is helpful for this. The relevant circles must cross the unit circle orthogonally, so the radii are $\sqrt{|c_1|^2 - 1}$ and $\sqrt{|c_2|^2 - 1}$. The point $p^{-1}(t)$ lies on the intersection of the two circles, and so can be found by an exercise in coordinate geometry. After some simplification we arrive at the following formulae:

$$s_1 = r_1^{1-t_1} \qquad s_2 = r_2^{t_2}$$

$$c_1 = \frac{s_1^2/\zeta_1 - \zeta_1}{s_1^2 - 1} \qquad c_2 = \frac{s_2^2/\zeta_2 - i\zeta_2}{s_2^2 - 1}$$

$$\alpha = \mathrm{Re}\left(\frac{c_1(\overline{c_1} - \overline{c_2})}{|c_1 - c_2|^2}\right) \qquad \beta = \sqrt{\frac{|c_1|^2 - 1}{|c_1 - c_2|^2} - \alpha^2},$$

then

$$p^{-1}(t) = (1 - \alpha + i\beta)c_1 + (\alpha - i\beta)c_2.$$

Unfortunately this formula is not meaningful when $t_1 = 1$ or $t_2 = 0$, but those cases can be handled in an *ad hoc* way.

hyperbolic/HX_check.mpl: check_square_diffeo_H()

$\square$

4.5. The hyperbolic family is universal. [sec-H-universal]

Theorem 4.5.1. [thm-H-universal]
Let X be a cromulent surface. Then there is a unique number $b \in (0,1)$ such that X is isomorphic to $HX(b)$, and the isomorphism is also unique.

The rest of this section will constitute the proof. The threads will be gathered together in Proposition 4.5.14.

Theorem 3.7.2 says that every cromulent surface is isomorphic to $PX(a)$ for some a, so we may assume that $X = PX(a)$. We therefore have a curve system as in Definition 2.4.4, and a fundamental domain $PF_{16}(a)$ as in Proposition 3.3.1.

Definition 4.5.2. [defn-lm-nu-Dl]
As before we define $\lambda, \nu \colon \Delta \to \Delta$ by $\lambda(z) = iz$ and $\nu(z) = \overline{z}$. These maps clearly satisfy $\lambda^4 = \nu^2 = (\lambda\nu)^2 = 1$.

Lemma 4.5.3. [lem-p-Dl-X]
There is a unique covering map $p \colon \Delta \to X$ such that $p(0) = v_0$ and $p'(0)$ is a positive multiple of $c_5'(0)$. Moreover, this satisfies $\lambda p = p\lambda$ and $\nu p = p\nu$.

Proof. The uniformization theorem for Riemann surfaces shows that the universal cover of X is conformally equivalent to Δ, so we can choose a covering map $p_0 \colon \Delta \to X$. We can then choose a point $a \in \Delta$ with $p_0(a) = v_0$, and put $m(z) = (z + a)/(1 + \overline{a}z)$. Then m is an automorphism of Δ with $m(0) = a$, so the composite $p_1 = p_0 \circ m$ is a covering with $p_1(0) = v_0$. Now $p_1'(0) = re^{i\theta}\, c_5'(0)$ for some $r > 0$ and $\theta \in \mathbb{R}$, and the map $p(z) = p_1(z/e^{i\theta})$ is a covering with the required properties. If q is another such map, then standard theory of coverings gives an automorphism $f \colon \Delta \to \Delta$ such that $q = pf$. Our condition on p and q implies that $f(0) = 0$ and $f'(0) > 0$. However, any conformal automorphism of Δ has the form $f(z) = \alpha(z - \beta)/(1 - \overline{\beta}z)$ for some α, β with $|\alpha| = 1$ and $|\beta| < 1$. The condition $f(0) = 0$ gives $\beta = 0$, and then the condition $f'(0) > 0$ gives $\alpha = 1$, so f is the identity and $p = q$ as claimed.

The maps $\lambda^{-1}p\lambda$ and $\nu^{-1}p\nu$ are easily seen to satisfy the defining conditions for p, so $\lambda p = p\lambda$ and $\nu p = p\nu$. $\square$

Proposition 4.5.4. [prop-classify-a]
There is a unique system of points $v_i^* \in \Delta$ (for $0 \leq i \leq 13$) and continuous maps $c_j^* \colon \mathbb{R} \to \Delta$ (for $0 \leq j \leq 8$) such that the following hold:

(a) $v_0^* = 0$
(b) *For all i we have $p(v_i^*) = v_i$, and for all j and t we have $p(c_j^*(t)) = c_j(t)$.*
(c) *Whenever a number t appears in the i'th column of the j'th row of the table below, we have $c_j^*(t) = v_i^*$.*

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0			0	$\frac{\pi}{2}$			$\frac{\pi}{4}$							
1	0	π					$\frac{\pi}{2}$		$-\frac{\pi}{2}$					
2	0							$\frac{\pi}{2}$		$-\frac{\pi}{2}$				
3				$\frac{\pi}{2}$		$-\frac{\pi}{2}$						0		π
4			$-\frac{\pi}{2}$		$\frac{\pi}{2}$						0		$-\pi$	
5	0											π		
6	0										π			
7		0												
8		0												

(d) *The action of λ and ν on the points v_i^* is partially given by the following table:*

	v_0^*	v_1^*	v_2^*	v_3^*	v_4^*	v_5^*	v_6^*	v_7^*	v_8^*	v_9^*	v_{10}^*	v_{11}^*	v_{12}^*	v_{13}^*
λ	v_0^*			v_4^*		v_3^*	v_7^*	v_8^*	v_9^*	v_6^*		v_{10}^*		
ν	v_0^*			v_5^*		v_3^*	v_9^*	v_8^*	v_7^*	v_6^*		v_{11}^*		
$\lambda\nu$	v_0^*	v_1^*	v_3^*	v_2^*	v_5^*	v_4^*	v_6^*	v_9^*	v_8^*	v_7^*	v_{11}^*	v_{10}^*	v_{13}^*	v_{12}^*

(e) *The action of λ and ν on the curves c_j^* is partially given by the following table:*

	$c_0^*(t)$	$c_1^*(t)$	$c_2^*(t)$	$c_3^*(t)$	$c_4^*(t)$	$c_5^*(t)$	$c_6^*(t)$	$c_7^*(t)$	$c_8^*(t)$
λ		$c_2^*(t)$	$c_1^*(-t)$	$c_4^*(t)$		$c_6^*(t)$	$c_5^*(-t)$		
ν		$c_2^*(-t)$	$c_1^*(-t)$	$c_3^*(-t)$		$c_5^*(t)$	$c_6^*(-t)$		
$\lambda\nu$	$c_0^*(\pi/2-t)$	$c_1^*(t)$	$c_2^*(-t)$	$c_4^*(-t)$	$c_3^*(-t)$	$c_6^*(t)$	$c_5^*(t)$	$c_8^*(t)$	$c_7^*(t)$

Proof.

(0) We define $v_0^* = 0$ and note that this is fixed by λ and ν, and that $p(v_0^*) = v_0$ by the definition of p.

(1) For $j \in \{1, 2, 5, 6\}$ we define c_j^* to be the unique continuous lift of c_j that satisfies $c_j^*(0) = v_0^*$. The claimed formulae for $\lambda(c_j^*(t))$ and $\nu(c_j^*(t))$ then hold by an evident uniqueness argument. For example, we have seen previously that $\lambda(c_2(t)) = c_1(-t)$ in $PX(a)$, so the maps $t \mapsto \lambda(c_2^*(t))$ and $t \mapsto c_1^*(-t)$ are both continuous lifts of the map $t \mapsto c_1(-t)$. Both give v_0^* when $t = 0$, so they must agree for all t.

(2) We define

$$v_1^* = c_1^*(\pi) \qquad v_6^* = c_1^*(\pi/2) \qquad v_8^* = c_1^*(-\pi/2)$$
$$v_7^* = c_2^*(\pi/2) \qquad v_9^* = c_2^*(-\pi/2)$$
$$v_{10}^* = c_6^*(\pi)$$
$$v_{11}^* = c_5^*(\pi).$$

By taking $t = \pi$ or $t = \pm\pi/2$ in (1) we see that λ and ν act on $\{v_1^*, v_6^*, v_7^*, v_8^*, v_9^*, v_{10}^*, v_{11}^*\}$ as indicated in (d). Because the maps $c_j : \mathbb{R} \to PX(a)$ form a curve system, we see that $p(v_i^*) = v_i$ in all these cases.

(3) For $j \in \{3, 4, 7, 8\}$ we let c_j^* denote the unique lift of $c_j(t)$ satisfying the following initial condition:

$$c_3^*(0) = v_{11}^* \qquad c_4^*(0) = v_{10}^* \qquad c_7^*(0) = c_8^*(0) = v_1^*.$$

The claimed formulae for $\lambda(c_j^*(t))$ and $\nu(c_j^*(t))$ then hold by an evident uniqueness argument.

(4) We define

$$v_{13}^* = c_3^*(\pi) \qquad v_3^* = c_3^*(\pi/2) \qquad v_5^* = c_3^*(-\pi/2)$$
$$v_{12}^* = c_4^*(-\pi) \qquad v_4^* = c_4^*(\pi/2) \qquad v_2^* = c_4^*(-\pi/2).$$

By taking $t = \pi$ or $t = \pm\pi/2$ in (3) we see that λ and ν act on $\{v_2^*, v_3^*, v_4^*, v_5^*, v_{12}^*, v_{13}^*\}$ as indicated in (d). Because the maps $c_j : \mathbb{R} \to PX(a)$ form a curve system, we see that $p(v_i^*) = v_i$ in all these cases.

(5) We let c_0^* denote the unique lift of c_0 satisfying $c_0^*(\pi/4) = v_6^*$. By the usual uniqueness argument, so have $\lambda\nu(c_0^*(t)) = c_0^*(\pi/2 - t)$.

(6) Now all of (a), (b) and (c) is true by construction except for the identities $c_0^*(0) = v_2^*$ and $c_0^*(\pi/2) = v_3^*$. All parts of (d) and (e) have also been established. For the remaining facts, we consider the fundamental domain $PF_{16}(a)$ in $PX(a)$. We have two paths in $PF_{16}(a)$ from v_0 to v_3: one given by $c_5([0, \pi])$ followed by $c_3([0, \pi/2])$, and the other by $c_1([0, \pi/2])$ followed by $c_0([\pi/4, \pi/2])$. If we lift the first path starting with v_0^* then the endpoint is $c_3^*(\pi/2) = v_3^*$, and if we lift the second then the endpoint is $c_0^*(\pi/2)$. Now $PF_{16}(a)$ is homeomorphic to a square and so is contractible. In particular, our two paths are homotopic relative to the endpoints, and it follows that the two lifts have the same endpoints, so $c_0^*(\pi/2) = v_3$ as claimed. By applying the map $\lambda\nu$ we deduce that $c_0^*(0) = v_2$.

$\square$

Proposition 4.5.5. *For $0 \leq j \leq 8$ there is an antiholomorphic involution $\theta_j \colon \Delta \to \Delta$ such that c_j^* gives a diffeomorphism from $\mathbb{R}$ to the geodesic $C_j^* = \{z \in \Delta \mid \theta_j(z) = z\}$. Specifically, for $j \in \{1, 2, 5, 6\}$ we have*

$$\begin{aligned} \theta_1 &= \lambda\nu & C_1^* &= (-1,1).e^{i\pi/4} \\ \theta_2 &= \lambda^3\nu & C_2^* &= (-1,1).e^{-i\pi/4} \\ \theta_5 &= \nu & C_5^* &= (-1,1) \\ \theta_6 &= \lambda^2\nu & C_6^* &= (-1,1).i. \end{aligned}$$

Proof. Put $C_j = c_j(\mathbb{R}) \subset PX(a)$. We have seen previously that in each case there is an antiholomorphic involution $\rho_j \in G$ such that $\rho_j(c_j(t)) = c_j(t)$ for all $t \in \mathbb{R}$, and in fact C_j is a connected component of the set $PX(a)^{\rho_j} = \{z \in PX(a) \mid \rho_j(z) = z\}$. Moreover, C_j is diffeomorphic to S^1 and the map $c_j \colon \mathbb{R} \to C_j$ is a universal covering.

Next, standard covering theory shows that there is a unique continuous map $\theta_j \colon \Delta \to \Delta$ with $p\theta_j = \rho_j p$ and $\theta_j(c_j^*(0)) = c_j^*(0)$. As p is a holomorphic covering, the equation $p\theta_j = \rho_j p$ implies that θ_j is antiholomorphic. The map θ_j^2 covers $\rho_j^2 = 1$ and fixes $c_j^*(0)$; it follows that $\theta_j^2 = 1$. Now c_j^* and $\rho_j \circ c_j^*$ are both lifts of c_j with the same value at $t = 0$, so they must be the same, so $c_j^*(\mathbb{R}) \subseteq C_j^*$. We previously classified the antiholomorphic involutions on Δ, and using that classification we see that C_j^* is a geodesic in Δ and is diffeomorphic to $\mathbb{R}$. It follows that $p(C_j^*)$ is a connected subset of $PX(a)^{\rho_j}$ containing $p(c_j^*(0)) = c_j(0)$, so $p(C_j^*) \subseteq C_j$. Now p is a proper map with nonzero complex derivative everywhere in Δ. It follows that $p \colon C_j^* \to C_j$ is also proper with nonzero real derivative everywhere. This means that $p \colon C_j^* \to C_j$ is another universal covering, and by the uniqueness of universal coverings, we see that $c_j^* \colon \mathbb{R} \to C_j^*$ must be a diffeomorphism.

For the case $j = 5$, we have seen that $\nu(c_5^*(t)) = c_5^*(t)$ for all t and it follows that $\theta_5 = \nu$. We also have $\nu(z) = \overline{z}$ so $C_5^* = (-1, 1)$. The cases $j \in \{1, 2, 6\}$ are similar. ☐

Lemma 4.5.6. [lem-Dl-involutions]
For any point $v \in \Delta$, there is a unique holomorphic involution on Δ that fixes v.

Proof. As $\mathrm{Aut}(\Delta)$ acts transitively on Δ, we may assume that $v = 0$. In this case the map $z \mapsto -z$ is a holomorphic involution that fixes v. Let ϕ be any other holomorphic involution that fixes v. As ϕ is an automorphism of Δ we have $\phi(z) = \lambda(z - \alpha)/(1 - \overline{\alpha}z)$ for some λ, α with $|\lambda| = 1$ and $|\alpha| < 1$. As ϕ fixes $v = 0$ we have $\alpha = 0$, so $\phi(z) = \lambda z$. As ϕ is an involution we must have $\lambda = -1$. ☐

Proposition 4.5.7. [prop-Dl-kp]
Let κ be the unique holomorphic involution on Δ that fixes v_6^*. Then $p\kappa = \lambda\mu p$, and the action on the points v_i^* is partially given by the following table:

	v_0^*	v_1^*	v_2^*	v_3^*	v_4^*	v_5^*	v_6^*	v_7^*	v_8^*	v_9^*	v_{10}^*	v_{11}^*	v_{12}^*	v_{13}^*
κ	v_1^*	v_0^*	v_3^*	v_2^*			v_6^*				v_{13}^*	v_{12}^*	v_{11}^*	v_{10}^*

Also, the action on the curves c_j^* is partially given by the following table:

	$c_0^*(t)$	$c_1^*(t)$	$c_2^*(t)$	$c_3^*(t)$	$c_4^*(t)$	$c_5^*(t)$	$c_6^*(t)$	$c_7^*(t)$	$c_8^*(t)$
κ	$c_0^*(\pi/2 - t)$	$c_1^*(\pi - t)$		$c_4^*(t - \pi)$	$c_3^*(t + \pi)$	$c_8^*(t)$	$c_7^*(t)$	$c_6^*(t)$	$c_5^*(t)$

Proof. Given a point $z \in \Delta$, we let u be any path in Δ from v_6^* to z in Δ. We recall that $\lambda\mu(v_6) = v_6$, so the path $\lambda\mu pu$ starts at $p(v_6^*)$, so there is a unique lifting u' of $\lambda\mu pu$ with $u'(0) = v_6^*$. We define $\kappa(z) = u'(1)$. Standard covering theory shows that this is independent of the choice of u and gives the unique continuous map $\kappa \colon \Delta \to \Delta$ with $\kappa(v_6^*) = v_6^*$ and $p\kappa = \lambda\mu p$. As p is a holomorphic covering and holomorphy can be checked locally it follows that κ is holomorphic. The map κ^2 covers $(\lambda\mu)^2 = 1$ and fixes v_6^*; it follows that $\kappa^2 = 1$. Thus, κ is the unique holomorphic involution on Δ that fixes v_6^*.

The curves $\kappa(c_0^*(t))$ and $c_0^*(\pi/2 - t)$ both lift $c_0(\pi/2 - t)$ and pass through v_6^* at $t = \pi/4$, so they are the same. Taking $t \in \{0, \pi/2\}$ we deduce that κ exchanges v_2^* and v_3^*. Essentially the same argument gives $\kappa(c_1^*(t)) = c_1^*(\pi - t)$, and shows that κ exchanges v_0^* and v_1^*. Now that we know the action on the point $v_3^* = c_3^*(\pi/2)$ we can see that the curves $\kappa(c_3^*(t))$ and $c_4^*(t - \pi)$ both lift $c_4(t - \pi)$ and pass through v_2^* at $t = \pi/2$ so they are the same. As $\kappa^2 = 1$ we can also deduce that $\kappa(c_4^*(t)) = c_3^*(t + \pi)$. By taking $t = 0$ or

$t=\pm\pi$ we deduce that κ exchanges v_{10}^* and v_{13}^*, and also exchanges v_{11}^* and v_{12}^*. This just leaves the action on $c_j^*(t)$ for $j\in\{5,6,7,8\}$, which can be checked in the same way using v_0^* and v_1^* as basepoints. $\square$

Lemma 4.5.8. [lem-v-eleven-sign]
v_{11}^* is a positive real number, and v_6^* is a positive multiple of $e^{i\pi/4}=(1+i)/\sqrt{2}$.

Proof. We have seen that c_5^* gives a diffeomorphism from $\mathbb{R}$ to $(-1,1)$. We also have $p(c_5^*(t))=c_5(t)$ so $(c_5^*)'(0)=c_5'(0)/p'(0)$, and this is a positive real number by the definition of p. As c_5^* is a diffeomorphism the derivative cannot change sign, so it is a strictly increasing map. It follows that the point $v_{11}^*=c_5^*(\pi)$ lies on the positive real axis as claimed.

Next, we also know that c_1^* gives a diffeomorphism from $\mathbb{R}$ to $(-1,1).e^{i\pi/4}$. By examining the formula in Definition 3.2.1 we see that to first order in t we have

$$\begin{aligned} c_1(t)&=[te^{i\pi/4}/2:1:0:0]\\ c_5(t)&=[t\sqrt{a}/2:1:0:0], \end{aligned}$$

so $c_1'(0)$ is a positive multiple of $e^{i\pi/4}\,c_5'(0)$. Using this we see that c_5^* must carry $(0,\infty)$ to $(0,1).e^{i\pi/4}$. In particular, the point $v_6^*=c_1^*(\pi/2)$ is a positive multiple of $e^{i\pi/4}$ as claimed. $\square$

Lemma 4.5.9. [lem-right-circles]
Suppose that two circles in $\mathbb{R}^2$ meet at right angles. Let r_1 and r_2 be the radii, and let d be the distance between the centres; then $d^2=r_1^2+r_2^2$.

Proof. Elementary. $\square$

Proposition 4.5.10. [prop-b-H]
There is a unique number $b\in(0,1)$ such that

$$\begin{aligned}
v_0^* &= 0 & v_1^* &= \frac{1+i}{2}b_+\\
v_2^* &= \frac{b\,b_- - b_+}{i-b^2} & v_5^* &= -i\,v_2^*\\
v_3^* &= \frac{b\,b_- - b_+}{ib^2-1} & v_4^* &= i\,v_3^*\\
v_6^* &= \frac{1+i}{\sqrt{2}}\,\frac{\sqrt{2}-b_-}{b_+} & v_7^* &= i\,v_6^*\\
v_8^* &= -v_6^* & v_9^* &= -i\,v_6^*\\
v_{10}^* &= i(b_+-b) & v_{11}^* &= b_+-b\\
v_{12}^* &= (b+b_+)\frac{i+(i+2)b^2}{(b+b_+)^2+b^2} & v_{13}^* &= i\,\overline{v_{12}^*}
\end{aligned}$$

(where $b_\pm=\sqrt{1\pm b^2}$ as before). Moreover, the map κ is given by

$$\kappa(z)=\frac{b_+i-(1+i)z}{1+i-b_+z}.$$

Proof. We will use the curves $C_j^*\subset\Delta$ for $j\in\{0,1,3,5\}$. We have already seen that $C_5^*=(-1,1)$ and $C_1^*=(-1,1).e^{i\pi/4}$. The set C_3^* is a geodesic in Δ that does not pass through the origin, so it is the intersection of Δ with a circle centred outside Δ that crosses $\partial\Delta$ at right angles. (This is a standard fact of hyperbolic geometry.) We let b denote the radius of C_3^* (so $b>0$).

We next claim that

(a) The curves C_0^* and C_3^* cross at right angles at v_3^*
(b) The curves C_0^* and C_1^* cross at right angles at v_6^*
(c) The curves C_3^* and C_5^* cross at right angles at v_{11}^*.

Indeed, we see from Proposition 4.5.4 that the indicated curves cross at the indicated points, and that in all relevant cases we have $p(c_i^*(t))=c_i(t)$ and $p(v_j^*)=v_j$ in $PX(a)$. As p is a holomorphic covering it preserves angles, so the claim follows from Lemma 3.2.8.

As C_3^* meets the curve $C_5^* = (-1,1)$ at right angles at the point $v_{11}^* > 0$, we see that the centre of C_3^* must be on the positive real axis. As C_3^* also meets $\partial\Delta$ orthogonally, Lemma 4.5.9 shows that the centre is $\sqrt{1+b^2} = b_+$. It then follows that $v_{11}^* = b_+ - b$.

Now put $\omega = e^{i\pi/4} = (1+i)/\sqrt{2}$. By a similar argument, there is a number $c > 0$ such that $\widetilde{C}_0$ is a circular arc with centre $\sqrt{1+c^2}\omega$ and radius c, and we have $v_6^* = (\sqrt{1+c^2} - c)\omega$.

As $\widetilde{C}_0$ and $\widetilde{C}_3$ meet at right angles, we must have

$$b^2 + c^2 = |\sqrt{1+c^2}\omega - \sqrt{1+b^2}|^2 = \left(\frac{\sqrt{1+c^2}}{\sqrt{2}} - \sqrt{1+b^2}\right)^2 + \left(\frac{\sqrt{1+c^2}}{\sqrt{2}}\right)^2.$$

After some manipulation this gives $c^2 = (1-b^2)/(1+b^2)$. This ensures that $b < 1$, and we can take square roots to get $c = b_-/b_+$. We also get $\sqrt{1+c^2} = \sqrt{2}/b_+$ and so

$$v_6^* = (\sqrt{1+c^2} - c)\omega = \frac{1+i}{\sqrt{2}}\,\frac{\sqrt{2} - b_-}{b_+}$$

as claimed.

Now put $w = (b_+ - b\,b_-)/(1 - ib^2)$. One can check that

$$1 - |w|^2 = 2b\,b_-\,(b_+ - b\,b_-)/(1+b^4) > 0,$$

so $w \in \Delta$. Long but fairly straightforward calculations also show that $|w - b_+|^2 = b^2$ and $|w - \sqrt{1+c^2}\omega|^2 = c^2$, so $w \in \widetilde{C}_3 \cap \widetilde{C}_0$. It is standard that distinct geodesics in Δ meet in only one place, so we must have $v_3^* = w$.

Next, if we define

$$\kappa^*(z) = \frac{b_+ i - (1+i)z}{1 + i - b_+ z},$$

a straightforward calculation shows that this is a holomorphic involution on Δ that fixes v_6^*. However, Lemma 4.5.6 shows that there is only one such involution, so κ must be the same as κ^*

Above we have established formulae for v_0^*, v_3^*, v_6^* and v_{11}^* in terms of b. Propositions 4.5.4 and 4.5.7 also give

$$\begin{array}{llll} v_1^* = \kappa(v_0^*) & v_2^* = \kappa(v_3^*) & v_4^* = \lambda(v_3^*) & v_5^* = \nu(v_3^*) \\ v_7^* = \lambda(v_6^*) & v_8^* = \lambda^2(v_6^*) & v_9^* = \lambda^3(v_6^*) & \\ v_{10}^* = \lambda(v_{11}) & v_{12}^* = \kappa(v_{11}) & v_{13}^* = \lambda\nu(v_{12}). & \end{array}$$

Using these we can deduce the stated formulae for all points v_i^*. □

From now on we use the value of b coming from the previous Proposition. This gives maps $\lambda, \mu, \nu, \beta_0, \ldots, \beta_7$ generating a group Π as in Section 4. The maps λ and ν from that section are of course the same as the ones we have been using already in this section.

Definition 4.5.11. [defn-tPhi]

We say that a conformal or anticonformal automorphism $\phi\colon \Delta \to \Delta$ is *G-compatible* if there is an element $\phi_1 \in G$ such that the following diagram commutes:

$$\begin{tikzcd} \Delta \arrow[r, "\phi"] \arrow[d, "p"'] & \Delta \arrow[d, "p"] \\ X \arrow[r, "\phi_1"'] & X. \end{tikzcd}$$

We write $\widetilde{\Phi}$ for the group of all G-compatible automorphisms, and note that the construction $\phi \mapsto \phi_1$ gives a homomorphism $\widetilde{\Phi} \to G$. We write Φ for the kernel, which is just the group of automorphisms ϕ satisfying $p\phi = p$, or in other words deck transformations. Standard covering theory shows that p induces a conformal isomorphism $\Delta/\Phi \to PX(a)$.

Proposition 4.5.12. [prop-tPhi-gens]

The maps λ, μ, ν and κ are elements of $\widetilde{\Phi}$, with $\pi(\lambda) = \lambda$ and $\pi(\mu) = \mu$ and $\pi(\nu) = \nu$ and $\pi(\kappa) = \lambda\mu$.

Proof. The claims for λ, ν and κ are clear by construction. By the same argument that we used in Proposition 4.5.7, if we let μ' denote the unique holomorphic involution on Δ that fixes v_2, then $p\mu' = \mu p$, so $\mu' \in \widetilde{\Phi}$ with $\pi(\mu') = \mu$. However, if we define

$$\mu(z) = \frac{b_+ z - b^2 - i}{(b^2 - i)z - b_+}$$

as in Section 4, then straightforward calculation shows that μ is a holomorphic involution with $\mu(v_2^*) = v_2^*$, so μ is the same as μ'. $\square$

Proposition 4.5.13. [prop-Pi-in-Phi]
We have $\Pi \leq \Phi$ and $\widetilde{\Pi} \leq \widetilde{\Phi}$.

Proof. We first claim that $\beta_0 \in \Phi$. Note that $\lambda^2(v_{11}) = v_{11}$ in $PX(a)$, so the points $v_{11}^* = b_+ - b$ and $\lambda^2(v_{11}^*) = b - b_+$ have the same image under p. Recall that $\beta_0(z) = (b_+ z + 1)/(z + b_+)$; this implies that $\beta_0(\lambda^2(v_{11}^*)) = v_{11}^*$, and that β_0 restricts to give a strictly increasing automorphism of $(-1, 1)$.

We originally introduced p as the unique holomorphic covering map $\Delta \to PX(a)$ such that $p(0) = v_0$ and $p'(0)$ is a positive multiple of $c_5'(0)$. However, the same line of argument shows that p is also the unique holomorphic covering map $\Delta \to PX(a)$ such that $p(\lambda^2(v_{11}^*)) = v_{11}$ and $p'(\lambda^2(v_{11}^*))$ is a positive multiple of $c_5'(-\pi)$. The composite $p\beta_0$ has these properties, so $p\beta_0 = p$, so $\beta_0 \in \Phi$ as claimed.

It is also clear that Φ is normal in $\widetilde{\Phi}$ and $\lambda \in \widetilde{\Phi}$ so the conjugates $\beta_{2k} = \lambda^k \beta_0 \lambda^{-k}$ also lie in Φ.

We now recall that the elements $\lambda, \mu \in \widetilde{\Pi}$ lie in $\widetilde{\Phi}$ and satisfy $(\lambda\mu)^2 = \beta_7\beta_6$. As $(\lambda\mu)^2 = 1$ in G we can deduce that $\beta_7\beta_6 \in \Phi$. We saw above that $\beta_6 \in \Phi$, so $\beta_7 \in \Phi$. Using $\lambda^k \beta_7 \lambda^{-k} = \beta_{7+2k}$ we deduce that $\beta_j \in \Phi$ for all j, so $\Pi \leq \Phi$. As $\lambda, \mu, \nu \in \widetilde{\Phi}$ it also follows that $\widetilde{\Pi} \leq \widetilde{\Phi}$. $\square$

Proposition 4.5.14. [prop-H-universal]
We have $\Pi = \Phi$ and $\widetilde{\Pi} = \widetilde{\Phi}$, and the map $p\colon \Delta \to PX(a)$ induces an isomorphism $\overline{p}\colon HX(b) = \Delta/\Pi \to PX(a)$ of cromulent surfaces.

Proof. As $\Pi \leq \Phi$, we can factor the map p as

$$\Delta \xrightarrow{q} HX(b) = \Delta/\Pi \xrightarrow{\overline{p}} \Delta/\Phi \simeq PX(a).$$

As p and q are holomorphic coverings, we see that $\overline{p}$ is also a holomorphic covering. We have also seen that both $HX(b)$ and $PX(a)$ are compact, so $\overline{p}$ has degree $d < \infty$ say. It follows (by choosing compatible triangulations, say) that $\chi(HX(b)) = d\ \chi(PX(a))$ (where χ denotes the Euler characteristic). However, both $HX(b)$ and $PX(a)$ have genus $g = 2$ and therefore Euler characteristic $2 - 2g = -2$, so we must have $d = 1$, so $\overline{p}$ is an isomorphism of Riemann surfaces. By construction it is G-equivariant and sends v_i to v_i so it is an isomorphism of cromulent surfaces. $\square$

5. Relating the projective and hyperbolic families

[sec-P-H]

Recall that the projective and algebraic families are both universal, so for each $a \in (0, 1)$ there is a unique $b \in (0, 1)$ such that $PX(a)$ is isomorphic (in a unique way) to $HX(b)$, and *vice-versa*. In this section we will give two different methods for calculating this correspondence. The first method starts with b and calculates a, and the second works in the opposite direction.

5.1. Preliminaries. [sec-P-H-prelim]

The space $HX(b)$ is by definition a quotient of Δ. We have an isomorphism $HX(b) \to PX(a)$, and an isomorphism $PX(a)/\langle\lambda^2\rangle \to \mathbb{C}_\infty$ sending $j(w, z)$ to z. Composing these maps gives a map $p\colon \Delta \to \mathbb{C}_\infty$. Our main task is to calculate this map.

In Definition 4.2.3 we defined certain points $v_i \in \Delta$ (depending on b), which become the labelled points in $HX(b)$. In this section, we will write v_{Hi} for these points. Similarly, we will write c_{Hj} for the curves $\widetilde{c}_j\colon \mathbb{R} \to \Delta$ defined in Definition 4.3.1. Moreover, the images in $\mathbb{C}_\infty$ of the points $v_i \in PX(a)$ and the curves

$c_j\colon \mathbb{R} \to P(a)$ will be denoted by v_{Ci} and c_{Cj}. We are primarily interested in the corners of the fundamental domain, which we can tabulate as follows:

$$
\begin{aligned}
v_{H0} &= 0 & v_{C0} &= 0 \\
v_{H3} &= \frac{b\,b_- - b_+}{ib^2 - 1} & v_{C3} &= 1 \\
v_{H6} &= \frac{1+i}{\sqrt{2}}\,\frac{\sqrt{2} - b_-}{b_+} & v_{C6} &= i \\
v_{H11} &= b_+ - b & v_{C11} &= a.
\end{aligned}
$$

Because the map $HX(b) \to PX(a)$ is cromulent, we have $p(v_{Hi}) = v_{Ci}$ and $p(c_{Hj}(\mathbb{R})) = c_{Cj}(\mathbb{R})$. In particular, we have $v_{H0} = 0$ and $v_{C0} = 0$, so $p(0) = 0$. Using equivariance with respect to λ, we also see that $p(iz) = -p(z)$, so $p(-z) = p(z)$, and it follows that $p'(0) = 0$. This makes it inconvenient to work with p itself. Instead, we will work with a certain map of the form $p_1 = \phi p \psi$, where $\phi \in \mathrm{Aut}(\mathbb{C}_\infty)$ and $\psi \in \mathrm{Aut}(\Delta)$. This will be arranged so that $p_1(0) = 0$ and $p_1'(0) > 0$. Details are as follows:

Definition 5.1.1. [defn-schwarz-phi]
We define $\phi \in \mathrm{Aut}(\mathbb{C}_\infty)$ and $\psi \in \mathrm{Aut}(\Delta)$ by

$$
\begin{aligned}
\phi(z) &= \frac{i - z}{i + z} & \phi^{-1}(z) &= i\frac{1 - z}{1 + z} \\
\psi(z) &= \frac{1+i}{\sqrt{2}}\,\frac{\sqrt{2} - b_- - b_+ z}{b_+ (b_- - \sqrt{2}) z} & \psi^{-1}(z) &= -\frac{\sqrt{2} - b_- - (1-i) b_+ z/\sqrt{2}}{(1-i)(1 - b_-/\sqrt{2}) z - b_+}.
\end{aligned}
$$

We then define $p_1 = \phi \circ p \circ \psi\colon \Delta \to \mathbb{C}_\infty$. We also put $v_{HSi} = \psi^{-1}(v_{Hi}) \in \Delta$ and $v_{PSi} = \phi(v_{Ci}) \in \mathbb{C}_\infty$, so that $p_1(v_{HSi}) = v_{PSi}$. We define curves c_{HSj} and c_{PSj} in the same way. Finally, we put

$$
t = b^2 \frac{\sqrt{2} b_+ - 2 b b_-}{1 - b^2 + 2b^4} \qquad s = \frac{(b_+ b_- - \sqrt{2})(b - b^3)}{1 - b^2 + 2b^4}.
$$

Remark 5.1.2. In Maple, the maps ϕ and ψ are `schwarz_phi` and `schwarz_psi`, and the inverse maps are `schwarz_phi_inv` and `schwarz_psi_inv`. Maple notation for v_{HSi} and $c_{HSj}(t)$ is `v_HS[i]` and `c_HS[j](t)`, and similarly for v_{PSi} and $c_{PSj}(t)$. Maple notation for t and s is `t_schwarz` and `s_schwarz`. All of this is in `hyperbolic/schwarz.mpl`.

By direct calculation, we have

$$
\begin{aligned}
v_{HS0} &= \frac{\sqrt{2} - b_-}{b_+} & v_{PS0} &= 1 \\
v_{HS3} &= i\frac{b_-}{b_+ + \sqrt{2} b} & v_{PS3} &= i \\
v_{HS6} &= 0 & v_{PS6} &= 0 \\
v_{HS11} &= t + is & v_{PS11} &= \frac{i - a}{i + a}
\end{aligned}
$$

(where t and s are as in Definition 5.1.1).

`hyperbolic/schwarz_check.mpl: check_schwarz()`

Lemma 5.1.3. [lem-psi-edges]
ψ^{-1} acts as follows on the edges of HF_{16}:

- $\psi^{-1}(C_0) = (-1, 1).i$
- $\psi^{-1}(C_1) = (-1, 1)$
- *$\psi^{-1}(C_3)$ is the intersection of Δ with the circle of radius $\sqrt{2}b/b_-$ centred at ib_+/b_-*
- *$\psi^{-1}(C_5)$ is the intersection of Δ with the circle of radius $\sqrt{2}b_-/b_+$ centred at $(\sqrt{2} + b_-)/b_+$.*

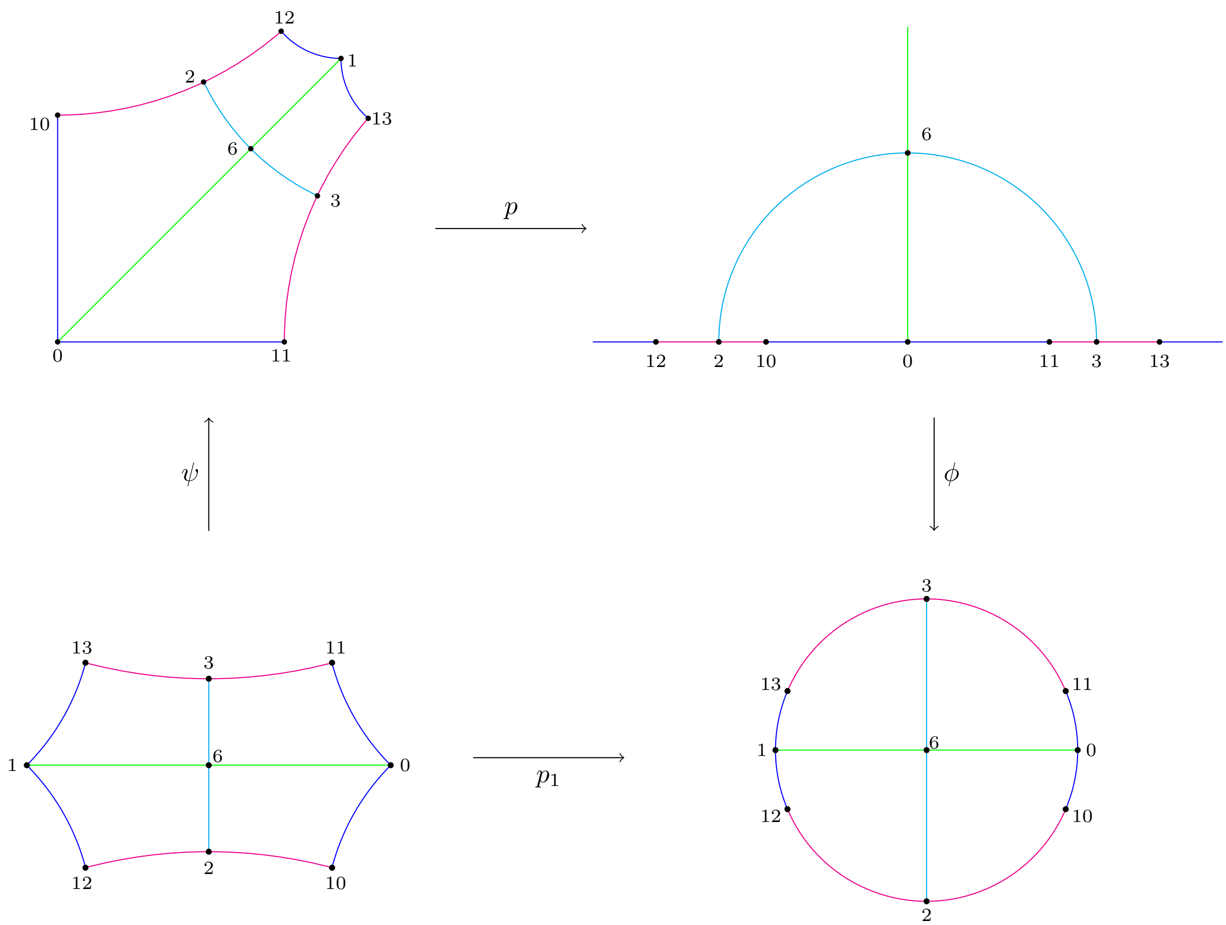
Proof. Let A_k denote the set that is claimed to be equal to $\psi^{-1}(C_k)$. In each case, it is easy to check that A_k is a geodesic. We also know that C_k is a geodesic and ψ is an isometry, so $\psi^{-1}(C_k)$ is also a geodesic.

Because of this, it will suffice to check that $|A_k \cap \psi^{-1}(C_k)| \geq 2$, which we can do by considering the points $\{\psi^{-1}(v_i) \mid i \in \{0, 3, 6, 11\}\}$.

`hyperbolic/schwarz_check.mpl: check_schwarz()`

□

We can illustrate the maps p, p_1, ϕ and ψ as follows.



Lemma 5.1.4. [lem-p-one-props]

$p_1(-z) = -p_1(z)$ and $p_1(\overline{z}) = \overline{p_1(z)}$. Thus, $p_1(z)$ has a Taylor series $\sum_i a_i z^{2i+1}$ with $a_i \in \mathbb{R}$.

Proof. We know that p is $\widetilde{\Pi}$-equivariant, so $p\beta_0\lambda\mu = \lambda\mu p$. Thus, if we put $\pi = \psi^{-1}\beta_0\lambda\mu\psi \in \text{Aut}(\Delta)$ and $\pi' = \phi\lambda\mu\phi^{-1} \in \text{Aut}(\mathbb{C}_\infty)$, we have $p_1\pi = \pi' p_1$. Direct calculation shows that $\beta_0\lambda\mu$ is the holomorphic involution fixing v_{H6}, so π is the holomorphic involution fixing $v_{HS6} = 0$, or in other words $\pi(z) = -z$. Direct calculation also gives $\pi'(z) = -z$, so $p_1(-z) = -p_1(z)$ as claimed.

Similarly, we have $p\lambda\nu = \lambda\nu p$. Thus, if we put $\xi = \psi^{-1}\lambda\nu\psi \in \text{Aut}(\Delta)$ and $\xi' = \phi\lambda\nu\phi^{-1} \in \text{Aut}(\mathbb{C}_\infty)$, we have $p_1\xi = \xi' p_1$. Here $\lambda\nu$ is the antiholomorphic involution of Δ that fixes v_{H0} and v_{H6}, so ξ is the antiholomorphic involution of Δ that fixes v_{HS0} and v_{HS6}, which gives $\xi(z) = \overline{z}$. Direct calculation also gives $\xi'(z) = \overline{z}$, so $p_1(\overline{z}) = \overline{p_1(z)}$ as claimed. □

Lemma 5.1.5. [lem-p-one-poles]
The set of poles of p_1 is $\psi^{-1}(\Pi.\{v_{H1}, v_{H9}\})$, and all these poles are simple. Moreover, the points $\pm ib_-/b_+$ are poles, and the corresponding residues are equal and are real.

Proof. We are interested in the preimage of ∞ under the composite

$$\Delta \xrightarrow[\simeq]{m_1=\psi} \Delta \xrightarrow{m_2} \Delta/\Pi = HX(b) \xrightarrow[\simeq]{m_3} PX(a) \xrightarrow{m_4} \mathbb{C}_\infty \xrightarrow[\simeq]{m_5=\phi} \mathbb{C}_\infty.$$

First, we have $m_5^{-1}\{\infty\} = \{\phi^{-1}(\infty)\} = \{-i\}$. The map $m_4 \colon PX(a) \to \mathbb{C}_\infty$ induces a bijection

$$PX(a)/\langle\lambda^2\rangle \to \mathbb{C}_\infty,$$

and it sends both v_{P7} and v_{P9} to $-i$, so $m_4^{-1}\{-i\} = \{v_{P7}, v_{P9}\}$. As m_3 is a cromulent isomorphism, it follows that

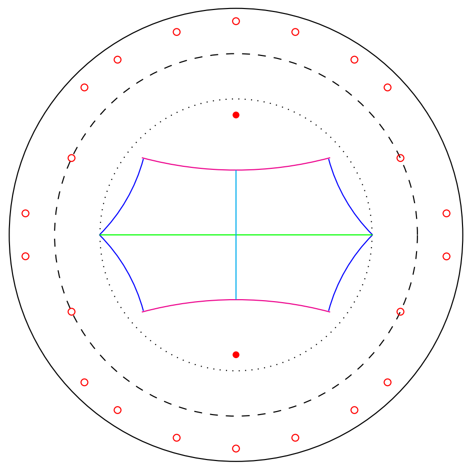
$$(m_3 m_2)^{-1}\{v_{P7}, v_{P9}\} = \Pi\{v_{H7}, v_{H9}\}.$$

It follows easily that the set of poles is as claimed. The points v_7 and v_9 in $PX(a)$ are not fixed by λ^2, so they are not branch points for the map $PX(a) \to \mathbb{C}_\infty$; it follows that all the poles are simple. Now put $\alpha = b_-/b_+ \in \mathbb{R}$. After unwinding the definitions and performing some algebraic simplification we find that $\psi^{-1}\beta_0(v_{H7}) = i\alpha$ and $\psi^{-1}\beta_2(v_{H9}) = -i\alpha$, which shows that the points $\pm i\alpha$ are poles. This means that there are constants $r_1, r_2 \in \mathbb{C}$ and a meromorphic function $q(z)$ on Δ such that q is holomorphic at $\pm i\alpha$ and

$$p_1(z) = \frac{r_1}{z - i\alpha} + \frac{r_2}{z + i\alpha} + q(z).$$

Moreover, the triple $(r_1, r_2, q(z))$ is characterised uniquely by these properties. Next, recall that $p_1(z) = -p_1(-z)$. This shows that $(r_1, r_2, q(z)) = (r_2, r_1, -q(-z))$. Similarly, the fact that $p_1(z) = \overline{p_1(\overline{z})}$ shows that $(r_1, r_2, q(z)) = (\overline{r_2}, \overline{r_1}, \overline{q(\overline{z})})$. This means that $r_1 = r_2 \in \mathbb{R}$ as claimed. $\square$

In the case $b = 0.75$, the poles can be illustrated as shown below. The inner dotted circle (with radius 0.6) is the smallest circle centred at the origin that contains $\psi^{-1}(HF_4)$. The only two poles inside this circle are $\pm ib_-/b_+$; these are shown as solid red dots. A further 22 poles are also shown; they are on or outside the outer dashed circle, which has radius 0.8. All remaining poles are even closer to the unit circle.



Lemma 5.1.6. [lem-p-one-branches]
$p_1'(v_{HS0}) = p_1'(v_{HS11}) = 0.$

Proof. This follows from the fact that the map $HX(b) \to PX(a) \to \mathbb{C}_\infty$ is λ^2-invariant, and v_0 and v_{11} are fixed by λ^2 in $HX(b)$. □

5.2. **Finding a from b.** [sec-a-from-b]

In this section, we describe an algorithm to calculate a from b. This algorithm is implemented by the methods of the class `H_to_P_map`, which is defined in the file `hyperbolic/H_to_P.mpl`. In more detail, if we want to take $b = 0.75$, we can enter

```
HP := `new/H_to_P_map`():
HP["set_a_H",0.75]:
HP["make_samples"]:
HP["find_p1"]:
HP["a_P"];
HP["p1"](z);
HP["err"];
```

The `find_p1` method takes about 22 seconds on a fairly capable PC. The line `HP["a_P"]` returns the value of a, which is about 0.1816. The line `HP["p1"](z)` returns a rational function of z, with poles only at the points $\pm ib_-/b_+$ mentioned in Lemma 5.1.5. When restricted to $\psi^{-1}(HF_4)$, this is a good approximation to $p_1(z)$. The line `HP["err"]` returns a measure of the quality of approximation, which is about 7×10^{-11} in this case. It could be improved by increasing the degree of polynomials and the number of sample points used in the algorithm; the code has options for this. The code also has methods to generate various different visualisations of the behaviour of p_1, and to analyse the errors in more detailed ways.

If one wants to perform the above calculation for several different values of b, and to compare the results with those obtained by the method of Section 5.4, then we can instead use the class `HP_table`, defined in `hyperbolic/HP_table.mpl`. For example, we can enter the following:

```
HPT := `new/HP_table`():
HPT["add_a_H",0.75];
HPT["add_a_H",0.76];
HPT["add_a_H",0.77];
```

This will perform the above calculation for the valuse $b = 0.75$, $b = 0.76$ and $b = 0.77$. The object of class `H_to_P_map` for $b = 0.75$ can then be retrieved as `HPT["H_to_P_maps"][0.75]`. After calcuating a sufficient range of values of b, one can enter `HPT["set_spline"]` and then `HPT["full_plot"]` to generate a plot of a against b.

Alternatively (as discussed in Section 9.4), one can read the file `build_data.mpl` and execute `build_data["HP_table` to perform all calculations for $b = 0.06$ to $b = 0.94$ in steps of 0.02, and to do various other related work. Here one may wish to enter `infolevel[genus2] := 7;` before starting the calculation; this will instruct Maple to print various progress reports as it proceeds.

Lemma 5.2.1. *There is a unique sequence of polynomials $p_{10}(z), p_{11}(z), p_{12}(z)$ and $p_{14}(z)$, such that:*

(a) *All the polynomials are odd, with real coefficients.*
(b) *The polynomials $p_{10}(z)$ to $p_{12}(z)$ have degree 13.*
(c) *The polynomial $p_{14}(z)$ has degree 15, and has the form $p_{14}(z) = z + O(z^3)$.*
(d) *For all k we have $p'_{1k}(v_{HS0}) = p'_{1k}(v_{HS11}) = 0$.*
(e) *Values at v_{HS0}, v_{HS3} and v_{HS11} are as follows:*

	v_0	v_3	v_{11}
p_{10}	1	i	0
p_{11}	0	0	1
p_{12}	0	0	i
p_{14}	0	0	0

Proof. Put $\alpha = v_{HS0} \in \mathbb{R}$ and $\beta = v_{HS3}/i \in \mathbb{R}$ and $\gamma = v_{HS11}$. Let F denote the space of all odd polynomials of degree at most 13, and note that this has dimension 7. Note also that for $f \in F$ we automatically have $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(i\mathbb{R}) \subseteq i\mathbb{R}$ and $f'(i\mathbb{R}) \subseteq \mathbb{R}$, and the roots and their multiplicities are invariant under the maps $z \mapsto -z$ and $z \mapsto \overline{z}$. Using this, we see that the only possibility for $p_{14}(z)$ is the polynomial

$$p_{14}(z) = z(1 - z^2/\alpha^2)^2(1 + z^2/\beta^2)(1 - z^2/\gamma^2)^2(1 - z^2/\overline{\gamma}^2)^2.$$

Next, we can define $\epsilon\colon F \to \mathbb{R}^7$ by

$$\epsilon(f) = (f(\alpha),\ f(i\beta)/i,\ \mathrm{Re}(f(\gamma)),\ \mathrm{Im}(f(\gamma)),\ f'(\alpha), \mathrm{Re}(f'(\gamma)),\ \mathrm{Im}(f'(\gamma))).$$

If $\epsilon(f) = 0$ then f must be divisible by $p_{14}(z)$, but we can then compare degrees to see that $f = 0$. This means that ϵ is an injective linear map between spaces of dimension 7, so it is an isomorphism. The polynomials $p_{10}(z)$, $p_{11}(z)$ and $p_{12}(z)$ can be obtained by applying ϵ^{-1} to suitable vectors in $\mathbb{R}^7$. □

Definition 5.2.2. We put

$$p_{15}(z) = (z - ib_-/b_+)(z + ib_-/b_+) = z^2 + \frac{1 - b^2}{1 + b^2},$$

and then $p_{13}(z) = p_{14}(z)/p_{15}(z)$. Then, given $a \in \mathbb{R}^d$, we put

$$P(a)(z) = p_{10}(z) + \sum_{i=1}^{3} a_i p_{1i}(z) + p_{14}(z) \sum_{i=4}^{d} a_i z^{2(i-4)}.$$

We now choose $\alpha \in (0,1)$ which we believe is a reasonable approximation to the required value of a. It would not be too harmful to just take $\alpha = 1/2$. Alternatively, the code defines a polynomial `f(t)=schwarz_b_approx(t)` of degree ten, which is a good approximation to the function $a \mapsto b$; we can thus find α by solving $f(\alpha) = b$ numerically. We then put $z_0 = (i-\alpha)/(i+\alpha)$, which is the value of v_{PS11} corresponding to $a = \alpha$.

Now consider the map $p_1(z)$. Let a_1 and a_2 be the real and imaginary parts of $p_1(v_{HS11})$, and let a_3 be the unique real constant such $p_1(z) - a_3 p_{13}(z)$ has residue zero at ib_-/b_+. We then find that the function $p_1(z) - p_{10}(z) - \sum_{i=1}^{3} a_i p_{1i}(z)$ is holomorphic on a disc Δ' centred at 0 that includes all of HF_4. Moreover, it is odd, with real Taylor coefficients. By considering it order of vanishing at the various points v_{HSj}, we see that is the product of $p_{14}(z)$ with an even function that is also holomorphic on Δ'. This means that when d is sufficiently large, $p_1(z)$ can be well approximated by $P(a)(z)$ for some $a \in \mathbb{R}^d$. To find a, we note that

$$p_1(C_{HS3} \cup C_{HS5}) = C_{PS3} \cup C_{PS5} = S^1.$$

We therefore choose a reasonably large number n (say $n = 200$) and a list of closely spaced points $s = (s_1, \dots, s_n)$ lying in $\psi^{-1}(HF_4 \cap (C_{H3} \cup C_{H5}))$. We then define $\eta\colon \mathbb{R}^d \to \mathbb{R}^n$ by

$$\eta(a)_j = |P(a)(s_j)|^2 - 1.$$

It is not hard to see that this has the form

$$\eta(a)_j = |(Ma + c)_j|^2 - 1$$

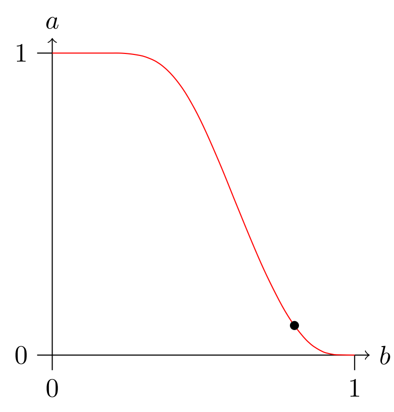
for some matrix $M \in M_{nd}(\mathbb{C})$ and some vector $c \in \mathbb{C}^n$ which can be precomputed. Using this, we get

$$\frac{\partial}{\partial a_k} \eta(a)_j = 2\mathrm{Re}(\overline{M_{jk}}(Ma + c)_j).$$

This makes it easy to minimise $\|\eta(a)\|^2$ by an iterative process.

In the case $b \simeq 0.80053190489236$ which is relevant for uniformising EX^*, we have taken $d = 50$ and $n = 300$, and have ended up with errors $|P(a)(s_j)|^2 - 1$ that are less than 10^{-23}.

Calculations using the above method give the following graph of a as a function of b (with the marked point representing EX^*):



The middle section of this graph can be obtained by the methods of this section, or those of Section 5.4. However, the methods of Section 5.4 behave poorly when (a,b) approaches $(1,0)$.

An optimist might hope for an explicit formula for the above graph, perhaps involving the elliptic integrals from Section 3, the constants s_k in Definition 4.3.1, or other standard special functions or quantities found elsewhere in this document. We have performed a fairly extensive experimental search for such formulae, but without success. Of course one can find polynomials of high degree fitting the graphs to any desired accuracy, but the answers are not illuminating. It might be helpful if we had a meaningful interpretation of the endpoints $(a,b) = (1,0)$ and $(a,b) = (0,1)$, perhaps in terms of a Deligne-Mumford compactification of an appropriate moduli space of stable curves, but we have not investigated this seriously.

5.3. **Recollections on the Schwarzian derivative.** [sec-schwarz]

We now start to discuss our second method, where we are given $a \in (0,1)$ and we try to find $b \in (0,1)$ such that $PX(a) \simeq HX(b)$. This method is based on the Schwarzian derivative, whose definition and properties we now recall.

Definition 5.3.1. Let f be a nonconstant meromorphic function on a connected domain $U \subseteq \mathbb{C}$. The Schwarzian derivative $S(f)$ is defined by

$$S(f) = (f''/f')' - \tfrac{1}{2}(f''/f')^2 = f'''/f' - \tfrac{3}{2}(f''/f')^2.$$

Proposition 5.3.2. [prop-vanishing-schwarzian]

We have $S(f) = 0$ if and only if there are constants $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ and $f(z) = (az + b)/(cz + d)$ on U (or in other words, f is a Möbius function).

Proof. This is standard. One can check by direct calculation that functions $f(z) = (az + b)/(cz + d)$ have $S(f) = 0$. Conversely, suppose that f is meromorphic on U with $S(f) = 0$. Then the function $g = f''/f'$ satisfies $g' = g^2/2$. If $g = 0$ then we deduce that $f'' = 0$ so f has the form $f(z) = az + b$ for some a and b, as required. Otherwise, we can consider the meromorphic function $1/g$ and we deduce that $(1/g)' = -g'/g^2 = -1/2$, which gives $g(z) = -2/(z + \delta)$ for some constant δ, or in other words $(z + \delta)f''(z) = -2f'(z)$. From this it follows that the function $(z + \delta)^2 f'(z)$ has zero derivative and so is constant. This gives that $f'(z) = \gamma/(z + \delta)^2$ and then $f(z) = \beta - \gamma/(z + \delta)$ for some constants β and δ, and this can evidently be rewritten in the form $(az + b)/(cz + d)$. □

Proposition 5.3.3. *Given meromorphic functions $V \xrightarrow{g} U \xrightarrow{f} \mathbb{C}$ we have*

$$S(f \circ g) = (S(f) \circ g) \cdot (g')^2 + S(g).$$

(We call this the Schwarzian chain rule.*) In particular, if f is a Möbius function then $S(f \circ g) = S(g)$.*

Proof. This is also standard. We can use the ordinary chain rule repeatedly to express the first three derivatives of $f \circ g$ in terms of those of f and g, and the rest is pure algebra. □

We next recall the relationship between the Schwarzian derivative and certain types of second order linear differential equations. Again, almost all of this is well-known, but we give a self-contained account to serve as a convenient basis for discussing some additional points that are less standard.

Proposition 5.3.4. [prop-schwarzian-solutions]

Let U be a simply connected open subset of $\mathbb{C}$, and let s be a holomorphic function on U. Put

$$F = \{ \text{ meromorphic functions } f \text{ on } U \text{ such that } f'' + \tfrac{1}{2}sf = 0\}$$
$$G = \{ \text{ nonconstant meromorphic functions } g \text{ on } U \text{ such that } S(g) = s\}.$$

Then

(a) *Every function in F is actually holomorphic.*
(b) *For any $z_0 \in U$ and any $u_0, u_1 \in \mathbb{C}$ there is a unique function $f \in F$ such that $f(z) = u_0 + u_1(z - z_0) + O((z - z_0)^2)$. In particular, F has dimension two over $\mathbb{C}$.*
(c) *G is precisely the set of functions of the form f_1/f_0, where f_0 and f_1 are linearly independent elements of F.*
(d) *For any $z_0 \in U$ and any $v_0, v_1, v_2 \in \mathbb{C}$ with $v_1 \neq 0$ there is a unique function $g \in G$ such that $g(z) = v_0 + v_1(z - z_0) + v_2(z - z_0)^2 + O((z - z_0)^3)$.*

Proof. (a) This is Lemma 5.3.5.
(b) This is Corollary 5.3.7.
(c) Combine Lemma 5.3.8 and Corollary 5.3.10.

□

Lemma 5.3.5. [lem-F-holomorphic]

Every function in F is holomorphic.

Proof. Consider an element $f \in F$ and a point $z_0 \in U$. If f is not holomorphic at z_0, then it must have a pole of order $d > 0$ at z_0, so f'' has a pole of order $d+2$, whereas $\frac{1}{2}sf$ has a pole of order at most d (or is holomorphic). This contradicts the equation $f'' + \frac{1}{2}sf = 0$. $\square$

Lemma 5.3.6. [`lem-F-disc`]
Let U be the open disc centred at z_0 with radius $r > 0$, and let s be a holomorphic function on U. Define F and G as above. Then for any $u_0, u_1 \in \mathbb{C}$ there is a unique function $f \in F$ such that $f(z) = u_0 + u_1(z - z_0) + O((z - z_0)^2)$. In particular, F has dimension two over $\mathbb{C}$.

Proof. We can expand s as a power series, say $s(z) = \sum_k s_k(z - z_0)^k$. Put $a_0 = u_0$ and $a_1 = u_1$, then define a_k recursively for $k > 1$ by

$$a_k = \frac{-1}{2k(k-1)} \sum_{j=0}^{k-2} a_j s_{k-2-j}.$$

It is then easy to see that the formal power series $f_0(z) = \sum_k a_k(z-z_0)^k$ is the unique one with $f_0''+\frac{1}{2}sf_0 = 0$ and $f_0(z) = u_0 + u_1(z - z_0) + O((z - z_0)^2)$. Moreover, the coefficients a_k grow at a rate comparable to that of the coefficients s_k, so they have the same radius of convergence. Thus, the above expression defines a holomorphic function on U. $\square$

Corollary 5.3.7. [`cor-F-disc`]
Claim (b) in Proposition 5.3.4 holds for an arbitrary simply connected domain U.

Proof. This follows by analytic continuation. In more detail, for any $z \in U$ we can choose a path γ from z_0 to z in U. We can then choose closely spaced points $\gamma(t_i)$ and radii $r_i > 0$ such that $t_0 = 1$ and $t_N = 1$ and the disc U_i of radius r_i centred at $\gamma(t_i)$ is contained in U and contains $\gamma(t_{i+1})$. We then let f_0 be the unique holomorphic function on U_0 with $f_0'' + \frac{1}{2}sf_0 = 0$ and $f_0(z) = u_0 + u_1(z - z_0) + O((z - z_0)^2)$. Using this as a starting point, we let f_k denote the unique holomorphic function on U_k that satisfies $f_k'' + \frac{1}{2}sf_k = 0$ and agrees with f_{k-1} to second order at $\gamma(t_k)$. We then define $f(z) = f_N(z)$. It is easy to check that this does not depend on the precise choice of points t_i, nor does it change if we move γ by a small homotopy fixing the endpoints. As U is simply connected any two paths from z_0 to z are homotopic relative to endpoints, and any homotopy can be broken down into small homotopies. It follows that $f(z)$ is independent of all choices, and it defines an element of F with the required behaviour near z_0. Any other such element will agree with f at least on U_0, but then it must agree everywhere on U by analytic continuation. $\square$

Lemma 5.3.8. [`lem-quotient-schwarzian`]
Suppose that f_0 and f_1 are linearly independent elements of F, so the quotient $g = f_1/f_0$ is a nonconstant meromorphic function. Then $S(g) = s$.

Proof. Put $W = f_0f_1' - f_0'f_1$ (the Wronskian of f_0 and f_1). This is easily seen to satisfy $W' = 0$, so it is constant. The quotient $f = f_1/f_0$ has $f' = W/f_0^2$, so $f'' = -2Wf_0'/f_0^3$, so $f''/f' = -2f_0'/f_0$. From this we find that $S(f) = s$ as claimed. $\square$

Lemma 5.3.9. [`lem-G-transitive`]
If $g, h \in G$ then $g = m \circ h$ for some Möbius function m.

Proof. As h is nonconstant and meromorphic, we can choose a small disc $V \subseteq U$ such that h is holomorphic on V and h' is nonzero everywhere in V. After shrinking V if necessary, we can then assume that the map $h\colon V \to h(V)$ is a conformal isomorphism. Put $m = g \circ h^{-1}$, which is meromorphic on $h(V)$ and satisfies $g = m \circ h$. The Schwarzian chain rule gives $S(g) = (S(m) \circ h) \cdot (h')^2 + S(h)$ on V. However, $S(g) = S(h) = s$ and h' is nowhere zero, so $S(m) \circ h = 0$ on V, so $S(m) = 0$ on $h(V)$. It follows that m is a Möbius function. The functions g and $m \circ h$ are both meromorphic and they agree on a disc so they must agree everywhere in U. $\square$

Corollary 5.3.10. [`cor-quotient-schwarzian`]
Every element $g \in G$ can be written as $g = f_1/f_0$ for some linearly independent pair of elements $f_0, f_1 \in F$.

Proof. Let e_0 and e_1 be any basis for F, and put $h = e_1/e_0$. Lemma 5.3.8 tells us that $h \in G$, so Lemma 5.3.9 tells us that $g = (ah + b)/(ch + d)$ for some a, b, c, d with $ad - bc \neq 0$. This means that the functions $f_1 = ae_1 + be_0$ and $f_0 = ce_1 + de_0$ are linearly independent elements of F with $g = f_1/f_0$. $\square$

Lemma 5.3.11. [lem-mobius-straighten]

Suppose that f is holomorphic at z_0, with $f'(z_0) \neq 0$. Then there is a unique Möbius function m such that $m(f(z)) = z - z_0 + O((z - z_0)^3)$. Specifically, if

$$f(z) = u_0 + u_1(z - z_0) + u_2(z - z_0)^2 + O((z - z_0)^3)$$

then

$$m(z) = \frac{u_1(z - u_0)}{u_2(z - u_0) + u_1^2}.$$

Proof. If we define $m(z)$ as above, then it is straightforward to check that $m(f(z)) = z - z_0 + O((z - z_0)^3)$. If n is another Möbius function with $n(f(z)) = z - z_0 + O((z - z_0)^3)$ then the function $k = n \circ m^{-1}$ must have the form $k(z) = (az + b)/(cz + d)$ for some a, b, c, d, but also $k(z) = z + O(z^3)$. In particular, we have $k(0) = 0$, which gives $b = 0$. We then have $k'(0) = 1$, which gives $a = d$. After cancelling we may assume that $a = d = 1$. Finally, we have $k''(0) = 0$, so $c = 0$, so k is the identity as required. $\square$

Corollary 5.3.12. [cor-mobius-straighten]

For any $z_0 \in U$ and any $v_0, v_1, v_2 \in \mathbb{C}$ with $v_1 \neq 0$ there is a unique function $g \in G$ such that $g(z) = v_0 + v_1(z - z_0) + v_2(z - z_0)^2 + O((z - z_0)^3)$.

Proof. It will be harmless, and notationally convenient, to assume that $z_0 = 0$. By Corollary 5.3.7 we can choose $f_0, f_1 \in F$ with $f_0(z) = 1 + O(z^2)$ and $f_1(z) = z + O(z^2)$. The function $g_0 = f_1/f_0$ now lies in G and has $g_0'(0) \neq 0$, so we can find a Möbius function m_1 such that the function $g_1(z) = m_1(g_0(z))$ satisfies $g_1(z) = z + O(z^3)$. We then put

$$m_2(z) = (v_0v_1 - (v_1^2 - v_0v_2)z)/(v_1 - v_2z)$$

and $g(z) = m_2(g_1(z))$. $\square$

The next result refers to circles in $\mathbb{C}_\infty$. Here we regard straight lines as circles of infinite radius. With this convention, it is well-known that Möbius functions send circles to circles.

Proposition 5.3.13. [prop-image-circle]

Suppose that a real interval (a, b) is contained in U, and that $f'(t) \neq 0$ for all $t \in (a, b)$, so $S(f)$ is holomorphic on (a, b). Then the following are equivalent:

(a) *$f((a, b))$ is contained in some circle $C \subset \mathbb{C}_\infty$.*
(b) *$S(f)$ is real on (a, b)*

Proof. Suppose that (a) holds. We can choose a Möbius function m such that $m(C) = \mathbb{R}$, and put $g = m \circ f$, so $g((a, b)) \subseteq \mathbb{R}$. From the definition of $S(g)$ it is clear that $S(g)$ is real on (a, b), but the Schwarzian chain rule shows that $S(f) = S(g)$, so (b) holds.

Conversely, suppose that the function $s = S(f)$ is real on (a, b). Choose a point $t_0 \in (a, b)$. Lemma 5.3.11 gives us a Möbius function m such that the function $g = m \circ f$ has $g(t) = (t - t_0) + O((t - t_0)^3)$. Note also that $S(g)$ is again equal to s. Now put $h(z) = \overline{g(\overline{z})}$, and note that this is again meromorphic. Using power series representations we can see that $S(h) = r$, where $r(z) = \overline{s(\overline{z})}$. Now r is also holomorphic, and agrees with s on (a, b), so it must agree with s everywhere in U. This means that g and h are both elements of G, so Lemma 5.3.9 gives us a Möbius function n with $g = n \circ h$. However, both $g(t)$ and $h(t)$ are of the form $(t - t_0) + O((t - t_0)^3)$. It follows that $n(z) = z + O(z^3)$, and thus that n is the identity, so $g = h$. This means that $g((a, b)) \subseteq \mathbb{R}$, so $f((a, b)) \subseteq m^{-1}(\mathbb{R})$. Moreover, as m is a Möbius function, the set $m^{-1}(\mathbb{R})$ is a circle as required. $\square$

Proposition 5.3.14. *Let U be the disc of radius r centred at z_0, and let $s(z)$ be a function that is holomorphic on $U \setminus \{z_0\}$ and has Laurent expansion $\sum_{k \geq -2} s_k(z - z_0)^k$ with $s_{-2} = 3/8$. Let U' be obtained from U by removing a line segment from z_0 to the edge, and let $\xi(z)$ be a holomorphic branch of $(z - z_0)^{1/4}$ on U'. Let F' be the space of holomorphic solutions of $f'' + \frac{1}{2}sf = 0$ on U'. Then*

- *There is a unique holomorphic function e_0 on U such that $e_0(0) = 1$ and the function $f_0 = e_0\xi$ lies in F'.*
- *There is a unique holomorphic function e_1 on U such that $e_1(0) = 1$ and the function $f_1 = e_1\xi^3$ lies in F'.*

Proof. It will be harmless, and notationally convenient, to assume that $z_0 = 0$.

By assumption we have $\xi(z)^4 = z$, so $4\xi^3\xi' = 1$, so $\xi'(z) = 1/(4\xi(z)^3) = \xi(z)/(4z)$.

Consider a function of the form $f = e\xi$, where $e(z) = \sum_{k\geq 0} a_k z^k$ (and we take $a_k = 0$ for $k < 0$). We then have

$$f'(z) = e'(z)\xi(z) + e(z)\xi'(z) = (e'(z) + \tfrac{1}{4}e(z)z^{-1})\xi(z).$$

By a similar argument, we have

$$f''(z) = (e''(z) + \tfrac{1}{2}e'(z)z^{-1} - \tfrac{3}{16}e(z)z^{-2})\xi(z).$$

Thus, the equation $f'' + \frac{1}{2}sf = 0$ is equivalent to

$$e''(z) + \tfrac{1}{2}z^{-1}e'(z) + \tfrac{1}{2}t(z)e(z) = 0,$$

where

$$t(z) = s(z) - \tfrac{3}{8}z^{-2} = \sum_{k\geq -1} s_k z^k.$$

Looking at the coefficient of z^k, we get

$$(k + \tfrac{3}{2})(k+2)a_{k+2} + \tfrac{1}{2}\sum_{j=0}^{k+1} s_{k-j}a_j = 0.$$

For $k \leq -2$ this is trivially satisfied, and for $k \geq -1$ it can be rearranged to express a_{k+2} in terms of $a_0, \dots, a_{k+1}$. It follows that there is a unique power series solution with $a_0 = 1$. The rate of growth of the coefficients a_k is comparable with that of the coefficients s_k, so the series $\sum_k a_k z^k$ converges to give the claimed function $e_0(z)$.

Now consider instead a function of the form $f = e\xi^3$. We find in the same way that

$$f''(z) = (e''(z) + \tfrac{3}{2}e'(z)z^{-1} - \tfrac{3}{16}e(z)z^{-2})\xi(z)^3,$$

and thus that the equation $f'' + \frac{1}{2}sf = 0$ is equivalent to

$$e''(z) + \tfrac{3}{2}z^{-1}e'(z) + \tfrac{1}{2}t(z)e(z) = 0,$$

or

$$(k + \tfrac{5}{2})(k+2)a_{k+2} + \tfrac{1}{2}\sum_{j=0}^{k+1} s_{k-j}a_j = 0.$$

Just as in the previous case, there is a unique solution, as claimed. The functions f_0 and f_1 are linearly independent (as we can see by considering their rate of growth near $z = 0$), and we have seen that F' has dimension two, so they must form a basis. □

5.4. **Application to cromulent surfaces.** [sec-b-from-a]

For each $a \in (0,1)$ we have shown that there is a unique $b \in (0,1)$ such that $PX(a) \simeq HX(b)$ as cromulent surfaces, and in fact there is a unique cromulent isomorphism $HX(b) \to PX(a)$. As in Section 5.1, we write p for the canonical map

$$\Delta \to \Delta/\Pi = HX(b) \to PX(a) \to PX(a)/\langle\lambda\rangle^2 \to \mathbb{C}_\infty,$$

and we also consider the map $p_1 = \phi p \psi\colon \Delta \to \mathbb{C}_\infty$.

In this section, we describe an algorithm to calculate b from a. This algorithm is implemented by the methods of the class `P_to_H_map`, which is defined in the file `hyperbolic/P_to_H.mpl`. In more detail, if we want to take $a = 0.1$, we can enter

```
PH := `new/P_to_H_map`():
PH["set_a_P",0.1]:
PH["add_charts"]:
PH["find_p1_inv"];
PH["a_H"];
PH["p1_inv"](z);
PH["err"];
```

The line `PH["a_H"]` returns the value of b, which is about 0.7994. The line `PH["p1_inv"](z)` returns a polynomial in z, which is a good approximation to $p_1^{-1}(z)$. The line `HP["err"]` returns a measure of the quality of approximation, which is about 10^{-21} in this case.

If one wants to perform the above calculation for several different values of a, and to compare the results with those obtained by the method of Section 5.2, then we can instead use the class `HP_table`, defined in `hyperbolic/HP_table.mpl`. For example, we can enter the following:

```
HPT := `new/HP_table`():
HPT["add_a_P",0.1];
HPT["add_a_P",0.2];
HPT["add_a_P",0.3];
```

This will perform the above calculation for the valuse $a = 0.1$, $a = 0.2$ and $a = 0.3$. The object of class `P_to_H_map` for $a = 0.1$ can then be retrieved as `HPT["P_to_H_maps"][0.1]`.

The function `build_data["HP_table"]()` (defined in `build_data.mpl`) does the calculation for a from 0.06 to 0.94 in steps of 0.02 (as well as implementing the method of Section 5.2 and performing various other work).

We can think of the inverse of p as giving a multivalued function $p^{-1}\colon \mathbb{C}_\infty \to \Delta$. It is a key point that the Schwarzian derivative $S(p^{-1})$ is single-valued, and in fact is a rational function whose properties we can understand quite explicitly. To explain this, we will use the following definitions

Definition 5.4.1. [defn-schwarz-s]

We put

$$\begin{aligned} B &= \{v_{Ci} \mid i \in \{0,1,10,11,12,13\}\} \\ &= \{0,\infty,a,-a,1/a,-1/a\} \subset \mathbb{C}_\infty \\ B_0 &= B \setminus \{\infty\} = \{0,a,-a,1/a,-1/a\} \\ r_a(z) &= \prod_{u\in B_0}(z-u) = z^5 - (a^2+a^{-2})z^3 + z \\ s_0(z) &= \frac{-3z^3/2}{r_a(z)} + \frac{3}{8}\sum_{u\in B_0}\frac{1}{(z-u)^2} \\ s_1(z) &= \frac{z}{r_a(z)}. \end{aligned}$$

We have seen $r_a(z)$ before; the definition is repeated for ease of reference and to display the connection with B and B_0. Recall also that B is the set of critical values of p, so p restricts to give a covering map $\Delta \setminus p^{-1}(B) \to \mathbb{C}_\infty \setminus B$. For any sufficiently small connected open set $U \subset \mathbb{C}_\infty \setminus B$, we can choose a holomorphic map $f\colon U \to \Delta$ with $pf = 1_U$ (so f is a local branch of p^{-1}); we then define $S(p^{-1})_U = S(f)$. This is independent of the choice of f, because any other choice has the form $m \circ f$ for some Möbius map $m \in \langle \Pi, \lambda^2\rangle$, and the Schwarzian chain rule gives $S(m \circ f) = S(f)$. Given this invariance, it is clear that $S(p^{-1})_U = S(p^{-1})_V$ on $U \cap V$. We can therefore patch these local functions together to get a meromorphic function $S(p^{-1})$ on $\mathbb{C}_\infty \setminus B$.

Proposition 5.4.2. [prop-S-p-inv]

There is a constant $d \in \mathbb{R}$ such that $S(p^{-1}) = s_0 + ds_1$ (where s_0 and s_1 are as in Definition 5.4.1).

Proof. For convenience, we write $s(z) = S(p^{-1})(z)$. Put

$$d(z) = (s(z) - s_0(z))/s_1(z) = (s(z) - s_0(z))r_a(z)/z.$$

The claim is that this is a real constant.

First let U and f be as in our definition of $s(z)$. The equation $pf = 1_U$ implies that f' is nonzero everywhere in U, so the Schwarzian derivative $s = f'''/f' - \frac{3}{2}(f''/f')^2$ is holomorphic in U. This shows that all singularities of s must lie in B.

Now consider a point $u \in B_0$, and choose $\widetilde{u} \in \Delta$ with $p(\widetilde{u}) = u$. By a standard argument with branched double covers, we have $p(z) = u + c(z - \widetilde{u})^2 + O((z - \widetilde{u})^3)$ for some $c \neq 0$. It follows that on any small open set close to u, we have $p^{-1}(z) = ((z-u)/c)^{1/2} + O((z-u)^{3/2})$. Computing the Schwarzian derivative from this approximation gives $s(z) = \frac{3}{8}(z-u)^{-2} + O((z-u)^{-1})$, which matches the behaviour of $s_0(z)$. We therefore see that $s(z) - s_0(z)$ has at worst a simple pole at u. As this holds for all $u \in B_0$, we see that the product $e(z) = (s(z) - s_0(z))r_a(z) = z\,d(z)$ is holomorphic everywhere in $\mathbb{C}$, so $d(z)$ has at worst a simple pole at $z = 0$.

Next, recall that there is an element $\mu \in \widetilde{\Pi}$ that satisfies $p(\mu(z)) = 1/p(z)$ for all $z \in \Delta$. In other words, if we define $\mu_0 \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $\mu_0(z) = 1/z$, then we have $\mu_0 \circ p = p \circ \mu$, and so $p^{-1} \circ \mu_0 = \mu \circ p^{-1}$. Here μ and μ_0 are Möbius maps so the Schwarzian chain rule gives $(S(p^{-1}) \circ \mu_0)\ (\mu_0')^2 = S(p^{-1})$, and thus $s(z^{-1}) = z^4 s(z)$. A similar argument with λ gives $s(-z) = s(z)$. By direct calculation we also see that

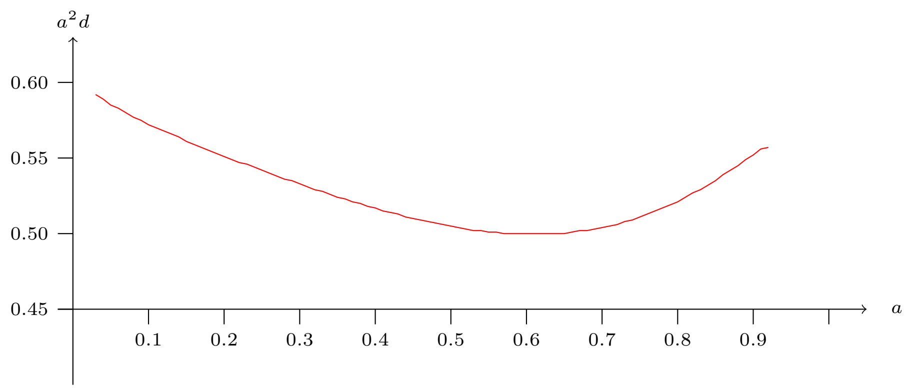
$$s_0(z^{-1}) = z^4 s_0(z^{-1}) \qquad\qquad s_0(-z) = s_0(z)$$
$$s_1(z^{-1}) = z^4 s_1(z^{-1}) \qquad\qquad s_1(-z) = s_1(z).$$

It follows that $d(z^{-1}) = d(-z) = d(z)$. As d is even and has at worst a simple pole at 0, it must actually be holomorphic at 0. It is also holomorphic elsewhere in $\mathbb{C}$ and it satisfies $d(z^{-1}) = d(z)$ so it must be bounded on $\mathbb{C}$ and thus constant.

Finally, recall that p is equivariant with respect to the map ν, or in other words $p(\overline{z}) = \overline{p(z)}$. This means that p is real on $\mathbb{R} \cap \Delta$, so p^{-1} can be chosen to be real on $\mathbb{R}$, so $S(p^{-1})$ is real on $\mathbb{R}$. From this it is clear that $d \in \mathbb{R}$. $\square$

Remark 5.4.3. Maple notation for d and $s(z)$ is `d_p_inv` and `S_p_inv`.

Remark 5.4.4. We can regard the parameter d in the proposition as a function of a or of b. Numerical calculations (by a method to be described below) suggest that d grows like a^{-2} as $a \to 0$, and give the following graph of $a^2 d$ against a:



We can now try to find p^{-1} and p by power series methods. As before, it turns out to be convenient to focus on p_1^{-1} rather than p^{-1}, because p_1^{-1} is unbranched at the origin, and the associated power series has a reasonably large radius of convergence.

Proposition 5.4.5. [prop-p-one]

We have $S(p_1^{-1}) = s_0^* + d s_1^*$, where d is the same real constant as in Proposition 5.4.2, and

$$s_0^*(z) = \frac{192a^4z^2(1+z^2)^2 - 9(1-a^4)^2(1-z^2)^4}{2(1-z^2)^2((1+a^2)^2(1-z^2)^2 + 16a^2z^2)^2} \qquad s_1^*(z) = \frac{4a^2}{(1+a^2)^2(1-z^2)^2 + 16a^2z^2}.$$

The set of poles of $S(p_1^{-1})$ is

$$\psi^{-1}(B) = \{v_{PSi} \mid i \in \{0, 1, 10, 11, 12, 13\}\} = \left\{\pm 1, \frac{i-a}{i+a}, \frac{i+a}{i-a}, \frac{a-i}{a+i}, \frac{a+i}{a-i}\right\}.$$

Proof. Recall that $p_1 = \phi p \psi$, so $p_1^{-1} = \psi^{-1}p^{-1}\phi^{-1}$. Here ϕ^{-1} and ψ^{-1} are both Möbius maps, so $S(\phi^{-1}) = 0$ and $S(\psi^{-1}) = 0$. The Schwarzian chain rule therefore gives

$$S(p_1^{-1}) = S(\psi^{-1} \circ p^{-1} \circ \phi^{-1}) = (S(p^{-1}) \circ \phi^{-1}) \cdot ((\phi^{-1})')^2.$$

Here $S(p^{-1})$ is given by Proposition 5.4.2, and

$$(\phi^{-1})'(z) = -\frac{2i}{(1+z)^2}.$$

It is now a lengthy but straightforward calculation to show that this is equivalent to the claimed formula. We also see that the poles of $S(p_1^{-1})$ are the images under ϕ of the poles of $S(p^{-1})$, together with the poles of $(\phi^{-1})'$. We saw previously that $S(p^{-1})$ has poles only in the set

$$\{0, \pm a, \pm 1/a\} = \{v_{C0}, v_{C10}, v_{C11}, v_{C12}, v_{C13}\}.$$

These points v_{Ci} are mapped by ϕ to the corresponding points v_{PSi}, and the only pole of $(\phi^{-1})'$ is at $-1 = v_{PS1}$, so the poles of $S(p_1^{-1})$ are as claimed. □

Remark 5.4.6. Maple notation for $s_0^*(z)$ and $s_1^*(z)$ is `S0_p1_inv(z)` and `S1_p1_inv(z)`.

Remark 5.4.7. [rem-heun]

We can also consider the map $p_2 = \xi \circ p_1$, where $\xi(z) = 2/(z^2 + z^{-2})$. If we choose domains so that the maps $U \xrightarrow{p_1} V \xrightarrow{\xi} W$ are invertible, then we can use the schwarzian chain rule to obtain a formula for $S(p_2^{-1})$ on W. It turns out that this is a rational function with poles (of various orders) at 0, 1, -1 and $-(1+a^2)/(1-6a^2+a^4)$. Because there are only four poles, the basic solutions to the differential equation $f'' + S(p_2^{-1})f/2$ can be expressed as products of certain factors $(z-\alpha)^{n/8}$ with suitable instances of the Heun G-function. Further details are given in the Maple code, but we will not discuss them here, because we did not find this representation to be useful. However, we believe that this is the closest possible connection with special functions that have previously been named and studied.

`hyperbolic/schwarz_check.mpl: check_heun()`

Definition 5.4.8. [defn-star-U]

We put

$$U = \mathbb{C} \setminus \{t\,\psi^{-1}(u) \mid t \geq 1,\ u \in B\}.$$

In other words, U is the domain obtained from $\mathbb{C}$ by deleting rays from the points of $\psi^{-1}(B)$ to ∞. This is simply connected and contains $i\mathbb{R} \cup \Delta$, and the maps s_0^* and s_1^* are holomorphic on U.

Proposition 5.4.9. [prop-p-one-section]

There is a unique holomorphic map $g\colon U \to \Delta$ satisfying $g(0) = 0$ and $p_1 g = 1$. This satisfies $g(\overline{z}) = \overline{g(z)}$ and $g(-z) = -g(z)$. Moreover, $g(z)$ can be written in the form $c\,f_1(z)/f_0(z)$, where

- *The maps f_k are holomorphic on U and satisfy $f_k'' + \frac{1}{2}(s_0^* + ds_1^*)f_k = 0$.*
- *f_0 is an even function with $f_0(0) = 1$ and $f_0(\overline{z}) = \overline{f_0(z)}$.*
- *f_1 is an odd function with $f_1'(0) = 1$ and $f_1(\overline{z}) = \overline{f_1(z)}$.*
- *c is a positive real number.*

Proof. As U contains none of the critical values of p_1, we see that the map $p_1\colon p_1^{-1}(U) \to U$ is a covering. We have also seen that $p_1(0) = 0$. As U is simply connected, standard covering theory tells us that there is a unique section $g\colon U \to \Delta$ with $p_1 g = 1$ and $g(0) = 0$. As p_1 is holomorphic, it is easy to check that the same holds for g. As $p_1(-z) = -p_1(z)$ and $p_1((-1,1)) \subseteq \mathbb{R}$ we see that $g(-z) = -g(z)$ and $g(\mathbb{R}) \subseteq (-1,1)$.

Now put $s^* = S(p_1^{-1})$, which has the form $s_0^* + ds_1^*$ as in Proposition 5.4.5. Let F denote the space of holomorphic functions on U that satisfy $e'' + \frac{1}{2}s^* e = 0$. Proposition 5.3.4 tells us that there are unique functions $f_0, f_1 \in F$ with $f_0(z) = 1 + O(z^2)$ and $f_1(z) = z + O(z^2)$, and these functions form a basis for F. From the definition of s_0^* and s_1^* we see that $s^*(z) = -z$. It follows that the maps $z \mapsto f_0(-z)$ and $z \mapsto -f_1(-z)$ satisfy the defining conditions of f_0 and f_1 respectively; so f_0 is even and f_1 is odd. Similarly, we see that $s^*(z) = \overline{s^*(\overline{z})}$, and it follows that $f_k(z) = \overline{f_k(\overline{z})}$. Proposition 5.3.4 also tells us that g is the quotient of two linearly independent elements of F, so there are constants A, B, C and D such that $g = (Af_0 + Bf_1)/(Cf_0 + Df_1)$ and $AD - BC \neq 0$. As $g(0) = 0$ we must have $A = 0$, so $Cf_0 + Df_1 = Bf_1/g$.

As both g and f_1 are odd we see that $Cf_0 + Df_1$ is even, so $D = 0$. Thus, if we put $c = B/C$ we have $g = c\, f_1/f_0$. By considering derivatives at the origin, we see that c is real and positive. ☐

5.5. **Methods for explicit calculation.** [sec-schwarz-methods]

The parameters b and d depend on a; in this section we will call them $\beta(a)$ and $\delta(a)$. While Proposition 5.4.9 is very satisfactory, we cannot immediately use it for computation, because we do not know the values of $\beta(a)$ and $\delta(a)$. We will describe some calculations that we can do with an arbitrary pair (b, d), which will enable us to test whether $(b, d) \simeq (\beta(a), \delta(a))$, and to improve the degree of approximation if necessary. First, we define $s^*(z) = s_0^*(z) + d s_1^*(z)$. Next, for any open set $V \subset \mathbb{C}$, we put

$$F^*(V) = \{f \in \mathrm{Hol}(V) \mid f'' + s^* f/2 = 0\}$$
$$G^*(V) = \{g \in \mathrm{Mer}(V) \mid S(g) = s^*\},$$

and note that these are described by Proposition 5.3.4. In particular, we can consider $F^*(U)$ and $G^*(U)$, where U is as in Definition 5.4.8. Just as in Proposition 5.4.9, we see that there is a unique basis $\{f_0, f_1\}$ for $F^*(U)$ such that $f_k(z) = z^k + O(z^2)$. Both the power series solution method in Lemma 5.3.6 and the analytic continuation method in Corollary 5.3.7 are straightforwardly computable, so we can calculate $f_k(z)$ for any $z \in U$. (To calculate $f_k(u)$ when $|u| = 1$, we have typically computed power series solutions $f_{kj}(z)$ centred at $ju/10$ for $0 \le j \le 10$, and compared then by evaluating $f_{kj}((j + 1)u/10)$ and $f'_{kj}((j + 1)u/10)$. This works provided that u is not too close to any of the branch points. If it is very close to a branch point, then we need to take smaller steps as we approach it.) We also put $g_0(z) = f_1(z)/f_0(z)$. It is not hard to see that this is odd with real coefficients, so $g_0(i\mathbb{R}) \subseteq i\mathbb{R}$.

Definition 5.5.1. [defn-circle-fit]

Suppose that

$$g_0(ie^{it})/i = u_0 + iu_1 t + u_2 t^2 + O(t^3).$$

Using the fact that $g_0(-\overline{z}) = -\overline{g(z)}$, we see that the coefficients u_j are real. We put

$$b = \frac{u_1^2}{\sqrt{8u_0u_1(u_0u_2 + u_1^2) + u_1^4}}$$
$$c = \sqrt{\frac{u_2}{u_0(u_0u_2 + u_1^2)}}$$
$$g(z) = c\, g_0(z).$$

These formulae are embedded in the Maple function `series_circle_fit(u0,u1,u2)`.

Proposition 5.5.2. *If $d = \delta(a)$ then $b = \beta(a)$ and $g(z)$ is the function described in Proposition 5.4.9. Moreover, for any z_0 on the arc of S^1 between 1 and $(i - a)/(i + a)$, we have $Im(\psi(g(z_0))) = 0$.*

Proof. Because $d = \delta(a)$, we see that f_0 and f_1 are as described in Proposition 5.4.9. Put $b^* = \beta(a)$, and let c^* and g^* be the number and the function denoted by c and g in Proposition 5.4.9, so $g^* = c^* g_0$; our task is to prove that $b^* = b$ and $c^* = c$ and $g^* = g$.

We saw in Lemma 5.1.3 that C_{HS3} is part of the circle of radius $R = \sqrt{2}b^*/b^*_-$ centred at the point $iA = ib^*_+/b^*_-$. We also have $p_1(C_{HS3}) \subseteq C_{PS3} \subseteq S^1$, and it follows that $g^*(ie^{it}) \subseteq C_{HS3}$ for small t, or in other words $|A - c^* g_0(ie^{it})/i|^2 = R^2$. This gives

$$R^2 = (A - c^* u_0 - c^* u_2 t^2)^2 + u_1^2 t^2 + O(t^3),$$

and we can expand out and compare coefficients to get

$$R^2 = (A - c^* u_0)^2$$
$$u_1^2 = 2(A - c^* u_0)c^* u_2.$$

Note also that the definitions $R = \sqrt{2}b^*/b^*_-$ and $A = b^*/b^*_-$ imply $A^2 - R^2 = 1$ (corresponding to the fact that C_{HS3} crosses the unit circle orthogonally). It is now an exercise in algebra to solve these equations giving

$$c^* = \sqrt{\frac{u_2}{u_0(u_0u_2+u_1^2)}} = c$$
$$A = \frac{u_0u_2+\frac{1}{2}u_1^2}{\sqrt{u_0u_2(u_0u_2+u_1^2)}}$$
$$R = \frac{\frac{1}{2}u_1^2}{\sqrt{u_0u_2(u_0u_2+u_1^2)}}.$$

From the $c^* = c$ we deduce that $g^* = g$. We also have $A = \sqrt{2}b^*/\sqrt{1-(b^*)^2}$, which gives $b^* = A/\sqrt{2+A^2}$; after some further manipulation this gives $b^* = b$.

Finally, let L denote the arc between 1 and $(i-a)/(i+a)$. This is part of $C_{PS5} = p_1(C_{HS5}) = p\psi^{-1}(C_{H5})$, but $C_{H5} = (-1,1)$; it follows that $\psi(g(L)) \subseteq (-1,1)$, so $\mathrm{Im}(\psi(g(L))) = 0$.

`hyperbolic/schwarz_check.mpl: check_schwarz()`

□

We thus arrive at the following method. Given a we fix a point z_0 on S^1 between 1 and $(i-a)/(i+a)$. We then choose d and compute $g_0(z)$. The power series for $g_0(ie^{it})/i$ gives the coefficients u_0, u_1 and u_2, from which we compute b, c and $g(z)$. We then put $\epsilon(d) = \mathrm{Im}(\psi(z_0))$. If $d = \delta(a)$ then we will have $\epsilon(d) = 0$. If $\epsilon(d) \neq 0$ then we can adjust our value of d and try again. As evaluation of $\epsilon(d)$ is quite expensive, it is important to use an efficient search algorithm. It is also useful to retain a lot of information generated in the course of the calculation which we have found to be awkward when using Maple's built in `fsolve` function. We have therefore used our own implementation of Brent's method [2] (closely following the Matlab implementation by John Burkardt [3]). This gives us a value of d such that $\epsilon(d) = 0$ to high accuracy. We have not given a general proof that $\epsilon(d) = 0$ implies $d = \delta(a)$, but in any given case it is easy to perform additional checks to verify that this is the case; for example, we can feed our new value of b into the method of Section 5.2 and check that everything is consistent.

Maple commands for the above algorithm were given at the beginning of Section 5.4. For more detail, see the comments in the code.

5.6. **Holomorphic forms.** [sec-hol-forms]

Suppose we have constructed an isomorphism $f\colon HX(b) \to PX(a)$, given generically by

$$f(z) = j(q(z), p(z)).$$

Recall that Proposition 3.3.3 gives a basis $\{\omega_0, \omega_1\}$ for $\Omega^1(PX(a))$. Let $m(z)$ be the function on Δ given by $f^*(\omega_0) = m(z)\,dz$. Note that this is holomorphic on Δ and so is given by a power series that converges everywhere on Δ, unlike the functions $p(z)$ and $p_1(z)$ which have infinitely many poles. In this section we will investigate the properties of $m(z)$.

Proposition 5.6.1. [prop-m-props]
The function $m(z)$ has the following automorphy properties:

(a) *For $\gamma \in \Pi$ we have $m(z) = m(\gamma(z))\,\gamma'(z)$.*
(b) *$m(z) = \overline{m(iz)}$ (so $m(z)$ is a power series in z^4).*
(c) *$m(\overline{z}) = \overline{m(z)}$ (so the power series for $m(z)$ has real coefficients).*
(d)
$$m(\mu(z)) = -\frac{b^2(1-b^2)\,m(z)}{((i-b^2)z+b_+)^2\,p(z)}.$$

Moreover, for a suitable branch of the square root we have

$$m(z) = p'(z)\,r_a(p(z))^{-1/2}$$

(where $r_a(z) = z(z-a)(z+a)(z-1/a)(z+1/a)$ as in Definition 3.1.1).

Note that property (d) means that p can be computed from m.

Proof. (a) For $\gamma \in \Pi$ we have $f\gamma = f \colon \Delta \to PX(a)$ and so

$$m(\gamma(z))\gamma'(z)\,dz = \gamma^*(m(z)\,dz) = (f\gamma)^*(\omega_0) = f^*(\omega_0) = m(z)\,dz.$$

(b) We have $f\lambda = \lambda f \colon \Delta \to PX(a)$, and $\lambda^* dz = i\,dz$ on Δ whereas $\lambda^*\omega_0 = i\,\omega_0$ on $PX(a)$ (by Proposition 3.3.3). It follows easily from this that $m(iz) = m(\lambda(z)) = m(z)$.
(c) This follows in the same way, using the fact that $\nu^\#(dz) = dz$ and $\nu^\#(\omega_0) = \omega_0$.
(d) Recall that $\mu^*(\omega_0) = \omega_1$, and on $PX_0(a)$ we have $\omega_1 = z\,\omega_0$, which means that

$$f^*(\omega_1) = p(z)\,f^*(\omega_0) = p(z)m(z)\,dz.$$

On the other hand, as $f\mu = \mu f$ we have

$$f^*(\omega_1) = \mu^*(\omega_0) = m(\mu(z))\mu'(z)\,dz.$$

A calculation gives $\mu'(z) = -b^2(1-b^2)/((i-b^2)z+b_+)^2$. Putting this together gives the claimed equation.

Finally, ω_0 is given on $PX_0(a)$ by dz/w, and this gives $f^*(\omega_0) = p'(z)\,dz/q(z)$. Also, as $(p(z),q(z)) \in PX_0(a)$ we have $q(z) = \pm\sqrt{r_a(p(z))}$. ☐

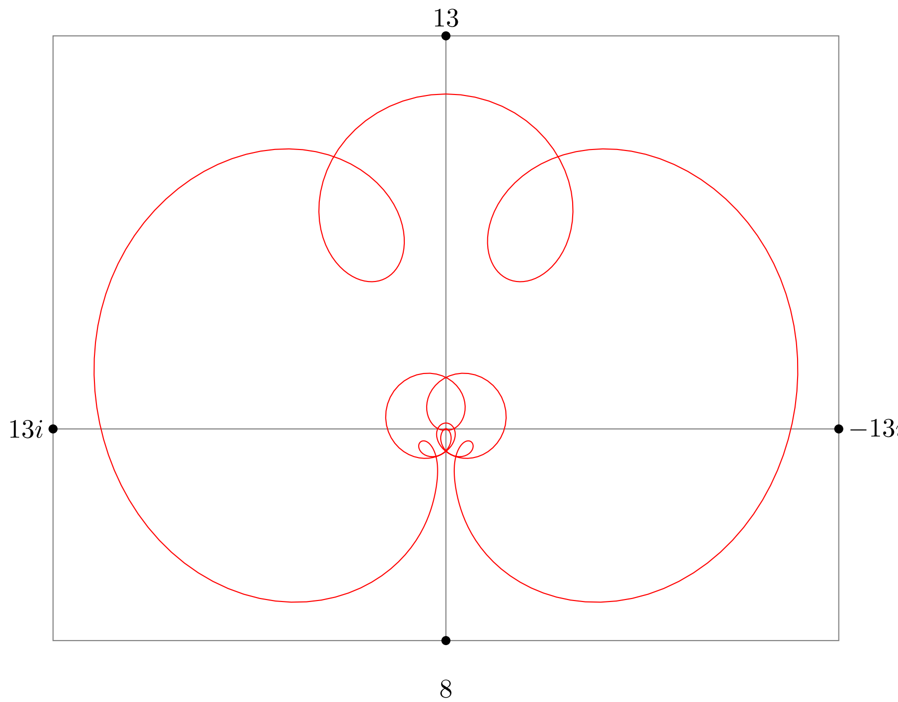
Using the methods of Sections 5.2 and 5.4, we can calculate $p_1(z)$ quite accurately on a domain including $\psi^{-1}(HF_{16}(b))$, and this lets us calculate $p(z)$ on $HF_{16}(b)$. Given an arbitrary point $z \in \Delta$ we can use the method in Remark 4.3.5 to find $\gamma \in \Pi$ such that $\gamma(z) \in HF_1(b)$, then we can find $\beta \in G$ such that $\beta\gamma(z) \in HF_{16}(b)$. Using the automorphy properties of m we can then find $m(z)$ from $m(\beta\gamma(z))$. Given an object `HP` of the class `H_to_P_map`, the method `HP["m_piecewise",z]` will calculate $m(z)$ by the above algorithm.

To obtain the power series for $m(z)$, it is best to calculate $m(s)$ for all s in some finite subset $S \subset \mathbb{C}$, and then find a polynomial $m_0(z) = \sum_{i=0}^{d} a_i z^{4i}$ which minimises $\sum_s \|m_0(s) - m(s)\|^2$. As this objective function depends quadratically on the coefficients $a_i \in \mathbb{R}$, the minimisation problem reduces easily to a matrix calculation. We have generally taken

$$S = \{re^{k\pi i/(2n)} \mid 0 \leq k < n\}$$

for some radius $r \in (0,1)$ (say $r = 0.95$) and some integer $n > 0$ (say $n = 400$). Note that the relation $m(iz) = m(z)$ makes it natural to consider only sample points in the first quadrant, and the maximum principle of complex analysis makes it reasonable to consider only sample points on the boundary of the region where we want our approximation to be accurate. This algorithm is implemented by the method `HP["find_m_series",r,n,d]`.

The following plot shows the curve $m(0.93e^{it})$ with the parameters a and b that are relevant for EX^*. (The real axis is drawn vertically.)



We will mention one other approach to the calculation of $m(z)$, but we will not go into great detail because we have not found it to be computationally efficient.

Definition 5.6.2. [defn-tensor-sections]
Put
$$A_k = \{f \in \mathrm{Hol}(\Delta) \mid f(z) = f(\gamma(z))\gamma'(z)^k \text{ for all } \gamma \in \Pi \text{ and } z \in \Delta\}.$$
Multiplication by $dz^{\otimes k}$ identifies A_k with the space of holomorphic sections of the k'th tensor power of the cotangent bundle of $HX(b)$. In particular, we have $A_1 = \Omega^1(HX(b))$.

Lemma 5.6.3. [lem-riemann-roch]
$$\dim_{\mathbb{C}}(A_k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ 2k-1 & \text{if } k > 1. \end{cases}$$

Proof. This is a standard consequence of the Riemann-Roch theorem (see [10, Section IV.1], for example). In more detail, that theorem tells us that for any line bundle $\mathcal{L}$ over a compact Riemann surface Z of genus g, we have
$$\dim(H^0(Z;\mathcal{L})) - \dim(H^0(Z;\Omega^1 \otimes \mathcal{L}^*)) = \deg(\mathcal{L}) + 1 - g.$$
Now take $Z = HX(b)$ (so $g = 2$) and put $f(n) = \dim(H^0(HX(b);(\Omega^1)^{\otimes n}))$ for $n \in \mathbb{Z}$. We have seen that $\{\omega_0, \omega_1\}$ is a basis for $H^0(HX(b);\Omega^1)$ over $\mathbb{C}$, so $f(1) = 2$. On any compact Riemann surface, the only holomorphic $\mathbb{C}$-valued functions are constant, so $f(0) = 1$. The Riemann-Roch theorem gives $f(n) - f(1-n) = n\deg(\Omega^1) - 1$. Taking $n = 1$ gives $\deg(\Omega^1) = 2$, so we get $f(n) - f(1-n) = 2n - 1$. Moreover, as $\deg(\Omega^1) > 0$ we have $f(n) = 0$ for $n < 0$. Thus, when $n > 1$ we have $f(1-n) = 0$ and so $f(n) = 2n-1$ as required. $\square$

Definition 5.6.4. [defn-automorphic-series]
For $j \geq 0$ and $k \geq 2$ we put
$$p_{jk}(z) = \sum_{\gamma \in \Pi} \gamma(z)^j \gamma'(z)^k.$$

It is a standard theorem that the above series is absolutely uniformly convergent on compact subsets of Δ. We will give a proof that includes explicit bounds in the case of interest.

Definition 5.6.5. [defn-automorphic-area]
For any subset $A \subseteq \Pi$ we put

$$a(A) = \text{area}\left(\bigcup_{\gamma\in A} \gamma(HF_1(b))\right) = \sum_{\gamma\in A} \text{area}(\gamma(HF_1(b))).$$

(Here areas are defined in terms of the Euclidean metric on Δ, not the hyperbolic metric.) We also write $a(\gamma) = a(\{\gamma\}) = \text{area}(\gamma(HF_1(b)))$. In particular, $a(1)$ is just the area of $HF_1(b)$.

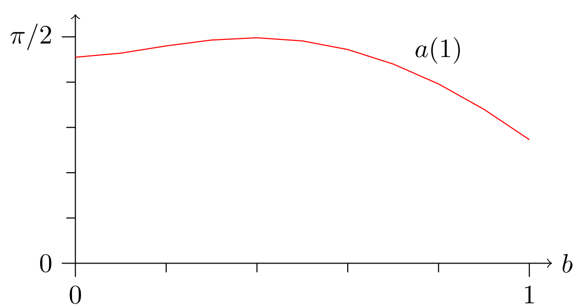
Remark 5.6.6. [rem-automorphic-area]
We can find $a(1)$ as follows. It is easy to see that $a(1)$ is 8 times the area of $HF_8(b)$. The boundary of this region consists of straight lines joining $v_0 = 0$ to v_1 and v_{11}, together with an arc of the circle C_3 joining v_{11} to v_{13}, and an arc of the circle C_7 joining v_1 to v_{13}. Recall that C_3 has centre $a_3 = b_+$ and radius $r_3 = \sqrt{|a_3|^2 - 1} = b$, whereas C_7 has centre $a_7 = b_+^{-1} + \frac{1}{2} i b_+$ and radius $r_7 = \sqrt{|a_7|^2 - 1} = \frac{1}{2} b_-^2 b_+^{-1}$. One can check that $v_{11} = a_3 - r_3$ and $v_1 = a_7 - r_7$ and

$$v_{13} = a_3 - r_3 e^{-i\theta} = a_7 - i\, r_7 e^{-i\theta}$$

where $\theta = 2\arctan(b(b_+ - b))$. Using this, it is easy to parameterise the boundary of $HF_8(b)$ and use Green's theorem (in the form $\text{area}(D) = -\oint_{\partial D} y\, dx$) to calculate the area. We eventually arrive at the following formula:

$$a(1) = 3 + b^2 - \frac{2bb_-^2}{b_+} - \frac{b_-^4}{b_+^2}\frac{\pi}{2} - \frac{3b^4 + 6b^2 - 1}{1 + b^2}\theta.$$

The graph is as follows:



hyperbolic/HX_check.mpl: check_F1_area()

Proposition 5.6.7. [prop-automorphic-conv]
Fix $k \geq 2$ and $\delta > 0$, and let K be the closed disc of radius $1 - \delta$ centred at the origin. Then the series defining $p_{jk}(z)$ converges absolutely and uniformly on K. More precisely, for any subset $A \subseteq \Pi$ and any $z \in K$ we have

$$\left| p_{jk}(z) - \sum_{\gamma\in A} \gamma(z)^j \gamma'(z)^k \right| \leq \sum_{\gamma\in\Pi\setminus A} |\gamma'(z)|^k \leq \delta^{-2k} a(\Pi \setminus A)/a(1) = \delta^{-2k}(\pi - a(A))/a(1).$$

Proof. Put $F = HF_1(b)$. Note that the images $\{\gamma(F) \mid \gamma \in \Pi\}$ cover Δ, and any two of these images have intersection of measure zero, so

$$\sum_{\gamma\in\Pi} a(\gamma) = \text{area}(\Delta) = \pi.$$

Next, if we write $\gamma(z)$ in the form $(az + b)/(cz + d)$ with $ad - bc = 1$, then we find that $\gamma'(z) = (cz + d)^{-2}$, so the Jacobian determinant is $|cz + d|^{-4}$. It follows that $a(\gamma) = \iint_F |cz + d|^{-4}$. Corollary 4.3.7 tells us that $|cz + d| \leq (\sqrt{2} + 1)|c|$, so $|c|^{-4} \leq (\sqrt{2} + 1)^4 |cz + d|^{-4}$, so $|c|^{-4} a(1) \leq a(\gamma)$.

Now let z be any point in K. Clearly $|\gamma(z)^j| \leq 1$. We have $|cz+d|^{-2k} = |c|^{-2k}|z+d/c|^{-2k}$. Corollary 4.3.7 also gives $|d/c| \geq 1$ so for $z \in K$ we have $|z + d/c|^{-2k} \leq \delta^{-2k}$. The same result also gives $|c| \geq 1$ and $k \geq 2$

by assumption so $|c|^{-2k} \leq |c|^{-4} \leq a(\gamma)/a(1)$. Putting this together gives $|\gamma(z)^j \gamma'(z)^k| \leq \delta^{-2k} a(\gamma)/a(1)$. Taking the sum over γ gives

$$\sum_{\gamma \notin A} |\gamma'(z)|^{-k} \leq \delta^{-2k} a(\Pi \setminus A)/a(1).$$

□

Corollary 5.6.8. [cor-automorphic-series]
$p_{jk} \in A_k$ for all j and k.

Proof. The proposition shows that the series for $p_{jk}(z)$ is absolutely uniformly convergent on compact subsets of Δ. This validates the following manipulation:

$$\begin{aligned} p_{jk}(\delta(z))\delta'(z)^k &= \sum_{\gamma \in \Pi} (\gamma\delta)(z)^j \gamma'(\delta(z))^k \delta'(z)^k \\ &= \sum_{\gamma \in \Pi} (\gamma\delta)(z)^j (\gamma\delta)'(z)^k \\ &= \sum_{\epsilon \in \Pi} \epsilon(z)^j \epsilon'(z)^k = p_{jk}(z). \end{aligned}$$

□

Next, the isomorphism $A_2 = \Gamma(HX(b); \Omega^{\otimes 2})$ gives rise to an action of G on A_2; we will write $\gamma^\bullet f$ for the action of γ on f. Note that the functions $m(z)$ and

$$n(z) = (\mu^\bullet m)(z) = m(\mu(z))\mu'(z)$$

give a basis for A_1. It follows that the functions m^2, n^2 and mn are linearly independent in A_2, and A_2 has dimension 3 by Lemma 5.6.3, so the indicated elements must give a basis. In particular, we have $p_{02} = a_0 m^2 + a_1 n^2 + a_2 mn$ for some constants a_i. One can check that p_{02}, m^2 and n^2 are fixed by λ whereas mn is negated, so $a_2 = 0$. Using the automorphy properties of m together with the relation $\beta_6\mu(v_1) = v_0$ one can check that $m(v_1) = n(v_0) = 0$ and

$$n(v_1)/m(v_0) = (\beta_6\mu)'(v_1) = 2i/(1 - b^2).$$

From this we obtain $a_1 = a^* a_0$, where

$$a^* = -\tfrac{1}{4}(1 - b^2)p_{02}(v_1)/p_{02}(v_0).$$

From this it follows that $p_{02} - a^*\mu^\bullet p_{02}$ is a constant multiple of m^2. Thus, if we can calculate p_{02} effectively, then we can recover the function m up to a constant multiplier, without using the methods in Sections 5.2 and 5.4. We can then find the map p from the relation $p = (\mu^\bullet m)/m$, noting that the unknown constant cancels out.

However, in practice we need an extremely large number of terms to calculate p_{02} accurately, so this is not an efficient approach. Various tactics are available to streamline the calculation, but they are not sufficient to change the conclusion.

The above algorithm is implemented by the class automorphy_system, which is declared in the file hyperbolic/automorphic.mpl. In more detail, we can enter the following:

```
AS := `new/automorphy_system`();
AS["a_H"] := a_H0;
AS["poly_deg"] := 100;
AS["band_number"] := 4;
AS["set_p0_series",2]:
AS["set_m_series"]:
AS["m_series"](z);
```

The last line will give a polynomial $m^*(z)$ of degree 100 which approximates $m(z)/m(0)$. It is based on a calculation of $p_{02}(z)$ obtained by summing over a certain subset of Π. More specifically, we can put $B_0 = \{1\}$ and

$$B_1 = \{\gamma \in \Pi \mid \gamma(HF_1(b)) \cap HF_1(b) \neq \emptyset\},$$

then we can define $B_n = B_1.B_{n-1}$ recursively. There are 25 elements in B_1, and it is not hard to list them explicitly, and then to give an algorithm which enumerates B_n for all n. The line

```
AS["band_number"] := 4;
```

specifies that sums should be taken over the set B_4, which has 156772 elements. We find that $|m(0)m^*(z) - m(z)| \leq 10^{-5}$ for $|z| \leq 0.5$, but the error grows to about 10^{-3} when $|z| = 0.65$, and becomes very large when $|z| > 0.8$.

6. The embedded family

[sec-E]

6.1. Geometry behind the definition. [sec-E-geometry]

Definition 6.1.1. [defn-X]

Fix $a \in (0,1)$. For $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we put

$$\begin{aligned}
\rho(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 = \sum_i x_i^2 \\
f_1(x) &= 2x_2^2 + (x_4 - 1 - x_3/a)^2 \\
f_2(x) &= 2x_1^2 + (x_4 - 1 + x_3/a)^2 \\
f(x) &= f_1(x) f_2(x) \\
EX(a) &= \{x \in \mathbb{R}^4 \mid \rho(x) = 1 \text{ and } f(x) = f(-x)\}.
\end{aligned}$$

Straightforward algebra shows that this is the same as the definition in the Introduction.

Now put

$$\Omega_1^+ = \{x \in S^3 \mid f_1(x) = 0\}.$$

Recall that $f_1(x) = 2x_2^2 + (x_4 - 1 - x_3/a)^2$, and a sum of two real squares can only be zero if the individual terms are zero. It follows that

$$\begin{aligned}
\Omega_1^+ &= \{x \in S^3 \mid x_2 = 0 \text{ and } x_4 = 1 + x_3/a\} \\
&= \{(x_1, 0, x_3, 1 + x_3/a) \in \mathbb{R}^4 \mid x_1^2 + x_3^2 + (1 + x_3/a)^2 = 1\}.
\end{aligned}$$

This is the intersection of S^3 with a two-dimensional affine subspace of $\mathbb{R}^4$, so it is a circle. This circle passes through the point $(0,0,0,1)$, which corresponds to infinity under the stereographic projection map $s\colon S^3 \to \mathbb{R}^3 \cup \{\infty\}$ that we defined in the Introduction. This means that the image $s(\Omega_1^+)$ is a "circle through ∞", or in other words a straight line. In fact, one can check that

$$s(\Omega_1^+) = \{(x,y,z) \in \mathbb{R}^3 \mid y = 0, z = -a\}.$$

Similarly, the set

$$\Omega_2^+ = \{x \in S^3 \mid f_2(x) = 0\}$$

is another circle in S^3, with stereographic projection

$$s(\Omega_2^+) = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0, z = +a\}.$$

embedded/geometry_check.mpl: check_omega()

Note that the lines $s(\Omega_1^+)$ and $s(\Omega_2^+)$ are at right angles to each other, but they do not touch except at ∞. We can also put

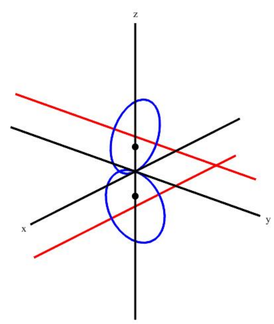
$$\Omega^+ = \{x \in S^3 \mid f(x) = 0\} = \{x \in S^3 \mid f_1(x) f_2(x) = 0\} = \Omega_1^+ \cup \Omega_2^+.$$

Now recall that the definition of X also involves $f(-x)$, so we should study the sets $\Omega_i^- = \{x \in S^3 \mid f_i(-x) = 0\}$ and $\Omega^- = \{x \in S^3 \mid f(-x) = 0\} = \Omega_1^- \cup \Omega_2^-$. The sets Ω_i^- are again circles in S^3, but they do not pass through $(0,0,0,1)$ so their stereographic projections are circles rather than straight lines. In fact one can check that

$$\begin{aligned}
s(\Omega_1^-) &= \{(x,0,z)\in\mathbb{R}^3 \mid x^2+(z-1/(2a))^2=1/(2a)^2\} \\
&= \text{ the circle of radius } 1/(2a) \text{ in the } (x,z)\text{-plane centred at } (0,0,1/(2a)) \\
s(\Omega_2^-) &= \{(0,y,z)\in\mathbb{R}^3 \mid y^2+(z+1/(2a))^2=1/(2a)^2\} \\
&= \text{ the circle of radius } 1/(2a) \text{ in the } (y,z)\text{-plane centred at } (0,0,-1/(2a)).
\end{aligned}$$

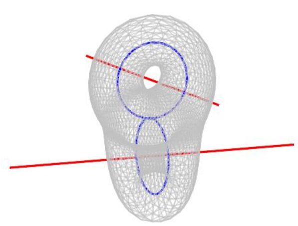
embedded/geometry_check.mpl: check_omega()

The picture for $a=1/\sqrt{2}$ is as follows:



(The set $s(\Omega^+)$ is shown in red, and the set $s(\Omega^-)$ is shown in blue.)

On Ω^+ we have $f(x)=0$ and $f(-x)>0$, whereas on Ω^- we have $f(x)>0$ and $f(-x)=0$. We defined $EX(a)=\{x\in S^3 \mid f(x)=f(-x)\}$, and we now see that this fits between Ω^+ and Ω^-. This can be displayed as follows:



It will be convenient to describe $EX(a)$ using the functions g and g_0 given below.

Definition 6.1.2. [defn-g]

We put

$$\begin{aligned}
g_0(x) &= (f(x)-f(-x))/8 + x_4(\rho(x)-1) \\
&= ((1+a^{-2})x_3^2-2)x_4 + a^{-1}(x_1^2-x_2^2)x_3 \\
g(x) &= (f(x)-f(-x))/8 - x_4(\rho(x)-1) \\
&= (a^{-2}-1)x_3^2x_4 - 2(x_1^2+x_2^2)x_4 - 2x_4^3 + a^{-1}(x_1^2-x_2^2)x_3.
\end{aligned}$$

(The advantage of $g_0(x)$ is that it has few terms, and the advantage of $g(x)$ is that it is a homogeneous cubic.)

`embedded/geometry_check.mpl: check_g()`

It is now straightforward to check that

$$EX(a) = \{x \in S^3 \mid g_0(x) = 0\} = \{x \in S^3 \mid g(x) = 0\}.$$

We put

$$\begin{aligned} \widetilde{A} &= \mathbb{R}[x_1, x_2, x_3, x_4] \\ A = \mathcal{O}_{EX(a)} &= \widetilde{A}/(\rho(x) - 1, g(x)), \end{aligned}$$

so A is the ring of polynomial functions on $EX(a)$.

Remark 6.1.3. Maple notation for the parameter a is a_E. The global variable `a_E0` is set to $1/\sqrt{2}$, and `a_E1` is a 100 digit approximation to that. Elements of $\mathbb{R}^4$ are represented in Maple as lists of length 4. The functions $\rho(x)$, $f_1(x)$, $f_2(x)$, $f(x)$, $g_0(x)$ and $g(x)$ are `rho(x)`, `f_1(x)`, `f_2(x)`, `f(x)`, `g_0(x)` and `g(x)`. The functions obtained from these by setting $a = 1/\sqrt{2}$ are `f_10(x)`, `f_20(x)`, `f0(x)`, `g_00(x)` and `g0(x)`. Note in particular the difference between `g_0(x)` and `g0(x)`.

Proposition 6.1.4. [`prop-X-smooth`]
The space $EX(a)$ is a compact smooth oriented embedded submanifold in S^3.

We will use the following notation:

Definition 6.1.5. [`defn-nx`]
We put

$$\begin{aligned} n(x) &= (\nabla g)_x = (\partial g/\partial x_1, \dots, \partial g/\partial x_4) \\ &= (2x_1(x_3/a - 2x_4),\ -2x_2(x_3/a + 2x_4), \\ &\qquad 2(a^{-2} - 1)x_3x_4 + a^{-1}(x_1^2 - x_2^2),\ (a^{-2} - 1)x_3^2 - 6x_4^2 - 2(x_1^2 + x_2^2)). \end{aligned}$$

(This is `dg(x)` in Maple.)

Proof. Define $p\colon \mathbb{R}^4 \to \mathbb{R}^2$ by $p(x) = (\rho(x) - 1, g(x))$, so $EX(a) = p^{-1}\{0\}$. We must first show that 0 is a regular value of p, or equivalently that the gradients of $\rho - 1$ and g are linearly independent at every point in $EX(a)$. Note here that the gradient of $\rho - 1$ at x is just $2x$, which is certainly nonzero at all points in $EX(a)$. Moreover, as g is a homogeneous cubic function we have $(2x).n(x) = 6g(x)$, which is zero on $EX(a)$, so the two gradients are orthogonal. It will thus be enough to show that $n(x)$ is nonzero everywhere on $EX(a)$. By direct expansion one can check that

$$\frac{1 - a^2}{16}(n(x)_1^2 + n(x)_2^2) + \frac{a}{2}(x_1^2 - x_2^2)n(x)_3 - \frac{1}{4}(x_1^2 + x_2^2)n(x)_4 = x_1^4 + x_2^4 + (\tfrac{5}{2} - a^2)(x_1^2 + x_2^2)x_4^2.$$

`embedded/geometry_check.mpl: check_g()`
`embedded/EX_check.mpl: check_smoothness()`

If $n(x) = 0$ then the left hand side vanishes so the right hand side must also vanish. After noting that $a \in (0, 1)$ so $\frac{5}{2} - a^2 > 0$, it follows that $x_1 = x_2 = 0$. After substituting this back into the equations for $n(x)$, we see that $x_3x_4 = 0$ and $(a^{-2} - 1)x_3^2 - 6x_4^2 = 0$, which easily implies that $x_3 = x_4 = 0$. Thus, the only place in $\mathbb{R}^4$ where $n(x) = 0$ is the origin, so in particular there are no points in $EX(a)$ with this property.

This completes the proof that 0 is a regular value of p, which implies in a standard way that $EX(a)$ is a smooth closed submanifold of $\mathbb{R}^4$. It is compact because it is closed in S^3. Now let ω_k denote the standard volume form in $\Lambda^k(\mathbb{R}^k)$. Standard exterior algebra now tells us that for each $x \in EX(a)$ there is a unique element $\alpha_x \in \Lambda^2(T_xEX(a)) < \Lambda^2(\mathbb{R}^4)$ such that $\alpha_x \wedge p^*(\omega_2) = \omega_4$. These forms give a smooth, nowhere vanishing section of $\Lambda^2(T)$ over $EX(a)$, and thus an orientation of $EX(a)$. $\square$

Remark 6.1.6. [`rem-move-to-X`]

The above considerations also give an efficient practical method to compute points on X numerically. For any $y \in \mathbb{R}^4$, we define

$$\sigma(y) = \frac{1}{\|y\|}\left(y - \frac{g(y)}{\|n(y)\|^2}n(y)\right).$$

One can show that if the distance from y to X is of order $\epsilon \ll 1$, then the distance from $\sigma(y)$ to X is of order ϵ^2. This implies that the sequence $(\sigma^k(x))_{k\geq 0}$ converges rapidly to a point $\sigma^\infty(x) \in X$. This is implemented by the Maple function `move_to_X(x)`.

Next, we use the metric and the orientation to give an almost complex structure on X. Explicitly, for each $x \in X$ there is a unique isometric linear map $J_x \colon T_xX \to T_xX$ such that $\langle u, J_xu\rangle = 0$ for all u, and $u \wedge J_xu$ is a positive multiple of α_x for all $u \neq 0$. This satisfies $J_x^2 = -1$ and so gives a complex structure on T_xX. All this can be verified easily after choosing an oriented orthonormal basis. As mentioned in the introduction, this can be integrated to give a complex structure on X. In more detail, if $U \subseteq X$ is open and $q \colon U \to \mathbb{C}$ is a smooth map, we say that q is *holomorphic* (or *conformal*) if for each $x \in U$, the derivative $Dq_x \colon T_xX \to \mathbb{C}$ is $\mathbb{C}$-linear. It is a nontrivial fact that for each $x \in X$ there is an open neighbourhood U of X and an injective holomorphic function $q \colon U \to \mathbb{C}$ such that $Dq_y \colon T_yX \to \mathbb{C}$ is an isomorphism for all $y \in U$. This has been known at least since the work of Korn and Lichtenstein in 1916. A modern proof is given in [6]. As we mentioned in the introduction, it can also be seen as a special case of the Newlander-Nirenberg theorem. It is clear that the transition map between any pair of holomorphic charts as above, is a function on an open domain in $\mathbb{C}$ that is holomorphic in the traditional sense. We can thus use these charts to regard X as a Riemann surface.

6.2. **The group action.** [sec-E-G]

We define linear isometric maps $\lambda, \mu, \nu : \mathbb{R}^4 \to \mathbb{R}^4$ as follows:

$$\begin{aligned}
\lambda(x_1, x_2, x_3, x_4) &= (-x_2, \quad x_1, \quad x_3, -x_4) \\
\mu(x_1, x_2, x_3, x_4) &= (\quad x_1, -x_2, -x_3, -x_4) \\
\nu(x_1, x_2, x_3, x_4) &= (\quad x_1, -x_2, \quad x_3, \quad x_4).
\end{aligned}$$

Maple notation for $g.x$ is `act_R4[g](x)`.

It is straightforward to check that

$$\lambda^4 = \mu^2 = \nu^2 = (\mu\nu)^2 = (\mu\lambda)^2 = (\nu\lambda)^2 = 1,$$

so we have an isometric action of our group G on $\mathbb{R}^4$.

`embedded/EX_check.mpl: check_R4_action()`

All three of the generators have determinant -1 and so reverse the orientation of $\mathbb{R}^4$. Note also that $\lambda^2\mu\nu(x) = -x$.

Directly from the definitions we see that

$$\begin{aligned}
\rho(\lambda(x)) = \rho(\mu(x)) = \rho(\nu(x)) &= \rho(x) \\
g(\lambda(x)) = g(\mu(x)) &= -g(x) \\
g(\nu(x)) &= g(x)
\end{aligned}$$

This means that λ, μ and ν send X to itself.

`embedded/EX_check.mpl: check_symmetry()`

Remark 6.2.1. [rem-heegard]

If we put $D_+ = \{x \in S^3 \mid g(x) \geq 0\}$ and $D_- = \{x \in S^3 \mid g(x) \leq 0\}$ then we have $D_+ \cup D_- = S^3$ and $D_+ \cap D_- = EX(a)$, and the map μ gives a homeomorphism between D_+ and D_-. Moreover, D_+ and D_- are handlebodies, with the figure eight curve Ω^- as a deformation retract of D_+, and Ω^+ as a deformation retract of D_-. This gives an unusually explicit Heegaard splitting of S^3 along $EX(a)$.

Recall that the orientation form $\alpha_x \in \Lambda^2(T_xX)$ is characterised by the equation $\alpha_x \wedge (\rho - 1, g)^*(\omega_2) = \omega_4$. As λ preserves $\rho - 1$ and changes the sign of g and ω_4, we deduce that $\lambda^*\alpha = \alpha$, so λ preserves orientation. As the complex structure on $EX(a)$ is determined by the metric and the orientation, it follows that λ is a conformal automorphism of $EX(a)$. By the same kind of logic, the map μ preserves orientation and is conformal, whereas ν reverses orientation and is anticonformal.

6.3. **Isotropy.** [sec-E-isotropy]

We now define points $v_i \in \mathbb{R}^4$ as follows:

$$
\begin{aligned}
v_0 &= (\ \ 0, \ \ 0, \ \ 1, \ \ 0) & v_6 &= (\ \ 1, \ \ 1, \ \ 0, \ \ 0)/\sqrt{2} \\
v_1 &= (\ \ 0, \ \ 0, -1, \ \ 0) & v_7 &= (-1, \ \ 1, \ \ 0, \ \ 0)/\sqrt{2} \\
v_2 &= (\ \ 1, \ \ 0, \ \ 0, \ \ 0) & v_8 &= (-1, -1, \ \ 0, \ \ 0)/\sqrt{2} \\
v_3 &= (\ \ 0, \ \ 1, \ \ 0, \ \ 0) & v_9 &= (\ \ 1, -1, \ \ 0, \ \ 0)/\sqrt{2} \\
v_4 &= (-1, \ \ 0, \ \ 0, \ \ 0) & v_{10} &= (0, 0, \sqrt{\frac{2a^2}{1+a^2}}, \sqrt{\frac{1-a^2}{1+a^2}}) \\
v_5 &= (\ \ 0, -1, \ \ 0, \ \ 0) & v_{11} &= (0, 0, \sqrt{\frac{2a^2}{1+a^2}}, -\sqrt{\frac{1-a^2}{1+a^2}}) \\
& & v_{12} &= (0, 0, -\sqrt{\frac{2a^2}{1+a^2}}, -\sqrt{\frac{1-a^2}{1+a^2}}) \\
& & v_{13} &= (0, 0, -\sqrt{\frac{2a^2}{1+a^2}}, \sqrt{\frac{1-a^2}{1+a^2}})
\end{aligned}
$$

These are `v_E[i]` in Maple.

Routine calculation shows that these points all lie in X, and that the action of G permutes them, according to the following rules:

$$
\begin{aligned}
\lambda &\mapsto (2\ 3\ 4\ 5)\ (6\ 7\ 8\ 9)\ (10\ 11)\ (12\ 13) \\
\mu &\mapsto (0\ 1)\ (3\ 5)\ (6\ 9)\ (7\ 8)\ (10\ 12)\ (11\ 13) \\
\nu &\mapsto (3\ 5)\ (6\ 9)\ (7\ 8).
\end{aligned}
$$

(For example, the cycle $(2\ 3\ 4\ 5)$ in λ means that $\lambda(v_2) = v_3$, $\lambda(v_3) = v_4$, $\lambda(v_4) = v_5$ and $\lambda(v_5) = v_2$.) This almost shows that we have a precromulent surface, except that we need to check that there are no further points with nontrivial stabiliser in D_8.

Proposition 6.3.1. [prop-no-more-isotropy]

If $x \in EX(a) \setminus \{v_0, \dots, v_{13}\}$ *then* $\mathrm{stab}_{D_8}(x) = 1$.

Proof. Consider for example a point $x \in EX(a)$ with $\lambda^2(x) = x$, or in other words $(-x_1, -x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$ or $x_1 = x_2 = 0$. The equations for $EX(a)$ become $x_3^2 + x_4^2 = 1$ and $x_4((a^{-2} - 1)x_3^2 - 2x_4^3) = 0$. If $x_4 = 0$ we must have $x = (0, 0, \pm 1, 0)$, so $x = v_0$ or $x = v_1$. Otherwise we must have $(a^{-2} - 1)x_3^2 = 2x_4^2$. In conjunction with $x_3^2 + x_4^2 = 1$ this gives $x_3^2 = 2a^2/(1+a^2)$ and $x_4^2 = (1-a^2)/(1+a^2)$ so $x \in \{v_{10}, v_{11}, v_{12}, v_{13}\}$. Note that if x is fixed by λ or $\lambda^3 = \lambda^{-1}$ then it is certainly fixed by λ^2, so again it is one of the v_i.

Similarly:

- We have $\mu(x) = x$ iff $x_2 = x_3 = x_4 = 0$, and $\lambda^2\mu(x) = x$ iff $x_1 = x_3 = x_4 = 0$. In these cases it is clear that $x \in \{v_2, v_3, v_4, v_5\}$.
- We have $\lambda\mu(x) = x$ iff $x_1 = x_2$ and $x_3 = 0$. In this context we have $g_0(x) = -2x_4$, so we must also have $x_4 = 0$, which makes it clear that $x \in \{v_6, v_8\}$.
- A similar argument shows that if $\lambda^3\mu(x) = x$ then $x \in \{v_7, v_9\}$.

`embedded/EX_check.mpl: check_fixed_points()`

□

Remark 6.3.2. [rem-tangent]

It will be useful to understand the tangent space $T_{v_0}X$ more explicitly. The formula in Definition 6.1.5 shows that the gradient of g at v_0 is

$$n(v_0) = n(e_3) = (0, 0, 0, a^{-2} - 1) = (a^{-2} - 1)e_4.$$

On the other hand, we have $\nabla(\rho-1)_{v_0} = 2v_0 = 2e_3$, so

$$(\rho-1,g)^*(\omega_2) = \nabla(\rho-1)_{v_0}\wedge\nabla(g)_{v_0} = 2(a^{-2}-1)e_3\wedge e_4.$$

The tangent space is the orthogonal complement to v_0 and $n(v_0)$, so it is spanned by the basis vectors e_1 and e_2 (which are orthonormal). The orientation form α_{v_0} must therefore be $(e_1\wedge e_2)/(2(a^{-2}-1))$, which is a positive multiple of $e_1\wedge e_2$, so $J_{v_0}(e_1) = e_2$. In other words, the map $(x+iy)\mapsto(x,y,0,0)$ gives an isometric $\mathbb{C}$-linear isomorphism $\mathbb{C}\to T_{v_0}X$. It is also clear from this that λ acts on $T_{v_0}EX(a)$ as multiplication by i.

6.4. **Associated complex varieties.** [sec-E-complex]

For $x\in\mathbb{C}^4$ we again put $\rho(x) = \sum_j x_j^2$ (not $\sum_j |x_j|^2$). We then put

$$\begin{aligned}
CEX(a) &= \{x\in\mathbb{C}^4 \mid g(x)=0,\ \rho(x)=1\}\\
PEX(a) &= \{[x]\in\mathbb{C}P^3 \mid g(x)=0\}\\
PEX'(a) &= \{[x]\in\mathbb{C}P^3 \mid g(x)=0,\ \rho(x)\neq 0\}.
\end{aligned}$$

It is clear that $CEX(a)$ is an affine variety, and $PEX(a)$ is a projective variety, and $PEX'(a)$ is a quasiprojective open subvariety of $PEX(a)$. The map $x\mapsto[x]$ gives a double covering $CEX(a)\to PEX'(a)$. We can identify $EX(a)$ with the set of real points in $CEX(a)$.

Proposition 6.4.1. *If $a\neq 1/\sqrt{2}$, then $CEX(a)$, $PEX(a)$ and $PEX'(a)$ are all smooth.*

Proof. As $PEX'(a)$ is open in $PEX(a)$ and $CEX(a)$ is a double cover of $PEX'(a)$, it will suffice to treat the case of $PEX(a)$. The partial derivatives of g are

$$\begin{aligned}
n_1 &= \partial g/\partial x_1 = \quad 2x_1(x_3/a - 2x_4)\\
n_2 &= \partial g/\partial x_2 = -2x_2(x_3/a + 2x_4)\\
n_3 &= \partial g/\partial x_3 = 2(a^{-2}-1)x_3x_4 + (x_1^2 - x_2^2)/a\\
n_4 &= \partial g/\partial x_4 = (a^{-2}-1)x_3^2 - 2x_1^2 - 2x_2^2 - 6x_4^2.
\end{aligned}$$

Put

$$U_i = \{x\in\mathbb{C}^4 \mid x_i = 1,\ n_1=n_2=n_3=n_4=0\}.$$

By well-known arguments, it will suffice to check that the sets U_i are all empty. Consider a point $x\in U_1$. The equation $n_1=0$ gives $x_3 = 2ax_4$, and we can substitute this in $n_2=0$ to get $x_2x_4=0$. If $x_2\neq 0$ then this gives $x_4=0$ so $x_3=0$, and the relation $n_3=0$ gives a contradiction. We must therefore have $x_2=0$. Putting $x_1=1$ and $x_2=0$ and $x_3=4ax_4$ in $n_3=n_4=0$ we get $2a^2=1$ and $2x_4^2=-1$. We are assuming explicitly that $a\neq 1/\sqrt{2}$ and implicitly that $a\in(0,1)$, so this is impossible. We conclude that $U_1=\emptyset$, and a similar argument gives $U_2=\emptyset$. Now consider $x\in U_3$. If we had $x_1\neq 0$ then x/x_1 would be in U_1, which is impossible. Thus $x_1=0$, and similarly $x_2=0$, and $x_3=1$ by assumption. Substituting this into $n_3=n_4=0$ gives a contradiction. A similar argument works for U_4.

```
embedded/EX_check.mpl: check_PEX_smoothness()
```

∎

Proposition 6.4.2. *None of the surfaces $CEX(1/\sqrt{2})$, $PEX(1/\sqrt{2})$ or $PEX'(1/\sqrt{2})$ is smooth. Moreover, $PEX(1/\sqrt{2})$ is isomorphic to the singular Cayley cubic with equation*

$$X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4 = 0.$$

Remark 6.4.3. Although the isomorphism with the Cayley cubic is interesting, it interacts poorly with the underlying real structure, and so is not too helpful for studying the cromulent surface $EX(1/\sqrt{2})$.

Proof. At the point $w = (\sqrt{-2},0,\sqrt{2},1)$ we find that $\rho(w)=1$ and $g(w)=0$ and all partial derivatives of g also vanish. It follows that w is a singular point in $CEX(1/\sqrt{2})$ and that $[w]$ is a singular point in $PEX'(1/\sqrt{2})\subset PEX(1/\sqrt{2})$. The same holds for all points in the G-orbit of w. (One can check that $\nu(w)=w$ and $\lambda^2\mu\nu(w) = -w$ and $|G.w|=8$ and $|G.[w]|=4$.)

Now put

$$X_1 = x_4 + x_3/\sqrt{2} + x_1\sqrt{-2} \qquad X_2 = x_4 + x_3/\sqrt{2} - x_1\sqrt{-2}$$
$$X_3 = x_4 - x_3/\sqrt{2} + x_2\sqrt{-2} \qquad X_4 = x_4 - x_3/\sqrt{2} - x_2\sqrt{-2}.$$

We find that

$$X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4 = -2g(x),$$

so this gives an isomorphism with the Cayley cubic.

`embedded/roothalf/cayley_surface_check.mpl: check_cayley_surface();`

□

6.5. The ring of functions. [sec-E-functions]

Definition 6.5.1. [defn-yzu]
We put

$$y_1 = x_3 \qquad y_2 = (x_2^2 - x_1^2 - (a^{-1} + a)x_3x_4)/(2a)$$
$$z_1 = y_1^2 \qquad z_2 = y_2^2$$

and

$$u_1 = (1 - 2ay_2)/2 - \tfrac{1}{2}(y_2 - a)(y_2 - a^{-1})y_1^2$$
$$u_2 = (1 + 2ay_2)/2 - \tfrac{1}{2}(y_2 + a)(y_2 + a^{-1})y_1^2$$
$$u_3 = 4u_1u_2 = (1 - z_1 - z_1z_2)^2 - z_2((a + a^{-1})z_1 - 2a)^2$$
$$u_4 = u_1 + u_2 = 1 - z_1 - z_1z_2$$

`embedded/invariants_check.mpl: check_invariants()`

Proposition 6.5.2. [prop-OX-basis]
The ring A of polynomial functions on $EX(a)$ can be described as

$$A = \mathbb{R}[y_1, y_2][x_1, x_2]/(x_1^2 - u_1, x_2^2 - u_2),$$

with $x_3 = y_1$ and $x_4 = -y_1y_2$. The set

$$M = \{x_1^i x_2^j y_1^k y_2^l \mid i, j, k, l \in \{0, 1\}\}$$

is a basis for A over the subring $\mathbb{R}[z_1, z_2]$.

Proof. We have $x_3 = y_1$ by definition, and the relation $g_0(x) = 0$ can easily be rearranged as $x_4 = -y_1y_2$. We also have

$$x_1^2 + x_2^2 = 1 - x_3^2 - x_4^2 = 1 - y_1^2 - y_1^2y_2^2$$
$$x_1^2 - x_2^2 = -2ay_2 - (a + a^{-1})x_3x_4 = -2ay_2 + (a + a^{-1})y_1^2y_2.$$

Adding these equations gives $x_1^2 = u_1$, and subtracting them gives $x_2^2 = u_2$. We now see that $A = \mathbb{R}[y_1, y_2]/(x_i^2 - u_i)$. It is clear from this that $\{1, x_1, x_2, x_1x_2\}$ is a basis for A over $\mathbb{R}[y_1, y_2]$, and $\{1, y_1, y_2, y_1y_2\}$ is a basis for $\mathbb{R}[y_1, y_2]$ over $\mathbb{R}[z_1, z_2]$, so M is a basis for A over $\mathbb{R}[z_1, z_2]$. □

Remark 6.5.3. The global symbols `x`, `y`, `z` (and various others) are protected (by a call to the `protect` command in the file `Rn.mpl`). This prevents the user from assigning values to these symbols, which is necessary in for our use of Gröbner bases to work properly. Variables such as `x0` or `X` can be used instead of `x` for storing points in $\mathbb{R}^4$. The variable `xx` is set equal to `[x[1],x[2],x[3],x[4]]` (a list of length 4, whose entries are unassigned symbols). Similarly, `yy` is `[y[1],y[2]]`, and `zz` is `[z[1],z[2]]`. Note although $z_1 = x_3^2$, the symbol `z[1]` does not have the value `x[3]^2`. Instead, there is another variable `zx[1]` with the value `x[3]^2`. If we have an expression `m` involving `z[1]` and `z[2]`, we can convert it to an expression in `x[1]` to `x[4]` using the syntax `subs({z[1]=zx[1],z[2]=zx[2]},m)` or `eval(subs({z=zx},m))`. Some esoteric features of Maple mean that the second form will not work correctly without `eval()`. Similarly, there are variables `yx[i]` which contain expressions for y_i in terms of x_j, and variables `uy[1]`, `uy[2]`, `uz[3]` and `uz[4]` which contain expressions for u_i in terms of y_j or z_j. We can also regard the rule $x \mapsto y$ as giving a function $\mathbb{R}^4 \to \mathbb{R}^2$. Maple notation for this function is `y_proj(x)`, and `z_proj(x)` is similar.

Remark 6.5.4. The functions `NF_x`, `NF_y` and `NF_z` can be used to simplify elements of the ring A, by reducing them modulo a suitable Gröbner basis. The function `NF_x` will convert any expression to one that involves only the variables x_i, whereas `NF_y` converts x's to y's as far as possible, and similarly for `NF_z`.

It is straightforward to check that G acts on A as follows:

$$\begin{aligned}
\lambda^*(x_1) &= -x_2 & \mu^*(x_1) &= x_1 & \nu^*(x_1) &= x_1 \\
\lambda^*(x_2) &= x_1 & \mu^*(x_2) &= -x_2 & \nu^*(x_2) &= -x_2 \\
\lambda^*(x_3) &= x_3 & \mu^*(x_3) &= -x_3 & \nu^*(x_3) &= x_3 \\
\lambda^*(x_4) &= -x_4 & \mu^*(x_4) &= -x_4 & \nu^*(x_4) &= x_4 \\
\lambda^*(y_1) &= y_1 & \mu^*(y_1) &= -y_1 & \nu^*(y_1) &= y_1 \\
\lambda^*(y_2) &= -y_2 & \mu^*(y_2) &= y_2 & \nu^*(y_2) &= y_2 \\
\lambda^*(z_1) &= z_1 & \mu^*(z_1) &= z_1 & \nu^*(z_1) &= z_1 \\
\lambda^*(z_2) &= z_2 & \mu^*(z_2) &= z_2 & \nu^*(z_2) &= z_2.
\end{aligned}$$

In particular, the group acts as the identity on $\mathbb{R}[z_1, z_2]$, and permutes the set $M \cup (-M)$. Some of this can also be expressed in terms of the characters listed in Proposition 2.1.1:

$$\begin{aligned}
\gamma^*(x_3) &= \chi_2(\gamma)x_3 & \gamma^*(x_4) &= \chi_3(\gamma)x_4 \\
\gamma^*(y_1) &= \chi_2(\gamma)y_1 & \gamma^*(y_2) &= \chi_1(\gamma)y_2 \\
\gamma^*(x_1x_2) &= \chi_4(\gamma)x_1x_2.
\end{aligned}$$

This makes it easy to analyse the invariants for various subgroups of G. The most important cases are as follows:

Proposition 6.5.5. [`prop-invariants`]
$A^G = \mathbb{R}[z_1, z_2]$ and $A^{\langle\lambda^2,\nu\rangle} = \mathbb{R}[y_1, y_2]$.

Proof. Any element $a \in A$ can be written uniquely as $a = a_0 + a_1x_1 + a_2x_2 + a_3x_1x_2$ with $a_0, \dots, a_3 \in \mathbb{R}[y_1, y_2]$. We then find that

$$\begin{aligned}
(\lambda^2)^*(a) &= a_0 - a_1x_1 - a_2x_2 + a_3x_1x_2 \\
\nu^*(a) &= a_0 + a_1x_1 - a_2x_2 - a_3x_1x_2,
\end{aligned}$$

so a is invariant under $\langle\lambda^2, \nu\rangle$ if and only if $a_1 = a_2 = a_3 = 0$ and $a = a_0 \in \mathbb{R}[y_1, y_2]$. If this holds, we can write a uniquely as $b_0 + b_1y_1 + b_2y_2 + b_3y_1y_2$ with $b_0, \dots, b_3 \in \mathbb{R}[z_1, z_2]$. We then find that

$$\begin{aligned}
\lambda^*(a) &= b_0 + b_1y_1 - b_2y_2 - b_3y_1y_2 \\
\mu^*(a) &= b_0 - b_1y_1 + b_2y_2 - b_3y_1y_2,
\end{aligned}$$

so a is invariant under all of G if and only if $b_1 = b_2 = b_3 = 0$ and $a = b_0 \in \mathbb{R}[z_1, z_2]$. □

6.6. **The curve system.** [`sec-E-curves`]

In this section we will construct a curve system for $EX(a)$.

Definition 6.6.1. [`defn-slices`]
We put $X_k = \{x \in EX(a) \mid x_k = 0\}$.

Proposition 6.6.2. [`prop-slices-a`]

The fixed points of antiholomorphic elements of G are as follows:

$$\begin{aligned}
EX(a)^{\mu\nu} &= \{x \in EX(a) \mid x_3 = x_4 = 0\} = X_3 \cap X_4 \\
EX(a)^{\lambda\nu} &= \{x \in EX(a) \mid x_1 = x_2,\ x_4 = 0\} \subseteq X_4 \\
EX(a)^{\lambda^3\nu} &= \{x \in EX(a) \mid x_1 = -x_2,\ x_4 = 0\} \subseteq X_4 \\
EX(a)^{\lambda^2\nu} &= \{x \in EX(a) \mid x_1 = 0\} = X_1 \\
EX(a)^{\nu} &= \{x \in EX(a) \mid x_2 = 0\} = X_2.
\end{aligned}$$

Proof. Immediate from formulae for the action of the relevant group elements on $\mathbb{R}^4$. □

Lemma 6.6.3. [lem-a-order]
Put $a^* = \sqrt{(a^{-2}-1)/2}$, so $2at^2 + a - a^{-1} = 0$ iff $t = \pm a^*$. Then:

- *If $0 < a < 1/\sqrt{2}$ we have*
$$-a^* < 0 < a < \frac{1}{2a} < a^* < \frac{1}{a}.$$
- *If $a = 1/\sqrt{2}$ we have*
$$-a^* < 0 < a = \frac{1}{2a} = a^* < \frac{1}{a}.$$
- *If $1/\sqrt{2} < a < 1$ we have*
$$-a^* < 0 < a^* < \frac{1}{2a} < a < \frac{1}{a}.$$

Proof. Straightforward. □

Definition 6.6.4. [defn-T-alg]
For all $a \in (0,1)$ we put

$$\begin{aligned}
T^-_{\text{alg}} &= [-a^*, 0] \\
T^+_{\text{alg}} &= [\min(1/(2a), a^*), \max(1/(2a), a^*)] = \begin{cases} [1/(2a), a^*] & \text{if } 0 < a \leq 1/\sqrt{2} \\ [a^*, 1/(2a)] & \text{if } 1/\sqrt{2} \leq a < 1. \end{cases} \\
T_{\text{alg}} &= T^-_{\text{alg}} \cup T^+_{\text{alg}}
\end{aligned}$$

Note that $1/a$ is never in T_{alg}, and a is in T_{alg} if and only if $a = 1/\sqrt{2}$. Note also that T^+_{alg} and T^-_{alg} are nonempty disjoint closed sets.

Proposition 6.6.5. [prop-T-alg]
Consider a point $x \in X_1$ (so $x_1 = 0$), and define $y_1 = x_3$ and $y_2 = (x_2^2 - (a + a^{-1})x_3x_4)/(2a)$ as in Definition 6.5.1. Then

$$y_1^2(y_2 - a)(y_2 - a^{-1}) = -2a(y_2 - 1/(2a)) \tag{A}$$
$$x_2^2(y_2 - a)(y_2 - a^{-1}) = 2ay_2(y_2^2 - (a^*)^2) \tag{B}$$
$$\left(y_1^2 - \frac{2}{a^{-2}+1}\right)(y_2 - a)(y_2 - a^{-1}) = \frac{2}{a^{-2}+1}((a^*)^2 - y_2^2) \tag{C}$$

Thus $y_2 \in T^+_{\text{alg}} \amalg T^-_{\text{alg}}$, and if $y_2 \in T^-_{\text{alg}}$ then

$$y_1 \in \left[-1, -\sqrt{\frac{2}{a^{-2}+1}}\right] \amalg \left[\sqrt{\frac{2}{a^{-2}+1}}, 1\right].$$

Proof. Recall from Proposition 6.5.2 that $x_1^2 = u_1$, where

$$u_1 = (1 - 2ay_2)/2 - \tfrac{1}{2}(y_2 - a)(y_2 - a^{-1})y_1^2.$$

Here $x_1 = 0$ so $u_1 = 0$, and this can be rearranged to give equation (A). Next, as $x_1 = 0$ and $x_3 = y_1$ and $x_4 = -y_1y_2$ we have $x_2^2 = 1 - x_3^2 - x_4^2 = 1 - (1 + y_2^2)y_1^2$. We can thus take the equation

$$(y_2 - a)(y_2 - a^{-1}) = y_2^2 - (a + a^{-1})y_2 + 1 \tag{D}$$

and subtract $(1+y_2^2)$ times equation (A) to get equation (B). Alternatively, we can subtract $2/(a^{-2}+1)$ times (D) from (A) and rearrange slightly to get (C). Now consider the signs of the left and right hand sides of (A), bearing in mind Lemma 6.6.3; it follows that we must have $y_2 \in T_{\text{alg}}$. Suppose in fact that $y_2 \in T_{\text{alg}}^-$, so $-a^* \leq y_2 \leq 0$. On the left hand side of (C) the factors $y_2 - a^{\pm 1}$ are both strictly negative, and on the right hand side $(a^*)^2 - y_2^2 \geq 0$. It therefore follows from (D) that $y_1^2 \geq 2/(a^{-2}-1)$. On the other hand, we also have $y_1^2 = x_3^2 = 1 - x_2^2 - x_4^2 \leq 1$. The relation

$$y_1 \in \left[-1, -\sqrt{\frac{2}{a^{-2}+1}}\right] \amalg \left[\sqrt{\frac{2}{a^{-2}+1}}, 1\right]$$

is now clear. □

Remark 6.6.6. [rem-T-alg]
We have $\lambda(X_1) = \lambda^{-1}(X_1) = X_2$ and $\lambda^*(y_1) = y_1$ and $\lambda^*(y_2) = -y_2$. Using this we deduce that when $x \in X_2$ we have $-y_2 \in T_{\text{alg}}$, and if $-y_2 \in T_{\text{alg}}^-$ we again have

$$y_1 \in \left[-1, -\sqrt{\frac{2}{a^{-2}+1}}\right] \amalg \left[\sqrt{\frac{2}{a^{-2}+1}}, 1\right].$$

Definition 6.6.7. [defn-C-star]
We put

$$\begin{aligned}
C_0^* &= \{(\cos(t),\ \sin(t),\ 0,\ 0) \mid t \in \mathbb{R}\} \\
C_1^* &= \{(\ \ \sin(t)/\sqrt{2},\ \sin(t)/\sqrt{2},\ \cos(t),\ 0) \mid t \in \mathbb{R}\} \\
C_2^* &= \{(-\sin(t)/\sqrt{2},\ \sin(t)/\sqrt{2},\ \cos(t),\ 0) \mid t \in \mathbb{R}\} \\
C_3^* &= \{x \in X_1 \mid \quad y_2 \in T_{\text{alg}}^+\} \\
C_4^* &= \{x \in X_2 \mid -y_2 \in T_{\text{alg}}^+\} \\
C_5^* &= \{x \in X_2 \mid -y_2 \in T_{\text{alg}}^-,\ y_1 \geq 0\} \\
C_6^* &= \{x \in X_1 \mid \quad y_2 \in T_{\text{alg}}^-,\ y_1 \geq 0\} \\
C_7^* &= \{x \in X_2 \mid -y_2 \in T_{\text{alg}}^-,\ y_1 \leq 0\} \\
C_8^* &= \{x \in X_1 \mid \quad y_2 \in T_{\text{alg}}^-,\ y_1 \leq 0\}.
\end{aligned}$$

It is straightforward to check that $C_k = C_k^*$ for $k \in \{0,1,2\}$. Using Proposition 6.6.5 and Remark 6.6.6 we see that

$$\begin{aligned}
X^{\lambda^2\nu} &= X_1 = C_3^* \amalg C_6^* \amalg C_8^* \\
X^{\nu} &= X_2 = C_4^* \amalg C_5^* \amalg C_7^*.
\end{aligned}$$

Next, recall that C_3 is the component of $X^{\lambda^2\nu} = X_1$ containing v_{11}. One can check that $y_2 = a^*$ at v_{11}, so $v_{11} \in C_3^*$. By connectivity, it follows that $C_3 \subseteq C_3^*$. The same line of argument shows that $C_k \subseteq C_k^*$ for all $k \in \{3,\dots,8\}$, and C_k will be the same as C_k^* if C_k^* is connected. To prove that this holds we need to parameterise C_k^*. We will do this in two different ways.

Definition 6.6.8. [defn-c-alg]
We put

$$c_{\text{alg}}(t) = \left(0, \sqrt{\frac{(2at^2+a-a^{-1})t}{(t-a)(t-a^{-1})}}, \sqrt{\frac{1-2at}{(t-a)(t-a^{-1})}}, -t\sqrt{\frac{1-2at}{(t-a)(t-a^{-1})}}\right).$$

One can check using Lemma 6.6.3 that this defines a continuous map $c_{\text{alg}}\colon T_{\text{alg}} \to \mathbb{R}^4$ except in the case $a = 1/\sqrt{2}$, when the domain of c_{alg} is $T_{\text{alg}} \setminus \{a\} = T_{\text{alg}}^-$.

The map $c_{\text{alg}}(t)$ is defined in `embedded/extra_curves.mpl` as `c_algebraic(t)`.

We will show that the map c_{alg} is essentially inverse to $y_2\colon X_1 \to T_{\text{alg}}$. The formula is forced by the identities in Proposition 6.6.5. A precise statement of the key property is as follows.

Proposition 6.6.9. [prop-c-alg]
The image of c_{alg} is contained in X_1, and we have $y_2(c_{\mathrm{alg}}(t)) = t$. Conversely, consider a point $x \in X_1$, so $y_2 \in T_{\mathrm{alg}}$. Except in the case where $y_2 = a = 1/\sqrt{2}$, there is an element $\gamma \in \{1, \mu, \nu, \mu\nu\}$ such that $x = \gamma(c_{\mathrm{alg}}(y_2))$. If $x_2, x_3 \geq 0$ then we can take $\gamma = 1$.

Proof. First, it is straightforward to check that $\rho(c_{\mathrm{alg}}(t)) = 1$ and $g_0(c_{\mathrm{alg}}(t)) = 0$ so the image of c_{alg} is contained in X_1. When $x_3 \neq 0$ we have $y_2 = -x_4/x_3$, and using this we see that $y_2(c_{\mathrm{alg}}(t)) = t$ except possibly when $t = 1/(2a)$, but that case can be recovered by continuity.

For the converse, consider a point $x \in X_1$, and suppose we are not in the exceptional case where $y_2 = a = 1/\sqrt{2}$, so $(y_2 - a)(y_2 - a^{-1}) \neq 0$ and the identities in Proposition 6.6.5 can be rearranged to give formulae for x_2^2 and x_3^2 in terms of y_2. Recall also that $x_4 = -x_3 y_2$. Using this, we see that the point $x' = c_{\mathrm{alg}}(y_2)$ satisfies $x_2 = \pm x_2'$ and $x_3 = \pm x_3'$. Recall also that

$$\begin{aligned}\mu(0, x_2, x_3, x_4) &= (0, -x_2, -x_3, -x_4)\\ \nu(0, x_2, x_3, x_4) &= (0, -x_2, \quad x_3, \quad x_4),\end{aligned}$$

and that y_2 is invariant under μ and ν. The claim now follows easily. $\square$

It would be possible to prove $C_k = C_k^*$ using only c_{alg}, with a slight digression to cover the case $a = 1/\sqrt{2}$, but we prefer to use a full curve system instead.

Definition 6.6.10. [defn-E-curves]
We define maps $c_k \colon \mathbb{R} \to \mathbb{R}^4$ as follows.

$$\begin{aligned}
c_0(t) &= (\cos(t), \sin(t), 0, 0)\\
c_1(t) &= (\sin(t)/\sqrt{2}, \sin(t)/\sqrt{2}, \cos(t), 0)\\
c_2(t) &= \lambda(c_1(t))\\
p_3(t) &= (1+a^2)\sin(t)^2 + \sqrt{(1+a^2)(1-a^2+2a^2\sin(t)^2) + (1-a^2)^2\cos(t)^4}\\
c_3(t) &= \left(0, \sqrt{\frac{2(1-a^2)+4a^2\sin(t)^2}{p_3(t)}}\sin(t),\right.\\
&\qquad \left.\sqrt{\frac{2}{1+a^{-2}}}\cos(t), \sqrt{\frac{2}{1+a^{-2}}}\frac{a^{-1}-a+2a\sin(t)^2}{p_3(t)}\cos(t)\right)\\
c_4(t) &= \lambda(c_3(t))\\
\tau_5(t) &= -\sqrt{(a^{-2}-1)/2}\,\sin(t/2)^2\\
p_5(t) &= (\tau_5(t)-a)(\tau_5(t)-a^{-1})\\
c_5(t) &= \left(\frac{(a^{-2}-1)^{3/4}}{2^{5/4}}\sqrt{\frac{a(1+\sin(t/2)^2)}{p_5(t)}}\sin(t), 0, \sqrt{\frac{1-2a\tau_5(t)}{p_5(t)}}, -\tau_5(t)\sqrt{\frac{1-2a\tau_5(t)}{p_5(t)}}\right)\\
c_6(t) &= \lambda(c_5(t)); \quad c_7(t) = \mu(c_5(t)); \quad c_8(t) = \lambda(\mu(c_5(t))).
\end{aligned}$$

Maple notation for these is `c_E[k](t)`.

Remark 6.6.11. We have written $p_3(t)$ in a form which makes it clear that it is always strictly positive, and using this it is not hard to see that $c_3(t)$ is well-defined. It is also clear that $\tau_5(t) \leq 0$ and so $p_5(t) \geq 1$, which in turn implies that $c_5(t)$ is well-defined. The map c_5 is essentially $c_{\mathrm{alg}} \circ \tau_5$ except that c_{alg} involves nonnegative square roots of certain quantities, whereas c_5 uses different branches of these roots that are sometimes negative. A similar approach would be possible for c_3 but would run into problems as a passes through $1/\sqrt{2}$.

Proposition 6.6.12. [prop-E-curves]
The images of the above maps lie in $EX(a)$, and they give a curve system for $EX(a)$. Moreover, we have $c_k(\mathbb{R}) = C_k = C_k^*$ for all k.

Proof. We first need to check that $c_k(\mathbb{R}) \subseteq EX(a)$, or equivalently $\rho(c_k(t)) = 1$ and $g_0(c_k(t)) = 0$. This is easy for $k \in \{0,1,2\}$. The algebra is harder for $k \in \{3,5\}$ but there is no conceptual difficulty, it is just a lengthy exercise in trigonometric simplification. The remaining cases $k \in \{4,6,7,8\}$ follow from the cases $k \in \{3,5\}$ using the group action.

Next, it is elementary to check all the group transformation equations in axiom (c) of Definition 2.4.4, and to check all the identities $c_i(\theta) = v_j$ corresponding to the nonempty boxes in axiom (b).

We now address axiom (a). The cases $k \in \{0,1,2\}$ are again easy, and the cases $k \in \{4,6,7,8\}$ will follow from $k \in \{3,5\}$ by symmetry. Suppose that $c_3(t) = c_3(u)$. By looking at the third component, we see that $\cos(t) = \cos(u)$, and thus that $\sin(t) = \pm\sin(u)$, and thus that $p_3(t) = p_3(u)$. After recalling that $p_3 > 0$ and inspecting the second component we deduce that $\sin(t) = \sin(u)$, so $t - u \in 2\pi\mathbb{Z}$. Similarly, if $c_3'(t) = 0$ then by looking at the third component we see that $\sin(t) = 0$, so $t = n\pi$ for some $n \in \mathbb{Z}$. This means that the functions cos and $\sin^2$ are both constant to first order near t, so the same is true of p_3. By inspecting the second component of c_3 we deduce that sin must also be constant to first order, so $\cos(t) = 0$, which is impossible. This proves all claims for c_3.

For c_5, we first observe that the map $y_2 \colon EX(a) \to \mathbb{R}$ is invariant under μ and ν and satisfies $y_2(c_5(t)) = -\tau_5(t)$. Thus, if $c_5(t) = c_5(u)$ then $\tau_5(t) = \tau_5(u)$, which easily gives $\cos(t) = \cos(u)$. Given this, the rest of the argument is essentially the same as for c_3. This completes the proof of axiom (a).

Now note that $c_3(\mathbb{R})$ is contained in $X^{\lambda^2\nu} = X_1$ and is connected and contains $c_3(0) = v_{11}$, but C_3 is defined to be the component of v_{11} in $X^{\lambda^2\nu}$, so $c_3(\mathbb{R}) \subseteq C_3 \subseteq C_3^*$. Moreover, $y_2(c_3(\mathbb{R}))$ is a connected subset of T^+_{alg} containing both of the endpoints $a^* = y_2(c_3(0))$ and $1/(2a) = y_2(c_3(\pi/2))$, so $y_2(c_3(\mathbb{R})) = T^+_{\text{alg}}$. Thus, if $x \in C_3^*$ then there exists $t \in \mathbb{R}$ with $y_2(c_3(t)) = y_2(x)$, and it follows that $x = \gamma(c_3(t))$ for some $\gamma \in \{1, \mu, \nu, \mu\nu\}$. As $\mu(c_3(t)) = c_3(t+\pi)$ and $\nu(c_3(t)) = c_3(-t)$ we deduce that $x \in c_3(\mathbb{R})$. In conclusion, we have $c_3(\mathbb{R}) = C_3 = C_3^*$. Similar arguments give $c_k(\mathbb{R}) = C_k = C_k^*$ for all $k \in \{3,\dots,8\}$. Recall also that the sets C_3^*, C_6^* and C_8^* are disjoint (immediately from the definitions). Proposition 2.4.8 therefore guarantees that we have a curve system. □

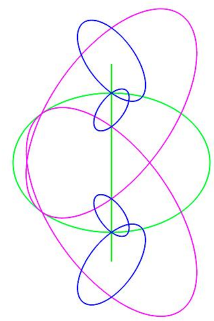
Proposition 6.6.13. [prop-slices]
$X_3 = C_0$ and $X_4 = C_0 \cup C_1 \cup C_2$.

Proof. First, it is straightforward to check that $X_3 \supseteq C_0$ and $X_4 \supseteq C_0 \cup C_1 \cup C_2$. For the converse, suppose that $x \in X_3$. Then the relation $x_4 = -y_1y_2 = -x_3y_2$ shows that x_4 vanishes as well as x_3, and we also have $\|x\| = 1$ so $x \in C_0$. Suppose instead that $x \in X_4 \setminus X_3$. As $x_4 = 0 \neq x_3$ the relation $g_0(x) = 0$ becomes $2(x_1 - x_2)(x_1 + x_2)x_3 = 0$ and we can divide by x_3 to get $x_1 = \pm x_2$. We also have $\|x\| = 1$ and it follows easily that $x \in C_1 \cup C_2$. □

Remark 6.6.14. [rem-curves-roothalf]
The formulae for c_3 and c_5 simplify significantly in the case $a = 1/\sqrt{2}$:

$$c_3(t) = \left(0, \sin(t), \sqrt{2/3}\cos(t), -\sqrt{1/3}\cos(t)\right)$$
$$c_5(t) = \left(-\sin(t), 0, 2^{3/2}, \cos(t) - 1\right)/\sqrt{10 - 2\cos(t)}.$$

The following picture shows all the curves $c_k(t)$ in that case.



6.7. **Fundamental domains.** [sec-E-fundamental]

In this section we define and study retractive fundamental domains in $EX(a)$ for certain subgroups of G. It is convenient to start with an easy case:

Definition 6.7.1. [defn-F-two]

We put $H_2 = \{1, \lambda^2\nu\}$ and recall that $C_2 = \{1, \lambda^2\}$, noting also that

$$\lambda^2(x) = (-x_1,\ -x_2,\ x_3,\ x_4)$$
$$\lambda^2\nu(x) = (-x_1,\quad x_2,\ x_3,\ x_4).$$

We then put $F_2 = \{x \in EX(a) \mid x_1 \geq 0\}$, and define $r_2 \colon EX(a) \to F_2$ by

$$r_2(x) = (|x_1|,\ x_2,\ x_3,\ x_4).$$

Proposition 6.7.2. [prop-F-two]

F_2 is a retractive fundamental domain for H_2, with retraction r_2. Moreover, F_2 is also a non-retractive fundamental domain for C_2.

Proof. Clear. □

We will see later that F_2 is homeomorphic to a disc with two holes.

Definition 6.7.3. [defn-F-four]

We put $H_4 = \{1, \lambda^2, \nu, \lambda^2\nu\}$, recalling that

$$\lambda^2(x) = (-x_1,\ -x_2,\ x_3,\ x_4)$$
$$\nu(x) = (\quad x_1,\ -x_2,\ x_3,\ x_4)$$
$$\lambda^2\nu(x) = (-x_1,\quad x_2,\ x_3,\ x_4).$$

We then put $F_4 = \{x \in EX(a) \mid x_1, x_2 \geq 0\}$, and define $r_4 \colon EX(a) \to F_4$ by

$$r_4(x) = (|x_1|,\ |x_2|,\ x_3, x_4).$$

We also put

$$F_4^* = \{y \in \mathbb{R}^2 \mid u_1, u_2 \geq 0\},$$

where u_1 and u_2 are defined in terms of y_1 and y_2 as in Definition 6.5.1:

$$u_1 = (1 - 2ay_2)/2 - \tfrac{1}{2}(y_2 - a)(y_2 - a^{-1})y_1^2$$
$$u_2 = (1 + 2ay_2)/2 - \tfrac{1}{2}(y_2 + a)(y_2 + a^{-1})y_1^2.$$

Proposition 6.7.4. [prop-F-four]
F_4 is a retractive fundamental domain for H_4, with retraction r_4. Moreover, there is a map $p_4\colon EX(a) \to F_4^*$ given by

$$p_4(x) = (x_3, (x_2^2 - x_1^2 - (a^{-1} + a)x_3x_4)/(2a)) = (y_1, y_2),$$

and a map $s_4\colon F_4^* \to F_4$ given by

$$s_4(y) = (\sqrt{u_1}, \sqrt{u_2}, y_1, -y_1y_2),$$

and these satisfy $p_4s_4 = 1$ and $s_4p_4 = r_4$. Thus, p_4 restricts to give a homeomorphism $F_4 \to F_4^*$ with inverse s_4.

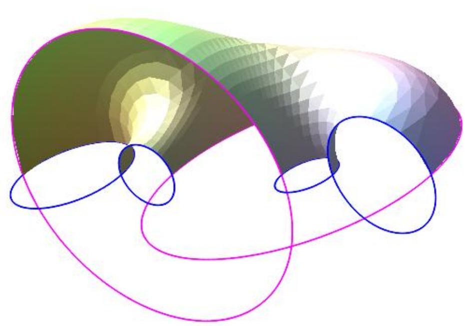
Maple notation for $p_4(x)$ and $s_4(t)$ is `y_proj(x)` and `y_lift(t)`.

Proof. It is clear that F_4 is a retractive fundamental domain for H_4, with retraction r_4. Recall that the ring of functions on $EX(a)$ is generated by y_1, y_2, x_1 and x_2, with $x_i^2 = u_i$ for $i = 1, 2$ and $x_3 = y_1$ and $x_4 = -y_1y_2$. It follows that u_1 and u_2 are nonnegative as functions on $EX(a)$, or equivalently that $p_4(EX(a)) \subseteq F_4^*$. It also follows that the stated formula gives a well-defined map $s_4\colon F_4^* \to F_4$, and it is straightforward to check that $p_4s_4 = 1$ and $s_4p_4 = r_4$.

`embedded/EX_check.mpl: check_E_F4()`

□

The following picture shows F_4 together with the curves $c_3, \dots, c_8$.

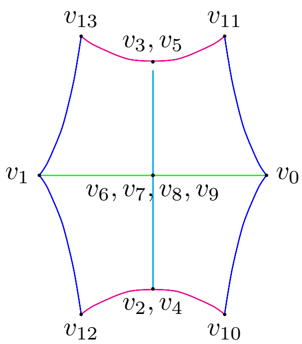


Formulae for the action of p_4 on some of the curves c_i are as follows:

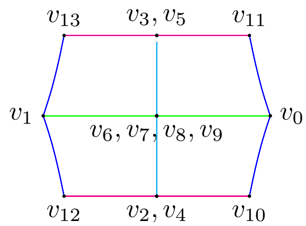
$$p_4(c_{\mathrm{alg}}(t)) = \left(\sqrt{\frac{1-2at}{(t-a)(t-a^{-1})}}, t\right)$$
$$p_4(c_0(t)) = (0, -\cos(2t)/(2a))$$
$$p_4(c_1(t)) = p_4(c_2(t)) = (\cos(t), 0).$$

Formulae for the remaining curves are not illuminating.

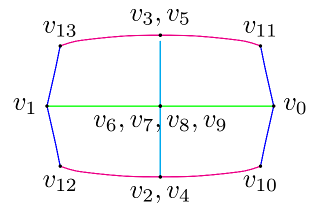
The following picture shows the set F_4^* for three different values of the parameter a, together with the images of the points v_i under the map p_4.



$a = 1/2$

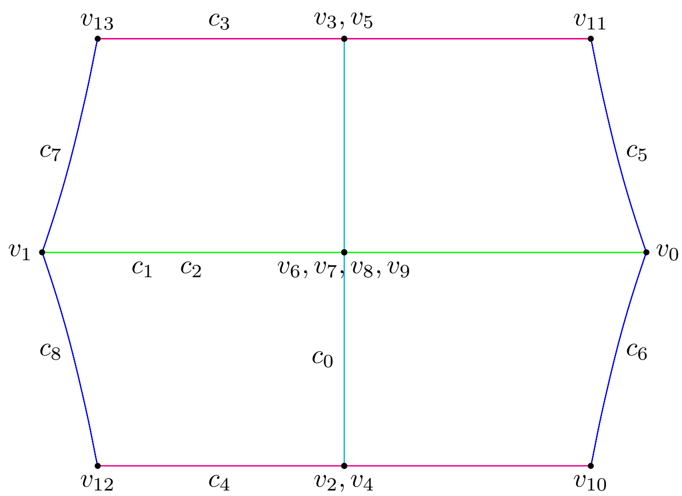


$a = 1/\sqrt{2}$

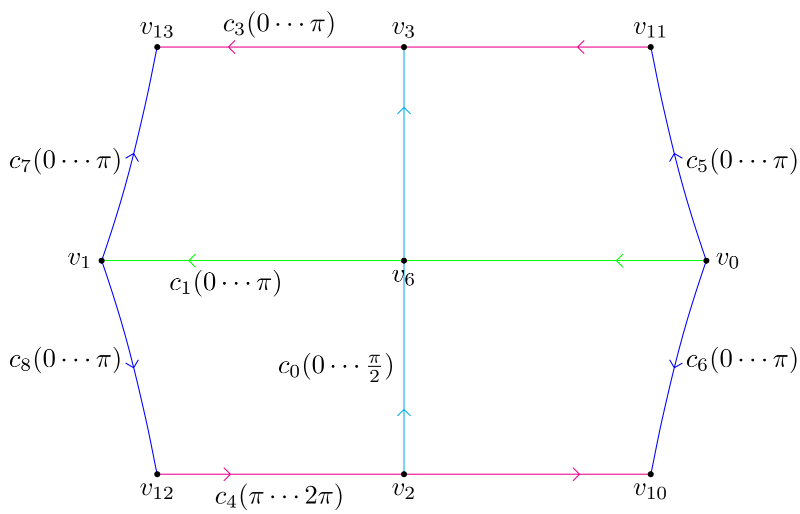


$a = 4/5$

Here is a more detailed version for the case $a = 1/\sqrt{2}$:



The above picture shows the images of all the points v_i, but it is also useful to restrict attention to those that lie in F_4:



The annotation $c_1(0 \cdots \pi)$ indicates that c_1 sends the interval $[0, \pi]$ to F_4. For values $t \in (\pi, 2\pi)$, the point $c_1(t)$ lies outside F_4, but it has the same H_4-orbit as some point $c_1(t')$ with $t' \in [0, \pi]$, so $p_4 c_1(t)$ will still lie on the middle horizontal line in the above diagram. The other annotations should be interpreted in the same way.

We now define a retractive fundamental domain for the full group G.

Definition 6.7.5. [defn-F-sixteen]
We put

$$F_{16} = \{x \in EX(a) \mid x_1, x_2, y_1, y_2 \geq 0\}$$
$$r_{16}(x) = \begin{cases} (|x_1|, |x_2|, |x_3|, -|x_4|) & \text{if } y_2 \geq 0 \\ (|x_2|, |x_1|, |x_3|, -|x_4|) & \text{if } y_2 \leq 0. \end{cases}$$
$$F_{16}^* = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1, z_2, u_3, u_4 \geq 0\}$$

Here y_1, y_2, z_1, z_2, u_3 and u_4 are as in Definition 6.5.1. Note that if $y_2 = 0$ then $x_4 = -y_1 y_2 = 0$ so the relation $y_2 = 0$ becomes $x_2^2 = x_1^2$, so $|x_1| = |x_2|$; this shows that r_{16} is well-defined.

Proposition 6.7.6. [prop-F-sixteen]
F_{16} is a retractive fundamental domain for G, with retraction r_{16}. Moreover, there is a map $p_{16}\colon EX(a) \to F_{16}^*$ given by

$$p_{16}(x) = (x_3^2, (x_2^2 - x_1^2 - (a^{-1} + a)x_3 x_4)^2/(4a^2)) = (z_1, z_2),$$

and a map $s_{16}\colon F_{16}^* \to F_{16}$ given by

$$s_{16}(z_1, z_2) = s_4(\sqrt{z_1}, \sqrt{z_2})$$

and these satisfy $p_{16}s_{16} = 1$ and $s_{16}p_{16} = r_{16}$. Thus, p_{16} restricts to give a homeomorphism $F_{16} \to F_{16}^*$ with inverse s_{16}.

Maple notation for $p_{16}(x)$ and $s_{16}(t)$ is `z_proj(x)` and `z_lift(t)`.

Proof. First, a straightforward check of cases shows that $r_{16}(EX(a)) \subseteq F_{16}$ and that r_{16} is the identity on F_{16}, so r_{16} is a retraction. We also claim that $r_{16}(\gamma(x)) = r_{16}(x)$ for all $x \in EX(a)$ and $\gamma \in G$. If $\gamma \in \langle \lambda^2, \mu, \nu \rangle$ then $|\gamma(x)_i| = |x_i|$ for all i and $y_2(\gamma(x)) = y_2(x)$ so everything is easy. This just leaves the case $\gamma = \lambda$. Here γ^* exchanges $|x_1|$ and $|x_2|$, and changes the sign of y_2, so we again have $r_{16}(\gamma(x)) = r_{16}(x)$. It follows that r_{16} induces a surjective map $EX(a)/G \to F_{16}$.

We now show that any point $x \in EX(a)$ can be moved into F_{16} by the action of G. First, after applying an element of H_4 we may assume that $x \in F_4$, so $x_1, x_2 \geq 0$. Now note that

$$\begin{aligned} \lambda\mu(x) &= (x_2, x_1, -x_3, x_4) & y_2(\lambda\mu(x)) &= -y_2(x) \\ \lambda\nu(x) &= (x_2, x_1, x_3, -x_4) & y_2(\lambda\nu(x)) &= -y_2(x) \\ \mu\nu(x) &= (x_1, x_2, -x_3, -x_4) & y_2(\mu\nu(x)) &= y_2(x). \end{aligned}$$

We can thus apply one of the maps $1, \lambda\mu, \lambda\nu, \mu\nu$ to move into F_{16}, as required. Note also that if $\gamma(x) = a \in F_{16}$ then we can apply r_{16} to deduce that $a = r_{16}(x)$, so $x = \gamma^{-1}(r_{16}(x))$. It follows that $r_{16}(x) = r_{16}(x')$ iff $Gx = Gx'$, so the induced map $EX(a)/G \to F_{16}$ is a bijective retraction and therefore a homeomorphism.

Now put

$$E = \{x \in F_{16} \mid x_1 = 0 \text{ or } x_2 = 0 \text{ or } y_1 = 0 \text{ or } y_2 = 0\}.$$

The set $F_{16} \setminus E$ is defined by strict inequalities and so is contained in the interior of F_{16}. To understand the structure of E, it is helpful to recall that $c_k(\mathbb{R}) = C_k = C_k^*$ for all k, where C_k^* was defined in Definition 6.6.7. Using this, we see that

$$E = c_0([\tfrac{\pi}{4}, \tfrac{\pi}{2}]) \cup c_1([0, \tfrac{\pi}{2}]) \cup c_3([0, \tfrac{\pi}{2}]) \cup c_5([0, \pi]).$$

Using this, we can show that arbitrarily close to every point in E there are points outside F_{16}. For example, if $x = c_0(t) = (\cos(t), \sin(t), 0, 0)$ with $\pi/4 \leq t \leq \pi/2$ then the vector $(0, 0, -1, -\cos(2t))$ is tangent to $EX(a)$ at x, and after moving a small distance in this direction we reach the region where $y_1 = x_3 < 0$. Similar arguments work for the other cases, so we see that E is precisely the boundary of F_{16}.

Now consider a point $x \in F_{16}$ and an element $\gamma \in G \setminus \{1\}$ such that $\gamma(x)$ also lies in F_{16}. By applying r_{16}, we see that $\gamma(x) = x$. In Section 6.6 we discussed the fixed sets $EX(a)^\gamma$ for all orientation-reversing elements of G, and in particular we saw that $EX(a)^\gamma$ is always contained in $\bigcup_{i=0}^{8} C_i$. Thus, if γ reverses orientation then $x \in E$. On the other hand, if γ preserves orientation then Proposition 6.3.1 tells us that

$x = v_i$ for some i with $0 \le i \le 13$. The only points of this type lying in F_{16} are v_0, v_3, v_6 and v_{11}, but we have

$$\begin{aligned} v_0 &= c_1(0) = c_5(0) \\ v_3 &= c_0(\pi/2) = c_3(\pi/2) \\ v_6 &= c_0(\pi/4) = c_1(\pi/2) \\ v_{11} &= c_3(\pi/2) = c_5(\pi) \end{aligned}$$

so these points are also in E. This proves that $\mathrm{int}(F_{16}) \cap \gamma(F_{16}) = \emptyset$. We conclude that F_{16} is a retractive fundamental domain, as claimed.

We now need to show that $p_{16}(EX(a)) \subseteq F_{16}^*$, or equivalently that $z_1, z_2, u_3, u_4 \ge 0$ as functions on $EX(a)$. This is clear from the identities $z_i = y_i^2$ and $u_3 = 4u_1u_2 = 4x_1^2x_2^2$ and $u_4 = u_1 + u_2 = x_1^2 + x_2^2$.

Now suppose we start with a point $z \in F_{16}^*$, and define $y_i = \sqrt{z_i}$ for $i = 1, 2$, and then

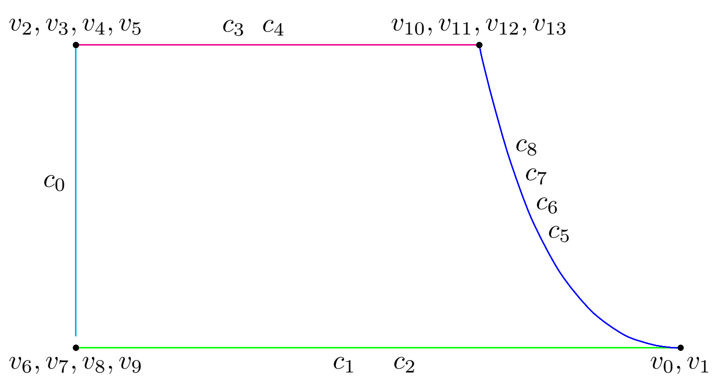
$$\begin{aligned} u_1 &= \tfrac{1}{2}(1 - y_2) - \tfrac{1}{2}(y_2 - a)(y_2 - a^{-1})y_1^2 \\ u_2 &= \tfrac{1}{2}(1 + y_2) - \tfrac{1}{2}(y_2 + a)(y_2 + a^{-1})y_1^2. \end{aligned}$$

We find that $u_1 + u_2 = 1 - z_1 - z_1z_2$ and $4u_1u_2 = (1 - z_1 - z_1z_2)^2 - z_2((a + a^{-1})z_1 - 2a)^2$; these are the quantities u_3 and u_4 that are assumed to be nonnegative by the definition of F_{16}^*. It follows from this by a check of cases that both u_1 and u_2 are nonnegative, so $y \in F_4^*$ and the point $x = s_4(y) = (\sqrt{u_1}, \sqrt{u_2}, y_1, -y_1y_2)$ is a well-defined element of F_4. It is clear by construction that in fact $x \in F_{16}$, so we have a well-defined map $s_{16} \colon F_{16}^* \to F_{16}$ with the claimed properties.

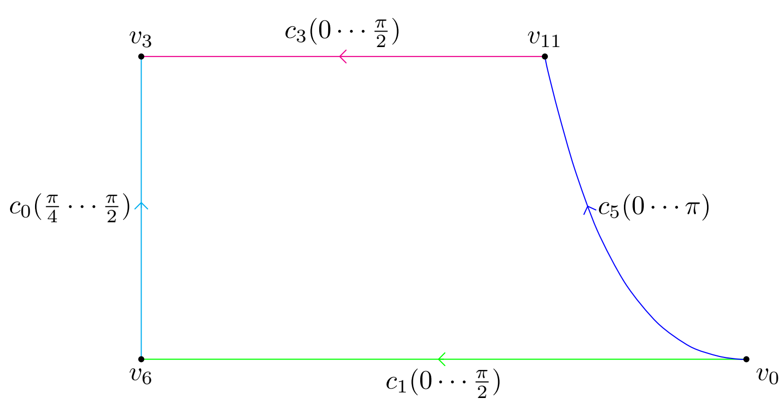
`embedded/EX_check.mpl: check_E_F16()`

□

The following picture shows the set F_{16}^* (where $a = 1/\sqrt{2}$) together with the images under p_{16} of the curves $c_i(t)$ (for $0 \le i \le 8$) and the points v_j (for $0 \le j \le 13$).



If we show only the curves and vertices lying in F_{16}, we obtain the following picture:



It is clear from these pictures that F_{16}^* is homeomorphic to the unit square. It is convenient to have an explicit homeomorphism, which is provided by the following result. (Later we will give other homeomorphisms that are specific to the case $a = 1/\sqrt{2}$, and have better properties.)

Proposition 6.7.7. [prop-square]
Put

$$w_1 = \frac{z_1(1 + (a + a^{-1})\sqrt{z_2} + z_2)}{1 + 2a\sqrt{z_2}} \qquad\qquad w_2 = \sqrt{z_2}\frac{2a(1 - z_1) + z_1\sqrt{z_2}}{1 - z_1 + (a^{-1} - a)z_1\sqrt{z_2}},$$

with the convention that $w_2 = 0$ at points where $1 - z_1 + (a^{-1} - a)z_1\sqrt{z_2} = 0$. Then the map $q\colon (z_1, z_2) \mapsto (w_1, w_2)$ gives a homeomorphism $F_{16}^* \to [0, 1]^2$.

Proof. In view of Proposition 6.7.6 we can identify F_{16}^* with $F_{16} \subseteq EX(a)$, so we have nonnegative functions y_1, y_2, u_1, u_2 as discussed previously and $\sqrt{z_2} = y_2$.

It is clear that w_1 is continuous and nonnegative. Next, note that the functions $1 - z_1 = u_4 + z_1z_2$ and $z_1\sqrt{z_2}$ are nonnegative on F_{16}^*, and the only place where they both vanish is $(1, 0)$. Away from that point we deduce that w_2 is continuous with $0 \leq w_2 \leq \sqrt{z_2}$, and these inequalities imply that w_2 is continuous at the exceptional point as well.

One can also check that $1 - w_1 = 2u_2/(1 + 2ay_2) \geq 0$ and $1 - w_2 = 2u_1/(1 - y_1^2 + (a^{-1} - a)y_1^2y_2) \geq 0$ so $w_1, w_2 \leq 1$. Thus, we have a well-defined and continuous map $q\colon F_{16}^* \to [0, 1]^2$.

Now suppose we start with a point $w \in [0, 1]^2$. We will assume that $w_1 < 1$; the case $w_1 = 1$ requires only minor modifications and is left to the reader. Consider the functions

$$\begin{aligned}
p_0(s) &= s((1 + w_1)s^2 + (a^{-1} + a + ((2a)^{-1} - 2a)w_1)s + (1 - w_1)) \\
p_1(s) &= ((2a)^{-1} + (a^{-1} - a)w_1)s^2 + \tfrac{1}{2}((a^{-2} - 1)(w_1 + 1) + 2(1 - w_1))s + (2a)^{-1}(1 - w_1) \\
p(s) &= p_0(s)/p_1(s).
\end{aligned}$$

All coefficients of powers of s in $p_0(s)$ and $p_1(s)$ are strictly positive, and $p_0(s)$ is cubic whereas $p_1(s)$ is quadratic. It follows that p defines a continuous function $[0, \infty) \to [0, \infty)$, with $p(0) = 0$ and $p(s) \to \infty$ as $s \to \infty$. We claim that this is strictly increasing. Indeed, by standard algebra one can check that $p'(s) = p_1(s)^{-2}\sum_{i=0}^4 m_is^i$, where

$$\begin{aligned}
m_0 &= \tfrac{1}{2}a^{-1}(1 - w_1)^2 \\
m_1 &= (1 - w_1)((\tfrac{3}{2}a^{-2} - 1)w_1 + (a^{-2} + 1)(1 - w_1)) \\
m_2 &= \tfrac{1}{2}a^{-3}(1 - a^2)(1 - w_1) + a^{-3}(1 - a^2)(\tfrac{3}{2} - a^2)w_1^2 + \\
&\qquad \tfrac{1}{4}a^{-3}(1 + 2a^2)(5 - a^2)w_1(1 - w_1) + \tfrac{1}{2}a^{-1}(5 + a^2)(1 - w_1)^2 \\
m_3 &= a^{-2}(1 + w_1)(2(1 - a^2)w_1 + (1 + a^2)(1 - w_1)) \\
m_4 &= \tfrac{1}{2}a^{-1}(1 + w_1)(1 + 2(1 - a^2)w_1).
\end{aligned}$$

We have written these coefficients in a form that makes it clear that they are positive. It follows that $p'(s) > 0$ for $s \geq 0$, as claimed. It follows that there is a unique number $y_2 \geq 0$ with $p(y_2) = w_2$. We put

$$y_1 = \sqrt{\frac{w_1(1 + 2ay_2)}{(y_2 + a)(y_2 + a^{-1})}}$$

and $y = (y_1, y_2) \in [0, \infty)^2$. We then define u_1 and u_2 in terms of y in the usual way. If we substitute the above value for y_1, then straightforward algebra gives

$$\begin{aligned}
u_1 &= (1 - p(y_2))\frac{((1 - a^2)w_1 + \frac{1}{2})y_2^2 + (\frac{1}{2}(1 - w_1)(a + a^{-1}) + w_1(a + a^{-1}))y_2 + (1 - w_1)/2}{(y_2 + a)(y_2 + a^{-1})} \\
u_2 &= (1 + 2ay_2)(1 - w_1)/2.
\end{aligned}$$

After recalling that $y_2 \geq 0$ and $p(y_2) = w_2 \in [0,1]$ it follows that $u_1, u_2 \geq 0$, so y lies in F_4^* and the point $z = (y_1^2, y_2^2)$ lies in F_{16}^*. Now note that $y_2 \geq 0$ and put

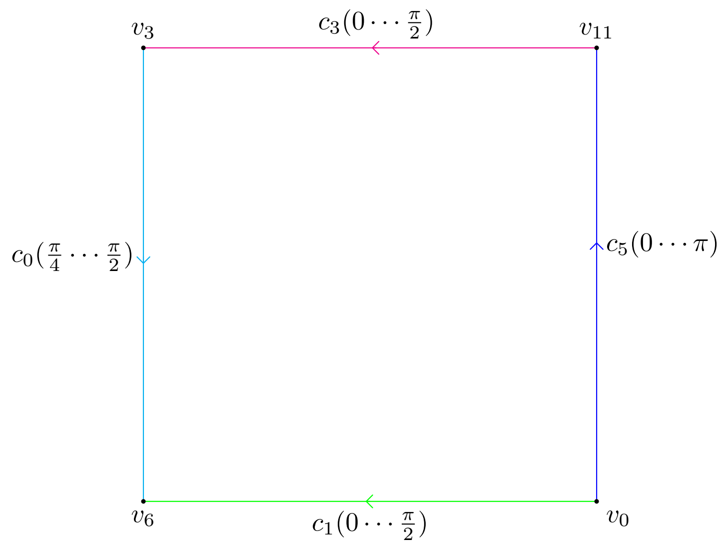
$$w_1' = \frac{z_1(1 + (a + a^{-1})\sqrt{z_2} + z_2)}{1 + 2a\sqrt{z_2}} \qquad\qquad w_2' = \sqrt{z_2}\frac{2a(1 - z_1) + z_1\sqrt{z_2}}{1 - z_1 + (a^{-1} - a)z_1\sqrt{z_2}},$$

so $q(z) = (w_1', w_2')$. If we substitute in our definition of y_1 and simplify we get $w_1' = w_1$ and $w_2' = p(y_2)$, but $p(y_2) = w_2$, so $q(z) = w$. We leave it to the reader to check that all steps in this construction are forced, so z is the unique point in F_{16}^* with $q(z) = w$. This means that $q\colon F_{16}^* \to [0,1]^2$ is a continuous bijection between compact Hausdorff spaces, so it is a homeomorphism.

```
embedded/invariants_check.mpl: check_invariants()
```

□

The following picture shows the images under $q \circ p_{16}$ of the curves $c_i(t)$ and the points v_j.



Some relevant formulae are as follows:

$$\begin{aligned}
qp_{16}(v_i) &= (1,0) && \text{for } 0 \leq i \leq 1 \\
qp_{16}(v_i) &= (0,1) && \text{for } 2 \leq i \leq 5 \\
qp_{16}(v_i) &= (0,0) && \text{for } 6 \leq i \leq 9 \\
qp_{16}(v_i) &= (1,1) && \text{for } 10 \leq i \leq 13
\end{aligned}$$

$$\begin{aligned}
qp_{16}(c_0(t)) &= (0,\ |\cos(2t)|) \\
qp_{16}(c_1(t)) = qp_{16}(c_2(t)) &= (\cos(t)^2,\ 0) \\
qp_{16}(c_{\text{alg}}(t)) &= \begin{cases} \left(a^{-1}\frac{(1-2at)(t+a)(t+a^{-1})}{(1+2at)(t-a)(t-a^{-1})},\ 1\right) & \text{if } t \in T_{\text{alg}}^+ \\ \left(1,\ -at\frac{3-2a^2-4at}{2-2a^2-(3a+2a^3)t}\right) & \text{if } t \in T_{\text{alg}}^-. \end{cases}
\end{aligned}$$

Proposition 6.7.8. [prop-E-cromulent]
$EX(a)$ is a cromulent surface.

Proof. Condition (a) in Definition 1.0.4 is proved in Section 6.2, and conditions (b) and (c) are proved in Section 6.3. For condition (d), we can take F' to be the interior of F_{16}. □

6.8. **Additional points and curves.** [sec-extra-curves]
The surface $EX(a)$ has some additional points and curves that are not part of the precromulent structure but are nevertheless useful for some purposes (such as our analysis of torus quotients of EX^* in Section 7.4). All claims in this section are checked as follows:

```
embedded/extra_vertices_check.mpl: check_extra_vertices()
embedded/extra_curves_check.mpl: check_extra_curves()
```

Definition 6.8.1. [defn-c-nine]

We put

$$c_9(t) = \left(\sqrt{\frac{1-a^2}{2(1+a^2)}} \sin(t),\ \sqrt{\frac{1-a^2}{2(1+a^2)}} \sin(t),\ \sqrt{\frac{2a^2}{1+a^2}},\ -\sqrt{\frac{1-a^2}{1+a^2}} \cos(t) \right),$$

then

$$c_{10}(t) = \lambda(c_9(t)) \qquad c_{11}(t) = \mu(c_9(t)) \qquad c_{12}(t) = \lambda\mu(c_9(t)).$$

It is straightforward to check that c_9 lands in $EX(a)$, so the same is true for c_{10}, c_{11} and c_{12}. One can also check that $\lambda^2 c_9(t) = c_9(-t)$ and $\lambda\nu(c_9(t)) = c_9(\pi - t)$. It follows that we cannot get anything interestingly new by applying further group elements to the above curves. In the case $a = 1/\sqrt{2}$, we have

$$c_9(t) = \left(\frac{\sin(t)}{\sqrt{6}},\ \frac{\sin(t)}{\sqrt{6}},\ \sqrt{\frac{2}{3}},\ -\frac{\cos(t)}{\sqrt{3}} \right).$$

Definition 6.8.2. [defn-c-thirteen]

We put

$$c_{13}(t) = \left(\frac{\cos(t)}{\sqrt{2}} \left(1 - \frac{\sin(t)}{\sqrt{2/(1-a^2) - \cos(t)^2}} \right), \quad \frac{\cos(t)}{\sqrt{2}} \left(1 + \frac{\sin(t)}{\sqrt{2/(1-a^2) - \cos(t)^2}} \right), \right.$$
$$\left. \frac{\sqrt{2}a}{\sqrt{1+a^2}} \sin(t), \quad \frac{-\sqrt{2}\sin(t)^2}{\sqrt{1+a^2}\sqrt{2/(1-a^2) - \cos(t)^2}} \right),$$

then

$$c_{14}(t) = \lambda(c_{13}(t)) \qquad c_{15}(t) = \nu(c_{13}(t)) \qquad c_{16}(t) = \lambda\nu(c_{13}(t)).$$

In the case $a = 1/\sqrt{2}$ this becomes

$$c_{13}(t) = \left(\frac{\cos(t)}{\sqrt{2}} \left(1 - \frac{\sin(t)}{\sqrt{4 - \cos(t)^2}} \right), \frac{\cos(t)}{\sqrt{2}} \left(1 + \frac{\sin(t)}{\sqrt{4 - \cos(t)^2}} \right), \sqrt{2/3}\sin(t), \frac{-2\sin(t)^2}{\sqrt{3}\sqrt{4 - \cos(t)^2}} \right).$$

It is straightforward to check that c_{13} lands in $EX(a)$, so the same is true for c_{14}, c_{15} and c_{16}. One can also check that $\lambda\mu(c_{13}(t)) = c_{13}(-t)$ and $\lambda^2(c_{13}(t)) = c_{13}(\pi - t)$. It again follows that we cannot get anything interestingly new by applying further group elements.

Definition 6.8.3. [defn-c-seventeen]

If $a \leq 1/\sqrt{2}$, we put

$$c_{17}(t) = \left(\sqrt{\frac{1-2a^2}{3(1+2a^2)}} \sin(t),\ -\cos(t),\ -\frac{4a}{\sqrt{6(1+2a^2)}} \sin(t),\ \frac{2}{\sqrt{6(1+2a^2)}} \sin(t) \right),$$

then

$$c_{18}(t) = \lambda(c_{13}(t)) \qquad c_{19}(t) = \lambda^2(c_{13}(t)) \qquad c_{20}(t) = \lambda^3(c_{13}(t)).$$

One can again check that this produces curves in $EX(a)$ satisfying $\nu c_{17}(t) = c_{17}(\pi - t)$ and $\lambda^2\mu(c_{17}(t)) = c_{17}(-t)$. In the case $a = 1/\sqrt{2}$, we find that $c_{17}(t) = c_3(-\frac{\pi}{2} - t)$, and similarly c_{18}, c_{19} and c_{20} are just reparametrisations of the lower numbered curves.

Note that for $k \in \{0, 1, 2, 17, \dots, 20\}$, the image $C_k = c_k(\mathbb{R})$ is a great circle. The homogeneous polynomial $g(x)$ defines a cubic surface in the projective space $\mathbb{P}^3$, which is smooth except when $a = 1/\sqrt{2}$. A famous theorem of Cayley and Salmon says that any smooth cubic surface contains precisely 27 linearly embedded copies of $\mathbb{P}^1$ (when counted with appropriate multiplicities); see [7, Theorem 9.1.13] for a modern treatment. In our case, the above great circles give seven copies of $\mathbb{P}^1$. One can check that the remaining copies come in ten complex conjugate pairs, and so do not correspond to great circles in the real variety $EX(a)$. In the case $a = 1/\sqrt{2}$ everything degenerates and we have only five great circles and two additional conjugate pairs of $\mathbb{P}^1$'s. Some or all of these must have multiplicity greater than one, but we have not investigated this.

`embedded/cayley_check.mpl: check_cayley()`

For $k \in \{9, \dots, 12\}$ the image is again the intersection of S^3 with a two-dimensional subspace of $\mathbb{R}^4$, but in these cases it is an affine subspace rather than a vector subspace. We suspect that again there are no more curves of this type contained in $EX(a)$, but we have not proved this.

Remark 6.8.4. The curves $c_i(t)$ for $0 \leq i \leq 16$ are represented in Maple as `c_E[i](t)`. However, for $17 \leq i \leq 20$, the curves $c_i(t)$ are not defined when $a > 1/\sqrt{2}$, and this makes it inconvenient to use the same framework. Instead, these curves are represented in Maple by the functions `c_cayley[j](t)` for $1 \leq j \leq 4$.

We next introduce some additional points v_i for $14 \leq i \leq 45$. For this, it is convenient to enumerate the elements of G as follows:

$$\begin{array}{cccc}
\gamma_0 = 1 & \gamma_1 = \lambda & \gamma_2 = \lambda^2 & \gamma_3 = \lambda^3 \\
\gamma_4 = \mu & \gamma_5 = \lambda\mu & \gamma_6 = \lambda^2\mu & \gamma_7 = \lambda^3\mu \\
\gamma_8 = \nu & \gamma_9 = \lambda\nu & \gamma_{10} = \lambda^2\nu & \gamma_{11} = \lambda^3\nu \\
\gamma_{12} = \mu\nu & \gamma_{13} = \lambda\mu\nu & \gamma_{14} = \lambda^2\mu\nu & \gamma_{15} = \lambda^3\mu\nu.
\end{array}$$

Definition 6.8.5. We put

$$v_{14} = \left(\sqrt{\frac{1-a^2}{2(1+a^2)}},\ \sqrt{\frac{1-a^2}{2(1+a^2)}},\ \sqrt{\frac{2a^2}{1+a^2}},\ 0\right),$$

then $v_{14+i} = \gamma_i(v_{14})$ for $0 \leq i < 8$. If $a \leq 1/\sqrt{2}$ we also put

$$\begin{aligned}
v_{22} &= \left(\sqrt{\frac{1-2a^2}{3(1+2a^2)}},\ 0,\ \sqrt{\frac{8a^2}{3(1+2a^2)}},\ -\sqrt{\frac{2}{3(1+2a^2)}}\right) \\
v_{30} &= \left(\sqrt{\frac{1-2a^2}{4(1+a^2)}},\ \sqrt{\frac{1-2a^2}{4(1+a^2)}},\ \sqrt{\frac{2a^2}{1+a^2}},\ \sqrt{\frac{1}{2(1+a^2)}}\right),
\end{aligned}$$

then

$$\begin{aligned}
v_{22+i} &= \gamma_i(v_{22}) && (0 \leq i < 8) \\
v_{30+i} &= \gamma_i(v_{30}) && (0 \leq i < 16).
\end{aligned}$$

Straightforward calculations show that these points lie in $EX(a)$. We have $\lambda\nu(v_{14}) = v_{14}$ and $\nu(v_{22}) = v_{22}$, which implies that $\{v_0, \dots, v_{45}\}$ is closed under the action of G. One can check that

$$\begin{aligned}
C_1 \cap C_9 &= \{v_{14}, v_{16}\} \\
C_5 \cap C_{19} &= \{v_{22}\} \\
C_9 \cap C_{20} &= \{v_{30}\}.
\end{aligned}$$

This is the main justification for considering these extra points.

6.9. **Charts.** [`sec-E-charts`]

In this section we discuss three different kinds of charts for $EX(a)$.

Our first construction is simple and works at every point of $EX(a)$. It only gives an approximate chart, but that is sufficient for many purposes.

Definition 6.9.1. [`defn-quadratic-chart`]

Let x be a point in $EX(a)$, and let T be the tangent space to $EX(a)$ at x. We define $\phi\colon T \to \mathbb{R}^4$ by

$$\phi(t) = \left(1 - \frac{\|t\|^2}{2}\right) x + t - \frac{n(t).x}{\|n(x)\|^2} n(x).$$

(Here $n(x)$ is the gradient of g, as in Definition 6.1.5.) We call this the *quadratic approximate chart* at x. Maple notation for $\phi(t)$ is `quadratic_chart(x,t)` (defined in `embedded/geometry.mpl`).

Proposition 6.9.2. [prop-quadratic-chart]

We have

$$\begin{aligned}\phi(t) &= x + t + O(\|t\|^2)\\ g(\phi(t)) &= O(\|t\|^3)\\ \rho(\phi(t)) &= 1 + O(\|t\|^4).\end{aligned}$$

Proof. As the map $n\colon \mathbb{R}^4 \to \mathbb{R}^4$ is homogeneous quadratic, we see that $\phi(t) = x + t + O(\|t\|^2)$. Next, as x, $n(x)$ and t are mutually orthogonal we have

$$\begin{aligned}\rho(\phi(t)) &= \left(1 - \frac{\|t\|^2}{2}\right)^2 + \|t\|^2 + \left(\frac{n(t).x}{\|n(x)\|^2}\right)^2 \|n(x)\|^2\\ &= 1 + \frac{\|t\|^4}{4} + \frac{(n(t).x)^2}{\|n(x)\|^2}.\end{aligned}$$

Using the fact that n is homogeneous quadratic again, we see that this is $1 + O(\|t\|^4)$.

Next, using the fact that g is homogeneous cubic we find that

$$g(a+b) = g(a) + n(a).b + n(b).a + g(b).$$

We apply this with $a = \left(1 - \frac{\|t\|^2}{2}\right) x$ and $b = t - \frac{n(t).x}{\|n(x)\|^2} n(x)$, neglecting terms of order $\|t\|^3$ everywhere. As $g(x) = 0$ we have $g(a) = 0$. Moreover, b is $O(\|t\|)$, so $g(b)$ is negligible, and when calculating $n(a).b$ we can neglect terms in $n(a)$ that are quadratic in t. This leaves $n(a) \simeq n(x)$, and $n(x)$ is normal to t, so

$$n(x).b \simeq -\frac{n(t).x}{\|n(x)\|^2} n(x).n(x) = -n(t).x.$$

Similarly, as n is quadratic, we can neglect terms in b that are $O(\|t\|^2)$ when calculating $n(b)$. This gives $n(b) \simeq n(t)$ and so $n(b).a \simeq n(t).a \simeq n(t).x$. Altogether, we have

$$g(\phi(t)) = g(a) + n(a).b + n(b).a + g(b) \simeq 0 - n(t).x + n(t).x + 0 = 0.$$

`embedded/geometry_check.mpl: check_quadratic_chart()`

□

Next, recall from Section 2.5 that each of the maps $c_k\colon \mathbb{R} \to EX(a)$ can be extended in a canonical way to give a holomorphic map $\widetilde{c}_k$ defined on a neighbourhood of $\mathbb{R}$ in $\mathbb{C}$.

The file `embedded/annular_charts.mpl` gives a formula for $\widetilde{c}_0(t+iu)$ modulo u^4, and formulae for $\widetilde{c}_1(t+iu)$ and $\widetilde{c}_2(t+iu)$ modulo u^3, but we will not reproduce them here. Given a fixed value of a and $t_0 \in \mathbb{R}$ it is also not hard to compute power series for $\widetilde{c}_k(t_0 + t + iu)$ to reasonably high order, and similar methods can be used to produce series for conformal charts centred at points that do not lie on any of the curves C_k. We postpone a more detailed discussion to Section 7.3, where we focus on the case $a = 1/\sqrt{2}$.

We next describe a different class of charts that will be useful for triangulating $EX(a)$. It is inspired by the definition of barycentric coordinates for spherical triangles described in [14]. Related code is in the file `embedded/barycentric.mpl`.

Definition 6.9.3. [defn-barycentric]

Let a_0, a_1 and a_2 be distinct points on $EX(a)$. For any $x \in EX(a)$ we let $n(x)$ denote the gradient of g at $EX(a)$, and we put

$$\widetilde{p}(x) = (\det(x, n(x), a_1, a_2),\ \det(x, n(x), a_2, a_0),\ \det(x, n(x), a_0, a_1)) \in \mathbb{R}^3.$$

If $\sum_i \widetilde{p}(x)_i \neq 0$, we put

$$p(x) = \widetilde{p}(x) / \sum_i \widetilde{p}(x)_i.$$

This clearly lies in the set $\mathbb{R}^3_1 = \{t \in \mathbb{R}^3 \mid \sum_i t_i = 1\}$, which contains the simplex Δ_2. If we need to emphasise the dependence on the points a_i, we will write $p_{a_0,a_1,a_2}(x)$ rather than $p(x)$. We call the components of $p(x)$ the *barycentric coordinates* of x (with respect to the a_i).

Definition 6.9.4. For $x \in EX(a)$ we put $T'_x EX(a) = x + T_x EX(a) \subset \mathbb{R}^4$, and call this the *affine tangent space* to $EX(a)$ at x. In a small neighbourhood of x, this is of course a good approximation to $EX(a)$ itself. We write $\pi'_x(y)$ for the closest point in $T'_x EX(a)$ to y. This can be computed as

$$\pi'_x(y) = x + y - \langle y, x \rangle x - \langle y, n(x) \rangle n(x)/\|n(x)\|^2.$$

The next result motivates the term "barycentric coordinates".

Lemma 6.9.5. *If $p(x)$ is defined, then it is the unique element $t \in \mathbb{R}^3_1$ such that $x = \sum_i t_i \pi'_x(a_i)$.*

Proof. First, we write $\widetilde{u} = \widetilde{p}(x)$, so

$$\begin{aligned}
\widetilde{u}_0 &= \det(x, n(x), a_1, a_2) \\
\widetilde{u}_1 &= \det(x, n(x), a_2, a_0) \\
\widetilde{u}_2 &= \det(x, n(x), a_0, a_1).
\end{aligned}$$

As $p(x)$ is assumed to be defined, we must have $\widetilde{u}_0 + \widetilde{u}_1 + \widetilde{u}_2 \neq 0$, and in particular $(\widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_2) \neq (0, 0, 0)$.

Next, the list $a_0, a_1, a_2, x, n(x)$ consists of five vectors in $\mathbb{R}^4$, so there must be a linear relation

$$s_0 a_0 + s_1 a_1 + s_2 a_2 + s_3 x + s_4 n(x) = 0$$

where not all of the coefficients s_i are zero. As x and $n(x)$ are nonzero and orthogonal, it follows easily that $(s_0, s_1, s_2) \neq (0, 0, 0)$.

Now apply the map $y \mapsto \det(x, n(x), a_0, y)$ to our linear relation. Only the second and third terms contribute anything, and we deduce that $s_1 \widetilde{u}_2 - s_2 \widetilde{u}_1 = 0$. Similarly, we can apply the map $y \mapsto \det(x, n(x), a_1, y)$ to see that $s_0 \widetilde{u}_2 - s_2 \widetilde{u}_0 = 0$, and we can apply the map $y \mapsto \det(x, n(x), a_2, y)$ to see that $s_0 \widetilde{u}_1 - s_1 \widetilde{u}_0 = 0$. We have already seen that the vectors $(\widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_2)$ and (s_0, s_1, s_2) are nonzero, and the above relations imply that they are unit multiples of each other. It follows that the vector $u = p(x) = \widetilde{u}/\sum_i \widetilde{u}_i$ is also a unit multiple of (s_0, s_1, s_2). The definition of the coefficients s_i implies that

$$s_0 a_0 + s_1 a_1 + s_2 a_2 \in \operatorname{span}(x, n(x)) = (T_x EX(a))^\perp,$$

and we now see that $\sum_i u_i a_i \in (T_x EX(a))^\perp$ as well. From this it follows easily that $x = \sum_i u_i \pi'_x(a_i)$ as claimed.

All that is left is to check that the numbers u_i are uniquely characterised by the above property. Suppose there is another vector $u' \in \mathbb{R}^3_1$ with $x = \sum_i u'_i \pi'_x(a_i)$. It follows that the vector $r = u' - u$ has $\sum_i r_i = 0$ and $\sum_i r_i a_i \in (T_x EX(a))^\perp = \operatorname{span}(x, n(x))$, so there exist scalars r_3, r_4 such that

$$r_0 a_0 + r_1 a_1 + r_2 a_2 + r_3 x + r_4 n(x) = 0.$$

Just as before we deduce that $r_i \widetilde{u}_j - r_j \widetilde{u}_i = 0$ for all $i, j \in \{0, 1, 2\}$, so (r_0, r_1, r_2) is a multiple of $(\widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_2)$. As $\sum_i r_i = 0$ but $\sum_i \widetilde{u}_i \neq 0$, we see that the multiplier must be zero, so $u' = u$ as claimed. $\square$

From the above characterisation, we can show that barycentric coordinates for adjacent triangles are equal on the shared edge, in the following sense.

Corollary 6.9.6. [cor-edge-barycentric]
Suppose we have points $a_0, a_1, b, b', x \in EX(a)$ such that the vectors $u = p_{a_0,a_1,b}(x)$ and $u' = p_{a_0,a_1,b'}(x)$ are both defined, and that $u_2 = 0$. Then $u' = u$ (and in particular, u'_2 is also zero).

Proof. We can apply the lemma to see that $x = u_0 \pi'_x(a_0) + u_1 \pi'_x(a_1)$. On the other hand, u' is uniquely characterised by the fact that $x = u'_0 \pi'_x(a_0) + u'_1 \pi'_x(a_1) + u'_2 \pi'_x(b')$. The claim is clear from this. $\square$

Remark 6.9.7. A similar uniqueness argument shows that if $p_{a_0,a_1,a_2}(a_i)$ is defined, then it must be equal to the i'th standard basis vector e_i. More precisely, it is clear from the definitions that $\widetilde{p}_{a_0,a_1,a_2}(a_i)_j = 0$ for all $j \neq i$, so either $\widetilde{p}_{a_0,a_1,a_2}(a_i)_i \neq 0$ (and $p_{a_0,a_1,a_2}(a_i) = e_i$) or $\widetilde{p}_{a_0,a_1,a_2}(a_i)_i = 0$ (and $p_{a_0,a_1,a_2}(a_i)$ is undefined).

Definition 6.9.8. We put

$$T_0(a_0, a_1, a_2) = \{x \in EX(a) \mid p_{a_0,a_1,a_2}(x) \text{ is defined and lies in } \Delta_2\}.$$

Typically, if the a_i are close together and in general position, there will be a single connected component $T(a_0, a_1, a_2) \subseteq T_0(a_0, a_1, a_2)$ that contains all the points a_i, and the map p_{a_0,a_1,a_2} will restrict to give a diffeomorphism $T(a_0, a_1, a_2) \to \Delta_2$. We can use maps of this form to give a triangulation of $EX(a)$, with compatibility along edges given by Corollary 6.9.6.

Remark 6.9.9. [rem-barycentric-inverse]
The inverse of the map $p\colon T(a_0, a_1, a_2) \to \Delta_2$ can be computed efficiently by a kind of Newton-Raphson method. If x is reasonably close to $p^{-1}(t)$ then the first order Taylor approximation at x to the map $p\colon EX(a) \to \mathbb{R}^3_1$ will be an affine isomorphism $p_*\colon T'_x EX(a) \to \mathbb{R}^3_1$, so we can define $\kappa(x) = \sigma^\infty(p_*^{-1}(t))$, where σ^∞ is as in Remark 6.1.6. As an initial approximation we can take $x_0 = \sigma^\infty(\sum_i t_i a_i)$, and then the sequence $(\kappa^n(x_0))_{n\geq 0}$ converges rapidly to $p^{-1}(t)$. If we need to do this for a large number of different points t, we can also speed up the method by precomputing various coefficients that depend only on the points a_i.

`embedded/barycentric_check.mpl: check_barycentric()`

One could attempt to triangulate EX^* using the points v_i as vertices, but it turns out that the resulting simplices are too large for barycentric coordinates to work properly. We have tried several different triangulations with smaller simplices. In one of these, we introduce a new set of points a_{ij} for $0 \leq i \leq 6$ and $0 \leq j \leq 4$, and use them as vertices for a triangulation of F_{16} with 48 triangles. We can then use the group action to obtain a triangulation of all of EX^*, with 768 triangles. It is convenient to use points with good rationality properties, as described in Section 7.6. This ensures that at least the first steps of our calculations can be specified exactly. It is also convenient to choose points such that the various edges have lengths that do not differ by too large a factor. We use the following corner points:

$$\left(\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}, 0, 0\right) \quad (0,1,0,0) \quad (0,0,1,0) \quad \left(0, 0, \tfrac{1}{3}\sqrt{6}, -\tfrac{1}{3}\sqrt{3}\right)$$

Plus the following additional edge points:

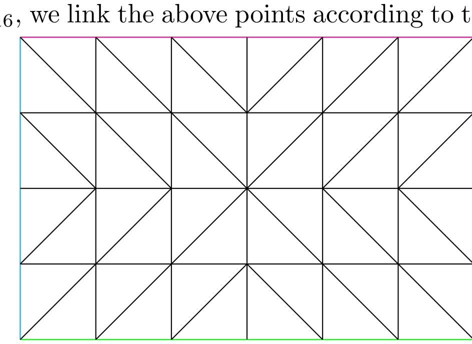
$$\left(\tfrac{407}{745}, \tfrac{624}{745}, 0, 0\right) \quad \left(\tfrac{5}{13}, \tfrac{12}{13}, 0, 0\right) \quad \left(\tfrac{9}{41}, \tfrac{40}{41}, 0, 0\right)$$
$$\left(\tfrac{1}{60}\sqrt{195}, 0, \tfrac{1}{4}\sqrt{15}, -\tfrac{1}{60}\sqrt{30}\right) \quad \left(\tfrac{3}{70}\sqrt{55}, 0, \tfrac{2}{7}\sqrt{10}, -\tfrac{9}{70}\sqrt{5}\right) \quad \left(\tfrac{1}{68}\sqrt{238}, 0, \tfrac{1}{12}\sqrt{102}, -\tfrac{7}{102}\sqrt{51}\right)$$

$$\left(\tfrac{23}{34}, \tfrac{23}{34}, \tfrac{7}{34}\sqrt{2}, 0\right) \quad \left(\tfrac{79}{130}, \tfrac{79}{130}, \tfrac{47}{130}\sqrt{2}, 0\right) \quad \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\sqrt{2}, 0\right) \quad \left(\tfrac{79}{202}, \tfrac{79}{202}, \tfrac{119}{202}\sqrt{2}, 0\right) \quad \left(\tfrac{7}{34}, \tfrac{7}{34}, \tfrac{23}{34}\sqrt{2}, 0\right)$$
$$\left(0, \tfrac{59}{62}, \tfrac{11}{62}\sqrt{2}, -\tfrac{11}{62}\right) \quad \left(0, \tfrac{11}{13}, \tfrac{4}{13}\sqrt{2}, -\tfrac{4}{13}\right) \quad \left(0, \tfrac{61}{86}, \tfrac{35}{86}\sqrt{2}, -\tfrac{35}{86}\right) \quad \left(0, \tfrac{1}{2}, \tfrac{1}{2}\sqrt{2}, -\tfrac{1}{2}\right) \quad \left(0, \tfrac{11}{38}, \tfrac{21}{38}\sqrt{2}, -\tfrac{21}{38}\right)$$

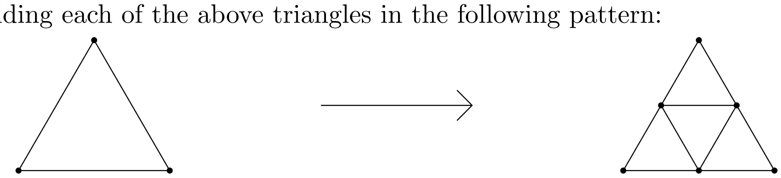
Plus the following points in the interior:

$$\left(\tfrac{92124}{152207}, \tfrac{232883}{304414}, \tfrac{31}{202}\sqrt{2}, -\tfrac{11005}{304414}\right) \quad \left(\tfrac{15}{38}, \tfrac{86}{95}, \tfrac{1}{10}\sqrt{2}, -\tfrac{13}{190}\right) \quad \left(\tfrac{4793}{21846}, \tfrac{72098}{76461}, \tfrac{23}{154}\sqrt{2}, -\tfrac{20585}{152922}\right)$$
$$\left(\tfrac{1287}{2425}, \tfrac{344}{485}, \tfrac{8}{25}\sqrt{2}, -\tfrac{248}{2425}\right) \quad \left(\tfrac{214099}{533478}, \tfrac{217940}{266739}, \tfrac{73}{274}\sqrt{2}, -\tfrac{91469}{533478}\right) \quad \left(\tfrac{1121}{4510}, \tfrac{1864}{2255}, \tfrac{25}{82}\sqrt{2}, -\tfrac{237}{902}\right)$$
$$\left(\tfrac{7267}{15458}, \tfrac{8565}{15458}, \tfrac{125}{262}\sqrt{2}, -\tfrac{1000}{7729}\right) \quad \left(\tfrac{37323}{100798}, \tfrac{29390}{50399}, \tfrac{95}{202}\sqrt{2}, -\tfrac{28595}{100798}\right) \quad \left(\tfrac{22828}{111879}, \tfrac{1572959}{2461338}, \tfrac{907}{2046}\sqrt{2}, -\tfrac{975025}{2461338}\right)$$
$$\left(\tfrac{92619}{240218}, \tfrac{224276}{600545}, \tfrac{359}{610}\sqrt{2}, -\tfrac{166217}{1201090}\right) \quad \left(\tfrac{5389}{14982}, \tfrac{27671}{74910}, \tfrac{1301}{2270}\sqrt{2}, -\tfrac{10408}{37455}\right) \quad \left(\tfrac{342221}{1435238}, \tfrac{4932765}{10046666}, \tfrac{1549}{3038}\sqrt{2}, -\tfrac{2143816}{5023333}\right)$$
$$\left(\tfrac{143}{486}, \tfrac{298}{1701}, \tfrac{83}{126}\sqrt{2}, -\tfrac{415}{3402}\right) \quad \left(\tfrac{42732}{137557}, \tfrac{20435}{137557}, \tfrac{280}{457}\sqrt{2}, -\tfrac{49720}{137557}\right) \quad \left(\tfrac{23}{119}, \tfrac{4}{17}, \tfrac{4}{7}\sqrt{2}, -\tfrac{60}{119}\right)$$

To create a triangulation of F_{16}, we link the above points according to the following combinatorial scheme:



This has been arranged to ensure that no edge in the triangulation has endpoints on two different sides of F_{16}, which would cause trouble in certain numerical algorithms. We have also used a finer triangulation obtained by subdividing each of the above triangles in the following pattern:



This gives a grid with 192 faces in F_{16}.

Information about this kind of triangulation can be encoded in an object of the class `E_grid`, which is declared in the file `embedded/E_domain.mpl`. This class extends the `grid` class, which is declared in `domain/grid.mpl`. (There used to be parallel classes `H_grid` and `P_grid`, but we found that various algorithms based on triangulations were not very effective, so we have not maintained that code.) In particular, the 192 face triangulation described above is encoded in this form and stored in the file `embedded/roothalf/split_rational_grid_wx_30.mpl` in the `data` directory. One can thus enter

```
read(cat(data_dir,"/embedded/roothalf/split_rational_grid_wx_30.mpl"));
G := eval(split_rational_grid_wx_30):
G["num_points"];
```

This will print 117, indicating that the triangulation has 117 vertices lying in F_{16}. As well as the obvious data about the vertices, edges and faces of the triangulation, the object `G` also contains extensive information about 175 sample points in each of the 192 faces. This can be used for computing integrals over EX^*, as will be explained in Section 7.7. All of this information is computed to 100 decimal places. Because of this, the file is rather large (about 53MB).

One can also regenerate the object `G` using the function `build_data["grid"]()` defined in `build_data.mpl`. See Section 9.4 for more discussion of this framework.

6.10. **Curvature and the Laplacian.** [sec-E-curvature]

We now discuss the Gaussian curvature of $EX(a)$. Most treatments of this invariant are formulated in terms of local coordinates on the manifold, but for us it is more useful to have a formula in terms of the coordinates x_i for the ambient space $\mathbb{R}^4$. We have not been able to find a reference for the formula given below, although it would be surprising if it did not appear somewhere. Most of our work will be valid for $EX(a)$ for all a, but we will focus on the case $a = 1/\sqrt{2}$ for simplicity.

Let $n(x) \in \mathbb{R}^4$ be the gradient of g at x, and let $m(x) \in M_4(\mathbb{R})$ be the Hessian, so

$$\begin{aligned} n(x)_i &= \partial g(x)/\partial x_i \\ m(x)_{ij} &= \partial^2 g(x)/\partial x_i\,\partial x_j. \end{aligned}$$

Explicitly, for $a = 1/\sqrt{2}$ we have

$$n(x) = \begin{bmatrix} -4x_1x_4 + 2\sqrt{2}x_1x_3 \\ -4x_2x_4 - 2\sqrt{2}x_2x_3 \\ 2x_3x_4 + \sqrt{2}x_1^2 - \sqrt{2}x_2^2 \\ -2x_1^2 - 2x_2^2 + x_3^2 - 6x_4^2 \end{bmatrix}$$

$$m(x) = \begin{bmatrix} -4x_4 + 2\sqrt{2}x_3 & 0 & 2\sqrt{2}x_1 & -4x_1 \\ 0 & -4x_4 - 2\sqrt{2}x_3 & -2\sqrt{2}x_2 & -4x_2 \\ 2\sqrt{2}x_1 & -2\sqrt{2}x_2 & 2x_4 & 2x_3 \\ -4x_1 & -4x_2 & 2x_3 & -12x_4 \end{bmatrix}.$$

Next, we let ξ be the usual isomorphism $\Lambda^3(\mathbb{R}^4) \to \mathbb{R}^4$, given by

$$\begin{aligned} \xi(e_1 \wedge e_2 \wedge e_3) &= e_4 \\ \xi(e_1 \wedge e_2 \wedge e_4) &= -e_3 \\ \xi(e_1 \wedge e_3 \wedge e_4) &= e_2 \\ \xi(e_2 \wedge e_3 \wedge e_4) &= -e_1, \end{aligned}$$

so

$$\xi^{-1}(u) \wedge v = \langle u, v\rangle e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

We then let $p(x) \in M_4(\mathbb{R})$ denote the unique matrix such that

$$p(x)y = \xi(x \wedge n(x) \wedge m(x)y)$$

for all $y \in \mathbb{R}^4$.

Theorem 6.10.1. [thm-curvature]
The Gaussian curvature of EX^* at x is $1 + \mathrm{trace}(p(x)^2)/\|n(x)\|^2$.

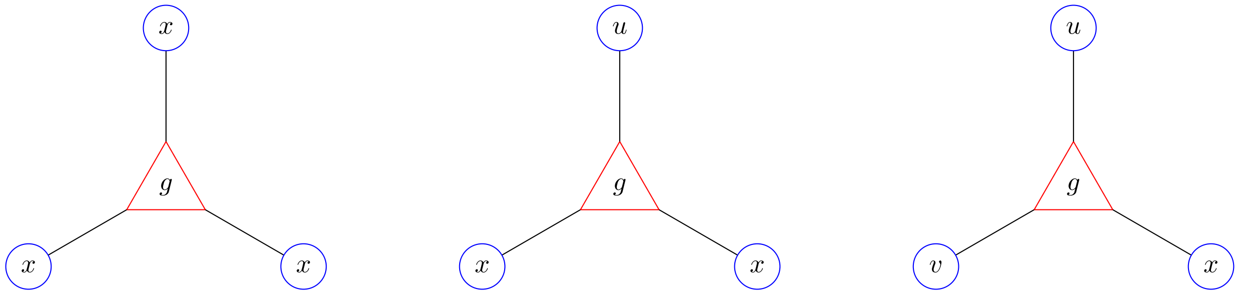
The proof will follow after some preliminaries.

Definition 6.10.2. We put

$$g_{ijk} = \frac{1}{6}\frac{\partial^3 g(x)}{\partial x_i \partial x_j \partial x_k}.$$

It is clear that g_{ijk} is invariant under permutations of the three indices. As g is a homogeneous cubic, we see that g_{ijk} is constant, and that $g(x) = \sum_{ijk} g_{ijk} x_i x_j x_k$. By differentiating this we obtain $n(x)_p = 3\sum_{j,k} g_{pjk} x_j x_k$ and $m(x)_{pq} = 6\sum_k g_{pqk} x_k$.

Remark 6.10.3. Some of our calculations in this section are most easily understood in terms of Penrose diagrams. The paper [13] is a good reference for these from a mathematical perspective. The above expressions for $g(x)$, $n(x)$ and $m(x)$ can be expressed graphically as follows:



$g(x)$ $\qquad$ $n(x).u/3$ $\qquad$ $(m(x)u).v/6$

We next need a little exterior algebra. To keep everything straight, we need to spell out some conventions.

Definition 6.10.4. [defn-exterior]
For any vector space V, we will identify $\lambda^k(V)$ with a subspace of $V^{\otimes k}$ in such a way that $v_1 \wedge \dots \wedge v_k$ becomes

$$\sum_{\sigma \in \Sigma_k} \epsilon(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}.$$

If V has an inner product, we give $V^{\otimes k}$ the inner product such that

$$\langle v_1 \otimes \dots \otimes v_k,\ w_1 \otimes \dots \otimes w_k \rangle = \prod_i \langle v_i, w_i \rangle.$$

However, we give the subspace $\lambda^k(V)$ the alternative inner product $\langle \alpha, \beta \rangle' = \langle \alpha, \beta \rangle / k!$; this has the property that

$$\langle v_1 \wedge \dots \wedge v_k,\ w_1 \wedge \dots \wedge w_k \rangle'$$

is the determinant of the matrix of inner products $\langle v_i, w_j \rangle$.

Now suppose that V has dimension d and we have a given volume form $\omega \in \lambda^d(V)$ with $\langle \omega, \omega \rangle' = 1$. We define the Hodge operator $*\colon \lambda^k(V) \to \lambda^{d-k}(V)$ by the property

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle' \omega.$$

Lemma 6.10.5. [lem-hodge]
Suppose that $x \in EX(a)$, and let (u, v) be any oriented orthonormal basis for $T_x EX^*$. Let ϵ be the usual totally antisymmetric tensor:

$$\epsilon_{ijkl} = \begin{cases} +1 & \text{if } (i,j,k,l) \text{ is an even permutation of } (1,2,3,4) \\ -1 & \text{if } (i,j,k,l) \text{ is an odd permutation of } (1,2,3,4) \\ 0 & \text{if } (i,j,k,l) \text{ is not a permutation of } (1,2,3,4). \end{cases}$$

Then

$$u \wedge v = u \otimes v - v \otimes u = \frac{1}{2\|n\|} \sum_{ijkl} \epsilon_{ijkl} x_i n(x)_j e_k \wedge e_l.$$

Proof. This is standard, except that we need to check that our conventions give the indicated factor of two.

The vectors x and $w = n(x)/\|n(x)\|$ form an orthonormal basis for $T_x EX(a)^\perp$, and it follows that the map $\phi\colon \alpha \mapsto x \wedge w \wedge \alpha$ gives an isomorphism $\lambda^2(T_x EX(a)) \to \lambda^4(\mathbb{R}^4) = \mathbb{R}.\omega_4$. As (u, v) is an oriented orthonormal basis for $T_x EX(a)$, we have $\phi(u \wedge v) = \omega_4$. Now put

$$\theta = \frac{1}{2\|n\|} \sum_{ijkl} \epsilon_{ijkl} x_i n(x)_j e_k \wedge e_l = \frac{1}{2} \sum_{ijkl} \epsilon_{ijkl} x_i w_j e_k \wedge e_l.$$

From the definitions we have

$$x \wedge w \wedge e_k \wedge e_l = \sum_{p,q} \epsilon_{pqkl} x_p w_q \omega_4.$$

This gives

$$\phi(\theta) = x \wedge w \wedge \theta = \frac{1}{2} \sum_{ijklpq} \epsilon_{ijkl} \epsilon_{pqkl} x_i x_p w_j w_q \omega_4.$$

One can also check that

$$\sum_{kl} \epsilon_{ijkl} \epsilon_{pqkl} = 2\delta_{ip}\delta_{jq} - 2\delta_{iq}\delta_{jp}.$$

(For example, if $i = p = 1$ and $j = q = 2$ then the terms for $(k, l) = (3, 4)$ and $(k, l) = (4, 3)$ both contribute $+1$, and all other terms are zero so the sum is $+2$. On the other hand, if $i = q = 1$ and $j = p = 2$ then the terms for $(k, l) = (3, 4)$ and $(k, l) = (4, 3)$ both contribute -1, and all other terms are zero so the sum is -2.) This gives

$$\begin{aligned}
\phi(\theta) &= \sum_{ijpq} (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) x_i x_p w_j w_q \omega_4 \\
&= \left(\sum_{ij} x_i^2 w_j^2 - \sum_{ij} x_i w_i x_j w_j \right) \omega_4 \\
&= (\|x\|^2 \|w\|^2 - \langle x, w\rangle^2) \omega_4 = \omega_4.
\end{aligned}$$

As ϕ is an isomorphism, this means that $u \wedge v = \theta$. □

Proof of Theorem 6.10.1. Put $r = \|n(x)\|$. Let N be the span of x and $n(x)$, so the tangent space to EX^* at x is the space $T = N^\perp$. Define a quadratic map $\psi_0\colon T \to N$ by

$$\psi_0(t) = (1 - \|t\|^2/2)x - (n(t).x)n(x)/r^2,$$

and put

$$X' = \text{ graph of } \psi_0 = \{t + \psi_0(t) \mid t \in T\}.$$

Proposition 6.9.2 tells us that X' agrees with EX^* to second order near x. As curvature depends only on second derivatives, the curvature of EX^* at x is the same as that of X'.

Now choose an orthonormal basis (u, v) for T, oriented so that $x \wedge n(x) \wedge u \wedge v$ is a positive multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Define $\psi\colon \mathbb{R}^2 \to N$ by $\psi(a, b) = \psi_0(au + bv)$. It will be enough to calculate the curvature of the graph of ψ at the point where $a = b = 0$.

There is a well-known formula for the curvature of the graph of a function $\psi(a, b)$ from $\mathbb{R}^2$ to $\mathbb{R}$ (rather than N): we put

$$E = 1 + \frac{\partial\psi}{\partial a}.\frac{\partial\psi}{\partial a} \qquad F = \frac{\partial\psi}{\partial a}.\frac{\partial\psi}{\partial b} \qquad G = 1 + \frac{\partial\psi}{\partial b}.\frac{\partial\psi}{\partial b}$$

$$L = \frac{\partial^2\psi}{\partial^2 a} \qquad M = \frac{\partial^2\psi}{\partial a\,\partial b} \qquad N = \frac{\partial^2\psi}{\partial^2 b},$$

where everything is evaluated at $(0, 0)$. The curvature is then

$$K = \frac{LN - M^2}{EG - F^2}.$$

Essentially the same argument works in our case, except that now L, M and N are vectors, and the formula is

$$K = \frac{L.N - M.M}{EG - F^2}.$$

In our case ψ is constant plus quadratic and so the first order derivatives vanish at $(a, b) = (0, 0)$, which gives $E = G = 1$ and $F = 0$, so $K = L.N - M.M$.

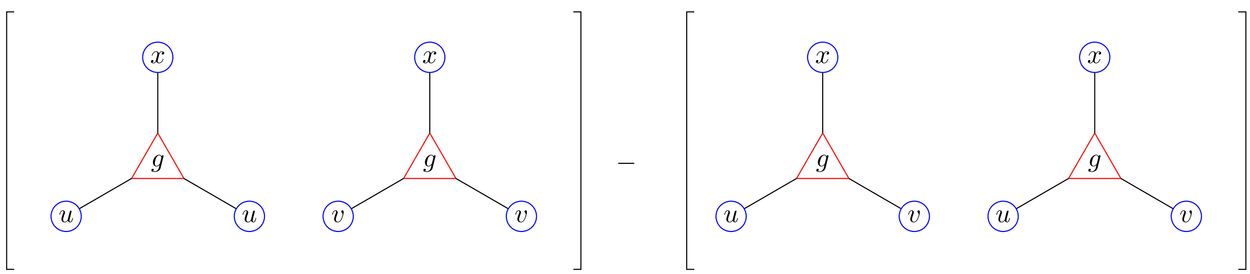
Next, using the fact that $\|t\|^2$ and $n(t)$ are homogeneous quadratic functions of t, we see that

$$\begin{aligned} L &= -x - 2(x.n(u))n(x)/r^2 \\ N &= -x - 2(x.n(v))n(x)/r^2 \\ M &= -(x.m(u)v)n(x)/r^2. \end{aligned}$$

Here $x.n(x) = 3g(x) = 0$, and $n(u)$ can be written as $m(u)u/2$, and similarly for v. This gives

$$\begin{aligned} K &= L.N - M.M \\ &= x.x + \frac{n(x).n(x)}{r^4}\left(4\, x.n(u)\, x.n(v) - (x.m(u)v)^2\right) \\ &= 1 + \frac{1}{r^2}\left((x.m(u)u)(x.m(v)v) - (x.m(u)v)^2\right). \end{aligned}$$

We put $P = (x.m(u)u)(x.m(v)v) - (x.m(u)v)^2$ so that $K = 1 + P/r^2$. Note that $P/36$ can be expressed as the following difference of Penrose diagrams:



We now consider various elements of the space $(\mathbb{R}^4)^{\otimes 4}$. We give this the obvious inner product so that elements of the form $e_i \otimes e_j \otimes e_k \otimes e_l$ give an orthonormal basis. Any permutation $\sigma \in \Sigma_4$ gives an automorphism α_σ of $(\mathbb{R}^4)^{\otimes 4}$ by permuting the tensor factors; for example, we have

$$\alpha_{(2;3)}.(u_1 \otimes u_2 \otimes u_3 \otimes u_4) = u_1 \otimes u_3 \otimes u_2 \otimes u_4.$$

We put

$$\begin{aligned} A &= \sum_i x_i g_{ijk} e_j \otimes e_k \in (\mathbb{R}^4)^{\otimes 2} \\ B &= u \otimes v - v \otimes u \in (\mathbb{R}^4)^{\otimes 2} \\ C &= u \otimes u \otimes v \otimes v - u \otimes v \otimes u \otimes v \in (\mathbb{R}^4)^{\otimes 4}. \end{aligned}$$

It is now not hard to see that $P = 36\langle A \otimes A, C\rangle$. We claim that also

$$P = 18\langle A \otimes A, \alpha_{(2\ 3)}(B \otimes B)\rangle = 18\langle \alpha_{(2\ 3)}(A \otimes A), B \otimes B\rangle.$$

Indeed, we have

$$\begin{aligned} \alpha_{(2\ 3)}(B \otimes B) &= \alpha_{(2\ 3)}(u \otimes v \otimes u \otimes v - u \otimes v \otimes v \otimes u - v \otimes u \otimes u \otimes v + v \otimes u \otimes v \otimes u) \\ &= u \otimes u \otimes v \otimes v - u \otimes v \otimes v \otimes u - v \otimes u \otimes u \otimes v + v \otimes v \otimes u \otimes u. \end{aligned}$$

Now $A \otimes A$ is invariant under the permutations $(1\ 2)$, $(3\ 4)$ and $(1\ 3)(2\ 4)$. Using this we see that

$$\begin{aligned} \langle A \otimes A, u \otimes v \otimes v \otimes u\rangle &= \langle A \otimes A, u \otimes v \otimes u \otimes v\rangle \\ \langle A \otimes A, v \otimes u \otimes u \otimes v\rangle &= \langle A \otimes A, u \otimes v \otimes u \otimes v\rangle \\ \langle A \otimes A, v \otimes v \otimes u \otimes u\rangle &= \langle A \otimes A, u \otimes u \otimes v \otimes v\rangle. \end{aligned}$$

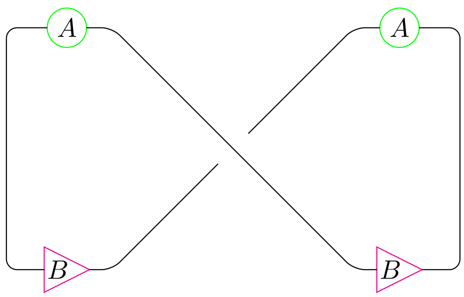
The claim follows easily from this. It can be rewritten as

$$P = 18 \sum_{i,j,k,l} A_{ij} A_{kl} B_{ik} B_{jl}.$$

We now define a matrix Q with $Q_{jk} = \sum_i A_{ij} B_{ik}$. After noting that $A_{kl} = A_{lk}$ and $B_{jl} = -B_{lj}$, the above expression can be rewritten again as

$$P = -18 \sum_{j,k} Q_{jk} Q_{kj} = -18\mathrm{trace}(Q^2).$$

Some of the above can be represented graphically as follows:



$P/18$

Q

Note that our choice of symbols reflects the fact that A is symmetric and B is not.

Now put

$$D = x \wedge n(x) = x \otimes n(x) - n(x) \otimes x,$$

so

$$D_{ij} = 3 \sum_{k,l} (g_{jkl} x_i x_k x_l - g_{ikl} x_j x_k x_l).$$

Lemma 6.10.5 tells us that

$$B_{ik} = \sum_{l,m} \epsilon_{iklm} D_{lm}/(2r).$$

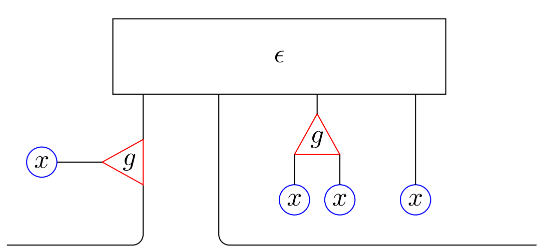
Now

$$Q_{jk} = \sum_i A_{ij} B_{ik} = \frac{1}{2r} \sum_{i,l,m} \epsilon_{iklm} A_{ij} D_{lm} = \frac{3}{2r} \sum_{h,i,l,m,n,p} \epsilon_{iklm} g_{hij} x_h (g_{lnp} x_m x_n x_p - g_{mnp} x_l x_n x_p).$$

Note that the two terms in brackets are essentially the same except that l and m are exchanged, but ϵ_{iklm} is also antisymmetric in l and m. We can thus drop one of the terms and introduce a factor of 2 giving

$$\begin{aligned} Q_{jk} &= \frac{3}{r} \sum_{h,i,l,m,n,p} \epsilon_{iklm} g_{hij} g_{lnp} x_h x_m x_n x_p \\ &= \frac{1}{6r} \sum_{i,l,m} \epsilon_{iklm} n(x)_l \; m(x)_{ij} = \pm p(x)_{jk}/(6r). \end{aligned}$$

Equivalently, we have the following Penrose diagram for $rQ/3$:



We now have

$$P = -18\mathrm{trace}(Q^2) = -\mathrm{trace}(p(x)^2)/(2r^2),$$

and so

$$K = 1 + P/r^2 = 1 - \mathrm{trace}(p(x)^2)/(2r^4).$$

`embedded/curvature_check.mpl: check_EX_curvature()`

□

Remark 6.10.6. [rem-curvature-z]

The full formula for K in terms of the variables x_i is too large to be given here. However, K is invariant under the group action, and so can be expressed in terms of the functions z_1 and z_2 from Section 6.5. Even that expression is somewhat unwieldy for general a, but when $a = 1/\sqrt{2}$ one can check that the formula is as follows:

$$K = 1 + 8\frac{2z_2 - 1}{(2 - z_1)^2(1 + z_2)^2}.$$

It will also be useful for us to have an expression for the Laplace-Beltrami operator Δ on EX^* in terms of the ambient coordinates x_i.

Definition 6.10.7. As before, we write $r = \|n(x)\|$, so

$$r^2 = n(x).n(x) = 9\sum_{ijklm} g_{ijm}g_{klm}x_ix_jx_kx_l.$$

We also define

$$r' = \mathrm{trace}(m(x)) = 6\sum_{i,j} g_{ijj}x_i$$

$$r'' = n(x)^T m(x)n(x) = 54\sum_{ijklmnp} g_{ijk}g_{klm}g_{mnp}x_ix_jx_lx_nx_p$$

Definition 6.10.8. [defn-Delta-prime]

Let U' be an open subset of $\mathbb{R}^4 \setminus \{0\}$. We define a differential operator $\Delta' \colon C^\infty(U') \to C^\infty(U')$ by

$$\Delta'(p) = \sum_i \frac{\partial^2 p}{\partial x_i^2} - \sum_{i,j} x_ix_j\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{1}{r^2}\sum_{i,j} n(x)_i n(x)_j \frac{\partial^2 p}{\partial x_i\partial x_j} - 2\sum_i x_i\frac{\partial p}{\partial x_i} + \left(\frac{r''}{r^4} - \frac{r'}{r^2}\right)\sum_i n(x)_i\frac{\partial p}{\partial x_i}$$

Proposition 6.10.9. [prop-Delta-prime]

If we put $U = U' \cap EX^*$, then the following diagram commutes:

$$\begin{tikzcd} C^\infty(U') \arrow[r, "\Delta'"] \arrow[d, "res"'] & C^\infty(U') \arrow[d, "res"] \\ C^\infty(U) \arrow[r, "\Delta"'] & C^\infty(U). \end{tikzcd}$$

Proof. Consider a smooth function $p \in C^\infty(U')$ and a point $x \in EX^*$. Choose a chart $\phi \colon \mathbb{R}^2 \to EX^*$ with $\phi(0,0) = x$. We will write a and b for coordinates on $\mathbb{R}^2$. The chart gives a Riemannian metric on $\mathbb{R}^2$, corresponding to the matrix $M = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$, where

$$E = \frac{\partial\phi}{\partial a}.\frac{\partial\phi}{\partial a} \qquad F = \frac{\partial\phi}{\partial a}.\frac{\partial\phi}{\partial b} \qquad G = \frac{\partial\phi}{\partial b}.\frac{\partial\phi}{\partial b}.$$

We will use the standard formula

$$(\Delta p)\circ\phi = \det(M)^{-1/2}\mathrm{div}\left(\det(M)^{1/2}M^{-1}\mathrm{grad}(p\circ\phi)\right).$$

We will only be using this formula at the point $(a,b) = (0,0)$, so it will be harmless to replace M by an approximation involving only terms that are constant or linear in a and b. Similarly, ϕ need not be an exact chart, so long as it is quadratically close to EX^*. We can thus follow Proposition 6.9.2 and define ϕ as below:

$$\psi_0(t) = (1 - \|t\|^2/2)x - (n(t).x)n(x)/r^2 \qquad \phi_0(t) = t + \psi_0(t)$$
$$\psi(a,b) = \psi_0(au+bv) \qquad \phi(a,b) = \phi_0(au+bv).$$

By routine calculation we have

$$\frac{\partial\phi}{\partial a} = u - ax - (Aa + Cb)n(x)$$
$$\frac{\partial\phi}{\partial b} = v - bx - (Bb + Ca)n(x)$$

where

$$A = \frac{6}{r^2}\sum_{ijk} g_{ijk}x_iu_ju_k$$
$$B = \frac{6}{r^2}\sum_{ijk} g_{ijk}x_iv_jv_k$$
$$C = \frac{6}{r^2}\sum_{ijk} g_{ijk}x_iu_jv_k.$$

Because x, $n(x)$, u and v are orthogonal, it follows that we have $E = G = 1$ and $F = 0$ up to quadratic corrections. We may thus take M to be the identity matrix, and deduce that $(\Delta\ p)(x)$ is the value at $(0,0)$ of $\left(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2}\right)(p\circ\phi)$. After using the chain rule twice, we see that this is the same as $P + Q$, where

$$P = \sum_{i,j}\frac{\partial^2 p}{\partial x_i\,\partial x_j}\left(\frac{\partial\phi_i}{\partial a}\frac{\partial\phi_j}{\partial a} + \frac{\partial\phi_i}{\partial b}\frac{\partial\phi_j}{\partial b}\right)$$
$$Q = \sum_i \frac{\partial p}{\partial x_i}\left(\frac{\partial^2\phi_i}{\partial a^2} + \frac{\partial^2\phi_i}{\partial b^2}\right).$$

Previously we recorded formulae for $\partial\phi/\partial a$ and $\partial\phi/\partial b$; when $a = b = 0$ they just give u and v. Thus, we have

$$P = \sum_{i,j}\frac{\partial^2\phi}{\partial x_i\,\partial x_j}\left(u_iu_j + v_iv_j\right).$$

Now, the numbers $u_iu_j + v_iv_j$ are the matrix entries for the orthogonal projection onto the tangent space T, which is the identity minus the projection onto the normal space $N = T^\perp = \mathrm{span}(x, n(x))$. We thus have

$$u_iu_j + v_iv_j = \delta_{ij} - x_ix_j - n(x)_in(x)_j/r^2.$$

Using this we obtain

$$P = \sum_i \frac{\partial^2 p}{\partial x_i^2} - \sum_{i,j} x_ix_j\frac{\partial^2 p}{\partial x_i\partial x_j} - \frac{1}{r^2}\sum_{i,j} n(x)_in(x)_j\frac{\partial^2 p}{\partial x_i\partial x_j}.$$

This accounts for the first three terms in $\Delta'(p)$.

We now turn to Q. We have

$$\frac{\partial^2\phi}{\partial a^2} = -x - A\,n(x)$$
$$\frac{\partial^2\phi}{\partial b^2} = -x - B\,n(x)$$

and it follows that

$$Q = (-2x - (A+B)n(x)).\nabla(p).$$

Now

$$A + B = \frac{6}{r^2}\sum_{i,j,k} g_{ijk}x_i(u_ju_k + v_jv_k),$$

and we can again use the relation $u_j u_k + v_j v_k = \delta_{jk} - x_j x_k - n(x)_j n(x)_k / r^2$ to eliminate u and v, giving

$$A + B = \frac{6}{r^2}\left(\sum_{i,j} g_{ijj} x_i - \sum_{i,j,k} g_{ijk} x_i x_j x_k - \frac{1}{r^2}\sum_{i,j,k} g_{ijk} x_i n(x)_j n(x)_k\right).$$

Here $\sum_{i,j,k} g_{ijk} x_i x_j x_k$ is $g(x)$, which is zero because $x \in EX^*$. We also have

$$\sum_{i,j} g_{ijj} x_i = r'/6$$
$$\sum_{i,j,k} g_{ijk} x_i n(x)_j n(x)_k = r''/6.$$

Putting this together gives

$$A + B = \frac{r'}{r^2} - \frac{r''}{r^4},$$

and this identifies Q with the remaining terms in $\Delta'(p)$.

```
embedded/geometry_check.mpl: check_laplacian_a()
embedded/geometry_check.mpl: check_laplacian_b()
embedded/geometry_check.mpl: check_laplacian_z()
```

□

Remark 6.10.10. The uniformization theorem for Riemann surfaces says that EX^* is conformally equivalent to the quotient of the open unit disc by a discrete group of automorphisms that preserve the hyperbolic metric. Using this, we deduce that the original metric on EX^* can be conformally rescaled so that the new metric has constant curvature -1. Let g denote the original metric, with curvature K, and consider a rescaled metric $g^* = e^{2f} g$. It is then known that the corresponding curvature is $K^* = (K - \Delta(f))/e^{2f}$. We therefore want to find a G-invariant function f such that $K = \Delta(f) - e^{2f}$. There is a lot of freedom to do this locally, but not globally. Thus, the most natural approach is to try to minimize $\int_{EX^*}(1 + (K - \Delta(f))/e^{2f})^2$ as f ranges over some finite-dimensional space of invariant functions. To carry this forward, we need some theory of integration on EX^*, which will be treated in the next section.

7. The surface EX^*

[sec-roothalf]

We now focus on the surface $EX^* = EX(1/\sqrt{2})$. We start by recording explicitly a number of formulae that are obtained by substituting $a = 1/\sqrt{2}$ in the results of Section 6. Maple notation for all these things is obtained by appending the character `0` to the corresponding notation in Section 6: the points v_i are `v_E0[i]`, the curves $c_j(t)$ are `c_E0[j](t)` and so on.

We have

$$EX^* = \{x \in \mathbb{R}^4 \mid \rho(x) = 1,\ g(x) = 0\} = \{x \in \mathbb{R}^4 \mid \rho(x) = 1,\ g_0(x) = 0\},$$

where

$$g(x) = x_3^2 x_4 - 2x_4^3 - (2(x_1^2 + x_2^2))x_4 + \sqrt{2}(x_1^2 - x_2^2)x_3$$
$$g_0(x) = (3x_3^2 - 2)x_4 + \sqrt{2}(x_1^2 - x_2^2)x_3.$$

The gradient of $g(x)$ is

$$n(x) = \left(2x_1(\sqrt{2}x_3 - 2x_4),\ -2x_2(\sqrt{2}x_3 + 2x_4),\ 2x_3x_4 + \sqrt{2}(x_1^2 - x_2^2),\ -2x_1^2 - 2x_2^2 + x_3^2 - 6x_4^2\right).$$

We put $y_1 = x_3$ and $y_2 = (x_2^2 - x_1^2)/\sqrt{2} - \frac{3}{2}x_3x_4$ and $z_i = y_i^2$. We find that

$$x_1^2 = u_1 = \tfrac{1}{2}(1 - \sqrt{2}y_2)(1 - y_1^2(1 - y_2/\sqrt{2}))$$
$$x_2^2 = u_2 = \tfrac{1}{2}(1 + \sqrt{2}y_2)(1 - y_1^2(1 + y_2/\sqrt{2}))$$
$$4x_1^2x_2^2 = u_3 = (1 - 2z_2)((1 - z_1)^2 - z_1^2 z_2/2)$$
$$x_1^2 + x_2^2 = u_4 = 1 - z_1 - z_1 z_2.$$

The ring of G-invariant polynomial functions on EX^* is $\mathbb{R}[z_1, z_2]$. In particular, we have

$$\|n(x)\|^2 = 4(1 - z_1/2)^2(1 + z_2),$$

and the curvature is

$$K = 1 - 2\frac{1 - 2z_2}{(1 - z_1/2)^2(1 + z_2)^2}.$$

We can reduce polynomials to normal form using the functions `NF_x0`, `NF_y0` and `NF_z0`. There are also functions `FNF_y0` and `FNF_z0` which deal intelligently with rational functions as well as polynomials.

The isotropy points are

$$\begin{aligned}
v_0 &= (0, 0, 1, 0) & v_6 &= (1, 1, 0, 0)/\sqrt{2} \\
v_1 &= (0, 0, -1, 0) & v_7 &= (-1, 1, 0, 0)/\sqrt{2} \\
v_2 &= (1, 0, 0, 0) & v_8 &= (-1, -1, 0, 0)/\sqrt{2} \\
v_3 &= (0, 1, 0, 0) & v_9 &= (1, -1, 0, 0)/\sqrt{2} \\
v_4 &= (-1, 0, 0, 0) & v_{10} &= (0, 0, \sqrt{2/3}, \sqrt{1/3}) \\
v_5 &= (0, -1, 0, 0) & v_{11} &= (0, 0, \sqrt{2/3}, -\sqrt{1/3}) \\
& & v_{12} &= (0, 0, -\sqrt{2/3}, -\sqrt{1/3}) \\
& & v_{13} &= (0, 0, -\sqrt{2/3}, \sqrt{1/3}).
\end{aligned}$$

The curve system is as follows:

$$\begin{aligned}
c_0(t) &= (\cos(t), \sin(t), 0, 0) \\
c_1(t) &= \left(\sin(t)/\sqrt{2}, \sin(t)/\sqrt{2}, \cos(t), 0\right) \\
c_2(t) &= \lambda(c_1(t)) \\
c_3(t) &= \left(0, \sin(t), \sqrt{2/3}\cos(t), -\sqrt{1/3}\cos(t)\right) \\
c_4(t) &= \lambda(c_3(t)) \\
c_5(t) &= \left(-\sin(t), 0, 2\sqrt{2}, \cos(t) - 1\right)/\sqrt{10 - 2\cos(t)} \\
c_6(t) &= \lambda(c_5(t)) \\
c_7(t) &= \mu(c_5(t)) \\
c_8(t) &= \lambda\mu(c_5(t)).
\end{aligned}$$

In this case, the curves $c_5, \dots, c_8$ have some additional properties. If we put

$$h(x) = x_3x_4/\sqrt{2} + x_1^2 + x_4^2,$$

we find that $h = 0$ on $C_5 \cup C_7$. Note here that $h(x)$ is a homogeneous quadratic. It can be diagonalised as $h(x) = \sum_{i=1}^4 m_i \langle u_i, x\rangle^2$, where

$$\begin{aligned}
m_1 &= 1 & u_1 &= (1, 0, 0, 0) \\
m_2 &= 0 & u_2 &= (0, 1, 0, 0) \\
m_3 &= \frac{2 - \sqrt{6}}{4} \simeq -0.11 & u_3 &= \left(0, 0, \sqrt{1/2 + 1/\sqrt{6}}, -\sqrt{1/2 - 1/\sqrt{6}}\right) \\
m_4 &= \frac{2 - \sqrt{6}}{4} \simeq -0.11 & u_4 &= \left(0, 0, \sqrt{1/2 - 1/\sqrt{6}}, \sqrt{1/2 + 1/\sqrt{6}}\right).
\end{aligned}$$

The vectors u_i here form an oriented orthonormal basis for $\mathbb{R}^4$.

`embedded/roothalf/E_roothalf_check.mpl: check_oval()`

One can also check that there is an alternative parametrisation of C_5 as follows:

$$c_5^{\mathrm{alt}}(t) = \left(\frac{\sin(t)}{\beta},\ 0,\ \frac{1+\beta^2}{12}\left(\sqrt{1-\sin(t)^2/\beta^4} + \cos(t)/\beta^2\right),\ \frac{\cos(t) - \sqrt{1-\sin(t)^2/\beta^4}}{2\sqrt{3}} \right),$$

where $\beta = \sqrt{2}+\sqrt{3} \simeq 3.15$. Note here that $\beta^4 \simeq 98$, so we have a good approximation

$$c_5^{\mathrm{alt}}(t) \simeq \left(\frac{\sin(t)}{\beta},\ 0,\ \frac{1+\beta^2}{12}\left(1 + \cos(t)/\beta^2\right),\ \frac{\cos(t) - 1}{2\sqrt{3}} \right),$$

showing that C_5 is close to an ellipse. If we put

$$c_6^{\mathrm{alt}}(t) = \lambda(c_5^{\mathrm{alt}}(t)) \qquad c_7^{\mathrm{alt}}(t) = \mu(c_5^{\mathrm{alt}}(t)) \qquad c_8^{\mathrm{alt}}(t) = \lambda\mu(c_5^{\mathrm{alt}}(t)),$$

and $c_k^{\mathrm{alt}} = c_k$ for $0 \leq k \leq 4$, then one can check that this gives an alternative curve system.

`embedded/roothalf/E_roothalf_check.mpl: check_c_alt()`

Proposition 7.0.11. [`prop-roothalf-fundamental`]

In the case $a = 1/\sqrt{2}$, we have

$$F_4^* = \{y \in \mathbb{R}^2 \mid |y_2| \leq 1/\sqrt{2},\ |y_1| \leq (1 + |y_2|/\sqrt{2})^{-1/2}\}$$
$$F_{16}^* = \{z \in \mathbb{R}^2 \mid 0 \leq z_2 \leq 1/2,\ 0 \leq z_1 \leq (1 + \sqrt{z_2/2})^{-1}\}.$$

Proof. From the definitions it is clear that

$$F_4^* = \{y \in \mathbb{R}^2 \mid (y_1^2, y_2^2) \in F_{16}^*\};$$

using this, we can reduce the first claim to the second one. Put

$$F'_{16} = \{z \in \mathbb{R}^2 \mid 0 \leq z_2 \leq 1/2,\ 0 \leq z_1 \leq (1 + \sqrt{z_2/2})^{-1}\},$$

so we need to show that $F_{16}^* = F'_{16}$.

Recall that $F_{16}^* = \{z \in \mathbb{R}^2 \mid z_1, z_2, u_3, u_4 \geq 0\}$, and that in the present case $a = 1/\sqrt{2}$ we have

$$u_3 = (1-2z_2)((1-z_1)^2 - z_1^2 z_2/2)$$
$$u_4 = 1 - z_1 - z_1 z_2.$$

We assume implicitly throughout that $z_1, z_2 \geq 0$. Put $w_1 = 1 - 2z_2$ and $w_2 = (1-z_1)^2 - z_1^2 z_2/2$, so $u_3 = w_1 w_2$. We will leave to the reader all the cases where any of the quantities z_1, z_2, w_1, w_2, u_3 or u_4 are zero.

First suppose that $z \in F_{16}^*$, so $u_3, u_4 > 0$. As $u_3 = w_1 w_2$ we see that w_1 and w_2 have the same sign. One can check that

$$\tfrac{1}{2} z_1^2 z_2 w_1 + w_2 = u_4(u_4 + 2z_1 z_2) > 0,$$

so w_1 and w_2 must both be positive. As $w_1 > 0$ we have $0 < z_2 < 1/2$. As $u_4 > 0$ we have $1 - z_1 > z_1 z_2 > 0$, and as $w_2 > 0$ we have $(1-z_1)^2 > z_1^2 z_2/2$; it follows that $1 - z_1 > z_1\sqrt{z_2/2}$, and thus that $0 < z_1 < 1/(1+\sqrt{z_2/2})$ as claimed.

Conversely, suppose that $0 < z_1 < 1/(1+\sqrt{z_2/2})$ (so in particular $z_1 < 1$) and $0 < z_2 < 1/2$. We can reverse the above arguments to see that $w_1, w_2 > 0$ and so $u_3 = w_1 w_2 > 0$. Next, one can check that

$$((1 - \tfrac{1}{2}z_2)(1+z_2)(1-z_1) + \tfrac{1}{2}z_2(4+z_2))u_4 = \tfrac{1}{2}z_2 w_1 + (1+z_2)^2 w_2.$$

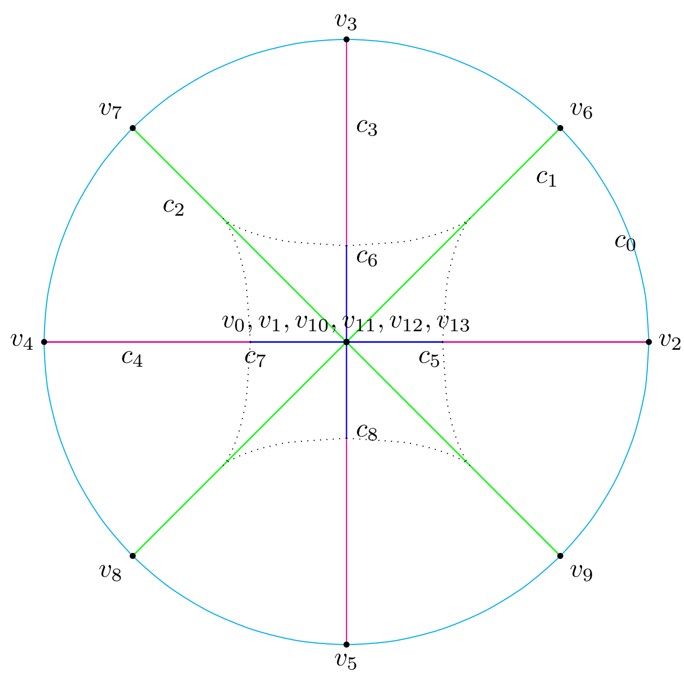
The coefficient of u_4 is strictly positive, as are all the terms on the right hand side, so we deduce that $u_4 > 0$, which means that $z \in F_{16}^*$. □

7.1. **Linear projections.** [sec-disc]

In this section we study the images of our points and curves under three different orthogonal projections $\mathbb{R}^4 \to \mathbb{R}^2$. These do not have any great theoretical significance, but they provide some insight into the geometry. Many additional details are given in the Maple code, especially the files `embedded/disc_proj.mpl` and `embedded/roothalf/zeta.mpl` and `embedded/roothalf/crease.mpl`. The projections that we consider are

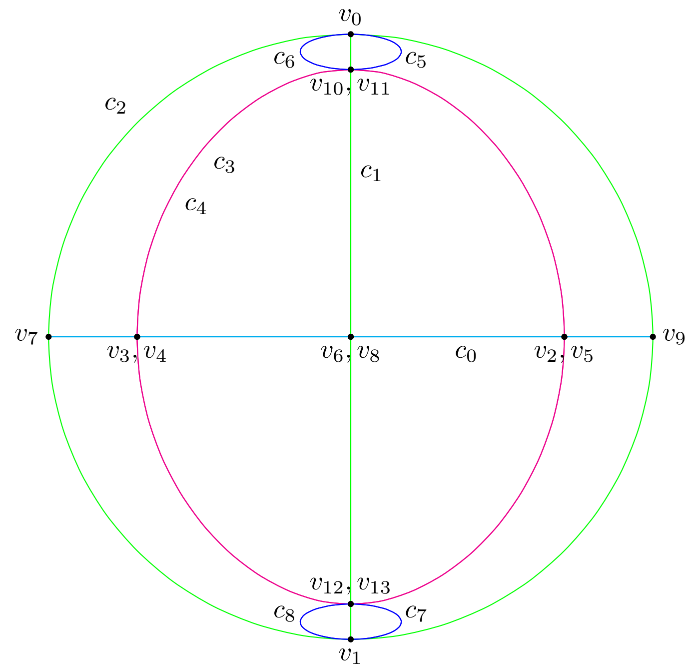
$$\pi(x) = (x_1, x_2)$$
$$\delta(x) = ((x_1 - x_2)/\sqrt{2}, x_3)$$
$$\zeta(x) = ((x_3 - x_4)/\sqrt{2}, x_2).$$

The picture for π is as follows:



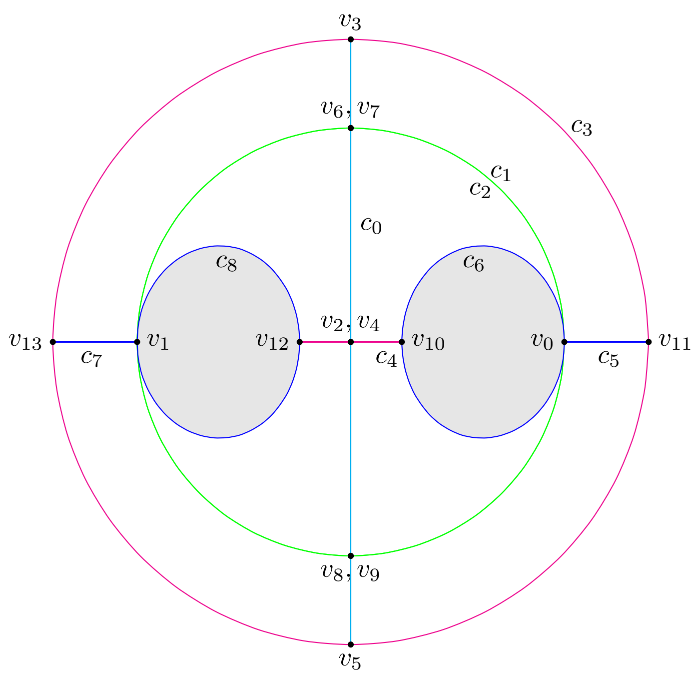
The set of singular values of π is the union of the unit circle with the dotted curve. Points inside the dotted curve have six preimages in EX^*, whereas those between the dotted curve and the unit circle have two preimages, but there is no tidy formula for these.

The effect of δ on the curves c_i and vertices v_j can be displayed as follows:



Here again the image of δ is the full unit disc. For most points in the disc, the preimage consists of two points that are exchanged by the action of $\lambda^{-1}\nu$, and one can give a nice formula for these points.

For the map ζ, we have the following picture:



In this case, the shaded regions are not part of the image, so the image is homeomorphic to a disc with two holes. It can be identified with $EX^*/\langle\lambda^2\nu\rangle$, and is combinatorially equivalent to the space Net_4^+ discussed in Section 2.6.

7.2. **Homeomorphisms with the square.** [sec-roothalf-square]

Proposition 7.2.1. *There is a homeomorphism $\tau\colon EX^*/G \to [0,1]^2$ given by*

$$\tau(x) = (2z_1 - z_1^2 + \tfrac{1}{2}z_1^2 z_2,\ 2z_2).$$

Moreover, we can define a homeomorphism $\tau^*\colon [0,1]^2 \to F_{16}$ *by*

$$\tau^*(t_1,t_2) = \left(\sqrt{\frac{(1-\sqrt{t_2})(\sqrt{t_3}+\sqrt{t_2})}{2(2+\sqrt{t_2})}},\ \sqrt{\frac{(1+\sqrt{t_2})(\sqrt{t_3}-\sqrt{t_2})}{2(2-\sqrt{t_2})}},\ \sqrt{2}\sqrt{\frac{2-\sqrt{t_3}}{4-t_2}},\ -\sqrt{t_2}\sqrt{\frac{2-\sqrt{t_3}}{4-t_2}}\right),$$

where $t_3 = t_1t_2 + 4(1-t_1)$. *This is inverse to* $\tau|_{F_{16}}$.

Maple notation for $\tau(x)$ and $\tau^*(t)$ is `t_proj(x)` and `t_lift(t)`.

Proof. First, recall that the map $p_{16}\colon x \mapsto z$ gives a homeomorphism from $F_{16} \simeq EX^*/G$ to the set

$$F_{16}^* = \{z \in \mathbb{R}^2 \mid 0 \le z_2 \le 1/2,\ 0 \le z_1 \le (1+\sqrt{z_2/2})^{-1}\}.$$

We have $\tau = \sigma \circ p_{16}$, where

$$\sigma(z) = (2z_1 - z_1^2 + \tfrac{1}{2}z_1^2 z_2,\ 2z_2).$$

An elementary exercise shows that for fixed $z_2 \in [0,1/2]$, the map $z_1 \mapsto 2z_1 - z_1^2 + \frac{1}{2}z_1^2 z_2$ gives a bijection from the interval $[0,(1+\sqrt{z_2/2})^{-1}]$ to $[0,1]$. This implies that σ gives a continuous bijection from F_{16}^* to $[0,1]^2$. It follows that τ gives a continuous bijection from F_{16} to $[0,1]^2$. All the spaces involved are compact and Hausdorff, so continuous bijections are homeomorphisms.

Next, it is clear that the definition of τ^* involves only strictly positive denominators, and square roots of nonnegative quantities, so it gives a well-defined and continuous map from $[0,1]^2$ to $\mathbb{R}^4$. Routine simplification gives $\rho(\tau^*(t)) = 1$ and $g(\tau^*(t)) = 0$, so $\tau^*(t) \in EX^*$. Recall also that $F_{16} = \{x \in EX^* \mid x_1,x_2,y_1,y_2 \ge 0\}$, where $y_1 = x_3$ and y_2 is given generically by $-x_4/x_3$. This makes it clear that the image of τ^* is contained in F_{16}. It is now easy to check that $\tau\tau^* = 1$, so τ^* is the inverse of τ.

`embedded/roothalf/E_roothalf_check.mpl: check_t_proj()`

□

The map τ is clearly smooth, but the inverse map fails to be smooth on the boundary of $[0,1]^2$. This is a necessary consequence of the fact that τ comes from a G-invariant smooth function defined on all of EX^*, but it is often awkward. For example, we can try to use τ to convert integrals over F_{16} to integrals over $[0,1]^2$, but the singular boundary behaviour makes it difficult to obtain accurate results, even with adaptive quadrature methods. We will therefore describe a different map, which has a different set of advantages and disadvantages.

Definition 7.2.2. We define $\delta\colon EX^* \to \mathbb{R}^2$ by

$$\begin{aligned}
\alpha_0(x) &= x_3 - x_4/\sqrt{2} + x_1^2 + x_4^2 + x_3(x_4 - x_2)/\sqrt{2} \\
\alpha_1(x) &= (\tfrac{3}{\sqrt{8}} - 1)x_1 + x_2 - x_3 - \sqrt{2}x_4 \\
\alpha_2(x) &= x_1 - \tfrac{3}{4}\sqrt{3}x_3 + (3 - \tfrac{3}{4}\sqrt{6})x_4 \\
\delta(x) &= \left(x_3\alpha_0(x) - x_2^2x_4,\ (x_2 - x_1)\alpha_1(x) + x_4\alpha_2(x)\right).
\end{aligned}$$

Proposition 7.2.3. [prop-square-diffeo]
The map δ *gives a diffeomorphism* $F_{16} \to [0,1]^2$, *which satisfies*

$$\delta(C_0) \subseteq 0 \times \mathbb{R} \qquad \delta(C_1) \subseteq \mathbb{R} \times 0 \qquad \delta(C_3) \subseteq \mathbb{R} \times 1 \qquad \delta(C_5) \subseteq 1 \times \mathbb{R}.$$

In order to prove this, we will need to consider the Jacobian of δ. It will be convenient to formulate the required discussion more generally.

Definition 7.2.4. Consider a map $f\colon EX^* \to \mathbb{R}^2$, and a point $a \in EX^*$. Choose an oriented orthonormal basis (u,v) for T_aEX^*, giving vectors $f_*(u), f_*(v) \in \mathbb{R}^2$. It is easy to see that the determinant $\det(f_*(u), f_*(v))$ is independent of the choice of (u,v). We write $j(f)(a)$ for this determinant, and we call $j(f)$ the *Jacobian* of f.

Lemma 7.2.5. [lem-jacobian]
Suppose that there are functions f_1, f_2 *defined on some neighbourhood of* EX^* *in* $\mathbb{R}^4$, *such that* $f(x) = (f_1(x), f_2(x))$. *Put*

$$\widetilde{\jmath}(f) = \det(x, n, \nabla f_1, \nabla f_2)$$

(where as usual $n = \nabla g$). Then $j(f) = \widetilde{\jmath}(f)/\|n\|$.

Proof. Fix a point $a \in EX^*$, and choose an oriented orthonormal basis (u, v) for the corresponding tangent space, as before. We write n for $n(a)$, and w_i for the value of ∇f_i at a. Now $(a, n/\|n\|, u, v)$ is an oriented orthonormal basis for $\mathbb{R}^4$, so we can write

$$w_i = \alpha_i a + \beta_i n/\|n\| + \gamma_i u + \delta_i v$$

for some scalars $\alpha_i, \ldots, \delta_i$. We have

$$\begin{aligned} f_*(u) &= (u.w_1,\ u.w_2) = (\gamma_1, \gamma_2) \\ f_*(v) &= (v.w_1,\ v.w_2) = (\delta_1, \delta_2), \end{aligned}$$

so $j(f)(a) = \gamma_1\delta_2 - \gamma_2\delta_1$. We need to show that this is the same as

$$D = \det(a, n/\|n\|, w_1, w_2).$$

Because $(a, n/\|n\|, u, v)$ is an oriented orthonormal basis for $\mathbb{R}^4$, we find that

$$D = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_1 & \beta_1 & \delta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \delta_2 & \gamma_2 \end{bmatrix} = \gamma_1\delta_2 - \gamma_2\delta_1$$

as required. □

Proof of Proposition 7.2.3. Put $h(x) = x_3x_4/\sqrt{2} + x_1^2 + x_4^2$. As we have noted previously, this vanishes on C_5. By direct expansion of polynomials, we find the following:

- If $x = (x_1, x_2, 0, 0)$ then $\delta(x)_1 = 0$.
- If $x = (x_1, x_1, x_3, x_4)$ then $\delta(x)_2 = 0$.
- If $x = (0, x_2, x_3, -x_3/\sqrt{2})$ then $\delta(x)_2 = \rho(x)$.
- If $x = (x_1, 0, x_3, x_4)$ then $\delta(x)_1 = \rho(x) - (1 - x_3)h(x)$.

We can compare this with the definitions of the curves $c_k(t)$ for $k \in \{0, 1, 3, 5\}$, remembering that $\rho(x) = 1$ for $x \in EX^*$; we find that the images $\delta(C_k)$ are as claimed. It is also straightforward to check that δ sends v_6, v_0, v_3 and v_{11} to $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ respectively.

Next, we can use Lemma 7.2.5 to obtain a formula for the Jacobian $j(\delta)(x)$. By numerical evaluation and plotting, we find that $j(f) > 0.1$ everywhere in F_{16}. (More precisely, the minimum is approximately 0.1079, attained at a point $c_1(t_0)$ for some t_0 with $0 < |t_0 - 5\pi/16| < 0.002$.)

Now put $E_0 = C_0 \cap F_{16}$, and note that δ_2 gives a map from E_0 to $\mathbb{R}$ with $\delta_2(v_6) = 0$ and $\delta_2(v_3) = 1$. As $j(\delta) > 0$ on F_{16} we see that this restricted map has no critical points, so it must give a diffeomorphism $E_0 \to [0, 1]$. By applying the same line of argument to the other edges of F_{16}, we find that δ gives a bijection $\partial F_{16} \to \partial[0, 1]^2$.

Now consider a point $b \in (0, 1)^2$, and put $A = \{x \in F_{16} \mid \delta(x) = b\}$. As $\delta(\partial F_{16}) = \partial[0, 1]^2$, we see that $A \cap \partial F_{16} = \emptyset$. Using the fact that $j(\delta) > 0$ on F_{16}, we see that A is discrete in F_{16}, and therefore finite, say $A = \{a_1, \ldots, a_n\}$. The fact that $j(\delta) > 0$ on F_{16} also means that δ gives an orientation preserving homeomorphism from some neighbourhood of a_i to some neighbourhood of b, and therefore induces an isomorphism

$$H_2(F_{16}, F_{16} \setminus \{a_i\}) \to H_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{b\}) \simeq \mathbb{Z}$$

of homology groups. We also have a commutative diagram

$$\begin{tikzcd} H_2(F_{16}, \partial F_{16}) \arrow[r] \arrow[d, "\delta_*"'] & H_2(F_{16}, F_{16} \setminus A) \arrow[d, "\delta_*"] \\ H_2(\mathbb{R}^2, \partial[0, 1]^2) \arrow[r, "\simeq"'] & H_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{b\}) \end{tikzcd}$$

Because F_{16} and $\mathbb{R}^2$ are contractible, and $\delta \colon \partial F_{16} \to \partial[0, 1]^2$ is a homeomorphism, we see that the left hand map is an isomorphism. Standard methods also show that the bottom map is an isomorphism, with both groups being isomorphic to $\mathbb{Z}$. On the other hand, $H_2(F_{16}, A^c)$ splits as the sum of the groups $H_2(F_{16}, F_{16} \setminus \{a_i\}) \simeq \mathbb{Z}$ indexed by the points of A, and the map

$$H_2(F_{16}, F_{16} \setminus A) \to \bigoplus_{i=1}^{n} (F_{16}, F_{16} \setminus \{a_i\})$$

is just the diagonal map $\mathbb{Z} \to \mathbb{Z}^n$. We have seen that δ_* acts as the identity on each summand, and this can only be consistent if $n = 1$. It follows that δ gives a bijection $F_{16} \to [0,1]^2$, as claimed.

embedded/roothalf/square_diffeo_check.mpl: check_square_diffeo_E0()

□

7.3. **Charts.** [sec-roothalf-charts]

Recall from Section 2.5 that each of the maps $c_k\colon \mathbb{R} \to EX^*$ can be extended in a canonical way to give a holomorphic map defined on a neighbourhood of $\mathbb{R}$ in $\mathbb{C}$. There is no case where we know a closed formula for such a holomorphic extension. However, it is not too hard to calculate high order power series approximations. For example, we have found a map $c_0^*\colon \mathbb{C} \to \mathbb{R}^4$ such that

- Each component $c_0^*(t+iu)_n$ (for $1 \leq n \leq 4$) is a polynomial of total degree at most 44 in t and u, with coefficients in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
- The polynomials $\rho(c_0^*(t+iu)) - 1$ and $g(c_0^*(t+iu))$ lie in $(t,u)^{45}$, so $c_0^*(t+iu)$ lies very close to EX^* when (t,u) is small.
- If we put $a = \partial c_0^*(t+iu)/\partial t$ and $b = \partial c_0^*(t+iu)/\partial u$ then $\langle a, b\rangle$ and $\langle a, a\rangle - \langle b, b\rangle$ lie in $(t,u)^{44}$, so c_0^* is very close to being conformal.
- $c_0^*|_{\mathbb{R}}$ is the 44th order Taylor approximation to c_0 at $t = 0$.

These conditions imply that $c_0^*(z)$ agrees with the holomorphic extension $\widetilde{c}_0(z)$ to order 44 at $z = 0$.

Calculations of this kind are implemented by methods of the class E_chart, which is defined in the file embedded/roothalf/E_atlas.mpl. Specifically, we can calculate the above chart as follows:

```
C := `new/E_chart`():
C["curve_set_exact",0,0]:
C["curve_set_degree_exact",45]:
C["p"]([t,u]);
```

Proposition 2.5.1 can also be used to define charts at points that do not lie on the curves C_k:

Proposition 7.3.1. [prop-frame-chart]

Suppose that $a \in EX^*$, and that (u, v) is an oriented orthonormal basis for the tangent space T_aEX^*. Then there is a unique local conformal chart ϕ with $u.\phi(t) = t$ and $v.\phi(t) = 0$ for small $t \in \mathbb{R}$.

Proof. We can define a real analytic map $\pi\colon EX^* \to \mathbb{C}$ by $\pi(x) = u.x + iv.x$ (so $\pi(a) = 0$). This induces an isomorphism $T_aEX^* \to \mathbb{C}$, so it is locally invertible, and we can define $c(t) = \pi^{-1}(t)$ for small $t \in \mathbb{R}$. Proposition 2.5.1 now gives a holomorphic map ϕ that is defined on a neighbourhood of 0 in $\mathbb{C}$ and extends c. □

It is again fairly straightforward to find polynomial approximations to ϕ, of any desired order. We can start with $\phi_1(t+is) = a + tu + sv$. Suppose we have defined ϕ_d of degree d such that

$$\begin{aligned}
\rho(\phi_d(t+is)) &= 1 \pmod{(s,t)^{d+1}} \\
g(\phi_d(t+is)) &= 0 \pmod{(s,t)^{d+1}} \\
\langle u, \phi_d(t)\rangle &= t \pmod{(s,t)^{d+1}} \\
\langle v, \phi_d(t)\rangle &= 0 \pmod{(s,t)^{d+1}} \\
\langle \partial_t\phi_d(t+is), \partial_u\phi_d(t+is)\rangle &= 0 \pmod{(s,t)^{d}} \\
\langle \partial_t\phi_d(t+is), \partial_t\phi_d(t+is)\rangle - \langle \partial_u\phi_d(t+is), \partial_u\phi_d(t+is)\rangle &= 0 \pmod{(s,t)^{d}}.
\end{aligned}$$

(Note that we assume a lower degree of accuracy in the last two conditions, which is natural because they involve a derivative.) We then take

$$p_{d+1}(t+is) = p_d(t+is) + \sum_{j=0}^{d+1} \alpha_j t^j s^{d+1-j},$$

where $\alpha_j \in \mathbb{R}^4$. It is not hard to see that p_{d+1} satisfies the required conditions with one more degree of accuracy iff the coefficients α_j satisfy a certain system of inhomogeneous linear equations. The abstract theory tells us that these equations must be uniquely solvable, and of course that is easily verified in any explicit computation. We have implemented a version of this using numerical approximations for the power series coefficients. (As usual, we work with 100 digit precision by default.) If `x0` is a point in EX^*, we can enter the following to find a chart of polynomial degree 20:

```
C := `new/E_chart`():
C["centre_set_numeric",x0]:
C["centre_set_degree_numeric",20]:
C["p"]([t,u]);
```

We have found charts centred at many different points of EX^*. The real problem is to patch them together by some kind of analytic continuation. The only way we have succeeded in doing this is via a hyperbolic rescaling of the metric, as we will discuss in Section 8.1. (We have attempted various more direct approaches to numerical analytic continuation, but the results were not robust, and the literature suggests that we should not expect otherwise.)

7.4. **Torus quotients.** [sec-torus-quotients]

We saw in Section 2.3 that for any cromulent surface X, the quotients $X/\langle\mu\rangle$ and $X/\langle\lambda\mu\rangle$ are tori. In Section 3.5, we gave a detailed analysis of these quotients for the projective family. In the present section, we study $EX^*/\langle\mu\rangle$ and $EX^*/\langle\lambda\mu\rangle$. If we were very optimistic we might hope for explicit conformal isomorphisms between these quotients and suitable elliptic curves, but we have not achieved that. However, we will write down reasonably simple formulae for homeomorphisms from $EX^*/\langle\mu\rangle$ and $EX^*/\langle\lambda\mu\rangle$ to $S^1 \times S^1$, which have all the expected equivariance properties and homological properties, and which do not deviate too far from being conformal.

Definition 7.4.1. [defn-AR]

We recall from Proposition 7.0.11 that $|y_1| \leq 1$ and $|y_2| \leq 1/\sqrt{2}$ on EX^*, so we can define functions $r_1, r_2 \colon EX^* \to \mathbb{R}^+$ by $r_1 = \sqrt{1 - y_2/\sqrt{2}}$ and $r_2 = \sqrt{1 + y_2/\sqrt{2}}$. We write AR for the extension of $A = \mathcal{O}_{EX^*}$ generated by r_1 and r_2, and note that AR is freely generated by the set

$$\{x_1^i x_2^j r_1^k r_2^l \mid 0 \leq i, j, k, l \leq 1\}$$

as a module over $\mathbb{R}[y_1, y_2]$. We also write KR for the field of fractions of AR, which is freely generated by the same set as a module over $\mathbb{R}(y_1, y_2)$.

Remark 7.4.2. [rem-KR-subfields]

The field KR has automorphisms α_1 and α_2 which act as the identity on the subring A and satisfy

$$\alpha_1(r_1) = -r_1 \qquad \alpha_1(r_2) = r_2$$
$$\alpha_2(r_1) = r_1 \qquad \alpha_2(r_2) = -r_2.$$

The group $G' = \langle G, \alpha_1, \alpha_2\rangle$ has order 64, and it acts on KR. Some of the work in this section and the following section can be interpreted in terms of the Galois theory of this action. There is code related to this in the files `embedded/roothalf/group64.mpl` and `embedded/roothalf/KR_subfields.mpl`.

Definition 7.4.3. [defn-E-to-TTC]

We define $\tau_1 \colon EX^* \to S^1 \subset \mathbb{C}$ by

$$\tau_1(x) = \frac{y_1(1 - y_2/\sqrt{2}) - 1/\sqrt{2} + ix_1}{(1 - y_1/\sqrt{2})r_1}$$

A straightforward calculation, using the relations in Section 6.5, shows that $|\tau_1(x)|^2 = 1$.

We then define $\tau_i\colon EX^* \to S^1$ for $2 \leq i \leq 4$ by

$$\tau_2 = \tau_1\lambda^{-1} \qquad \tau_3 = \tau_1\mu^{-1} \qquad \tau_4 = \tau_1(\lambda\mu)^{-1}.$$

We define $q\colon EX^* \to (S^1)^4$ by

$$q(x) = (\tau_1(x), \tau_2(x), \tau_3(x), \tau_4(x)).$$

Remark 7.4.4. Although we have not succeeded in formulating a precise theorem in this direction, extensive experimental investigation suggests that the map τ_1 is much simpler and better behaved than any other map in the same homotopy class.

Remark 7.4.5. [rem-q-denom]
We can define a homeomorphism $S^1 \to \mathbb{R}_\infty$ by $x + iy \mapsto (1-x)/y$, or equivalently $e^{i\theta} \mapsto \tan(\theta/2)$. Composing τ_1 with this gives the map $\tau_1^*\colon EX^* \to \mathbb{R}_\infty$ with formula

$$\tau_1^*(x) = (1/\sqrt{2} + r_1)(1 - r_1y_1)/x_1.$$

More explicitly, if $x \in EX^*$ is such that the numerator and denominator of the above fraction are not both zero, then the fraction can be interpreted in an obvious way as an element of $\mathbb{R}_\infty$, and that element is $\tau_1^*(x)$. However, for $x \in C_6$, the numerator and denominator both vanish, so we can only evaluate $\tau_1^*(x)$ by first simplifying $\tau_1(x)$, or by taking a limit over nearby points. For some purposes it is convenient to work with τ_1^* instead of τ_1, but these kinds of degenerate cases cause significant trouble. We also put

$$q^*(x) = (\tau_1^*(x), \tau_2^*(x), \tau_3^*(x), \tau_4^*(x)) \in (\mathbb{R}_\infty)^4.$$

Remark 7.4.6. In Maple, we also need to distinguish explicitly between the circle in $\mathbb{C}$ and the circle in $\mathbb{R}^2$, and thus between the 4-torus in $\mathbb{C}^4$ and the 4-torus in $\mathbb{R}^8$. We thus have two versions of q, namely `E_to_TTC` (with values in $\mathbb{C}^4$) and `E_to_TT` (with values in $\mathbb{R}^8$). We also have `E_to_TTP`, corresponding to q^*. There are functions `TT_to_TTC` and so on, which convert between these representations.

Proposition 7.4.7. [prop-E-to-TTC]
The map $q\colon EX^* \to (S^1)^4$ is equivariant with respect to the G-action on $(S^1)^4$ given by

$$\lambda(z) = (\overline{z_2}, z_1, \overline{z_4}, z_3)$$
$$\mu(z) = (z_3, \overline{z_4}, z_1, \overline{z_2})$$
$$\nu(z) = (z_1, \overline{z_2}, z_3, \overline{z_4}).$$

Moreover, the induced map

$$q_*\colon H_1EX^* \to H_1((S^1)^4) = \mathbb{Z}^4$$

is the same as the isomorphism ψ from Proposition 2.7.1.

Proof. First, using the formulae

$$\nu(x) = (x_1, -x_2, x_3, x_4)$$
$$\lambda^2(x) = (-x_1, -x_2, x_3, x_4)$$

We see that $\tau_1(\nu(x)) = \tau_1(x)$ and $\tau_1(\lambda^2(x)) = \overline{\tau_1(x)} = \tau_1(x)^{-1}$. Using this and the structure of G we deduce that q is equivariant. Next, recall that the classes $\{[c_k] \mid 5 \leq k \leq 8\}$ give a basis for H_1EX^*, whereas the inclusions of the axes give a basis $\{e_k \mid 1 \leq k \leq 4\}$ for $H_1((S^1)^4)$. Recall also that if $u, v\colon S^1 \to S^1$ have $|u - v| < 2$ everywhere, then u and v are homotopic (by a straight line homotopy) in $\mathbb{C}^\times$, so u and v have the same winding numbers. By simplification and plotting, one can check that

$$|\tau_1(c_5(t)) - e^{it}| \leq 0.14$$
$$|\tau_1(c_6(t)) - 1| = 0$$
$$|\tau_1(c_7(t)) + 1| \leq 0.23$$
$$|\tau_1(c_8(t)) + 1| = 0.$$

It follows that the winding numbers of τ_1 composed with $c_5, \ldots, c_8$ are $1, 0, 0, 0$. Using the group action, we deduce that $q_*[c_5] = e_1$, and then that $q_*[c_{4+k}] = e_k$ for $1 \leq k \leq 4$. This proves that $q_*\colon H_1EX^* \to H_1((S^1)^4)$ is an isomorphism.

embedded/roothalf/torus_quotients_check.mpl: check_torus_T()

□

Proposition 7.4.8. [prop-q-inj]
The map $q\colon EX^* \to (S^1)^4$ is injective.

Proof. Let Q denote the image of the map

$$q^*\colon C((S^1)^4,\mathbb{R}) \to C(EX^*,\mathbb{R}),$$

and let Q^+ denote the set of strictly positive functions in Q. Note that if $f \in Q^+$ we have $f = f_0 \circ q$ for some $f_0 \in C((S^1)^4,\mathbb{R})$, and by compactness there exists constant $\epsilon > 0$ such that $f \geq \epsilon$. If we put $f_1 = \max(\epsilon, f_0) \in C((S^1)^4,\mathbb{R})$ then we still have $f = f_1 \circ q$, and from this it is clear that the functions $1/f = (1/f_1)\circ q$ and $\sqrt{f} = \sqrt{f_1}\circ q$ also lie in Q^+.

We regard $x_1,\dots,x_4$ and y_1,y_2,z_1,z_2,r_1,r_2 as functions on EX^*; we need to show that they lie in Q. We write

$$q(x) = (u_1+iu_2, u_3+iu_4, u_5+iu_6, u_7+iu_8);$$

this defines elements $u_1,\dots,u_8 \in Q$. The argument can be summarised by the following list of equations.

$$\begin{aligned}
a_1 &= (2-u_1u_5-u_2u_6-u_3u_7-u_4u_8)/4 && = z_1/(2-z_1)\\
z_1 &= 2a_1/(1+a_1) \in Q\\
a_2 &= (u_1^2+u_3^2+u_5^2+u_7^2)/4\\
a_3 &= (2a_1+a_2)/((1+a_1)(1+2a_1)) && = 1/(2-z_2)\\
z_2 &= 2-1/a_3 \in Q\\
r_1r_2 &= \sqrt{1-z_2/2} \in Q^+\\
r_1+r_2 &= \sqrt{2(1+r_1r_2)} \in Q^+\\
a_4 &= 1-z_2+r_1r_2 \in Q^+\\
a_5 &= (u_1+u_3-u_5-u_7)(r_1+r_2)r_1r_2/(2(1+a_1)a_4) && = y_1 = x_3\\
a_6 &= (u_3u_5-u_1u_7)r_1r_2(1-z_1/2)/2 && = -y_1y_2 = x_4\\
a_7 &= (1-z_2/2)((u_1^2-u_3^2)(1-y_1/\sqrt{2})^2+(u_5^2-u_7^2)(1+y_1/\sqrt{2})^2)\\
a_8 &= (u_1u_5+u_2u_6-u_3u_7-u_4u_8)/(1+a_1)\\
a_9 &= (a_7+a_8)/(\sqrt{2}(1+z_1z_2)) && = y_2\\
r_1 &= \sqrt{1-y_2/\sqrt{2}} \in Q^+\\
r_2 &= \sqrt{1+y_2/\sqrt{2}} \in Q^+\\
a_{10} &= u_2r_1(1-y_1/\sqrt{2}) && = x_1\\
a_{11} &= u_4r_2(1-y_1/\sqrt{2}) && = x_2.
\end{aligned}$$

The equations with a_i on the left are definitions. In each case they define a_i in terms of functions that are already known to lie in Q, so $a_i \in Q$. All other equations are claims that can be verified by straightforward (but sometimes lengthy) calculation in the ring AR. Along the way, we need to verify that certain denominators are strictly positive. By applying the Cauchy-Schwartz inequality to the unit vectors (u_{2i-1},u_{2i}), we see that a_1 takes values in $[0,1]$ (as does a_2); this validates the definition of a_3. We know that $0 \leq z_1 = y_1^2 \leq 1$ and $0 \leq z_2 = y_2^2 \leq 1/2$, and thus that $r_1, r_2 > 0$; this validates all other denominators. We also see that $r_1+r_2 > 0$, and it is straightforward to check that $(r_1+r_2)^2 = 2(1+r_1r_2)$, so $r_1+r_2 = \sqrt{2(1+r_1r_2)}$. At the end of the chain of equations we have seen that the functions x_i all lie in Q, and this clearly implies that q is injective.

embedded/roothalf/torus_quotients_check.mpl: check_torus_T()

□

Remark 7.4.9. [rem-q-inj]

We can give simpler formulae if we are willing to use denominators that sometimes vanish. Generically, one can check that

$$\begin{aligned}
x_1 &= -\sqrt{2}u_2u_6/(u_1u_6+u_2u_5) \\
x_2 &= -\sqrt{2}u_4u_8/(u_3u_8+u_4u_7) \\
x_3 &= \sqrt{2}(u_2-u_6)/(u_2+u_6) = \sqrt{2}(u_4-u_8)/(u_4+u_8) \\
y_2 &= \sqrt{2}(\alpha-\beta)/(\alpha+\beta),
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= (u_1-u_5)^2(1-u_3u_7-u_4u_8)^2 \\
\beta &= (u_3-u_7)^2(1-u_1u_5-u_2u_6)^2.
\end{aligned}$$

One can then check that q is injective by doing some additional work to cover the cases where one or more of the above denominators are zero. However, this is unpleasant.

Remark 7.4.10. The proof of Proposition 7.4.8 implicitly gives a map from the image of q back to EX^*. This is implemented in Maple as `TTC_to_E`. The simpler function defined in Remark 7.4.9 is `TTC_to_E_generic`.

Definition 7.4.11. [defn-qp-qm]
We define $q_+, q_- \colon EX^* \to S^1 \times S^1$ by

$$\begin{aligned}
q_+(x) &= (-\tau_1(x)\tau_3(x),\ -\tau_2(x)\tau_4(x)^{-1}) \\
q_-(x) &= (-\tau_1(x)\tau_4(x),\ -\tau_2(x)\tau_3(x))
\end{aligned}$$

In Maple these are `E_to_TCp` and `E_to_TCm`.
In terms of the variables x_1, x_2, y_1, y_2 one can check that

$$\begin{aligned}
q_+(x)_1 &= \frac{2ix_1 + y_1^2(1-\sqrt{2}y_2)/\sqrt{2} - y_2(1-y_1^2/2)}{\sqrt{2}(1-y_1^2/2)(1-y_2/\sqrt{2})} \\
q_+(x)_2 &= \frac{2ix_2y_1 + y_1^2(1+\sqrt{2}y_2) - (1-y_1^2/2)}{1-y_1^2/2} \\
q_-(x)_1 &= -\frac{(ix_1 + y_1(1-y_2/\sqrt{2}) - 1/\sqrt{2})(ix_2 - y_1(1+y_2/\sqrt{2}) - 1/\sqrt{2})}{(1-y_1^2/2)\sqrt{1-y_2^2/2}} \\
q_-(x)_2 &= -\frac{(ix_1 - y_1(1-y_2/\sqrt{2}) - 1/\sqrt{2})(ix_2 + y_1(1+y_2/\sqrt{2}) - 1/\sqrt{2})}{(1-y_1^2/2)\sqrt{1-y_2^2/2}}.
\end{aligned}$$

We will prove the following result:

Proposition 7.4.12. [prop-qp-qm]
The map q_+ induces a homeomorphism $EX^*/\langle\mu\rangle \to (S^1)^2$, and the map q_- induces a homeomorphism $EX^*/\langle\lambda\mu\rangle \to (S^1)^2$.

We can get most of the way by a fairly straightforward argument. We will show that the Jacobian of q_+ (suitably interpreted) is strictly positive away from the fixed points of μ, and that the Jacobian of q_- is strictly positive away from the fixed points of $\lambda\mu$. If the Jacobian of q_+ was strictly positive everywhere, we would be able to conclude that q_+ was a covering map, and everything would follow quite easily from the general theory of coverings. In reality we have only a branched covering, and we do not have complex structures with respect to which q_+ is conformal, so we cannot use the analytic theory of branched coverings. We will need some digressions to deal with this.

We first define the version of the Jacobian which we will use.

Definition 7.4.13. Suppose we have a smooth map $u\colon EX^* \to S^1$. We then have a real vector field

$$D(u) = (\nabla u)/(iu) = \nabla(\arg(u))$$

on EX^*. Now suppose we have a smooth map $u\colon EX^* \to (S^1)^2$. We then define $\tilde{\jmath}(u)\colon EX^* \to \mathbb{R}$ by $\tilde{\jmath}(u)(x) = \det(x, n(x), D(u_1), D(u_2))$, and we call this the *Jacobian* of u.

Remark 7.4.14. [rem-jacobian-formula]
By a tiny adaptation of Lemma 7.2.5, we see that the induced map $u_*\colon T_xEX^* \to T_{u(x)}(S^1)^2$ is an isomorphism provided that $\mathcal{J}(u)(x) \neq 0$, and that it preserves orientations provided that $\mathcal{J}(u)(x) > 0$. If we have an expression for u_i as a function of the variables x_i, then we can calculate ∇u_i by taking the vector of partial derivatives, and then projecting it orthogonally into the tangent space. However, this orthogonal projection will just alter our vector by multiples of x and $n(x)$, and this will leave the determinant $\mathcal{J}(u)(x)$ unchanged. Thus, we can just work with the original vector of partial derivatives.

Remark 7.4.15. Suppose that $u = v + iw$ with $v^2 + w^2 = 1$, and put $u^* = (1 - v)/w\colon EX^* \to \mathbb{R}_\infty$, as in Remark 7.4.5. Differentiating the relation $v^2 + w^2 = 1$ gives $v\nabla(v) + w\nabla(w) = 0$. Using this one can check that

$$D(u) = \frac{\nabla u}{iu} = \frac{2\nabla u^*}{1 + (u^*)^2}.$$

This form is sometimes easier to use.

Proposition 7.4.16. [prop-qp-J]
The Jacobian of q_+ is

$$\frac{4\sqrt{2}(1 - x_1^2)}{(1 - y_1^2/2)(1 - y_2/\sqrt{2})}.$$

This is zero at the points $v_2 = (1, 0, 0, 0)$ and $v_4 = -v_2$ (which are precisely the fixed points of μ). It is strictly positive everywhere else in EX^*.

Proof. The formula can be checked by computer calculation following the recipe described above. (It is somewhat miraculous that the final answer is so simple, as the intermediate calculations are enormous.) It is clear from the formula that the Jacobian vanishes iff $x_1 = \pm 1$, which forces $x_2 = x_3 = x_4 = 0$ because $\sum_i x_i^2 = \rho(x) = 1$.

`embedded/roothalf/torus_quotients_check.mpl: check_torus_jacobian()`

□

Proposition 7.4.17. [prop-qm-J]
The Jacobian of q_-^* is

$$4\sqrt{2}\frac{3/2 - (1 - y_1^2/2)(1 - y_2^2/2) - x_1x_2}{(1 - y_1^2/2)(1 - y_2^2/2)}.$$

This is zero at the points $v_6 = (1, 1, 0, 0)/\sqrt{2}$ and $v_8 = -v_6$ (which are precisely the fixed points of $\lambda\mu$). It is strictly positive everywhere else in EX^*.

Proof. The formula for the Jacobian can be checked by computer calculation following the recipe described above. The conclusion is that j is a positive multiple of $a - x_1x_2$, where $a = 3/2 - (1 - y_1^2/2)(1 - y_2^2/2)$. Now x_1^2 and x_2^2 can be rewritten as polynomials in y_1 and y_2, and using this we obtain

$$a^2 - (x_1x_2)^2 = y_2^2(1 + y_2^2/4) + y_1^2(1 - y_2^2)(1 + y_2^2/4) + \tfrac{3}{8}y_1^4y_2^2(1 - y_2^2/2).$$

It is visible that the right hand side is nonnegative, and it vanishes only where $y_1 = y_2 = 0$. It is easy to see that the only points with these values of y are v_6, v_7, v_8 and v_9. By going back to the original formula, we see that the Jacobian is zero at v_6 and v_8, but $8\sqrt{2}$ at v_7 and v_9. Moreover, the Jacobian is nowhere zero on the path-connected space $EX^* \setminus \{v_6, v_8\}$, so it cannot change sign; it is positive at v_7, so it must be positive everywhere.

`embedded/roothalf/torus_quotients_check.mpl: check_torus_jacobian()`

□

We next need to understand the preimages of a few points under the maps q_+ and q_-.

Proposition 7.4.18. [prop-q-preimages]

$$
\begin{aligned}
q_+^{-1}\{(\ \ 1,\ \ 1)\} &= \{v_0, v_1\} \\
q_+^{-1}\{((1+2\sqrt{2}i)/3, -1)\} &= \{v_2\} \\
q_+^{-1}\{((1-2\sqrt{2}i)/3, -1)\} &= \{v_4\} \\
q_-^{-1}\{(\ \ 1,\ \ 1)\} &= \{v_0, v_1\} \\
q_-^{-1}\{(\ \ i,\ \ i)\} &= \{v_6\} \\
q_-^{-1}\{(-i, -i)\} &= \{v_8\}.
\end{aligned}
$$

Proof. First recall that

$$
q_+(x)_2 = \frac{2ix_2y_1 + y_1^2(1+\sqrt{2}y_2) - (1-y_1^2/2)}{1-y_1^2/2}.
$$

By inspecting the imaginary part, we see that $q_+(x)$ can only be equal to $(1,1)$ or $((1+2\sqrt{2}i)/3, -1)$ if $x_2y_1 = 0$. The results in Section 6.6 show that this is only possible if $x \in C_0 \cup C_4 \cup C_5 \cup C_7$. One can check from the definitions that

$$
\begin{aligned}
q_+(c_0(t)) &= \left(\frac{\cos(t) + i/\sqrt{2}}{\cos(t) - i/\sqrt{2}},\ -1 \right) \\
q_+(c_4(t)) &= \left(\frac{i\sqrt{2} + \sin(t)}{i\sqrt{2} - \sin(t)},\ -1 \right) \\
q_+(c_5(t)) = q_+(c_7(t)) &= \left(\frac{\sin(t)^2 + 8\cos(t) + 4\sin(t)\sqrt{5-\cos(t)}i}{9-\cos(t)^2},\ 1 \right).
\end{aligned}
$$

This gives

$$
\mathrm{Im}(q_+(c_5(t))_1) = \mathrm{Im}(q_+(c_7(t))_1) = \frac{4\sin(t)\sqrt{5-\cos(t)}}{9-\cos(t)^2}.
$$

It follows easily that

$$
q_+^{-1}\{(1,1)\} \subseteq \{c_5(0), c_5(\pi), c_7(0), c_7(\pi)\} = \{v_0, v_1, v_{10}, v_{11}\}.
$$

By inspecting the definitions, we find that $q_+(v_0) = q_+(v_1) = (1,1)$ but $q_+(v_{10}) = (1,-1)$ and $q_+(v_{11}) = (-1,1)$. It follows that $q_+^{-1}\{(1,1)\} = \{v_0, v_1\}$ as claimed. Similarly, if $q_+(x) = ((1+2\sqrt{2}i)/3, -1)$ then we must have $x = c_0(s)$ for some s, or $x = c_4(t)$ for some t. Solving $(\cos(s)+i/\sqrt{2})/(\cos(s)-i/\sqrt{2}) = (1+2\sqrt{2}i)/3$ gives $\cos(s) = 1$, and solving $(i\sqrt{2}+\sin(t))/(i\sqrt{2}-\sin(t)) = (1+2\sqrt{2}i)/3$ gives $\sin(s) = -1$. We must therefore have $x = c_0(0)$ or $x = c_4(-\pi/2)$, and both of these are equal to v_2 as expected. A very similar argument gives $q_+^{-1}\{((1+2\sqrt{2}i)/3, -1)\} = \{v_4\}$.

Next, if $q_-(x)$ is $(1,1)$ or (i,i) or $(-i,-i)$ then we have $q_-(x)_1 - q_-(x)_2 = 0$. One can check from the definitions that

$$
q_-(x)_1 - q_-(x)_2 = 2\frac{(x_1y_1(1+y_2/\sqrt{2}) - x_2y_1(1-y_2/\sqrt{2}))i - y_1y_2}{(1-y_1^2/2)\sqrt{1-y_2^2/2}}.
$$

By inspecting the real part, we see that $y_1y_2 = 0$, but $y_1y_2 = -x_4$, so Proposition 6.6.13 tells us that $x \in C_0 \cup C_1 \cup C_2$. Now put

$$
\begin{aligned}
m_0(t) &= \frac{i\sqrt{2}(\sin(t)+\cos(t)) + \sin(2t) - 1}{\sqrt{4-\cos(2t)^2}} \\
m_1(t) &= \frac{i - \sin(t)}{i + \sin(t)} \\
m_2(t) &= \frac{i\sqrt{2}\cos(t) + 2\sin(t)}{i\sqrt{2}\cos(t) - 2\sin(t)}.
\end{aligned}
$$

One can directly that $q_-(c_k(t)) = (m_k(t), m_k(t))$ for $k = 0, 1$, but $q_-(c_2(t)) = (m_2(t), \overline{m_2(t)})$. We therefore need to solve $m_k(t) = 1$ and $m_k(t) = \pm i$.

It is easy to see that $\mathrm{Re}(m_0(t)) \leq 0$ for all t, so $m_0^{-1}\{1\} = \emptyset$. It is also easy to see that $m_1^{-1}\{1\} = m_2^{-1}\{1\} = \{0, \pi\}$, so

$$q_+^{-1}\{(1,1)\} = \{c_1(0), c_1(\pi), c_2(0), c_2(\pi)\} = \{v_0, v_1, v_1, v_0\} = \{v_0, v_1\}$$

as expected.

Next, for $m_0(t) = \pm i$ we need $\mathrm{Re}(m_0(t)) = 0$, which gives $\sin(2t) = -1$, so $t = \pi/4 \pmod{\pi}$. In fact we have $m_0(\pi/4) = i$ and $m_0(5\pi/4) = -i$, whereas $c_0(\pi/4) = v_6$ and $c_0(5\pi/4) = v_8$. Similarly, for $m_1(t) = \pm i$ we need $\sin(t) = \pm 1$, so $t = \pm\pi/2 \pmod{2\pi}$, whereas $c_1(\pi/2) = v_6$ and $c_1(-\pi/2) = v_8$. On the other hand, the relation $q_-(c_2(t)) = (m_2(t), \overline{m_2(t)})$ shows that $q_-(c_2(t))$ can never be equal to (i, i) or $(-i, -i)$. Putting this together, we see that $q_-^{-1}\{(i,i)\} = \{v_6\}$ and $q_-^{-1}\{(-i,-i)\} = \{v_8\}$, as expected. □

Proof of Proposition 7.4.12. First, it is straightforward to check that $q_+\mu = q_+$, so that q_+ induces a map $EX^*/\langle\mu\rangle \to (S^1)^2$.

Next, put

$$w_2 = q_+(v_2) = ((1 + 2\sqrt{2}i)/3, -1)$$
$$w_4 = q_+(v_4) = ((1 - 2\sqrt{2}i)/3, -1).$$

For any $u \in (S^1)^2 \setminus \{w_2, w_4\}$, Proposition 7.4.16 tells us that $q_+^{-1}\{u\}$ consists of points where the Jacobian is strictly positive. It follows (using the standard theory of degrees of maps of compact oriented manifolds) that the set $q_+^{-1}\{u\}$ is finite, of cardinality equal to the degree of q_+. This cardinality is two in the case $u = (1,1)$, so it must be two for all $u \notin \{w_2, w_4\}$. In these cases $q_+^{-1}\{u\}$ is contained in the set $EX^* \setminus \{v_2, v_4\}$ where μ acts freely, so $q_+^{-1}\{u\}$ must be a μ-orbit. Moreover, if $u = w_2$ or $u = w_4$ then Proposition 7.4.18 again tells us that $q_+^{-1}\{u\}$ is a (singleton) μ-orbit. It follows that the induced map $EX^*/\langle\mu\rangle \to (S^1)^2$ is a continuous bijection, and thus a homeomorphism (because the domain and codomain are compact and Hausdorff).

The proof for q_- is essentially the same. □

Remark 7.4.19. The Maple code contains a formula for the inverse of the map $q_+ \colon EX^*/\langle\mu\rangle \to (S^1)^2$. It also contains a method for computing the inverse of the map $q_- \colon EX^*/\langle\lambda\mu\rangle \to (S^1)^2$, which is not quite a formula because it involves solutions of a polynomial of degree four in one variable. These are given by the functions `TCp_to_E` and `TCm_to_E`, defined in `embedded/roothalf/torus_quotients.mpl`.

Remark 7.4.20. [rem-not-smooth]

Recall from Remark 2.3.4 that the smooth structures on $EX^*/\langle\mu\rangle$ and $EX^*/\langle\lambda\mu\rangle$ are subtle, so we cannot assume that the induced maps $EX^*/\langle\mu\rangle \to (S^1)^2$ and $EX^*/\langle\lambda\mu\rangle \to (S^1)^2$ are smooth. In fact, one can check that they are not. To do this, we need a chart ϕ centred at the branch point v_2 as in Section 7.3. One of the relevant functions is only implemented for vertices in F_{16}, so we find a chart centred at v_3 and apply λ^{-1}:

```
C := `new/E_chart`():
C["vertex_set_exact",3]:
C["curve_set_degree_exact",11]:
x0 := act_R4[LLL](C["p"]([t,u])):
s0 := simplify(multi_series(E_to_TCp(x0)[1],7,t,u));
```

This sets `s0` to a Taylor approximation to $q_+(\phi(t,u))_1$. If q_+ was smooth, it is not hard to see that `s0` would be expressible as a polynomial in the quantities

$$m = \mathrm{Re}((t + iu)^2) = t^2 - u^2$$
$$n = \mathrm{Im}((t + iu)^2) = 2tu,$$

and thus that the coefficients of t^6u^0 and t^0u^6 in `s0` would be negatives of each other. However, the above calculation gives the real parts of the relevant coefficients as $-8/405$ and $-8/243$, so the map is not in fact smooth. The same argument works for q_-, using a chart based at v_6, but in that case we already see a contradiction from the coefficients of t^2u^0 and t^0u^2.

7.5. **Sphere quotients.** [sec-sphere-quotients]

Remark 3.7.14 gives us a canonical conformal isomorphism $\widehat{p}\colon EX^*/\langle\lambda^2\rangle \to S^2$, but we do not know an exact formula for that. However, we will define a different homeomorphism $m\colon EX^*/\langle\lambda^2\rangle \to S^2$ which has many of the same properties as $\widehat{p}$. Specifically, m and $\widehat{p}$ are both equivariant for the same action of $G/\langle\lambda^2\rangle$ on S^2, and $m(v_i) = \widehat{p}(v_i)$ for $0 \le i \le 9$.

Definition 7.5.1. [defn-sphere-quotient-a]

For $x \in EX^*$, we put

$$\begin{aligned}
\widetilde{m}(x) &= \left(\sqrt{2}(1-y_1^2)y_2,\ 2x_1x_2,\ -2y_1\right)/(1+y_1^2) \in \mathbb{R}^3 \\
s(x) &= z_1^2 z_2(\tfrac{1}{2} - z_2)/(1+z_1)^2 \\
m(x) &= \widetilde{m}(x)/\sqrt{1-s(x)}.
\end{aligned}$$

(Maple notation for $m(x)$ is `E_to_S2(x)`.)

Proposition 7.5.2. [prop-sphere-quotient-a]

The above formula gives a map $m\colon EX^*/\langle\lambda^2\rangle \to S^2$. It satisfies

$$\begin{aligned}
m(v_0) &= (\ \ 0,\ \ 0, -1) \\
m(v_1) &= (\ \ 0,\ \ 0,\ \ 1) \\
m(v_2) = m(v_4) &= (-1,\ \ 0,\ \ 0) \\
m(v_3) = m(v_5) &= (\ \ 1,\ \ 0,\ \ 0) \\
m(v_6) = m(v_8) &= (\ \ 0,\ \ 1,\ \ 0) \\
m(v_7) = m(v_9) &= (\ \ 0, -1,\ \ 0) \\
m(v_{10}) &= (-1, 0, -2\sqrt{6})/5 \\
m(v_{11}) &= (\ \ 1, 0, -2\sqrt{6})/5 \\
m(v_{12}) &= (-1, 0,\ \ 2\sqrt{6})/5 \\
m(v_{13}) &= (\ \ 1, 0,\ \ 2\sqrt{6})/5.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
m_1(\lambda(x)) &= -m_1(x) & m_2(\lambda(x)) &= -m_2(x) & m_3(\lambda(x)) &= \ \ m_3(x) \\
m_1(\mu(x)) &= \ \ m_1(x) & m_2(\mu(x)) &= -m_2(x) & m_3(\mu(x)) &= -m_3(x) \\
m_1(\nu(x)) &= \ \ m_1(x) & m_2(\nu(x)) &= -m_2(x) & m_3(\nu(x)) &= \ \ m_3(x).
\end{aligned}$$

Proof. Follows directly from the definitions.

`embedded/roothalf/sphere_quotients_check.mpl: check_E_to_S2()`

∎

Remark 7.5.3. [rem-m-tilde]

One can check that the function $s(x)$ is zero at all the points v_i, and on $\bigcup_{i=0}^4 C_i$. Moreover, it is nonnegative and small everywhere, with a maximum value of about 0.0114. (An exact expression is recorded as `E_to_S2_s_max`.) Thus, the simpler function $\widetilde{m}(x)$ is a good approximation to $m(x)$.

Remark 7.5.4. [rem-m-a]

Recall that if $EX^* \simeq PX(a)$ then we have $\widehat{p}(v_{11}) = (2a, 0, a^2-1)/(a^2+1)$. If $a_0 = (\sqrt{3}-\sqrt{2})^2 \simeq 0.10102$ then we find that $(2a_0, 0, a_0^2-1)/(a_0^2+1) = (1, 0, -2\sqrt{6})/5 = m(v_{11})$. Thus, if we believed that m was close $\widehat{p}$ then we would expect that $EX^* \simeq PX(a)$ for some a that is close to a_0. In fact, the correct value of a is approximately 0.09836. It is perhaps surprising that this is so close to a_0, as m is quite far from being conformal.

Proposition 7.5.5. [prop-sphere-quotient-b]

The map $m\colon EX^*/\langle\lambda^2\rangle \to S^2$ is a homeomorphism.

As with Proposition 7.4.12, the main ingredient is the calculation of the Jacobian of m. Here we need a slightly different version of the Jacobian, because the codomain is S^2 rather than $\mathbb{R}^2$. Suppose that $m(a) = b$, so m gives a linear map $T_aEX^* \to T_bS^2$. Let (u, v) is an oriented orthonormal basis for T_aEX^*, and let (u', v') is an oriented orthonormal basis for T_aS^2. We can then form the matrix of m_* with respect to these bases, and $j(m)(a)$ is defined to be the determinant of that matrix.

Lemma 7.5.6. [lem-m-J]
Put $m_{i,j} = \partial m_i/\partial x_j$ and $n = \nabla(g)$ and

$$\widetilde{\jmath}(m) = -\frac{1}{2}\sum_{\sigma\in\Sigma_4}\sum_{\tau\in\Sigma_3}\epsilon(\sigma)\epsilon(\tau)x_{\sigma(1)}n_{\sigma(2)}m_{\tau(1),\sigma(3)}m_{\tau(2),\sigma(4)}m_{\tau(3)}.$$

Then $j(m) = \widetilde{\jmath}(m)/\|n\|$.

Proof. Fix a point x, and choose orthonormal bases (u, v) and (u', v') for $T_xEX(a)$ and $T_{m(x)}S^2$ as in the definition of $j(m)$. Note that with conventions as spelled out in Remark 3.7.14, the orientation conditions are that $\det(x, n/\|n\|, u, v) = 1$ and $\det(u', v', m) = -1$. Given this, it is not hard to see that $j(m) = -\det(m_*(u), m_*(v), m)$. Equivalently, if ω_d denotes the standard volume form for $\mathbb{R}^d$ and $\beta = u \wedge v$, then $j(m)$ is characterised by $m_*(\beta) \wedge m = -j(m)\,\omega_3$. Now Lemma 6.10.5 tells us that

$$\beta = \frac{1}{2\|n\|}\sum_{ijkl}\epsilon_{ijkl}x_in_je_k \wedge e_l,$$

so

$$\begin{aligned}
m_*(\beta) &= \frac{1}{2\|n\|}\sum_{ijkl}\epsilon_{ijkl}x_in_jm_*(e_k)\wedge m_*(e_l)\\
&= \frac{1}{2\|n\|}\sum_{ijklpq}\epsilon_{ijkl}x_in_jm_{p,k}m_{q,l}e_p\wedge e_q\\
m_*(\beta)\wedge m &= \frac{1}{2\|n\|}\sum_{ijklpqr}\epsilon_{ijkl}x_in_jm_{p,k}m_{q,l}m_re_p\wedge e_q\wedge e_r\\
&= \frac{1}{2\|n\|}\sum_{ijklpqr}\epsilon_{ijkl}\epsilon_{pqr}x_in_jm_{p,k}m_{q,l}m_r\omega_3.
\end{aligned}$$

The claim is clear from this. □

Corollary 7.5.7. [cor-m-J]
$\widetilde{\jmath}(m) = 8j_1/j_2^{3/2}$, where

$$\begin{aligned}
j_1 &= (1+z_1)((1-z_1)^2 - z_1^2z_2/2) + z_1^2z_2(\tfrac{1}{2} - z_2)(3+z_1)\\
j_2 &= 1 + 2z_1 + z_1^2(1 - z_2/2) + z_1^2z_2^2 \geq 1.
\end{aligned}$$

Moreover, we have $j_1 \geq 0$ everywhere, with $j_1 = 0$ only if

$$x \in (EX^*)^{\lambda^2} = \{v_0, v_1, v_{10}, v_{11}, v_{12}, v_{13}\}.$$

Proof. The lemma reduces the formula to a direct calculation, which can be checked by Maple. Next, we can write $j_1 = j_3j_4 + j_5$, where

$$\begin{aligned}
j_3 &= 1 + z_1 \geq 1\\
j_4 &= (1-z_1)^2 - z_1^2z_2/2\\
j_5 &= z_1^2z_2(\tfrac{1}{2} - z_2)(3+z_1) \geq 0.
\end{aligned}$$

Another standard calculation gives

$$(1/2 - z_2)j_4 = 2x_1^2x_2^2 \geq 0.$$

Recall that $1/2 - z_2 \geq 0$ everywhere, and $1/2 - z_2 > 0$ on a dense subset of EX^*; it follows that $j_4 \geq 0$ everywhere. It follows that $j_1 \geq 0$ everywhere, and that if $j_1 = 0$ then we must have $j_4 = j_5 = 0$. Note that $j_4 = 1$ when $z_1 = 0$, so we must have $z_1 > 0$ (and also $z_1 \leq 1$ as always). This lets us rearrange $j_4 = 0$

to give $z_2 = 2(z_1^{-1} - 1)^2$. After substituting this in the relation $j_5 = 0$ we see that $z_1 \in \{2/3, 1\}$ and so $z \in \{(2/3, 1/2), (1, 0)\}$. Each point in the z-plane corresponds to a G-orbit in EX^*, and it is straightforward to check that the relevant G-orbits are $\{v_{10}, v_{11}, v_{12}, v_{13}\}$ and $\{v_0, v_1\}$.

`embedded/roothalf/sphere_quotients_check.mpl: check_E_to_S2()`

$\square$

Lemma 7.5.8. [lem-sphere-quotient-preimages]
For all i we have $m^{-1}\{m(v_i)\} = \{v_i, \lambda^2(v_i)\}$. In particular, for $i \in \{0, 1, 10, 11, 12, 13\}$, we have $\lambda^2(v_i) = v_i$ and $m^{-1}\{m(v_i)\} = \{v_i\}$.

Proof. Recall that formulae for $m(v_i)$ were given in Proposition 7.5.2.
Now suppose that $x \in EX^*$ with $m(x) = u$.

(a) We have $u_1 = 0$ iff $(1 - y_1^2)y_2 = 0$ iff $x_3 = y_1 = \pm 1$ or $y_2 = 0$. Note that if $x_3 = \pm 1$ we must have $x \in \{e_3, -e_3\} = \{v_0, v_1\}$. On the other hand, we have $y_2 = 0$ iff $x \in C_1 \cup C_2$.
(b) We have $u_2 = 0$ iff $x_1 = 0$ or $x_2 = 0$, which means that $x \in \bigcup_{i=3}^8 C_i$.
(c) We have $u_3 = 0$ iff $x_3 = 0$ iff $x \in C_0$.

If $u = m(v_i)$ for some $i < 10$ then two of the coordinates u_p are zero, and it follows that $x \in C_r \cap C_s$ for some $r \neq s$, so $x = v_j$ for some j. A check of cases then shows that $x \in \{v_i, \lambda^2(v_i)\}$. Suppose instead that

$$m(x) = m(v_{11}) = (1, 0, -2\sqrt{6})/5.$$

As $m(x)_2 = 0$ we have $x_1 x_2 = 0$ and so $x \in \bigcup_{i=3}^8 C_i$. From the form of $m(x)_1$ and $m(x)_3$ it is also clear that $y_1, y_2 > 0$. Also, we have

$$\frac{\sqrt{2}(1 - y_1^2)y_2}{2y_1} = -\frac{m(x)_1}{m(x)_3} = \frac{1}{2\sqrt{6}},$$

so $y_2 = (2\sqrt{3}(y_1^{-1} - y_1))^{-1}$. Substituting this in the relation $m(x)_1^2 = 1/25$ and factoring leads to a relation

$$(2y_1^2 - 3)(3y_1^2 - 2)^2(8y_1^6 - 27y_1^4 + 30y_1^2 - 12) = 0.$$

One can check that the only root in the required range $0 < y_1 \leq 1$ is $y_1 = \sqrt{2/3}$, and this in turn gives $y_2 = (2\sqrt{3}(y_1^{-1} - y_1))^{-1} = 1/\sqrt{2}$. This gives $(x_3, x_4) = (y_1, -y_1 y_2) = (\sqrt{2/3}, -\sqrt{1/3})$, and as $x_3^2 + x_4^2 = 1$ we must have $x_1 = x_2 = 0$, so $x = v_{11}$ as required. The remaining cases $i \in \{10, 12, 13\}$ now follow using the group action. $\square$

Proof of Proposition 7.5.5. Put

$$U = S^2 \setminus \{m(v_i) \mid i \in \{0, 1, 10, 11, 12, 13\}\}.$$

We now see that for $u \in U$, the preimage $m^{-1}\{u\}$ contains only points where the Jacobian of m is strictly positive, so the number of points is equal to the degree of m. Taking $u = m(v_2)$ we see that $m^{-1}\{u\} = \{v_2, \lambda^2(v_2)\} = \{v_2, v_4\}$, so the degree is two. It follows that for all $u \in U$, the preimage consists of two points and is closed under the action of λ^2. The only points fixed by λ^2 are $\{v_0, v_1, v_{10}, v_{11}, v_{12}, v_{13}\}$, and these cannot lie in $m^{-1}\{u\}$, so $m^{-1}\{u\}$ must consist of a single λ^2-orbit. The same holds by Lemma 7.5.8 in the exceptional cases where $u \in S^2 \setminus U$. It follows that the induced map $EX^*/\langle\lambda^2\rangle \to S^2$ is a continuous bijection of compact Hausdorff spaces, so it is a homeomorphism. $\square$

Remark 7.5.9. Again put

$$U = S^2 \setminus \{m(v_i) \mid i \in \{0, 1, 10, 11, 12, 13\}\}.$$

In Section 8.2 we will explain how to define a function $u = C_+(Dm)/(C_+(Dm) + C_-(Dm)) \colon U \to [0, 1]$ which is zero at points where m is conformal, and one at points where m is anticonformal. We have not found a formula for u, but it is not hard to evaluate it at any given point. We find that $0 \leq u < 1/2$ everywhere in U, with $u(x) \to 1/2$ as $x \to v_0$ or $x \to v_1$. For $i \in \{10, 11, 12, 13\}$ we find that u oscillates between about 0.01 and 0.08 on any small circle surrounding v_i, so it does not extend continuously to v_i. On about 90% of the area of EX^* we have $u < 0.075$.

There are two other maps $EX^* \to S^2$ that have natural geometric descriptions, so it is reasonable to ask whether they are related to $\widehat{p}$. The first is the Hopf fibration $\eta\colon S^3 \to S^2$, which we can restrict to EX^*. (In Maple this is hopf_map, which is defined in Rn.mpl.) Note that $H_2(EX^*) \simeq H_2(S^2) \simeq \mathbb{Z}$, but $H_2(S^3) = 0$, which implies that the map $\eta_*\colon H_2(EX^*) \to H_2(S^2)$ is zero. In other words, the restricted map $\eta\colon EX^* \to S^2$ has degree zero, whereas $\widehat{p}$ has degree 2, and any other nonconstant conformal map has strictly positive degree. This means that η cannot be closely related to $\widehat{p}$.

```
Rn_check.mpl: check_hopf_map()
```

The second possibility is a variant of the Gauss map. We can identify $\mathbb{R}^4$ with the algebra $\mathbb{H}$ of quaternions, and $\mathbb{R}^3$ with the subspace $\mathbb{H}_0$ of purely imaginary quaternions. This allows us to interpret conjugation and multiplication of elements of $\mathbb{R}^4$. For $x \in X$ we recall that $n(x)$ is orthogonal to x, so the element $\gamma(x) = n(x)\overline{x}/\|n(x)\|$ is a purely imaginary quaternion of norm one. This defines a map $\gamma\colon X \to S^2$ (which is gauss_map in Maple). It depends on our conventions for identifying $\mathbb{R}^4$ with $\mathbb{H}$, but the dependence is easy to analyse. It is known that every special orthogonal automorphism of $\mathbb{H}$ has the form $x \mapsto ux\overline{v}$ for some $u, v \in S^3$, and every special orthogonal automorphism of $\mathbb{H}_0$ has the form $x \mapsto wx\overline{w}$. Changing conventions will replace γ by a map of the form $\gamma'(x) = w\gamma(ux\overline{v})\overline{w}$. By choosing paths in S^3 from u, v and w to the identity, we can produce a homotopy between γ and γ', so they at least have the same degree. One can calculate the degree by counting preimages of a regular value, with signs determined by the orientation behaviour. One can check that $\gamma^{-1}\{(1,1,0)/\sqrt{2}\} = \{v_6\}$, and that γ gives an orientation-reversing isomorphism of tangent spaces at this point. It follows that γ has degree -1, and cannot be homotopic to $\widehat{p}$.

```
embedded/geometry_check.mpl: check_gauss_map();
```

7.6. **Rational points.** [sec-rational]

In this section we study points in EX^* where the coordinates are rational or lie in some small extension of $\mathbb{Q}$. As well as being interesting for its own sake, it is useful to have a supply of points where we can easily do exact calculations rather than relying on numerical approximation.

Proposition 7.6.1. [prop-rational]

The set $EX^*(\mathbb{Q}) = EX^* \cap \mathbb{Q}^4$ is as follows:

(a) *For every $t \in \mathbb{Q}$, the point $(t^2-1, 2t, 0, 0)/(1+t^2)$ lies in $EX^*(\mathbb{Q}) \cap C_0$.*
(b) *The point $v_2 = (1,0,0,0)$ (corresponding to $t = \infty$) also lies in $EX^*(\mathbb{Q}) \cap C_0$.*
(c) *For any pair $(s,t) \in \mathbb{Q}^2$ with $2s^2 + t^2 = 1$, the point $(s,s,t,0)$ lies in $EX^*(\mathbb{Q}) \cap C_1$, and the point $(s,-s,t,0)$ lies in $EX^*(\mathbb{Q}) \cap C_2$.*
(d) *All points in $EX^*(\mathbb{Q})$ are of type (a), (b) or (c).*

Proof. Recall that the cubic defining equation is

$$(\tfrac{3}{2}x_3^2 - 2)x_4 + (x_1^2 - x_2^2)x_3/\sqrt{2} = 0.$$

Note here that 1 and $1/\sqrt{2}$ are linearly independent over $\mathbb{Q}$, and also that $\frac{3}{2}x_3^2 - 2 \neq 0$ for $x_3 \in \mathbb{Q}$. It follows that all rational solutions have $x_4 = 0$ and (either $x_3 = 0$ or $x_1 = \pm x_2$). If $x_3 = x_4 = 0$ then $x_1^2 + x_2^2 = 1$. If $x_1 = 1$ then we have case (a), otherwise we have case (b) with $t = x_2/(1-x_1)$. If $x_4 = 0$ and $x_1 = \pm x_2$ then we have case (c). $\square$

The pairs (s,t) as in (c) are well-understood in terms of the arithmetic of the field $\mathbb{Q}(\sqrt{-2})$, as we now recall briefly. For any element $x = t + s\sqrt{-2} \in \mathbb{Q}(\sqrt{-2})$, we put

$$N(x) = |x|^2 = 2s^2 + t^2 \in \mathbb{Q}.$$

This gives a homomorphism $\mathbb{Q}(\sqrt{-2})^\times \to \mathbb{Q}^\times$, and $EX^*(\mathbb{Q}) \cap C_1$ bijects with $\ker(N)$. If p is a prime congruent to 1 or 3 mod 8, then it is well-known that there is a unique pair of positive integers (a,b) such that $2a^2 + b^2 = p$. We put $\pi_p = b + a\sqrt{-2}$ and

$$u_p = \frac{\pi_p}{\overline{\pi_p}} = ((b^2 - 2a^2) + 2ab\sqrt{-2})/p.$$

Standard methods of algebraic number theory show that $\ker(N)$ is the product of $\{\pm 1\}$ with the free abelian group generated by these elements u_p. Moreover, the denominator of a product $\prod_p u_p^{n_p}$ is $\prod_p p^{|n_p|}$. Using this, we can enumerate all the solutions for which the denominator is less than some specified bound, and thus produce rational points that are closely spaced around C_1 and C_2.

Remark 7.6.2. The map in part (a) of Proposition 7.6.1 is `c_rational[0](t)`, and the map in part (c) is `c_rational[1]([s,t])`. The function `two_circle(n)` returns the list of all pairs $(s,t) \in \mathbb{Q}^2$ where $2s^2 + t^2 = 1$ and the denominators of s and t are less than or equal to n.

As the rational points do not cover much of EX^*, we instead consider the set

$$Q = \{x \in EX^* \mid x_1, x_2, x_4,\ x_3/\sqrt{2} \in \mathbb{Q}\}.$$

This set is again arithmetically simple, but has a richer structure than $EX^*(\mathbb{Q})$. We call the points in Q *quasirational*.

We first consider quasirational points which have nontrivial isotropy (and so lie in $\bigcup_{i=0}^{8} C_i$).

Proposition 7.6.3. [`prop-quasirational-isotropy`]

(a) *For every $t \in \mathbb{Q} \cup \{\infty\}$ we have a point*
$$\left(t^2 - 1, 2t, 0, 0\right)/(1 + t^2) \in Q \cap C_0 = EX^*(\mathbb{Q}) \cap C_0.$$

(b) *For every $t \in \mathbb{Q} \cup \{\infty\}$ we have points*
$$\left(1 - t^2 - 2t,\ 1 - t^2 - 2t,\ \sqrt{2}(1 - t^2 + 2t), 0\right)/(2(1 + t^2)) \in Q \cap C_1$$
$$\left(-(1 - t^2 - 2t),\ 1 - t^2 - 2t,\ \sqrt{2}(1 - t^2 + 2t), 0\right)/(2(1 + t^2)) \in Q \cap C_2.$$

(c) *For each $(s,t) \in \mathbb{Q}^2$ with $3s^2 + t^2 = 1$, we have points*
$$\left(0, t, \sqrt{2}s, -s\right) \in Q \cap C_3$$
$$\left(-t, 0, \sqrt{2}s, s\right) \in Q \cap C_4.$$

(d) *All quasirational points with nontrivial isotropy are accounted for by (a) to (c). In particular, we have $Q \cap \bigcup_{i=5}^{8} C_i = \emptyset$.*

Proof. First, it is straightforward to check that the constructions in (a) to (c) do in fact give quasirational points on the indicated curves.

Next, points in C_0 have $x_3 = 0$ so they are quasirational if and only if they are rational. Thus, we have seen already that all quasirational points on C_0 are as in (a).

Now let x be a quasirational point in C_1. We then have $x_1 = x_2 = m$ and $x_3 = \sqrt{2}n$ and $x_4 = 0$ for some rational numbers m and n. Put $p = n - m \in \mathbb{Q}$ and $q = n + m \in \mathbb{Q}$ so

$$x = \left((q-p)/2,\ (q-p)/2,\ (q+p)/\sqrt{2},\ 0\right).$$

For such points we have $g(x) = 0$ automatically, and $\rho(x) = p^2 + q^2$. It follows that $(p,q) = (2t, 1-t^2)/(1+t^2)$ for some $t \in \mathbb{Q} \cup \{\infty\}$, and we can use this to get the first formula in (b). The C_2 case follows from the C_1 case by the group action.

Next, one can check that (in the current case where $a = 1/\sqrt{2}$) we have

$$g(x_1, 0, x_3, x_4) = \sqrt{2}(x_3 - \sqrt{2}x_4)(x_1^2 + x_4^2 + x_3x_4/\sqrt{2}).$$

The first factor vanishes on C_4. The claim about C_4 in (c) follows easily from this, and the claim about C_3 can be deduced using the group action.

Now consider a point $x \in C_5$, so $x_2 = 0$ and the functions $r_0 = x_1^2 + x_3^2 + x_4^2 - 1$ and $r_1 = x_1^2 + x_4^2 + x_3x_4/\sqrt{2}$ also vanish at x. (Here r_1 is the second term in the above factorisation of g.) Now put $t = x_1^2$ and

$$u = x_1(x_4^2 - 5x_3x_4/\sqrt{2} - 1).$$

We claim that $u^2 = t^3 - 10t^2 + t$. In fact, one can check by direct expansion that

$$u^2 - (t^3 - 10t^2 + t) = a_0r_0 + a_1r_1 + a_2r_0r_1,$$

where

$$\begin{aligned}
a_0 &= -6\sqrt{2}(x_3 + \sqrt{2}x_4)x_4^3 \\
a_1 &= (1 - x_3^2 - x_4^2)(x_3^2 - 10x_4^2 + 9 + x_3x_4/\sqrt{2}) \\
a_2 &= x_3^2 + 2x_4^2 + x_3x_4/\sqrt{2} - x_1^2 + 9.
\end{aligned}$$

As $r_0 = r_1 = 0$, the claim follows. Note also that if x is quasirational then t and u will be rational. Now, the equation $u^2 = t^3 - 10t^2 + t$ describes an elliptic curve E over $\mathbb{Q}$, and algorithms to determine rational points on such curves are built in to the symbolic mathematics system Sage. Our curve can be described in extended Weierstrass form as

$$u^2 + \alpha_1 ut + \alpha_3 u = t^3 + \alpha_2 t^2 + \alpha_4 t + \alpha_6,$$

where

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6) = (0, -10, 0, 1, 0).$$

We can thus enter the following in Sage:

```
E = EllipticCurve([0,-10,0,1,0])
E.cremona_label()
E.rank()
E.torsion_points()
```

We learn that E is isomorphic to the curve labelled 96b2 in Cremona's database of elliptic curves [4], and that the rank is zero, so the rational points are all torsion points. We also learn that the only rational torsion points in the projective closure are $[0:0:1]$ and $[0:1:0]$, so the only rational point on the original affine curve is $(0,0)$. Thus, if x is quasirational then we must have $t = u = 0$, but that gives $x_1 = 0$. Substituting $x_1 = 0$ in the relation $r_1 = 0$ gives $x_4(x_4 + x_3/\sqrt{2}) = 0$, so $x_4 = 0$ or $x_4 = -x_3/\sqrt{2}$. Putting $x_1 = x_4 = 0$ in r_0 gives $x_3^2 = 1$; putting $x_1 = 0$ and $x_4 = -x_3/\sqrt{2}$ instead gives $x_3^2 = 2/3$. Neither of these is possible with $x_3/\sqrt{2} \in \mathbb{Q}$, so we see that there are no quasirational points in C_5. It follows using the group action that there are no quasirational points in $\bigcup_{i=5}^{8} C_i$.

`embedded/roothalf/rational_check.mpl: check_rational_elliptic()`

□

We can understand rational solutions to $3s^2 + t^2 = 1$ in terms of the arithmetic of the field $\mathbb{Q}(\sqrt{-3})$, and thus produce quasirational points that are closely spaced around C_3 and C_4. The story similar to that for $\mathbb{Q}(\sqrt{-2})$, and we will not give the details here.

Remark 7.6.4. The map in part (a) of Proposition 7.6.3 is `c_rational[0]`, and the maps in (b) and (c) are `c_quasirational[i]` for $i \in \{1, 2, 3, 4\}$. The function `three_circle(n)` returns the list of all pairs $(s,t) \in \mathbb{Q}^2$ where $3s^2 + t^2 = 1$ and the denominators of s and t are less than or equal to n.

We now consider quasirational points with trivial isotropy. We do not have a very good theory for these, but we have a reasonably efficient method for exhaustive search, as we now explain.

Proposition 7.6.5. [prop-quasirational-lift]
For any $x \in Q \cap F_{16}$, the numbers $s_1 = y_1/\sqrt{2} = x_3/\sqrt{2}$ and

$$s_2 = \sqrt{2}y_2 = x_2^2 - x_1^2 - 3x_3x_4/\sqrt{2}$$

lie in $\mathbb{Q} \cap [0,1]$. Conversely, suppose that $s \in (\mathbb{Q} \cap [0,1])^2$, and put

$$t_1 = s_1^2 \qquad t_2 = 6s_1^2 - 2 \qquad t_3 = -2s_2^2 - 4$$

$$p_1 = 2 + t_1t_3 \qquad p_2 = s_2t_2 \qquad p_3 = p_1 + p_2 \qquad p_4 = p_1 - p_2.$$

Then s arises from a quasirational point $x \in F_{16}$ if and only if p_3 and p_4 are perfect squares (and therefore nonnegative). If so, then

$$x = (\sqrt{p_3}/2,\ \sqrt{p_4}/2,\ \sqrt{2}s_1,\ -s_1s_2).$$

Proof. This is essentially a reformulation of Proposition 6.7.4, in the special case $a = 1/\sqrt{2}$. The formulae have been reorganised slightly to make rationality questions more visible, and to allow for efficient calculation as discussed below. $\square$

We wrote code in C to look for solutions using the above proposition. We ran the program on a cluster of machines with 64 bit processors. The native 64 bit integers are not large enough for our intermediate calculations, but fortunately the GCC compiler provides built in support for 128 bit integers encoded as pairs of native integers, and similarly for floating point numbers. (If we needed larger integers than 128 bits we would need to use an arbitrary precision library, which would come with a significant performance penalty.) We took $N = 2^{12} = 4096$ and enumerated the rational numbers of denominator at most N as a Farey sequence (of length $5100021 \simeq 5 \times 10^6$). Note that the numbers t_i in Proposition 7.6.5 depend only on a single rational number in the sequence, so we precomputed them in a single pass. We then looped through all possible pairs (s_1, s_2) and computed the numbers p_i. One can check that if $p_1 < 0$ at (s_1, s_2), then all pairs (s'_1, s'_2) with $s'_1 \geq s_1$ and $s'_2 \geq s_2$ can be disregarded. Using this we can cut down the number of pairs to be considered, but the order of magnitude is still 10^{13}, so the remaining steps must be highly optimised.

To check whether a rational number $p = a/b$ (possibly not in lowest terms) is a perfect square, we first test whether the 2-adic valuations of a and b are the same mod 2. Here the 2-adic valuation is the number of trailing zeros in the binary representation; this can be calculated in a single processor instruction for 64 bit integers, and is only slightly slower for 128 bit pairs. If this first test is passed, we divide a and b by appropriate powers of 2 to make them odd. This can be done as a bitwise shift rather than a division, so again it is very fast. Next, with the new a and b, it is not hard to check that a/b can only be a square if $a = b \pmod 8$. This can again be checked by bit masking rather than division, so it is very fast. If these fast tests are passed for both p_3 and p_4 (which is already relatively rare), we then start using some slower tests. We calculate the gcd of a and b and then divide by it to make a and b coprime. It seems that no algorithm is known that is usefully more efficient than the obvious one, so a significant number of divisions may be required. Now a/b will be a perfect square if and only if a and b are individually perfect squares. They are still odd, so we first test that they are congruent to 1 mod 8. If so, we reduce them modulo $3 \times 5 \times 7 \times 11 \times 13 \times 17 = 255255$, and look up in a precomputed table whether the result is a quadratic residue. Only about 0.63% of numbers pass these modular tests. If we get to this stage, we calculate the square root as a 128 bit floating point number, take the nearest integer, square it, and check whether the result is the same as the number we first thought of. If so, our number is obviously a perfect square. The converse is less obvious because of the possibility of rounding errors. However, one can check that our numerators and denominators are not much bigger than $N^7 = 2^{84}$ so 128 bit accuracy should be ample to avoid problems.

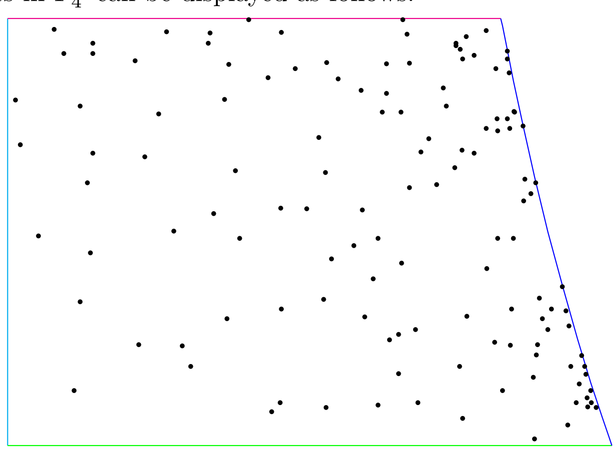
We found precisely 130 pairs s corresponding to quasirational points in the interior of F_{16}:

$(\frac{103}{154},\frac{65}{449})$	$(\frac{103}{554},\frac{2945}{3041})$	$(\frac{104}{1073},\frac{105}{233})$	$(\frac{1}{10},\frac{13}{19})$	$(\frac{1}{10},\frac{16}{17})$	$(\frac{1}{10},\frac{245}{267})$	$(\frac{11}{118},\frac{819}{1331})$	$(\frac{11}{1190},\frac{1664}{2057})$
$(\frac{1127}{2194},\frac{581}{731})$	$(\frac{1135}{2114},\frac{1088}{1137})$	$(\frac{1147}{2390},\frac{97}{961})$	$(\frac{1147}{2830},\frac{640}{1369})$	$(\frac{1151}{3722},\frac{11}{139})$	$(\frac{1223}{2330},\frac{16}{17})$	$(\frac{12}{29},\frac{247}{297})$	$(\frac{125}{262},\frac{16}{59})$
$(\frac{1255}{2146},\frac{1547}{1675})$	$(\frac{1260}{3599},\frac{559}{1009})$	$(\frac{127}{450},\frac{416}{417})$	$(\frac{1301}{2270},\frac{16}{33})$	$(\frac{13}{30},\frac{145}{1521})$	$(\frac{13}{30},\frac{16}{33})$	$(\frac{1389}{3038},\frac{176}{1049})$	$(\frac{1424}{3913},\frac{1571}{2179})$
$(\frac{148}{215},\frac{29}{323})$	$(\frac{151}{338},\frac{279}{1129})$	$(\frac{1549}{3038},\frac{2768}{3307})$	$(\frac{1603}{2350},\frac{19}{147})$	$(\frac{1611}{3878},\frac{355}{643})$	$(\frac{1648}{2875},\frac{2295}{3113})$	$(\frac{1672}{3329},\frac{1257}{2057})$	$(\frac{169}{322},\frac{475}{507})$
$(\frac{17}{106},\frac{1760}{2601})$	$(\frac{172}{511},\frac{15}{17})$	$(\frac{175}{298},\frac{2464}{2825})$	$(\frac{175}{362},\frac{112}{163})$	$(\frac{1792}{3635},\frac{2623}{3649})$	$(\frac{1809}{3050},\frac{32}{41})$	$(\frac{1829}{2750},\frac{97}{961})$	$(\frac{1864}{2843},\frac{145}{2993})$
$(\frac{191}{3490},\frac{1211}{1243})$	$(\frac{1939}{3350},\frac{19}{147})$	$(\frac{196}{335},\frac{2719}{3553})$	$(\frac{1964}{3085},\frac{31}{97})$	$(\frac{19}{70},\frac{16}{33})$	$(\frac{2093}{3534},\frac{115}{147})$	$(\frac{209}{338},\frac{819}{1331})$	$(\frac{211}{350},\frac{80}{107})$
$(\frac{23}{154},\frac{895}{993})$	$(\frac{23}{34},\frac{481}{2881})$	$(\frac{23}{50},\frac{32}{41})$	$(\frac{2435}{3854},\frac{16}{59})$	$(\frac{248}{401},\frac{95}{449})$	$(\frac{248}{665},\frac{29}{323})$	$(\frac{256}{377},\frac{35}{387})$	$(\frac{25}{82},\frac{237}{275})$
$(\frac{2636}{3887},\frac{445}{4003})$	$(\frac{280}{457},\frac{1243}{2107})$	$(\frac{28}{45},\frac{559}{1617})$	$(\frac{307}{830},\frac{267}{779})$	$(\frac{31}{202},\frac{355}{1507})$	$(\frac{313}{466},\frac{560}{2651})$	$(\frac{3}{14},\frac{5}{27})$	$(\frac{31}{50},\frac{355}{1507})$
$(\frac{316}{487},\frac{835}{2243})$	$(\frac{32}{49},\frac{767}{2433})$	$(\frac{341}{1070},\frac{97}{961})$	$(\frac{359}{610},\frac{463}{1969})$	$(\frac{364}{687},\frac{409}{441})$	$(\frac{36}{77},\frac{2575}{2673})$	$(\frac{367}{986},\frac{2176}{3401})$	$(\frac{37}{190},\frac{688}{1369})$
$(\frac{373}{630},\frac{16}{33})$	$(\frac{37}{70},\frac{5}{27})$	$(\frac{401}{650},\frac{16}{1067})$	$(\frac{403}{590},\frac{97}{961})$	$(\frac{404}{685},\frac{31}{97})$	$(\frac{415}{706},\frac{341}{459})$	$(\frac{427}{694},\frac{128}{803})$	$(\frac{43}{166},\frac{224}{251})$
$(\frac{432}{1681},\frac{1055}{3553})$	$(\frac{43}{94},\frac{480}{1849})$	$(\frac{4}{47},\frac{65}{193})$	$(\frac{455}{778},\frac{973}{1075})$	$(\frac{463}{1970},\frac{16}{17})$	$(\frac{468}{775},\frac{385}{673})$	$(\frac{47}{106},\frac{80}{97})$	$(\frac{4}{7},\frac{15}{17})$
$(\frac{48}{235},\frac{107}{459})$	$(\frac{501}{742},\frac{5}{27})$	$(\frac{52}{205},\frac{137}{169})$	$(\frac{555}{974},\frac{163}{675})$	$(\frac{560}{3163},\frac{951}{1225})$	$(\frac{56}{107},\frac{605}{931})$	$(\frac{569}{1042},\frac{2735}{2993})$	$(\frac{57}{130},\frac{32}{41})$
$(\frac{577}{1546},\frac{1680}{1873})$	$(\frac{600}{913},\frac{7}{25})$	$(\frac{61}{718},\frac{581}{731})$	$(\frac{65}{122},\frac{32}{507})$	$(\frac{65}{122},\frac{973}{1075})$	$(\frac{65}{274},\frac{2777}{2873})$	$(\frac{700}{1303},\frac{931}{3075})$	$(\frac{704}{1163},\frac{1435}{2299})$
$(\frac{7}{106},\frac{245}{267})$	$(\frac{71}{130},\frac{13}{19})$	$(\frac{7}{194},\frac{1855}{3777})$	$(\frac{72}{115},\frac{217}{729})$	$(\frac{73}{274},\frac{1253}{1947})$	$(\frac{735}{1594},\frac{1024}{2401})$	$(\frac{739}{1318},\frac{249}{601})$	$(\frac{7}{466},\frac{656}{931})$
$(\frac{75}{134},\frac{341}{459})$	$(\frac{75}{134},\frac{656}{675})$	$(\frac{76}{143},\frac{327}{473})$	$(\frac{767}{1834},\frac{235}{779})$	$(\frac{777}{2050},\frac{224}{513})$	$(\frac{7}{90},\frac{19}{147})$	$(\frac{80}{187},\frac{799}{2049})$	$(\frac{8}{25},\frac{1007}{1041})$
$(\frac{8}{25},\frac{31}{97})$	$(\frac{827}{2590},\frac{1963}{3531})$	$(\frac{83}{126},\frac{5}{27})$	$(\frac{839}{2170},\frac{736}{857})$	$(\frac{881}{1538},\frac{160}{209})$	$(\frac{907}{2046},\frac{1075}{1203})$	$(\frac{92}{381},\frac{1127}{2073})$	$(\frac{95}{202},\frac{1264}{1411})$
$(\frac{95}{202},\frac{301}{499})$	$(\frac{965}{2086},\frac{1072}{1075})$						

`embedded/roothalf/rational_check.mpl: check_rational()`

There do not appear to be any discernable patterns in the prime factorisations of these rational numbers. The corresponding points in EX^* are stored in the variable `inner_quasirational_points`.

The corresponding points in F_4^* can be displayed as follows:



(It may appear that some of the points lie on the boundary, but in fact they are just inside, by a distance of only about 5×10^{-4} in some cases.)

7.7. **Integration.** [sec-integration]

Later on we will need to integrate various functions on EX^*. Integration is most naturally defined for differential 2-forms. However, the metric and orientation on EX^* gives a volume form ω in a standard way, and we define the integral of a function f to be the integral of the form $f\omega$. In most cases of interest, f will be G-invariant so we can just integrate over F_{16} and multiply by 16.

Unfortunately, it seems to be difficult to compute such integrals accurately. Given an explicit smooth embedding $\phi\colon [0,1]^2 \to EX^*$, we can compute the Jacobian and then use standard numerical integration techniques to evaluate integrals over the image of ϕ. However, we have not succeeded in finding a family of such maps ϕ for which the Jacobians are explicitly computable and the images cover EX^* without overlap. We do know several different homeomorphisms $[0,1]^2 \to F_{16}$ that are diffeomorphisms away from the boundary, but it seems that the singular boundary behaviour destroys any possibility of using these maps for accurate integration. The best that we can do along these lines is to construct a barycentric triangulation of EX^* as in Section 6.9. This gives us a decomposition of EX^* into triangles T and diffeomorphisms $\phi\colon T \to \Delta_2$ where ϕ and its Jacobian are simple and explicit, but ϕ^{-1} is not. Nonetheless, we can compute $\phi^{-1}(a)$ and associated quantities for a large number of points $a \in T$. This leads to an approximate integration rule of the form $I(f) = \sum_{i=1}^n w_i\, f(a_i)$ for some points $a_1, \dots, a_n \in EX^*$ and weights $w_i \in \mathbb{R}$. We have carried out this process and obtained a rule for which we believe that $|I(f) - \int_{EX^*} f|/\|f\|_2$ is at most 10^{-25} or so for typical functions that we need to consider. The basis for this estimate will be explained after we have discussed some more theoretical ideas. Unfortunately, for this rule the number $n = 33600$ of sample points is very large, so we cannot compute integrals quickly. We will also discuss a method that gives less accurate integrals much more easily.

7.7.1. *A characterisation of the integration functional.* `[sec-int-props]`

Definition 7.7.1. `[defn-stokes]`
For any smooth one-form $\alpha = \sum_{j=1}^4 u_j\, dx_j$ on $\mathbb{R}^4$, we define a function $D(\alpha)\colon EX^* \to \mathbb{R}$ by

$$D(\alpha) = \|n\|^{-1} \sum_{ijkl} \epsilon_{ijkl} \frac{\partial u_j}{\partial x_i} x_k n_l.$$

Here n is the gradient of the function g in the definition of EX^*, and ϵ is the totally antisymmetric tensor. The operator D is called `stokes` in Maple; the definition is in `embedded/roothalf/forms.mpl`.

Lemma 7.7.2. `[lem-stokes]`
For any smooth one-form α on $\mathbb{R}^4$, we have

$$d(\alpha|_{EX^*}) = (d\alpha)|_{EX^*} = D(\alpha)\omega.$$

It follows that $D(\alpha)$ depends only on $\alpha|_{EX^*}$, and that $\int_{EX^*} D(\alpha)\omega = 0$.

Proof. We will freely use the metric to identify one-forms with vectors, so dx_i becomes the i'th basis vector e_i. We then have $d\alpha = \sum_{i,j} \frac{\partial u_j}{\partial x_i} e_i \wedge e_j$.

Now consider a point $x \in EX^*$. The vectors x and $n/\|n\|$ form an orthonormal basis for $(T_x EX^*)^\perp$, so multiplication by the form $\beta_x = x \wedge (n/\|n\|)$ gives an isometric isomorphism from $\Lambda^2 T_x EX^*$ to $\Lambda^4\mathbb{R}^4$. Thus, if we put $\epsilon = e_1 \wedge \dots \wedge e_4$, we must have $\omega_x \wedge \beta_x = \pm\epsilon$, and a glance at our orientation conventions shows that the sign is positive. However, it is clear from the definitions that $d\alpha \wedge \beta_x = D(\alpha)\epsilon$, so we must have $(d\alpha)|_{EX^*} = D(\alpha)\omega$ as claimed. It is standard that $(d\alpha)|_{EX^*} = d(\alpha|_{EX^*})$ so we can apply Stokes's Theorem to $\alpha|_{EX^*}$ to see that $\int_{EX^*} D(\alpha)\omega = 0$. □

Definition 7.7.3. `[defn-antiinvariant]`
We say that a one-form α is *antiinvariant* if $\lambda^*(\alpha) = \mu^*(\alpha) = \alpha$ and $\nu^*(\alpha) = -\alpha$. Equivalently, we must have $\gamma^*(\alpha) = \chi_7(\gamma)\alpha$ for all $\gamma \in G$, where χ_7 is as in Proposition 2.1.1, so $\gamma^*(\alpha) = \alpha$ whenever γ preserves orientation, and $\gamma^*(\alpha) = -\alpha$ whenever γ reverses orientation.

Proposition 7.7.4. `[prop-integration]`
The integration functional $I\colon C^\infty(EX^*) \to \mathbb{R}$ is the unique $\mathbb{R}$-linear map with the following properties.

(a) *For all $f \in C^\infty(EX^*)$ and all $\gamma \in G$ we have $I(\gamma^* f) = I(f)$.*
(b) *For the curvature map K we have $I(K) = -4\pi$.*
(c) *For all antiinvariant one-forms α we have $I(D(\alpha)) = 0$.*

Proof. First consider the following property:

(d) For all one-forms α we have $I(D(\alpha)) = 0$.

This clearly implies (c), and in fact (a) and (c) imply (d). To see this, note that the operator D involves division by ω; because of this, it satisfies $D(\gamma^*(\alpha)) = \chi_7(\gamma)\gamma^*(D(\alpha))$. Thus, for any one-form α, the form $\alpha' = |G|^{-1}\sum_\gamma \chi_7(\gamma)\gamma^*(\alpha)$ is antiinvariant, and if (a) holds then $I(D(\alpha)) = I(D(\alpha'))$, so if (c) holds then $I(D(\alpha)) = 0$.

For the integration functional property (a) is clear, and (b) is the Gauss-Bonnet theorem, and (d) follows from Lemma 7.7.2.

Now let I' be another functional with properties (a), (b) and (c), and therefore also (d). The de Rham Theorem tells us that the space of two-forms on EX^* modulo the image of d is isomorphic to the cohomology group $H^2(EX^*;\mathbb{R})$, which has dimension one. Integration gives a well-defined and nontrivial map from this quotient to $\mathbb{R}$, which is therefore an isomorphism. It is clear from this that I' must be equal to I. □

To use the above proposition, we need to have a good understanding of the antiinvariant forms.

Definition 7.7.5. We write Ξ for the set of antiinvariant polynomial 1-forms on EX^*. This is a module over the ring $B = A^G = \mathbb{R}[z_1, z_2]$. We will also consider the ring $B' = B[(2-z_1)^{-1}]$ and the module $\Xi' = B' \otimes_B \Xi$.

Recall that $0 \le z_1 \le 1$ on EX^*, so $2 - z_1$ is everywhere positive. It follows that B' can still be regarded as a ring of real analytic functions on EX^*.

Proposition 7.7.6. [prop-alpha-basis]
Ξ' is freely generated over B' by the following forms:

$$\begin{aligned}
\alpha_1 &= y_1(x_2\,dx_1 - x_1\,dx_2) \\
\alpha_2 &= z_1(1+z_2)y_1y_2(x_2\,dx_1 + x_1\,dx_2) - 2z_1y_2x_1x_2\,dx_3 + 2(1+z_1z_2)x_1x_2\,dx_4.
\end{aligned}$$

Proof. First, let Ω^* denote the free module over A on generators $dx_1, \dots, dx_4$. Let Θ be the submodule generated by the elements

$$\begin{aligned}
\theta_1 &= dg_0 \\
\theta_2 &= \tfrac{1}{2}d(\rho - 1) = \sum_i x_i\,dx_i;
\end{aligned}$$

then $\Omega = \Omega^*/\Theta$. There is an evident action of G on Ω^* that preserves Θ and is compatible with the standard action on Ω.

Next, for any $\mathbb{R}[G]$-module V we define $\Pi\colon V \to V$ by

$$\Pi(v) = |G|^{-1}\sum_{\gamma\in G}\chi_7(\gamma)\gamma^*(v).$$

It is standard that Π is B-linear with $\Pi^2 = \Pi$, and the image of Π is the subspace

$$V[\chi_7] = \{v \in V \mid \gamma^*(v) = \chi_7(\gamma)v \text{ for all } \gamma \in G\}.$$

We thus have $\Xi = \Omega[\chi_7] = \Pi(\Omega^*)/\Pi(\Theta)$.

Now put

$$M = \{x_1^i x_2^j y_1^k y_2^l \mid 0 \le i,j,k,l \le 1\},$$

and recall that this is a basis for $A = \mathcal{O}_{EX^*}$ over B. It follows that the group $\Omega^*[\chi_7] = \Pi(\Omega^*)$ is generated by elements of the form $\Pi(m\,dx_i)$ with $m \in M$ and $1 \le i \le 4$. A computer calculation shows that every nonzero element of this form is a constant multiple of one of the following forms:

$$\begin{aligned}
\beta_1 &= y_1(x_2\,dx_1 - x_1\,dx_2) \\
\beta_2 &= y_1y_2(x_2\,dx_1 + x_1\,dx_2) \\
\beta_3 &= x_1x_2y_2\,dx_3 \\
\beta_4 &= x_1x_2\,dx_4.
\end{aligned}$$

Thus, these form a basis for $\Omega^*[\chi_7]$ over B.

Similarly, $\Theta[\chi_7]$ is generated by elements of the form $\Pi(m\,\theta_i)$ with $m \in M$ and $i \in \{1,2\}$. Another computer calculation shows that every nonzero element of this form is a constant multiple of one of the following forms:

$$\begin{aligned}\alpha_3 &= x_1x_2\theta_1\\ \alpha_4 &= x_1x_2y_1y_2\theta_2.\end{aligned}$$

Thus, these give a basis for $\Theta[\chi_7]$ over B.

Now consider the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1(1+z_2) & -2z_1 & 2(1+z_1z_2) \\ \sqrt{2}(1-z_1)(1-2z_2) & z_1(1-2z_2) & z_1-2 & -2+3z_1-4z_1z_2 \\ (3z_1-2)z_2/(2\sqrt{2}) & (1-z_1-z_1z_2)/2 & z_1 & -z_1z_2 \end{bmatrix}.$$

By straightforward calculation in A, we have $\alpha_i = \sum_{j=1}^4 P_{ij}\beta_j$ for $i = 1,\dots,4$. We also have $\det(P) = 2-z_1$, which means that $\alpha_1,\dots,\alpha_4$ give a basis for $A'\otimes_A\Omega^*[\chi_7]$. It follows that α_1 and α_2 give a basis for $A'\otimes_A\Omega[\chi_7]$, as claimed.

`embedded/roothalf/forms_check.mpl: check_forms()`

□

The forms α_i and β_j are `alpha_form[i]` and `beta_form[j]` in Maple. The matrix P is `alpha_beta`.

Proposition 7.7.7. [prop-d-alpha]

The forms α_i satisfy

$$\begin{aligned}
d\alpha_1 &= (9z_1^2z_2 - 9z_1^2 + 2z_1z_2 + 9z_1 - 2)\|n\|^{-1}\omega\\
d\alpha_2 &= (45z_1^3z_2^2 - 45z_1^3z_2 + 78z_1^2z_2 - 20z_1z_2^2 - 12z_1^2 - 28z_1z_2 + 12z_1 - 8z_2)/\sqrt{2}\|n\|^{-1}\omega\\
dz_1\wedge\alpha_1 &= 2z_1(3z_1-2)(z_1z_2 - z_1 + 1)\|n\|^{-1}\omega\\
dz_1\wedge\alpha_2 &= \sqrt{2}z_1(9z_1^3z_2^2 - 9z_1^3z_2 - 12z_1^2z_2^2 + 26z_1^2z_2 + 4z_1z_2^2 - 4z_1^2 - 24z_1z_2 + 8z_1 + 8z_2 - 4)\|n\|^{-1}\omega\\
dz_2\wedge\alpha_1 &= 4z_1z_2(2z_2-1)\|n\|^{-1}\omega\\
dz_2\wedge\alpha_2 &= 2\sqrt{2}(3z_1^2z_2 - 2z_1z_2 + 2z_1 - 2)z_2(2z_2-1)\|n\|^{-1}\omega,
\end{aligned}$$

where

$$\|n\| = \left(\sum_i (\partial g/\partial x_i)^2\right)^{1/2} = (2-z_1)\sqrt{1+z_2}.$$

Proof. The ring $A = \mathcal{O}_{EX^*}$ is an integral domain, with field of fractions

$$K = \mathbb{R}(y_1,y_2)[x_1,x_2]/(x_1^2-u_1,\, x_2^2-u_2).$$

It will suffice to verify the above identities in $K\otimes_A\Omega^2$, which is the exterior square over K of the space $K\otimes_A\Omega^1$. From the above description of K, we have

$$dx_i = \frac{1}{2x_i}du_i = \frac{x_i}{2u_i}du_i \in K\{dy_1,\ dy_2\}.$$

Using this, it is not hard to see that $K\otimes_A\Omega^1$ is freely generated over K by dy_1 and dy_2. After rewriting everything in terms of this basis, all the above equations become straightforward.

`embedded/roothalf/forms_check.mpl: check_forms()`

□

Equations for rewriting forms in terms of dy_1 and dy_2 are in the list `forms_to_y`. The functions $(d\alpha_i)/\omega$ are `D_alpha[i]`, and the functions $(dz_i\wedge\alpha_j)/\omega$ are `dz_cross_alpha[i,j]`.

Using Proposition 7.7.7, we can calculate $D(f_1\alpha_1 + f_2\alpha_2)$ for any invariant smooth functions f_1 and f_2. This is implemented in Maple as `stokes_alpha([f1,f2])`, and it gives us a supply of functions f with $\int_{EX^*} f = 0$. Given an approximate integration functional I, we can test the accuracy of I by evaluating $I(f)/\sqrt{I(f^2)}$ for these functions f.

7.7.2. *Integration over triangles.* We next discuss integration over 2-simplices. Given any continuous function f on Δ_2, we write

$$\int_{\Delta_2} f = 2\int_{t_1=0}^{1}\int_{t_2=0}^{1-t_1} f(t_1,t_2,1-t_1-t_2)\,dt_2\,dt_1.$$

In other words, this is the integral with respect to the ordinary Lebesgue measure on Δ_2 normalised in such a way that the total area of Δ_2 is one. A standard exercise shows that

$$\int_{\Delta_2} t_1^i t_2^j t_3^k = 2\frac{i!\,j!\,k!}{(i+j+k+2)!}.$$

By an *n'th order quadrature rule* we mean a pair $Q=(a,w)$ with $a\in(\Delta_2)^n$ and $w\in\mathbb{R}^n$. Given any function f on Δ_2 we put $I_Q(f)=\sum_i w_i\,f(a_i)$. Given n distinct points a_i in Δ_2, and an n-dimensional space of functions P, there will typically be a unique vector w of weights such that $I_{(a,w)}(f)=\int_{\Delta_2} f$ for all $f\in P$, which can be found by solving a system of linear equations. It may or may not be the case that $I_{(a,w)}(f)$ is close to $\int_{\Delta_2} f$ for functions f not lying in P. In particular, if we take the points a_i to form a regularly spaced grid, then $I_{(a,w)}(f)$ is a rather poor approximation to $\int_{\Delta_2} f$ for general f. This was a surprise to the author, but is apparently well-known to numerical analysts. It is better to allow the points a_i to vary as well as the weights w_i. A naive dimension count then suggests that for any $3n$-dimensional space P there should be a unique n'th order quadrature rule Q that agrees with integration on P, but one has to solve a complex system of nonlinear equations to find Q. Things become somewhat simpler if P is preserved by the action of the symmetric group Σ_3 on P. Dunavant [8] developed an elegant theory for this case, which made it tractable to solve the relevant equations by computer. He found a quadrature rule of order 73 that integrates all polynomials of degree at most 20 exactly. It appears that rules obtained in this way have much better behaviour outside the space P on which they are exact by construction.

Later, Wandzurat and Xiao [18] gave a rule of order 175 that is exact to degree 30, and Xiao and Gimbutas gave a rule that is exact to degree 50. However, we have not been able to get correct answers from this last rule, so either we are misunderstanding the conventions or there is some kind of transcription error in the tables in the paper. We have therefore used the Wandzurat-Xiao rule instead.

Rules as above are represented in Maple by instances of the class `triangle_quadrature_rule`, which is declared in the file `quadrature/quadrature.mpl`. In the `quadrature` subdirectory of the `data` directory there is a file `wandzurat_xiao_30.mpl`. Reading this file creates an object representing the Wandzurat-Xiao rule, and assigns it to the variable `wandzurat_xiao_30`. There is also another file `dunavant_19.mpl` which implements the Dunavant rule (which is less accurate but faster).

Next, recall that in Section 6.9 we discussed a triangulation of the fundamental domain F_{16} using certain barycentric coordinate maps $p\colon T\xrightarrow{\simeq}\Delta_2$ for certain subsets $T\subseteq X$. For each point a_i in our quadrature rule, we can use Remark 6.9.9 to find $a_i'=p^{-1}(a_i)\in T$. The components of p are rational functions in the coordinates x_i, so it is straightforward to differentiate them and calculate the Jacobian of p at a_i', say u_i. If w_i is the i'th weight of our quadrature rule, then $\sum_i w_i u_i^{-1} f(a_i)$ is a good approximation to $\int_T f$, and we can take the sum over all simplices to get a functional $J(f)$ approximating $\int_{F_{16}} f$.

The Gauss-Bonnet theorem says that J of the curvature function K should be $(-4\pi)/|G|=-\pi/4$. With our 192-simplex triangulation we in fact have $|J(K)+\pi/4|<10^{-27}$. Similarly, for a function f of the form $D(z_1^i z_2^j\alpha_k)$ we would ideally have $J(f)=0$, and in fact we have $|J(f)|\leq 3\times10^{-27}\sqrt{J(f^2)}$ provided that $i+j\leq 10$. Subdivision makes a big difference here; with the original 48-simplex triangulation, errors are larger by a factor of 10^8 or so.

7.7.3. *Faster integration on EX^*.* After constructing a quadrature rule for EX^*, we can tabulate the integrals of monomials $z_1^i z_2^j$, which then gives a fast way of integrating arbitrary invariant polynomials. It is also useful to extend this slightly and include monomials $z_1^i z_2^j\|n\|^{-k}$ for $0\leq k\leq 4$ say. This is enough for some purposes, but in other cases we need to integrate more general functions (such as exponentials of polynomials) which cannot easily be expressed as linear combinations of some standard basis. It is therefore desirable to have an approximate integration functional of the form $I(f)=\sum_{i=1}^n w_i f(a_i)$ where n is not too large, but the approximation is reasonably accurate.

Suppose we fix n, and choose an n-dimensional subspace P of smooth invariant functions on EX^*. Let $p_1,\dots,p_n$ be a basis for P. For any n-tuple of points $a_i\in EX^*$, we let $\delta(\underline{a})$ denote the determinant of the matrix with entries $p_i(a_j)$. The literature on other quadrature problems suggests that we should aim to choose $\underline{a}$ so that $|\delta(\underline{a})|$ is as large as possible. Note that this problem is independent of the choice of basis $\{p_i\}$, because a different choice would just change δ by a constant factor. With the obvious kind of monomial bases, it works out that $\delta(\underline{a})$ is always extremely small, but it can be increased by many orders of magnitude if we choose the points a_i appropriately.

In our largest calculation of this kind, we took P to be the span of 256 monomials in z_1 and z_2, including all monomials of degree at most 21, plus some monomials of degree 22. For a randomly chosen set of 256 points it is typical that $\log_{10}|\delta| < -2800$ or so, but by numerical optimization we found a set with $\log_{10}|\delta| \simeq -2539$.

We next want to choose the weights w_i. One option is to set these weights such that $I(p_j) = \int_{EX^*} p_j$ for all j, which can be done by solving a system of linear equations. We then find that some of the weights are negative. This is an undesirable feature, leading to reduced accuracy when integrating functions outside the space P. We therefore extended our list of monomials to include all 300 monomials of degree at most 23, and then chose the weights w_i to minimise $\sum_j (I(p_j) - \int p_j)^2$ subject to the constraints $w_i \geq 0$. (This was done using Maple's `LSSolve()` command.) It turns out that there are 18 indices i such that $w_i = 0$, so we really only use 238 sample points. With these weights we have $|I(K) + \pi/4| \simeq 10^{-17.1}$, and if $f = D(z_1^i z_2^j \alpha_k)$ with $i + j \leq 10$ then $|I(f)| \leq 10^{-14.5}\sqrt{I(f^2)}$.

Quadrature rules as above are represented by instances of the class `E_quadrature_rule`, which is declared in the file `embedded/E_quadrature.mpl`. The specific rule described above is stored in the file `quadrature_frobenius_256a.m` in the directory `data/embedded/roothalf`. After reading that file, one can enter the following to integrate $1/(1+z_1)$ (for example):

```
Q := eval(quadrature_frobenius_256a):
Q["int_z",1/(1+z[1])];
```

One can test the accuracy of `Q` (as described above) using the methods `Q["curvature_error"]` and `Q["stokes_error",10]`. Various other methods are documented in the code.

One can regenerate the object `Q` using the function `build_data["E_quadrature_rule"]()` defined in the file `build_data.mpl`. However, there is not a very compelling reason to do this, as we can check that `Q` has the desired properties without needing to regenerate it. Also, the process is very slow, and may take several days to run.

8. Classifying EX^*

[sec-classify-roothalf]

Theorems 3.7.2 and 4.5.1 tell us that there are cromulent isomorphisms $HX(b) \xrightarrow{q} EX^* \xrightarrow{r} PX(a)$ for suitable values $a, b \in (0,1)$. In this section, we discuss numerical methods that enable us to calculate approximations to a, b, q and r. Note that the methods of Section 5 allow us to compute a and r from b and q, or *vice versa*. We have tried several different approaches. The most successful will be described first, in Section 8.1. We will then outline one other approach in Section 8.2.

Our current estimates are $a \simeq 0.0983562$ and $b \simeq 0.8005319$. We have some reason to hope that all the quoted digits are accurate, but we have not performed a rigorous error analysis.

8.1. **Hyperbolic rescaling.** [sec-rescaling]

Theorem 4.5.1 tells us that there is a conformal covering map $q\colon \Delta \to EX^*$, which induces an isomorphism $HX(b) \to EX^*$ for some b. Let g denote the Riemannian metric that EX^* inherits from $\mathbb{R}^4$, and let g_{hyp} denote the standard hyperbolic metric $ds^2 = 4|dz|^2/(1-|z|^2)^2$ on Δ. Recall that Remark 6.10.6 gives a formula for the Gaussian curvature $K(g)$, and it is a standard fact that $K(g_{\text{hyp}}) = -1$.

Proposition 8.1.1. [prop-rescaling]

There is a unique real analytic function f on EX^* such that $K(e^{2f}g) = -1$. Moreover, this function is G-invariant, and it satisfies $q^*(e^{2f}g) = g_{\text{hyp}}$. The curves $C_0, \dots, C_8 \subset EX^*$ are geodesics with respect to the metric $e^{2f}g$.

Remark 8.1.2. Note here that when we multiply the metric tensor g by e^{2f}, this multiplies lengths by e^f, and areas by e^{2f}.

The proof depends on the following formula:

Lemma 8.1.3. [lem-rescaled-curvature]
Let Z be a smooth oriented surface equipped with a Riemannian metric g. Let $\Delta = \Delta_g$ denote the associated Laplacian operator, and let $K(g)$ denote the Gaussian curvature. Then for any smooth function f on Z, we have

$$K(e^{2f}g) = (K(g) - \Delta(f))/e^{2f}.$$

Proof. See Chapter V of [16], for example. □

Proof of Proposition 8.1.1. Because q is a conformal covering, we see that $q^*(g)$ is a positive multiple of g_{hyp}, say $q^*(g) = e^{-2\widetilde{f}}g_{\text{hyp}}$ for some real analytic function $\widetilde{f}$ on Δ. Note that $\widetilde{\Pi}$ acts isometrically on Δ, and also (via $\widetilde{P}/\Pi = G$) on EX^*, and q is equivariant for these actions. It follows that $\widetilde{f}$ is invariant under $\widetilde{\Pi}$, so it has the form $\widetilde{f} = q^*(f)$ for some G-invariant real analytic function f on EX^*. Now put $g_1 = e^{2f}g$. We have $q^*K(g_1) = K(e^{2\widetilde{f}}q^*(g)) = K(g_{\text{hyp}}) = -1$, but q is surjective so $K(g_1) = -1$ as required. Now q is a local isometry from (Δ, g_{hyp}) to (EX^*, g_1), and it carries the geodesics $C_i \subset \Delta$ to the curves $C_i \subset EX^*$, so the latter must also be geodesic.

Now suppose we have a function u with $K(e^{2u}g_1) = -1$; we claim that $u = 0$. In the proof we will use the gradient operator ∇, the Laplacian operator Δ, and the integration operator $\int_{EX^*}$; these are all defined using the metric g_1. Using Lemma 8.1.3 we obtain $e^{2u} - 1 - \Delta(u) = 0$. It is a standard fact that for any functions $a, b \in C^\infty(EX^*)$ we have

$$\int_{EX^*} a\Delta(b) = -\int_{EX^*} \langle \nabla(a), \nabla(b)\rangle.$$

Using this, we get

$$\int_{EX^*} \big(u(e^{2u} - 1) + \|\nabla(u)\|^2\big) = \int_{EX^*} \big(u(e^{2u} - 1 + \Delta(u))\big) = 0.$$

By considering the cases $u \geq 0$ and $u \leq 0$ separately, we see that $u(e^{2u} - 1) \geq 0$, with equality only where $u = 0$. In view of this, the above integral formula shows that $u = 0$ everywhere. This shows that f is uniquely characterised by the fact that $K(e^{2f}g) = -1$. □

To go further, we will need to discuss the curves $c_k \colon \mathbb{R} \to \Delta$ as well as the curves $c_k \colon \mathbb{R} \to EX^*$. We will distinguish between them by writing c_{Hk} for the former, and c_{Ek} for the latter. Similarly, we will write v_{Hj} and v_{Ej} for the usual points in Δ and EX^*.

Corollary 8.1.4. [cor-side-lengths]
Let f be the unique function such that $K(e^{2f}g) = -1$, and put

$$L_0 = \int_{\pi/4}^{\pi/2} e^{f(c_{E0}(t))}\|c'_{E0}(t)\|\,dt$$

$$L_1 = \int_{0}^{\pi/2} e^{f(c_{E1}(t))}\|c'_{E1}(t)\|\,dt$$

$$L_3 = \int_{0}^{\pi/2} e^{f(c_{E3}(t))}\|c'_{E3}(t)\|\,dt$$

$$L_5 = \int_{0}^{\pi} e^{f(c_{E5}(t))}\|c'_{E5}(t)\|\,dt.$$

If b is the parameter such that $EX^* \simeq HX(b)$, and the points $v_{Hi} \in \Delta$ are defined in terms of b as in Definition 4.2.3, then we have

$$L_0 = d_{hyp}(v_{H3}, v_{H6}) = 2\,\mathrm{arctanh}((b_+ - \sqrt{2}b)/b_-)$$
$$L_1 = d_{hyp}(v_{H0}, v_{H6}) = 2\,\mathrm{arctanh}((\sqrt{2} - b_-)/b_+)$$
$$L_3 = d_{hyp}(v_{H3}, v_{H11}) = 2\,\mathrm{arctanh}((1 - b_-)/b)$$
$$L_5 = d_{hyp}(v_{H0}, v_{H11}) = 2\,\mathrm{arctanh}(b_+ - b).$$

Proof. The domain $HF_{16}(b) \subset \Delta$ is a hyperbolic polygon, with geodesic edges, and vertices v_{H0}, v_{H3}, v_{H6} and v_{H11}. The map q is a local isometry, which sends v_{Hi} to v_{Ei}. It follows that the geodesic distance from v_{Hi} to v_{Hj} in Δ is the same as the geodesic distance from v_{Ei} to v_{Ej} in EX^*. As the curves $C_i \subset EX^*$ are geodesic, the distances in EX^* are given by the indicated integrals.

The distances in Δ are given by the standard formula $d_{\text{hyp}}(z, w) = 2\,\text{arctanh}(m(z, w))$, where

$$m(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$

We therefore need to show that $m(v_{H3}, v_{H6}) = (b_+ - \sqrt{2}b)/b_-$ and so on. It is not hard to see that $(b_+ - \sqrt{2}b)/b_- \geq 0$, so it will suffice to show that $m(v_{H3}, v_{H6})^2 = ((b_+ - \sqrt{2}b)/b_-)^2$, and this can be done by straightforward algebraic manipulation. The same method works for the other three cases.

`hyperbolic/HX_check.mpl: check_side_lengths()`

□

We now describe an algorithm based on the above results. The full algorithm can be carried out by executing the function `build_data["EH_atlas",0]()` defined in `build_data.mpl`, which in turn invokes the functions `build_data["EH_atlas",i]()` for i from 1 to 4, each of which carries out a subset of the steps described below.

Everything is implemented by methods of the class `EH_atlas` (which is declared in the file `embedded/roothalf/EH_atl`
and some other related classes. We can enter

```
EHA := `new/EH_atlas`():
```

to create a new object of the required type.

First, we need a quadrature rule $Q(f) = \sum_i w_i f(a_i)$, which is intended to approximate $\int_{EX^*} f$ when f is G-invariant. It is not important for this approximation to be accurate, we just want the seminorm $\|f\|_Q = \sqrt{Q(f^2)}$ to be a reasonable measure of the size of f, at least for the functions f that arise in our calculations. After constructing a suitable instance `Q` of the class `E_quadrature_rule`, we can enter

```
EHA["quadrature_rule"] := eval(Q):
```

to attach the quadrature rule to the atlas.

Next, we need to choose a finite-dimensional subspace F of invariant, real analytic functions on EX^*, in which we will search for an approximation to the rescaling function f. Two possibilities are as follows:

$$F_{\text{poly}}(d) = \{ \text{ polynomials } p(z_1, z_2) \text{ of total degree at most } d\}$$
$$F_{\text{pade}}(d) = \left\{ \text{ rational functions } \frac{p(z_1, z_2)}{q(z_1, z_2)},\ \max(\deg(p), \deg(q)) \leq d,\ q(0, 0) = 1 \right\}.$$

We have found that F_{pade} works better than F_{poly} even if the degrees are chosen so that the total number of parameters is the same. We do not fully understand why, although it is reminiscent of the situation with Padé approximations in one variable.

One can search through the above spaces using one of the following methods:

```
EHA["find_rescale_poly",d];
EHA["find_rescale_poly_alt",d];
EHA["find_rescale_pade",d];
EHA["find_rescale_pade_alt",d];
```

It is best to call these methods repeatedly starting with $d = 2$. A measure of success is stored in the field `EHA["rescaling_error"]`, and this should decrease on each iteration. When this measure has settled down, one can increase d by one.

The first two of the above methods use $F_{\text{poly}}(d)$, and the second two use $F_{\text{pade}}(d)$. The alternative methods `find_rescale_poly` and `find_rescale_pade` are coded using a very direct translation of the mathematical problem that we seek to solve, so it is easy to check their correctness. The methods `find_rescale_poly_alt` and `find_rescale_pade_alt` are harder to understand but much more efficient: they precompute various vectors and matrices, and thereby convert the problem to linear algebra, as far as possible. The polynomial case works as follows:

- We enumerate the monomials of degree at most d as $m_1, \ldots, m_r$.

- The sample points for the quadrature rule are $u_1, \dots, u_n$, with weights $w_i \geq 0$. We set `rweights` to be the vector with entries $\sqrt{w_i}$, and we set `kpoints` to be the vector whose entries are the values of the curvature at the points u_i (computed using Remark 6.10.6). We also set `ones` to be the vector of length n whose entries are all one.
- Similarly, we set `mpoints` and `lpoints` to be matrices with entries $m_i(u_j)$ and $\Delta(m_i)(u_j)$ (computed using Proposition 6.10.9).
- Now if $f = \sum_j a_j m_j$, then the values of f and $K(g) - \Delta(f)$ are given by `mpoints.a` and `kpoints-lpoints.a`. This makes it easy to compute the objective function $FT(a) = Q(((K(g) - \Delta(f))/e^{2f} + 1)^2)$.
- More precisely, $FT(a)$ is $\sum_i F_i(a)^2$, where $F_i(a)$ is $\sqrt{w_i}$ times the value of $(K(g) - \Delta(f))/e^{2f} + 1$ at u_i. This is useful, because there are special algorithms (used by Maple's `LSSolve` command) for minimising a sum of squares.
- The above framework gives an efficient method for calculating the vector $F(a)$ and also the matrix of derivatives $\partial F_i(a)/\partial a_j$. These are the ingredients that we need in order to use the `LSSolve` command with the `objectivejacobian` option.

The rational case is more complicated. Here we have $f = g/h$, and the values of f and $\Delta(f)$ do not depend linearly on the coefficients of g and h. However, one can construct linear differential operators P_i and Q_j such that

$$\Delta(f) = \frac{P_1(g)}{h} + \frac{P_2(g)Q_2(h) + P_3(g)Q_3(h) + P_4(g)Q_4(h)}{h^2} + P_5(g)\frac{Q_2(h)Q_5(h) + Q_3(h)Q_6(h)}{h^3}.$$

(This is just a version of the quotient rule for second derivatives.) One can again precompute the values $P_i(m_k)(u_l)$ and $Q_j(m_k)(u_l)$ and thereby streamline the calculation of the objective function, and of its derivatives with respect to the coefficients of g and h.

After using these methods to find the rescaling function f, we can enter `EHA["log_rescale_z"](z)` to see f as an expression in z_1 and z_2, or `EHA["log_rescale_x"](x)` to see it as an expression in $x_1, \dots, x_4$.

Once we have found f, we define lengths L_i by the integrals specified in Corollary 8.1.4. We then use numerical methods to find b_5 such that $L_5 = 2\operatorname{arctanh}(2b_5^2 + 1 - 2b_5\sqrt{1 + b_5^2})$, and similarly for b_3, b_1 and b_0, using the formulae in Corollary 8.1.4. If our approximations are good, then b_0, b_1, b_3 and b_5 should all be close to the parameter b such that $EX^* \simeq HX(b)$. We can thus get an imperfect measure of the accuracy of our approximations from the differences $|b_i - b_j|$; these are at most $10^{-7.4}$ in our best attempt.

The above algorithm is implemented by the method `EHA["find_a_H"]`. The length L_k is stored as `EHA["curve_lengths"][k]`, and b_k is stored as `EHA["curve_a_H_estimates"][k]`. The average of these is `EHA["a_H"]`, and the maximum discrepancy between them is `EHA["a_H_discrepancy"]`.

Having found b, we can construct an isomorphism $HX(b) \to PX(a)$ by the methods of Sections 5.2 and 5.4. This gives objects of class `H_to_P_map` and `P_to_H_map`. These can be assigned to the fields `EHA["H_to_P_map"]` and `EHA["P_to_H_map"]`, in order to keep everything packaged together in a single object. These steps are not included in the function `build_data["EH_atlas",0]()`, but are instead in the functions `build_data["H_to_P_map"]()` and `build_data["P_to_H_map"]()`.

We next want to approximate the map $q\colon \Delta \to EX^*$.

Remark 8.1.5. [rem-H-to-E-method]

The broad outline of our method is as follows:

(a) It is not hard to see that $q(c_{kH}(t)) = c_{kE}(u_k(t))$ for a certain function $u_k(t)$, and to find Fourier series approximations to $u_k(t) - t$ by numerical integration of arc lengths.

(b) Given a point $a \in EX^*$, it is not too hard to find a polynomial map $p_a\colon \Delta \to \mathbb{R}^4$ which satisfies $p_a(0) = a$ and is a good approximation to an isometry $\Delta \to EX^*$, at least if we consider points close to the origin in Δ. (We will call these maps *hyperbolic charts*.) If $a \in C_k$ for some k then we can exploit information from (a) to find p_a; otherwise we use a more general method with power series. Experiment suggests that our value of $p_a(z)$ can only be trusted for fairly small values of $|z|$, perhaps $|z| < 0.1$ or so.

(c) We then show that there exists a Möbius map $m_a \in \operatorname{Aut}(\Delta)$ such that $p_a(z) \simeq q(m_a(z))$ for small z. Equivalently, for w close to $m_a(0)$ we have $q(w) \simeq p_a(m_a^{-1}(w))$. Thus, to find q, we should try to find p_a and m_a for a reasonable supply of points $a \in F_{16}$. Methods for p_a were discussed above, but we still need to consider m_a.

(d) Given nearby points $a, b \in EX^*$, we can find $p_a^{-1}(b)$ and $p_b^{-1}(a)$ numerically, and the values $d_{\text{hyp}}(0, p_a^{-1}(b))$ and $d_{\text{hyp}}(p_b^{-1}(a), 0)$ give two different estimates for the geodesic distance between a and b. (The discrepancy between them gives a check on the accuracy of our methods.)

(e) We now choose a reasonably fine grid of points $a_1, \ldots, a_N$ in F_{16}, and try to find the points $\beta_i = q^{-1}(a_i) \in HF_{16}(b)$. For any points a_i that lie in ∂F_{16}, we can do this using (a). For the remaining points, we note that when a_i is a neighbour of a_j, we can estimate the geodesic distance between them as in (d), and we should then have $d_{\text{hyp}}(\beta_i, \beta_j) = d(a_i, a_j)$. We therefore choose the points β_i to minimize a suitable measure of the overall discrepancy between the lengths $d_{\text{hyp}}(\beta_i, \beta_j)$ and $d(a_i, a_j)$.

(f) We now need the Möbius maps m_i such that $p_i = qm_i$. It is not hard to see that these must have the form

$$m_i(z) = \lambda_i \frac{z + \overline{\lambda_i}\beta_i}{1 - \lambda_i \overline{\beta_i} z} \qquad\qquad m_i^{-1}(w) = \overline{\lambda_i} \frac{w - \beta_i}{1 - \overline{\beta_i} w}.$$

If a_j is adjacent to a_i then we find that $m_i^{-1}(\beta_j)$ should be equal to $p_i^{-1}(a_j)$. We can again calculate $p_i^{-1}(a_j)$ numerically, and this gives

$$\lambda_i = (\beta_j - \beta_i)/(1 - \overline{\beta_i}\beta_j)/p_i^{-1}(a_j).$$

We can perform this calculation for every j such that a_j is adjacent to a_i, and then take a kind of average to get a final estimate for λ_i. (Of course, in the averaging process we impose the constraint $|\lambda_i| = 1$.)

(g) Now given a point $z \in \Delta$, we can approximate $q(z)$ as follows: we find $\gamma \in \widetilde{\Pi}$ such that $\gamma(z) \in F_{16}$, then find i such that $\gamma(z)$ is as close as possible to β_i, then take $q(z) = \gamma^{-1}(p_i(m_i^{-1}(\gamma(z))))$ (using the action of $\widetilde{\Pi}$ on EX^* via $\widetilde{\Pi}/\Pi = G$). We can use this method to calculate $q(z)$ for a large sample of points $z \in \Delta$, and then use numerical techniques to find an approximation to $q(x + iy)$ using rational functions in x and y.

We now discuss the above points (a) to (e) in more detail.

First, as q gives a cromulent isomorphism $HX(b) \to EX^*$, we must have $q(v_{Hi}) = v_{Ei}$ for all i, and $q(c_{Hk}(\mathbb{R})) = C_{Ek}$. As $q \circ c_{Hk} \colon \mathbb{R} \to C_{Ek}$ and $c_{Ek} \colon \mathbb{R} \to C_{Ek}$ are both 2π-periodic universal coverings, it is not hard to see that we must have $q(c_{Hk}(t)) = c_{Ek}(u_k(t))$ for some strictly increasing diffeomorphism $u_k \colon \mathbb{R} \to \mathbb{R}$ with $u_k(t + 2\pi) = u_k(t) + 2\pi$. This in turn means that the function $u_k(t) - t$ is 2π-periodic, so it can be represented by a Fourier series. The maps $c_k \colon \mathbb{R} \to \Delta$ were defined so as to have constant speed with respect to the hyperbolic metric; let that speed be s_k. As q is locally isometric, we can differentiate the relation $q(c_{Hk}(u_k^{-1}(t))) = c_{Ek}(t)$ to get

$$\frac{d}{dt} u_k^{-1}(t) = s_k^{-1} \|c'_{Ek}(t)\| e^{f(c_{Ek}(t))}.$$

We can integrate this numerically to find a Fourier series approximation to $u_k^{-1}(t) - t$. From this we can obtain a Fourier approximation to $u_k(t) - t$, and thus approximate formulae for $q(c_{Hk}(t))$. To give an idea of the size of the dominant terms, we have

$$\begin{aligned}
u_0(t) - t &\simeq 0.017 \sin(4t) \\
u_1(t) - t &\simeq -0.169 \sin(2t) + 0.010 \sin(4t) - 0.001 \sin(6t) \\
u_3(t) - t &\simeq -0.074 \sin(2t) + 0.002 \sin(4t) \\
u_5(t) - t &\simeq -0.362 \sin(t) + 0.026 \sin(2t) - 0.001 \sin(3t).
\end{aligned}$$

These are calculated by the method `EHA["find_u",d]` (where d controls the number of terms in the various Fourier series). After invoking this method, one can calculate $u_k(t)$ and $u_k^{-1}(t)$ as `EHA["u"][k](t)` and `EHA["u_inv"][k](t)`.

We next discuss point (b) in Remark 8.1.5.

Proposition 8.1.6. [prop-local-isometry]

Let Z be an oriented surface with a Riemannian metric of curvature -1. Let a be a point in Z, and let v be a nonzero tangent vector at a. Then there is a unique germ of an oriented local isometry $p\colon \Delta \to Z$ such that $p(0) = a$ and $p'(0) \in \mathbb{R}^+ v$.

Proof. Local existence of p is a theorem of Riemann; a convenient reference is [19, Theorem 2.4.11]. For uniqueness, it will suffice to prove the following: if u is a germ of an oriented local isometry $\Delta \to \Delta$ with $u(0) = 0$ and $u'(0) > 0$, then u is the identity. It is clear that u must act as a rotation on the tangent space $T_0\Delta$, so the condition $u'(0) > 0$ forces $u'(0) = 1$. The exponential map $\exp\colon T_0\Delta \to \Delta$ is characterised by its metric properties, so it commutes with u, and $T_0 u = 1$ so $u = 1$. □

Corollary 8.1.7. [cor-local-isometry]

Let $p\colon U \to EX^*$ be a hyperbolic chart (where U is a disc around 0 in Δ). Then there is a Möbius map $m \in \mathrm{Aut}(\Delta)$ such that $p = qm$ on U.

Proof. As $q\colon \Delta \to EX^*$ is a covering map and U is simply connected, we can choose $m\colon U \to \Delta$ such that $p = qm$. As both p and q are orientation-preserving isometries, the same is true of m. Now put $\alpha = m(0) \in \Delta$, and let λ denote the unit complex number such that $m'(0)$ is a positive multiple of $\overline{\lambda}$. Put $m_1(z) = (z + \lambda\alpha)/(\lambda + \overline{\alpha}z)$ (so $m_1^{-1}(z) = (z - \alpha)/(1 - \overline{\alpha}z)$). We find that m and m_1 are both orientation-preserving isometries of U into Δ such that $m'(0)$ and $m_1'(0)$ are positive multiples of each other. It follows (by the uniqueness clause in the Proposition) that $m = m_1$. □

Charts p as above are represented by instances of the class `EH_chart`, which is declared in the file `embedded/roothalf/EH_atlas.mpl`. It extends the class `E_chart`, which was discussed in Section 7.3. The algorithm to make a chart isometric is actually coded in the `isometrize` method of the `E_chart` class; this is invoked automatically by the methods that initialize instances of the `EH_chart` class. In more detail, the algorithm is as follows. We start with an approximate polynomial conformal chart $p_0\colon \mathbb{C} \to EX^*$ as in Proposition 7.3.1, which can be constructed by methods of the `E_chart` class. It is then not hard to show that there are unique numbers $a_1 \in \mathbb{R}^+$ and $a_2 \in \mathbb{C}$ such that the map $p_2(z) = p_0(a_1 z + a_2 z^2)$ is isometric to first order. We can then try to find a_3 such that the map $p_3(z) = p_2(z + a_3 z^3)$ is isometric to second order. This involves solving a system of inhomogeneous linear equations for the real and imaginary parts of a_3. As the curvature of g_1 is not exactly equal to -1, these equations will not usually be solvable. However, we can choose a_3 to minimize the mean square error in these equations, and then proceed to find coefficients a_4, a_5 and so in in the same way.

As mentioned previously, there a different method that is available for charts centred on one of the curves C_k. Suppose that $a = c_{Ek}(t_0)$, and that we have found a good approximation to the conformal chart p_0 with $p_0(t) = c_{Ek}(t_0 + t))$ for small $t \in \mathbb{R}$ (as discussed in Section 7.3). Put $t_1 = u_k^{-1}(t_0)$. The function $u_k\colon \mathbb{R} \to \mathbb{R}$ is real analytic, so it can be extended (using power series, for example) to give a holomorphic function on a neighbourhood of the point $t_1 = u_k^{-1}(t_0)$. Now put

$$\begin{aligned}
\lambda &= c'_{Hk}(t_1)/|c'_{Hk}(t_1)| \in S^1 \\
\alpha &= -c_{Hk}(t_1)/\lambda \\
\beta &= -\lambda\alpha \\
m(z) &= \lambda(z - \alpha)/(1 - \overline{\alpha}z),
\end{aligned}$$

so $m \in \mathrm{Aut}(\Delta)$ with $m(0) = c_{Hk}(t_1)$ and $m'(0) \in \mathbb{R}^+.c'_{Hk}(t_1)$. Finally, recall that s_k denotes the (constant) speed of the map $c_{Hk}\colon \mathbb{R} \to \Delta$.

Proposition 8.1.8. [prop-curve-chart]

With notation as above, the map

$$p(z) = p_0(u_k(t_1 + 2s_k^{-1}\operatorname{arctanh}(z)) - t_0)$$

is a hyperbolic chart at a. More specifically, we have $p(z) = q(m(z))$. In particular, we have $q^{-1}p(0) = m(0) = -\lambda\alpha = \beta$.

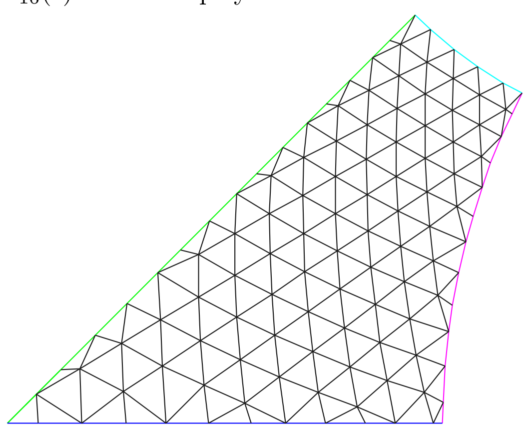
Proof. Note that $m^{-1}(c_{Hk}(\mathbb{R}))$ is a geodesic in Δ which is tangent to $\mathbb{R}$ at 0; it follows that $m^{-1}(c_{Hk}(\mathbb{R})) = (-1,1)$. The standard speed one parametrisation of $(-1,1)$ is $t \mapsto \tanh(t/2)$. It follows that $c_{Hk}(t_1+t) = m(\tanh(s_k t/2))$.

We now claim that $p(z) = q(m(z))$. Both $p(z)$ and $q(m(z))$ are holomorphic, so it will suffice to prove this for small real values of z. When z is real we see that $u_k(t_1 + 2s_k^{-1}\operatorname{arctanh}(z)) - t_0$ is also real, so

$$\begin{aligned} p(z) &= c_{Ek}(u_k(t_1 + 2s_k^{-1}\operatorname{arctanh}(z))) \\ &= q(c_{Hk}(t_1 + 2s_k^{-1}\operatorname{arctanh}(z))) = q(m(t)) \end{aligned}$$

as required. □

For $j \in \{0,3,6,11\}$ one can add an isometric chart centred at the point v_j to the atlas using the method `EHA["add_vertex_chart",j]`. Now suppose that $k \in \{0,1,3,5\}$, so the set $c_k^{-1}(F_{16})$ is a closed interval $[a,b]$ for some a and b. Then the method `EHA["add_curve_chart",k,t]` adds an isometric chart centred at $a + t(b-a)$ (so the natural domain for t is $[0,1]$). Finally, for a point x_0 in the interior of F_{16}, we can use the method `EHA["add_centre_chart",x0]` to add a chart centred at x_0. The function `build_data["EH_atlas",2]()` adds a total of 119 charts to the atlas created by `build_data["EH_atlas",1]()`. They are chosen so that the centres form an approximately equilateral triangular grid with respect to the rescaled hyperbolic metric. (To make everything fit, some triangles on the edge of F_{16} have to deviate strongly from being equilateral, but the ones in the interior are quite regular.) The corresponding points in $HF_{16}(b)$ can be displayed as follows:



All the charts are based on polynomials of degree 16. We find that, on the disc of radius 0.1, the entries in $p^*(g_1) - g_{\text{hyp}}$ are of absolute value less than 10^{-6}.

We next need to record the combinatorial structure of the above grid. If there is an edge joining the centre of chart i to the centre of chart j, we need to invoke the method `EHA["add_edge",i,j]`. (Charts are numbered from 0, in the order that they were added to the atlas.) This is also done by the function `build_data["EH_atlas",2]()`.

Next, for each chart p_i, we need to find the point $\beta_i = q^{-1}p_i(0) \in \Delta$. For charts that are centred on one of the curves C_k, this is given by Proposition 8.1.8. For the remaining charts, it is useful to start with a crude approximation, obtained by applying an essentially arbitrary diffeomorphism between the fundamental domains for EX^* and $HX(b)$. This is done using the method `EHA["set_beta_approx"]`. We then invoke the method `EHA["set_edge_lengths"]`. This calculates various quantities for each edge (i,j). In particular, it calculates the average of $d_{\text{hyp}}(0, q_i^{-1}(q_j(0)))$ and $d_{\text{hyp}}(0, q_j^{-1}(q_i(0)))$, which is an estimate of the hyperbolic distance in EX^* between the centres $q_i(0)$ and $q_j(0)$. Each edge is actually represented by an object `E` of class `EH_atlas_edge`, and this distance is stored as `E["EH_length"]`. On the other hand, `E["H_length"]` is set equal to $d_{\text{hyp}}(\beta_i, \beta_j)$, using the approximate values of β_i and β_j, which may be quite inaccurate at this stage. One can then invoke the method `EHA["optimize_beta"]` to adjust the values of β_i so as optimize the match between the edge lengths measured in Δ and in EX^*. The same method also calculates appropriate values for the parameters λ_i, and thus also for the Möbius maps m_i.

Now all the maps $p_i m_i^{-1}$ are approximations to q, and it is useful to test how well they agree with each other. The method `EHA["make_H_samples",N]` sets `EHA["H_samples"]` to be the list of all numbers

$z = (s + it)/N$ (with $s, t \in \mathbb{Z}$) that lie in $HF_{16}(b)$. The method `EHA["max_patching_error",r]` then does the following. For each point z_i in `EHA["H_samples"]`, it looks for charts p_j where $|z_i - \beta_j| < r$. Let k_i be the number of such charts. For each such chart, the method calculates $x_{ij} = p_j m_j^{-1}(z_i) \in EX^*$. These points should all be the same, so we let d_i denote the maximum euclidean distance between any two of them. The return value of the method is a triple (z_i, m_i, d_i), where d_i is maximal. If we take $r = 0.12$, we find that $m_i \geq 3$ and $d_i < 10^{-10.4}$ for all i. Thus, for an arbitrary point $z \in HF_{16}(b)$, it is safe to calculate $q(z)$ as $p_j m_j^{-1}(z)$, where j is chosen to minimize $d_{\text{hyp}}(z, \beta_j)$. We can then extend this over all of Δ by using the group action, as discussed earlier. This is implemented by the methods `EHA["q",[x,y]]` or `EHA["q_c",x+I*y]`.

We now want to find a function given by a single formula which is a good approximation to q on a reasonably large part of Δ, such as the disc of radius 0.9 centred at the origin. An obvious approach would be to approximate $q(x+iy)_k$ (for $1 \leq k \leq 4$) by a polynomial or rational function in x and y. This is implemented by the methods `set_q_approx_poly` and `set_q_approx_pade` of the class `EH_atlas`. However, results from this approach are poor. The approximating polynomials have extremely large coefficients (of different signs), even though $|q(x + iy)_k| \leq 1$, and the errors are fairly large even if we use polynomials or rational functions of high degree. It is better to consider the Fourier series on circles of fixed radius. To understand how this works, we first recall that q is equivariant with respect to $\langle \lambda, \nu \rangle$, which gives

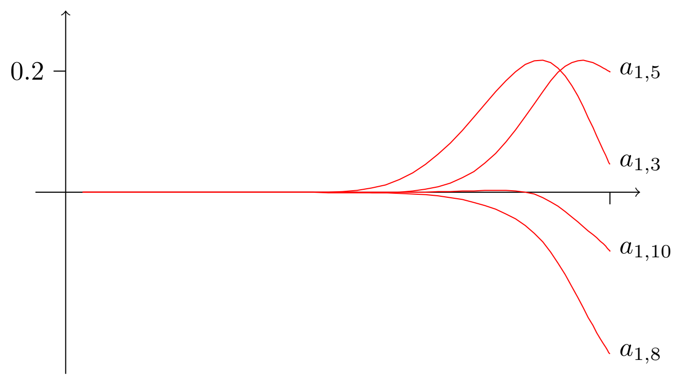
$$
\begin{aligned}
q_1(re^{i(\theta+\pi/2)}) &= -q_2(re^{i\theta}) & \qquad q_1(re^{-i\theta}) &= q_1(re^{i\theta}) \\
q_2(re^{i(\theta+\pi/2)}) &= q_1(re^{i\theta}) & q_2(re^{-i\theta}) &= -q_2(re^{i\theta}) \\
q_3(re^{i(\theta+\pi/2)}) &= q_3(re^{i\theta}) & q_3(re^{-i\theta}) &= q_3(re^{i\theta}) \\
q_4(re^{i(\theta+\pi/2)}) &= -q_4(re^{i\theta}) & q_4(re^{-i\theta}) &= q_4(re^{i\theta}).
\end{aligned}
$$

From this it follows that there are functions $a_{k,m}(r)$ such that

$$
\begin{aligned}
q_1(re^{i\theta}) &= \sum_m a_{1,m}(r)\cos((2m+1)\theta) \\
q_2(re^{i\theta}) &= \sum_m (-1)^m a_{1,m}(r)\sin((2m+1)\theta) \\
q_3(re^{i\theta}) &= \sum_m a_{3,m}(r)\cos(4m\theta) \\
q_4(re^{i\theta}) &= \sum_m a_{4,m}(r)\cos((4m+2)\theta).
\end{aligned}
$$

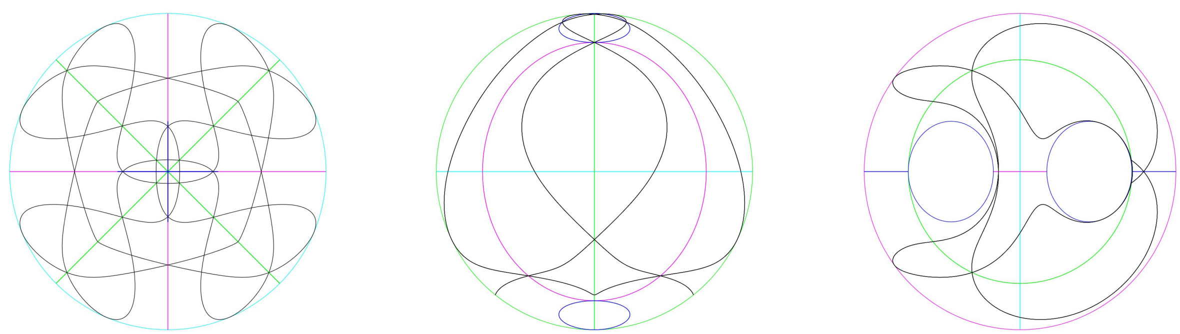
We can find approximations to the coefficients $a_{j,l}(r)$ by fixing $k \geq 0$, then calculating $q(re^{2\pi ij/2^k})$ for $0 \leq j < 2^k$, then taking a discrete Fourier transform. This algorithm is implemented by the method `EHA["set_q_approx_fourier",r_max,m,k]`, which calculates Fourier coefficients for `m` different radii, the largest being `r_max`.

It seems experimentally that for $r \leq 0.9$ we have $|a_{j,m}(r)| \leq 2$ for all j and m, and that $|a_{j,m}(r)|$ decreases quite rapidly with m. Thus, for a fixed value of r, the above representation is quite satisfactory. Now fix k and m, and consider $a_{k,m}(r)$ as a function of r. The following picture shows a typical sample of these functions, for $0 \leq r \leq 0.9$.

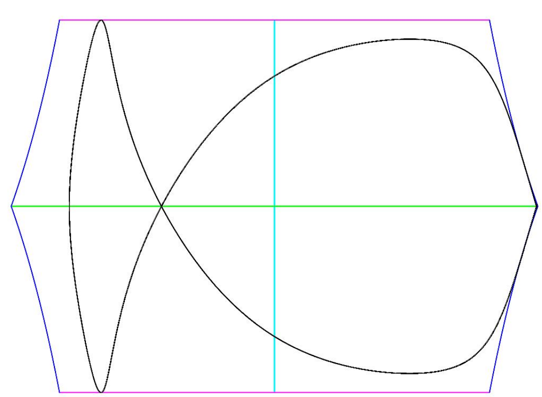


These functions are extremely flat for small values of r, and grow to modest size as r approaches 0.9. They cannot easily be approximated by polynomials or rational functions in r. We have tried various transformations, such as using the variable $s = 1/\log(1/r)$ instead of r, but none of these have yielded compelling results. It would be of interest to have a better theoretical understanding of the asymptotics of the functions $a_{k,m}(r)$, but so far we have not achieved that. However, approximation by cubic splines (which are also calculated by the `set_q_approx_fourier` method) is quite effective. An approximation to $q(x+iy)$ using these splines can be calculated by invoking `EHA["q_fourier",[x,y]]` or `EHA["q_fourier_c",x+I*y]`.

The following pictures show the images of $q(0.8e^{it})$ under the linear projections $\pi, \delta, \zeta \colon EX^* \to \mathbb{R}^2$ that were discussed in Section 7.1:



The following picture shows the image under the map $p_4 \colon x \mapsto y$ from Proposition 6.7.4:



Finally, we discuss the canonical conformal map $\widehat{p} \colon EX^* \to S^2$. The transformation properties of $\widehat{p}$ were discussed in Remark 3.7.14. By comparing these with the transformation properties of the basis in

Proposition 6.5.2, we see that $\widehat{p}(x)_i$ must be given by some G-invariant function $u_i(z_1, z_2)$ multiplied by $p^*(x)_i$, where

$$p^*(x) = \left(\sqrt{2}\, y_2,\ 2x_1x_2,\ -x_3)\right).$$

Remark 3.7.14 also records the values of $\widehat{p}(v_i)$ for $0 \leq i \leq 9$; these are equivalent to the conditions

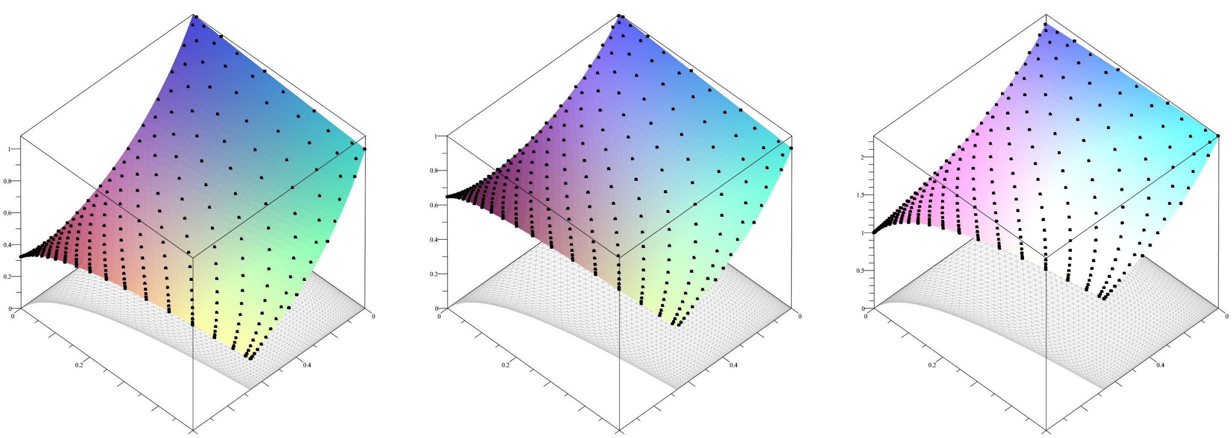
$$u_1(0, 1/2) = u_2(0, 0) = u_3(1, 0) = 1.$$

It turns out that the functions u_i can be approximated effectively by rational functions, using the method that we will now explain. We have already seen how to produce a list of points a_i^Δ in $HF_{16}(b)$ and calculate the images $a_i^E = q(a_i^\Delta)$ under the map $q\colon \Delta \to EX^*$. By applying the map $x \mapsto z$ to these, we obtain points $a_i^Z \in F_{16}^* \subset \mathbb{R}^2$. We have also seen how to calculate the isomorphism $HX(b) \to PX(a)$. By combining this with the projection $PX(a) \to \mathbb{C}_\infty$ and the stereographic projection map $\mathbb{C}_\infty \to S^2$ we obtain points $a_i^S \in S^2$. We must have $\widehat{p}(a_i^E) = a_i^S$, and using this, we can find the values $a_{ij}^U = u_j(a_i^Z) = \widehat{p}(a_i^E)_j / p^*(a_i^E)_j$ for $1 \leq j \leq 3$. Our problem is thus to find rational functions u_j^R such that $u_j^R(a_i^Z)$ is close to a_{ij}^U. In principle this should hold for all i, but we have found it best to discard the cases where $|p^*(a_i^E)_j| < 10^{-3}$, in order to avoid numerical instability.

We have found the following general approach to be effective.

Method 8.1.9. Suppose we have a finite set $S = \{s_1, \dots, s_n\}$, and a function $f\colon S \to \mathbb{R}$. We want to find an approximation $f \simeq g_1/g_2$, where g_1 and g_2 lie in some vector space $V \leq \mathrm{Map}(S, \mathbb{R})$. If n is large then it is not very tractable to minimize $\|f - g_1/g_2\|^2$, because this is nonlinear in the coefficients of g_2. However, it will often be adequate to minimize $\|fg_2 - g_1\|^2$ subject to a positive definite quadratic constraint on the size of g_2. To do this, let $\{v_1, \dots, v_m\}$ be a basis for V. Let M_1 be the matrix of values $v_j(s_i)$, and let M_2 be the matrix of values $f(s_i)v_j(s_i)$. We find the QR decompositions $M_i = Q_iR_i$, and put $N = Q_1^TQ_2$. We let a_2 denote a singular vector for N of maximal singular value with $\|a_2\| = 1$, and put $a_1 = Na_2$. We then put $b_i = R_i^{-1}a_i$ and $g_i = \sum_j b_{ij}v_j$. The function g_1/g_2 is then the desired approximation.

The above algorithm is implemented by the `find_p` method of the class `E_to_S_map`, which is declared in `embedded/roothalf/E_to_S.mpl`. This method must be passed an object of class `EH_atlas`, which encodes information about the points a_i^Δ and the maps $\Delta \to EX^*$ and $\Delta \to \mathbb{C}_\infty$. We have used this method to find rational approximations u_j^R where the numerator and denominator have total degree eight in z_1 and z_2. It turns out that these functions are quite tame, as shown in the graphs below.



All values lie between about 0.3 and 2.3. All coefficients in the numerators and denominators have absolute value at most one, and they appear to decrease quite rapidly with the total degree of the corresponding monomials. The full calculation can be carried out using the function `build_data["E_to_S_map"]()`, which is defined in `build_data.mpl`.

8.2. **Energy minimisation.** [`sec-energy`]

As explained in Remark 3.7.14, there is a canonical conformal map $\widehat{p}\colon EX^* \to S^2$. If we can find $\widehat{p}$, then all other information can easily be derived from that. One approach is to start with the map $\widehat{p}\colon EX^* \to S^2$

from Definition 7.5.1. This has the right equivariance properties and the right homotopy class, but it is not conformal. We can hope to adjust it by a numerical minimisation algorithm to make it conformal. For this, we need to recall the theory of Dirichlet energy.

Definition 8.2.1. Given matrices $P, Q \in M_2(\mathbb{R})$ we put

$$\langle P, Q\rangle = \sum_{i,j=1}^{2} P_{ij} Q_{ij} = \operatorname{trace}(P^T Q).$$

This is an inner product, with associated norm $\|P\|^2 = \sum_{i,j} P_{ij}^2$. We also put

$$C_+(P) = (P_{11} - P_{22})^2 + (P_{12} + P_{21})^2$$
$$C_-(P) = (P_{11} + P_{22})^2 + (P_{12} - P_{21})^2.$$

Remark 8.2.2. It is clear that $C_+(P) = 0$ iff $P = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for some $a, b \in \mathbb{R}$, or in other words P is a conformal matrix. Similarly, we have $C_-(P) = 0$ iff P is anticonformal.

Lemma 8.2.3. *For all $P \in M_2(\mathbb{R})$ we have*

$$\|P\|^2 = C_+(P) + 2\det(P) \geq 2\det(P)$$
$$\|P\|^2 = C_-(P) - 2\det(P) \geq -2\det(P),$$

so $\|P\|^2 \geq 2|\det(P)|$.

Proof. The equalities are direct calculations, and it is clear that $C_+(P) \geq 0$ and $C_-(P) \geq 0$. □

Corollary 8.2.4. [cor-invariants]
For $A, B \in SO(2)$ we have $\|APB\|^2 = \|P\|^2$ and $C_\pm(APB) = C_\pm(P)$ and $\det(APB) = \det(P)$.

Proof. We have $\det(A) = \det(B) = 1$ so $\det(APB) = \det(A)\det(P)\det(B) = \det(P)$. We also have $A^T A = B^T B = 1$ and $\operatorname{trace}(XY) = \operatorname{trace}(YX)$ so

$$\|APB\|^2 = \operatorname{trace}(B^T P^T A^T APB) = \operatorname{trace}(B^T P^T PB) = \operatorname{trace}(BB^T P^T P) = \operatorname{trace}(P^T P) = \|P\|^2.$$

We can now use $C_\pm(P) = \|P\|^2 \mp 2\det(P)$ to deduce that $C_\pm(APB) = C_\pm(P)$. □

Definition 8.2.5. Let V and W be oriented two-dimensional inner product spaces over $\mathbb{R}$, and let $\phi\colon V \to W$ be a linear map. We then choose oriented orthonormal bases v_1, v_2 for V and w_1, w_2 for W, and let P be the matrix such that

$$\phi(v_1) = P_{11} w_1 + P_{21} w_2$$
$$\phi(v_2) = P_{12} w_1 + P_{22} w_2.$$

We then put $\|\phi\|^2 = \|P\|^2$ and $\det(\phi) = \det(P)$ and $C_\pm(\phi) = C_\pm(P)$. This is independent of the choice of bases, by Corollary 8.2.4.

Definition 8.2.6. [defn-energy]
Let X and Y be connected oriented smooth closed surfaces with given Riemannian metrics. We give them the measures derived from the metric in the usual way. Consider a smooth map $f\colon X \to Y$. For each $x \in X$ we have a linear map $D_x f\colon T_x X \to T_{f(x)} Y$ between oriented two-dimensional inner product spaces, so we can define $\|D_x f\|^2$ and $\det(D_x f)$ and $C_\pm(D_x f)$. The *Dirichlet energy* of f is the integral over X of the scalar-valued function $x \mapsto \frac{1}{2}\|D_x f\|^2$. We write this as $E(f) = \int_X \frac{1}{2}\|Df\|^2$. We also define the *area* of f to be $A(f) = \int_X \det(Df)$.

Remark 8.2.7. In terms of differential forms, we can let ω_X and ω_Y denote the volume forms for X and Y, and then $f^*(\omega_Y) = \det(Df)\omega_X$. We can regard ω_X as a generator of the de Rham cohomology group $H^2(X)$, and similarly for Y. Integration gives an isomorphism from each of these cohomology groups to the reals. From this point of view it is clear that $A(f)$ depends only on the homotopy class of f. Moreover, if f is an orientation-preserving diffeomorphism then the standard change-of-variables formula shows that $A(f)$ is just the area of Y.

Proposition 8.2.8. [prop-dirichlet]
For any $f\colon X\to Y$ as above, we have $E(f)\geq A(f)$, with equality iff f is conformal.

Proof. This is clear from the identity $\|Df\|^2=2\det(Df)+C_+(Df)$. $\square$

One way to exploit this is via a discretised version. We triangulate the fundamental domain $F_{16}\subset EX^*$, then use the group action to obtain an equivariant triangulation of EX^*, with vertex set K_0 say. Taking the convex hulls of vertices of simplices gives a piecewise linear surface $X'\subset\mathbb{R}^4$ that lies close to EX^*. If we have a map $f\colon K_0\to S^2$, then we can extend it linearly to give a map $f'\colon X'\to\mathbb{R}^3$, which will usually land in $\mathbb{R}^3\setminus\{0\}$. There is an obvious retraction of $\mathbb{R}^3\setminus\{0\}$ onto S^2, and one can also construct a map from EX^* to X' that is close to the identity. After composing with these we get a map $f''\colon EX^*\to S^2$. One could attempt to minimise $E(f'')$, but that is analytically intractable. However, a slight modification of Definition 8.2.6 defines a quantity $E'(f')$ that is analogous to $E(f)$, and we can attempt to minimise that instead. We find that the rate of convergence is slow, and the resulting approximation is inaccurate close to the points $v_{10},\dots,v_{13}$ where the equivariance properties force the derivative of $\widehat{p}$ to be zero. One can improve the accuracy of the method by subdividing the triangulation, but this makes everything much slower. One could also use an approximation scheme that is better than linear interpolation, perhaps based on the various types of splines that are popular in computer graphics. However, we did not find an approach of this type that worked well for our purposes.

We could also avoid discretisation, and instead attempt to minimise the energy over some finite-dimensional space M of maps $EX^*\to S^2$. This should ideally be chosen so that it is easy to calculate $\|Df\|^2$ for $f\in M$, together with the derivatives of $\|Df\|^2$ with respect to suitable coordinates on M. We have not found a space M for which this works nicely.

9. Overview of the Maple code

[sec-maple]

9.1. **Directory structure.** The main directory for this project has subdirectories as follows:

- `latex:` LaTeXcode for this document. The subdirectory `tikz_includes` contains some files that were generated by Maple and are included in the main LaTeXdocument by `\input` commands. The code in the file `maple/plots.mpl` is relevant here.
- `images:` Image files (with extensions `.png` or `.jpg`), all generated by Maple. The code in the file `maple/plots.mpl` is relevant here.
- `plots:` Image files (with extensions `.m`) in Maple's internal format for plots. The code in the file `maple/plots.mpl` is relevant here.
- `maple:` This contains Maple code in various subdirectories, which will be described in more detail below. The files contain plain text, and have extension `.mpl`. Many files occur in pairs like `projective/ellquot.mpl` (which defines various functions related to elliptic curve quotients of $PX(a)$) an `projective/ellquot_check.mpl` (which defines procedures to check various assertions about those functions).
- `doc:` This contains various kinds of documentation of the Maple code, in HTML format. Some of the Maple code is object oriented (as will be discussed in Section 9.3 below) and some is not. For the object oriented code, there is automatically generated documentation of classes, fields and methods, similar to the standard javadoc framework for Java code. For the remaining code, there is an index of definitions of all defined symbols, with links to the defining files.
- `worksheets:` This contains Maple worksheets, with extension `.mw`. They all start with the following block:

```
restart;
interface(quiet=true):
olddir := currentdir("../maple"):
read("genus2.mpl"):
currentdir(olddir):
interface(quiet=false):
```

Executing this block will read in the file genus2.mpl, which will in turn read in many other files from the maple directory and its subdirectories. We have mostly used worksheets for development, and have moved code to the .mpl files when it has become stable.

- data: This contains files generated by Maple recording the results of certain complex calculations. There is a hierarchy of subdirectories parallel to those in the maple directory. Some are plain text files with extension .mpl, but most are in Maple's internal format and have extension .m. See Section 9.4 for information about how to recalculate these results.

The subdirectories of the maple directory are as follows:

- The top level directory contains some code about the general theory of precromulent surfaces (not tied to any of the three families), and some general utility code.
- projective: for code related to the projective family.
- hyperbolic: for code related to the hyperbolic family, and code for isomorphisms between hyperbolic and projective surfaces.
- embedded: for code related to the embedded family.
- embedded/roothalf: for code related to the special case $EX^* = EX(1/\sqrt{2})$.
- domain: for object oriented code dealing with triangulations of cromulent surfaces; this can be specialised to each of the three families. (At an earlier stage, we planned to do various substantial calculations using triangulations, but we eventually switched to different methods.)
- quadrature: object oriented code for quadrature rules on triangles.

9.2. **Checks.** [sec-checks]

Maple code that checks the correctness of various assertions is contained in files whose names end with _check. Unlike the other Maple files, these are not loaded automatically by the standard block at the top of the worksheets. One can read an individual file by entering a command like

```
read("../maple/projective/ellquot_check.mpl"):
```

Alternatively, one can enter

```
read("../maple/check_all.mpl"):
```

to load and run all possible checks (which takes a long time). The global variable checklist contains a list of all the checking functions that have been loaded. One can execute all of them by invoking the check_all() function. These functions will usually stop running if they encounter an assertion that fails, but this can be prevented by setting the global variable assert_stop_on_fail to false. Each checking function will print its name when it starts to run. Most functions will check a large number of assertions; if the global variable assert_verbosely is set to true, then a brief identifier will be printed for each assertion. (This happens after Maple has done the work of checking the assertion, but before it prints an error message if the assertion has failed.) The general framework for all this is set up by the files util.mpl and checks.mpl in the top Maple directory. The basic claim that the embedded, projective and hyperbolic families are precromulent has some special features; see the function check_precromulent() and associated comments in the file cromulent.mpl.

One might ask about the reliability, rigour and completeness of these checks.

First, we should explain that almost all checks cover assertions that are claimed to be exact. For the parts of the code that involve numerical approximation, we have also performed many checks, but we have not encapsulated them in a systematic framework.

(a) Some claims are of the form $u = 0$, where u is a constant expression. Usually u will be built from rational numbers by algebraic operations and by extraction of square roots of positive quantities. In some cases we evaluate trigonometric functions and rational multiples of π, and there are a few examples involving roots of polynomials of degree greater than two. In many cases, we can just use the command simplify(u) and the result will be zero. In some cases we need to use a more complicated command like

```
simplify(factor(expand(rationalize(u))))
```

We have been willing to assume that if a procedure like this returns zero, then u is genuinely equal to zero. One can check this by numerical evaluation of u. We have used 100 digit precision by default, and have not found any examples where the symbolic simplification functions seem to be incorrect.

(b) Some other claims are of the form $u = 0$, where u is an expression involving several constants and variables, which may be subject to certain constraints. The constants are of the type discussed in (a). Variables may be constrained to be real (using a command like `assume(t::real)`) or to lie in the unit interval (using a command like `assume(a_H>0 and a_H<1)`). Most expressions are built using algebraic operations, square roots and logarithms of quantities that can be shown to be positive, trigonometric functions and exponentials, and extraction of real parts of complex numbers. There are also some derivatives and integrals. There are obvious algorithms to deal with most of these things, and it seems unlikely that any bugs would have escaped detection. However, there are some expressions where Maple implicitly uses a significant amount of logic to determine that various terms are positive, and uses this to justify manipulations with roots and logarithms. We do not know what algorithms are used for this, but we have not detected any problems. All the relevant symbolic simplifications can be tested by numerical evaluation, either by plotting or by setting parameters to randomly chosen values.

(c) There is another kind of constraint that we did not mention under (b): variables can be subject to certain polynomial relations, which can be encoded using a Gröbner basis for the corresponding ideal. We only have examples where the polynomials have coefficients in $\mathbb{Q}(\sqrt{2})$, which is easy to handle. Often, we only need to check expressions of the form $u = 0$, where u is a rational function of the constrained variables. There are very standard algorithms for working with Gröbner bases, and it is highly unlikely that there could be any problems with expressions of this type. It is also common to have expressions u that involve square roots of polynomials that can be shown to be positive, and the roots are sometimes nested. The algorithms for this case are not quite as standard, but again we have detected no problems.

(d) A few expressions involve more sophisticated functions such as elliptic integrals and the Weierstrass $\wp$ function. It is here that we encountered the only significant bug that we have seen: Maple's numerical evaluation of $\wp'(z)$ is incorrect for certain ranges of arguments. On the other hand, when we first started working with $\wp'(z)$, it immediately became clear that something was causing inconsistent results, although it took time to locate the precise source of trouble. This raised our confidence that other bugs would also quickly become visible.

(e) There are a few cases where we have an indirect reason to know that an expression u should be zero, but we have not been able to persuade Maple to simplify u to zero. In these cases, we have written checking functions that rely completely on numerical methods, by evaluating u to 100 decimal places at various points in the parameter space. Alternatively, if u is a function on the surface EX^*, we can evaluate u exactly at all the quasirational points of EX^* (as discussed in Section 7.6).

(f) As well as the kinds of claims discussed in (a) to (e), we have various claims about more combinatorial structures, such as the groups G, Π and $\widetilde{\Pi}$. For these we have mostly written our own code, both to implement the definitions and to check the claimed properties. Thus, very little is hidden in the internals of Maple, and the sceptical reader can inspect all the relevant code.

9.3. **Object oriented Maple.** [sec-oo-maple]

For some of our work, it is natural to use an object oriented style of programming. For example, it is natural to have a class whose objects represent conformal charts on EX^*, and another class for atlases, and a class for quadrature rules, and so on. We have described a complex algorithm for calculating the canonical covering $\Delta \to EX^*$, and it is natural to implement the steps in this algorithm as methods of various classes. Maple does not natively support object oriented programming, but we have implemented our own framework using Maple's system of tables with user-defined indexing functions. Our framework was in fact developed some years ago for a rather different project, and adapted slightly for our current purposes. The relevant code is in the file `class.mpl` in the top Maple directory. Typical syntax is as follows:

- `` Q := `new/E_quadrature_rule`(); `` sets Q to be a new quadrature rule on $F_{16} \subset EX^*$.
- `Q["int_z",z[1]];` given a quadrature rule Q, returns the estimated integral of z_1 over F_{16}.
- `Q["curvature_error"];` given a quadrature rule Q, returns the difference between the estimated integral of the curvature and the correct value of $-\pi/4$.
- `A["num_charts"];` given an atlas A on EX^*, returns the number of charts.

Classes are declared using the `` `Class/Declare` `` function.

There is one notable place where we have chosen not to use the above framework. We have a lot of parallel structures for our three families of cromulent surfaces, for example the functions `c_E[k](t)`, `c_H[k](t)` and `c_P[k](t)` which encode the three curve systems. In some respects it would be natural to encapsulate these using a system of classes. However, that would lead to unwieldy notation for objects that we need to use extremely frequently, so we chose to avoid it.

9.4. **Building the data.** [sec-build]

This project involves some numerical computations, the results of which are stored in the `data` directory and its subdirectories. The file `build_data.mpl` (in the top Maple directory) defines various functions that can be used to perform these calculations. For example, the function `build_data["HP_table"]()` can be used to perform all the calculations described in Section 5, relating the projective and hyperbolic families. The result is encoded as an object of the class `HP_table` (declared in `hyperbolic/HP_table.mpl`). It can be saved in an appropriate place using the function `save_data["HP_table"]()`, and then reloaded later using the function `load_data["HP_table"]()`.

The functions `build_data["all"]()`, `save_data["all"]()` and `load_data["all"]()` work in the obvious way, building, saving or loading all of the required data. A full build of all data will take several days of computer time, at least. However, one can enter `set_toy_version(true)` before invoking `build_data["all"]`. This will cause Maple to do all calculations to lower accuracy, and finish in an hour or two. In this context, results will be saved to or loaded from the `data_toy` directory instead of the `data` directory.

The build process will generate a fairly large number of messages about the progress of the calculation. One can reduce the volume by setting `infolevel[genus2]` to a number less than the default value of 7, before invoking `build_data["all"]`.

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