A proof of the existence of the regular icosahedron.

Let $\lambda > 1$ be a real number, and consider the set X consisting of the 12 vectors

$$(0, \pm 1, \pm\lambda), \qquad (\pm 1, \pm\lambda, 0), \qquad (\pm\lambda, 0, \pm 1).$$

Let G be the group of symmetries of the set X, or equivalently of its convex hull in $\mathbb{R}^3$. For any choice of λ, show that 24 divides $|G|$, and show that G acts transitively on X. (Hint: find a subgroup of G of order 8 and another subgroup of G of order 3.)

Find all possible dot products $\mathbf{v}.\mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in X$. (Hint: since you have already shown that G acts transitively on X, it suffices to fix one particular choice of $\mathbf{v}$.) Check that for each $\mathbf{v} \in X$, there are five distinct vectors $\mathbf{w} \in X - \{\mathbf{v}\}$ such that $\mathbf{v}.\mathbf{w} > 0$. Call these 5 $\mathbf{w}$ the set $N_{\mathbf{v}}$ of neighbours of $\mathbf{v}$. Find a value for λ for which all of the products $\mathbf{v}.\mathbf{w}$ for $\mathbf{w} \in N_{\mathbf{v}}$ are equal. For this value of λ, show that the elements of $N_{\mathbf{v}}$ form a regular pentagon lying in a plane orthogonal to $\mathbf{v}$, whose centre lies on the line $t\mathbf{v}$. Deduce that for this particular value of λ, the group G contains an element of order 5, and hence $|G| \geq 120$, the order of the group of symmetries of the regular icosahedron.