ALL ISOMETRIES ARE AFFINE

N. P. STRICKLAND

Definition 1. An *isometry* of $\mathbb{R}^n$ is a bijective map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that preserves distances, ie $d(f(x), f(y)) = d(x, y)$ for all $x, y \in \mathbb{R}^n$. We write I_n for the group of all such isometries. We also write $FI_n := \{f \in I_n \mid f(0) = 0\}$.

It is clear that any orthogonal matrix $A \in O_n$ gives rise to an isometry $f_A\colon \mathbb{R}^n \to \mathbb{R}^n$ by $f_A(x) = Ax$. We will blur the distinction between A and f_A and thus write $O_n \subseteq I_n$. In fact $f_A(0) = 0$ so $O_n \subseteq FI_n$. Moreover, Proposition **??** tells us that $GL_n \cap I_n = O_n$.

For an example of an isometry not in O_n, let a be a point in $\mathbb{R}^n$ and define $T_a x = x + a$. This is clearly a bijection (with $T_a^{-1} = T_{-a}$) and

$$d(T_a x, T_a y) = \|(x + a) - (y + a)\| = \|x - y\| = d(x, y),$$

so it is an isometry. Isometries of this form are called *translations*. As $T_a T_b = T_{a+b}$, we see that the translations form an Abelian subgroup of I_n, which we denote by T_n. The assignment $a \mapsto T_a$ gives an isomorphism $\mathbb{R}^n \simeq T_n$. If $a \neq 0$ then $T_a^n = T_{na} \neq 1$ for all $n \neq 0$, so T_a has infinite order. Moreover, we have $T_a x \neq x$ for all x, so T_a has no fixed points.

Theorem 2. *Any distance preserving map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has the form $f(x) = Ax + b$ for some $A \in O_n$ and $b \in \mathbb{R}^n$. It is thus automatically bijective.*

The proof will be given after a number of preparatory lemmas.

Lemma 3. *For any $x, y \in \mathbb{R}^n$, the point $z = (x + y)/2$ is the unique point such that $d(x, z) = d(y, z) = d(x, y)/2$.*

Proof. It is trivial to check that $d(x, z) = d(y, z) = d(x, y)/2$, and it should be geometrically clear (at least in dimensions ≤ 3) that it is the unique point with this property. For a formal proof, suppose we also have $d(x, w) = d(y, w) = d(x, y)/2$. Put $x' = x - z = (x - y)/2$ and $y' = y - z = (y - x)/2 = -x'$ and $w' = w - z$ and $r = d(x, y)$. Then $\|x'\| = \|y'\| = r/2$, and also $\|x' - w'\| = \|y' - w'\| = r/2$. Thus $\langle x' - w', x' - w'\rangle = r^2/4 = \langle x', x'\rangle$, and by expanding this out we see that $\|w'\|^2 = 2\langle x', w'\rangle$. Similarly, we have $\|w'\|^2 = 2\langle y', w'\rangle$. By adding these equations together and using the fact that $y' = -x'$ we see that $\|w'\|^2 = 0$, so $w' = 0$, so $w = z$. ☐

Lemma 4. *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves distances then $f((x + y)/2) = (f(x) + f(y))/2$.*

Proof. Put $z = (x+y)/2$ and $z' = (f(x)+f(y))/2$ and $r = d(x, y)$. We then have $d(f(x), f(z)) = d(x, z) = r/2$ and $d(f(y), f(z)) = d(y, z) = r/2$ and $d(f(x), f(y)) = d(x, y) = r$. However, z' is the *unique* point with $d(f(x), z') = d(f(y), z') = d(f(x), f(y))/2$, so we must have $f(z) = z'$. In other words, $f((x + y)/2) = (f(x) + f(y))/2$, as claimed. ☐

Lemma 5. *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves distances and $f(0) = 0$ then $\langle f(x), f(y)\rangle = \langle x, y\rangle$.*

Proof. First, we have $f(0) = 0$ and thus

$$\|x\| = d(x, 0) = d(f(x), f(0)) = d(f(x), 0) = \|f(x)\|$$

for all $x \in \mathbb{R}^n$.

Next, we can take $y = 0$ in the previous lemma to see that $f(x/2) = f(x)/2$ for all $x \in \mathbb{R}^n$. We can then feed this back into the previous lemma to see that $f((x + y)/2) = f(x + y)/2$ and thus that

Date: February 6, 2020.

$f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$. If we take the inner product of this equation with itself, we find that

$$\|f(x+y)\|^2 = \|f(x)\|^2 + \|f(y)\|^2 + 2\langle f(x), f(y)\rangle.$$

Thus, we have

$$\begin{aligned} 2\langle f(x), f(y)\rangle &= \|f(x+y)\|^2 - \|f(x)\|^2 - \|f(y)\|^2 \\ &= \|x+y\|^2 - \|x\|^2 - \|y\|^2 \\ &= \langle x+y, x+y\rangle - \langle x, x\rangle - \langle y, y\rangle \\ &= 2\langle x, y\rangle, \end{aligned}$$

as required. □

Corollary 6. *If $f \colon \mathbb{R}^n \to \mathbb{R}^n$ preserves distances and $f(0) = 0$ then $f \in O_n$.*

Proof. Let $e_1, \dots, e_n$ be the usual basis of $\mathbb{R}^n$, and put $u_k = f(e_k)$. As f preserves inner products, we see that $\langle u_i, u_j\rangle = 0$ unless $i = j$, and $\langle u_i, u_i\rangle = 1$. Define a linear map $g \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x_1, \dots, x_n) = x_1 u_1 + \dots + x_n u_n.$$

We then have

$$\langle g(x), g(y)\rangle = \sum_{i,j} x_i y_j \langle u_i, u_j\rangle = \sum_i x_i y_i = \langle x, y\rangle,$$

so $g \in O_n$. I claim that $f = g$. To see this, consider the map $h = g^{-1} f$, which again preserves distances and sends 0 to 0, and thus preserves inner products. By construction, we have $h(e_k) = e_k$ for all k. As h preserves inner products, we have

$$\begin{aligned} x_k &= \langle x, e_k\rangle \\ &= \langle h(x), h(e_k)\rangle \\ &= \langle h(x), e_k\rangle \\ &= h(x)_k, \end{aligned}$$

so $x = h(x)$ for all $x \in \mathbb{R}^n$. Thus h is the identity, and $f = g$ as claimed. □

Proof of Theorem 2. Suppose that $f \colon \mathbb{R}^n \to \mathbb{R}^n$ preserves distances. Define $b = f(0)$ and $g(x) = f(x) - b$. Clearly g preserves distances and $g(0) = 0$. By the above Corollary, there is an orthogonal matrix A such that $g(x) = Ax$ for all x, so $f(x) = Ax + b$ for all x s claimed. It follows that f is a bijection, with $f^{-1}(y) = A^{-1}(y - b)$. □

We conclude this section by giving a simple criterion for when an isometry is the identity.

Definition 7. A list $u_0, \dots, u_n$ of $n+1$ points in $\mathbb{R}^n$ is in *general position* if the vectors $u_1 - u_0, \dots, u_n - u_0$ form a basis of $\mathbb{R}^n$.

Proposition 8. *If $u_0, \dots, u_n$ are in general position, $f \in I_n$ and $f(u_i) = u_i$ for all i, then $f = 1$.*

Proof. We have $f(x) = Ax + b$ for some A, b. It follows that

$$A(u_i - u_0) = (Au_i + b) - (Au_0 + b) = f(u_i) - f(u_0) = u_i - u_0$$

for all i. As the vectors $u_i - u_0$ form a basis, we deduce that $A = I$, so $f(x) = x + b$ for all x. In particular, $u_0 = f(u_0) = u_0 + b$, so $b = 0$. Thus $f(x) = x$ for all x as claimed. □