Commutative rings and nilpotents

- A commutative ring is a set R equipped with elements $0, 1$ and rules for addition, multiplication, satisfying all the usual rules including $ab = ba$.
- Examples: integers $\mathbb{Z}$, real numbers $\mathbb{R}$, modular numbers $\mathbb{Z}/n$, polynomials $\mathbb{Z}[x]$. Not $\mathbb{N}$ (no subtraction) or $M_n(\mathbb{R})$ (matrices; $ab \neq ba$).
- An element $x \in R$ is *nilpotent* if $x^{n+1} = 0$ for some $n \geq 0$. Example: 20^3 is divisible by 1000, so 20 is nilpotent in $\mathbb{Z}/1000$. We define $\mathrm{Nil}(R)$ to be the set of nilpotent elements, so $\mathrm{Nil}(\mathbb{Z}/1000) = \{0, 10, 20, \dots, 990\}$.
- If x and y are nilpotent then so is $x + y$. Example: if $x^3 = y^4 = 0$ then

$$\begin{aligned}(x+y)^6 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \\ &= (x^3 + 6x^2y + 15xy^2 + 20y^3)x^3 + (15x^2 + 6xy + y^2)y^4 \\ &= (\cdots)0 + (\cdots)0 = 0\end{aligned}$$

- Easier facts: 0 is nilpotent, and if x is nilpotent, then so is xy.
- An *ideal* is a subset $I \subseteq R$ such that (i) $0 \in I$ and (ii) when $x, y \in I$ we have $x + y \in I$ and (iii) when $x \in I$ and $y \in R$ we have $xy \in I$. Easy example: the even numbers form an ideal in $\mathbb{Z}$.
- The above facts show that $\mathrm{Nil}(R)$ is an ideal in R.