SOME PROOFS FOR LYAPUNOV THEORY

We first recall some background. Convergence of real numbers is defined as follows:

Definition 1. A sequence (x_n) of real numbers converges to a real number a if for all $\epsilon > 0$, there is a natural number N such that $|x_n - a| < \epsilon$ for all $n \geq N$.

We need to generalise this to cover sequences in $\mathbb{R}^2$ rather than $\mathbb{R}$.

Definition 2. Suppose we have a sequence of points $(x_n, y_n) \in \mathbb{R}^2$, and another point $(a, b) \in \mathbb{R}^2$. We say that the sequence converges to (a, b) if for all $\epsilon > 0$ there exists N such that $\|(x_n, y_n) - (a, b)\| < \epsilon$ for all $n \geq N$.

This can easily be related to convergence in $\mathbb{R}$:

Lemma 3. *Suppose we have a sequence of points $(x_n, y_n) \in \mathbb{R}^2$, and another point $(a, b) \in \mathbb{R}^2$. Then (x_n, y_n) converges to (a, b) if and only if x_n converges to a and y_n converges to b.*

Proof. Suppose that $(x_n, y_n) \to (a, b)$. Given $\epsilon > 0$, there exists N such that when $n \geq N$ we have $\|(x_n, y_n) - (a, b)\| < \epsilon$. However, we also have

$$|x_n - a| \leq \sqrt{|x_n - a|^2 + |y_n - b|^2} = \|(x_n, y_n) - (a, b)\|,$$

so $|x_n - a| < \epsilon$. This proves that $x_n \to a$, and similarly $y_n \to b$.

Conversely, suppose that $x_n \to a$ and $y_n \to b$. Given $\epsilon > 0$ we can choose L such that $|x_n - a| < \epsilon/\sqrt{2}$ for $n \geq L$, and we can choose M such that $|y_n - b| < \epsilon/\sqrt{2}$ for $n \geq M$. Put $N = \max(L, M)$. For $n \geq N$ we have $|x_n - a| < \epsilon/\sqrt{2}$ and $|y_n - b| < \epsilon/\sqrt{2}$, so

$$\|(x_n, y_n) - (a, b)\| = \sqrt{|x_n - a|^2 + |y_n - b|^2} < \sqrt{\epsilon^2/2 + \epsilon^2/2} = \epsilon,$$

as required. $\square$

The following result is standard:

Theorem 4 (Bolzano-Weierstrass)**.** *Let (x_n) be a sequence of real numbers, and suppose that there is a constant C such that $|x_n| \leq C$ for all n. Then there is a subsequence (x_{n_k}) (with $n_0 < n_1 < n_2 < \cdots$) and a real number a such that $x_{n_k} \to a$.*

We will need the two-dimensional version:

Corollary 5. *Let (x_n, y_n) be a sequence in $\mathbb{R}^2$, and suppose that there is a constant C such that $\|(x_n, y_n)\| \leq C$ for all n. Then there is a subsequence (x_{n_k}, y_{n_k}) (with $n_0 < n_1 < n_2 < \cdots$) and a point $(a, b) \in \mathbb{R}^2$ such that $(x_{n_k}, y_{n_k}) \to (a, b)$.*

Proof. By the original Bolzano-Weierstrass Theorem, we can find a real number a and an increasing sequence of indices p_k such that $x_{p_k} \to a$. Now the sequence y_{p_k} is again bounded by C, so we can find a real number b and an increasing sequence of indices q_j such that $y_{p_{q_j}} \to b$. In other words, if we put $n_j = p_{q_j}$ then $y_{n_j} \to b$. Moreover, the sequence (x_{n_j}) is a subsequence of (x_{m_i}), and $x_{m_i} \to a$ so $x_{n_j} \to a$. It follows that $(x_{n_j}, y_{n_j}) \to (a, b)$ as required. $\square$

Definition 6. Let F be a subset of $\mathbb{R}^2$. We say that F is *closed* if whenever (x_n, y_n) is a sequence in F that converges to a point $(a, b) \in \mathbb{R}^2$, we also have $(a, b) \in F$.

Example 7. Consider the sets

$$F_0 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$
$$F_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Then F_0 is closed but F_1 is not. For example, the sequence $(x_n, y_n) = (2^{-n}, 0)$ lies in F_1 and converges to the point $(a, b) = (1, 0)$, which does not lie in F_1; so F_1 is not closed. In general, sets defined using $=$ and $\leq$ will usually be closed, but sets defined using $<$ will often not be closed. However, this rule is not always reliable, so you should work from the official definition.

Definition 8. Given a point $u \in \mathbb{R}^2$, we define

$$B_\epsilon(u) = \{v \in \mathbb{R}^2 \mid \|u - v\| < \epsilon\}.$$

Definition 9. Let U be a subset of $\mathbb{R}^2$. We say that U is *open* if for every $u \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(u) \subseteq U$.

Example 10. Consider the sets

$$\begin{aligned}
U_0 &= \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\} \\
U_1 &= \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y \geq 0\} \\
U_2 &= \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}.
\end{aligned}$$

Then U_0 is open, but U_1 and U_2 are not. Indeed, the point $u = (1, 0)$ is an element of U_1, but for every $\epsilon > 0$ we have a point $v = (1, -\epsilon/2)$ which lies in $B_\epsilon(u)$ but not in U_1, so $B_\epsilon(u) \not\subseteq U_1$. This proves that U_1 is not open, and the same example also shows that U_2 is not open. In general, sets defined using $\neq$ and $<$ are usually open, but sets defined using $\leq$ will often not be open. However, this rule is not always reliable, so you should work from the official definition.

Remark 11. Most subsets of $\mathbb{R}^2$ are neither open nor closed. For example, the set U_1 described above is not open, and also it is not closed.

Proposition 12. *Let F and U be two subsets of $\mathbb{R}^2$ that are complements of each other, so that*

$$\begin{aligned}
U &= \{p \in \mathbb{R}^2 \mid p \notin F\} \\
F &= \{p \in \mathbb{R}^2 \mid p \notin U\}.
\end{aligned}$$

Then F is closed if and only if U is open.

Proof. We will prove that F is not closed if and only if U is not open. A little thought will convince you that this is equivalent to the claimed statement.

Suppose that F is not closed. This means that there is a sequence of points $v_n = (x_n, y_n) \in F$ and a point $u = (a, b) \notin F$ such that $v_n \to u$. As $u \notin F$ we instead have $u \in U$. Consider a number $\epsilon > 0$. As $v_n \to u$, there exists N such that $\|v_N - u\| < \epsilon$, or in other words $v_N \in B_\epsilon(u)$. Now $v_N \in F$, so $v_N \notin U$, so $B_\epsilon(u) \not\subseteq U$. As this holds for all $\epsilon > 0$, we see that U is not open.

Conversely, suppose that U is not open. This means that there is a point $u \in U$ such that there is no $\epsilon > 0$ with $B_\epsilon(u) \subseteq U$. In particular, $B_{1/n}(u)$ is not contained in U, so we can choose a point $v_n \in B_{1/n}(u)$ such that $v_n \notin U$, or equivalently $v_n \in F$. As $v_n \in B_{1/n}(u)$ we have $\|v_n - u\| < 1/n$, and this implies that $v_n \to u$. Now the points v_n form a sequence in F, which converges to the point u, which does not lie in F. This means that F is not closed. $\square$

We now consider a differential equation and its solutions. Let f and g be continuously differentiable functions on $\mathbb{R}^2$. For any point $(x_0, y_0) \in \mathbb{R}^2$, we would like to consider maps $x, y \colon \mathbb{R} \to \mathbb{R}^2$ such that $x(0) = x_0$ and $y(0) = y_0$ and

$$\begin{aligned}
\dot{x}(t) &= f(x(t), y(t)) \\
\dot{y}(t) &= g(x(t), y(t))
\end{aligned}$$

for all $t \in \mathbb{R}$. Unfortunately, this is not always possible.

Example 13. Consider the equations $\dot{x} = x^2$ and $\dot{y} = 0$ with $x = y = 1$ at $t = 0$. There is a solution $(x, y) = (1/(1 - t), 1)$, but this becomes undefined at $t = 1$. There is no solution that is defined and differentiable on all of $\mathbb{R}$.

However, the above example is essentially the worst thing that can happen, as shown by the following result.

Theorem 14. *For any point $(x_0, y_0) \in \mathbb{R}^2$, there is a unique pair of continuously differentiable functions $x, y\colon (\alpha, \beta) \to \mathbb{R}$ such that*

- *α is either $-\infty$ or a negative real number.*
- *β is either $+\infty$ or a positive real number.*
- *$x(0) = x_0$ and $y(0) = y_0$*
- *For all $t \in (\alpha, \beta)$ we have $\dot{x}(t) = f(x(t), y(t))$ and $\dot{y}(t) = g(x(t), y(t))$.*
- *Either $\alpha = -\infty$, or α is finite and $\|(x(t), y(t))\| \to \infty$ as $t \to \alpha$.*
- *Either $\beta = +\infty$, or β is finite and $\|(x(t), y(t))\| \to \infty$ as $t \to \beta$.*

This is a very important theorem, which we have implicitly been using throughout the course, but we will not prove it. A proof can be found in the recommended book by Teschl.

Assumption 15. For simplicity, we will consider only systems for which α is always $-\infty$ and β is always $+\infty$, so the solutions are defined for all $t \in \mathbb{R}$.

Definition 16. We put $\phi(t, (x_0, y_0)) = (x(t), y(t))$, where $x(t)$ and $y(t)$ give the unique solution as discussed above.

Theorem 17. *The function ϕ is continuously differentiable as a map from $\mathbb{R}^3$ to $\mathbb{R}^2$.*

Again, a proof can be found in the book by Teschl.

Assumption 18. From now on, we make the following additional assumptions:

(a) R is an open subset of $\mathbb{R}^2$.
(b) $e = (a, b)$ is a point in R where $f(e) = g(e) = 0$.
(c) V is a continuously differentiable function on R.
(d) $V(e) = 0$, and $V(u) > 0$ for all other points $u \in R$. (In other words, V is positive definite.)
(e) The function $W = V_x f + V_y g$ satisfies $W(e) = 0$, and $W(u) < 0$ for all other points $u \in R$. (In other words, W is negative definite.)

This means that V is a *strong Lyapunov function* for R and e.

Remark 19. Suppose we have a solution $u(t) = (x(t), y(t))$ to our equations. We then have

$$\begin{aligned} \frac{d}{dt}V(x(t), y(t)) &= V_x(x(t), y(t))\dot{x}(t) + V_y(x(t), y(t))\dot{y}(t) \\ &= V_x(x(t), y(t))f(x(t), y(t)) + V_y(x(t), y(t))g(x(t), y(t)) \\ &= W(x(t), y(t)) \leq 0, \end{aligned}$$

so $V(x(t), y(t))$ is a weakly decreasing function of t. However, this is only valid as long as the curve $u(t) = (x(t), y(t))$ stays inside R. Suppose that $t_0 < t_1 < t_2$ and $u(t_0), u(t_2) \in R$ but $u(t_1) \notin R$. Then the above argument is not valid when $t = t_1$, because $V(u(t_1))$ and $W(u(t_1))$ are undefined, so we cannot conclude that $u(t_2) \leq u(t_0)$.

Definition 20. The point e is *asymptotically stable* if for all $\epsilon > 0$ there exists $\epsilon' > 0$ such that whenever $\|u - e\| < \epsilon'$, we have $\|\phi(t, u) - e\| < \epsilon$ for all $t \geq 0$, and also $\phi(t, u) \to e$ as $t \to \infty$.

We will show that whenever there is a Lyapunov function as described above, the point e is automatically asymptotically stable.

Definition 21. We choose a number $\epsilon_0 > 0$ such that $B_{\epsilon_0}(e) \subseteq R$. (This is always possible, because R is assumed to be an open set containing e.) When $0 < \epsilon < \epsilon_0$, we put

$$\begin{aligned} U_\epsilon &= B_\epsilon(e) = \{u \in \mathbb{R}^2 \mid \|u - e\| < \epsilon\} \\ \overline{U}_\epsilon &= \{u \in \mathbb{R}^2 \mid \|u - e\| \leq \epsilon\}, \end{aligned}$$

so U_ϵ is open, $\overline{U}_\epsilon$ is closed, and $e \in U_\epsilon \subseteq \overline{U}_\epsilon \subseteq R$.

Definition 22. Given $\delta > 0$, we put $S^*_\delta = \{u \in R \mid V(u) \leq \delta\}$. This contains e, and it may split up into several different components. Suppose we have two points $u, v \in S^*_\delta$. We say that they are *connected in* S^*_δ if there is a continuous map $\gamma\colon [0,1] \to S^*_\delta$ with $\gamma(0) = u$ and $\gamma(1) = v$. We put

$$S_\delta = \{u \in S^*_\delta \mid u \text{ is connected to } e \text{ in } S^*_\delta\}.$$

Note that $e \in S_\delta \subseteq S^*_\delta \subseteq R$.

Example 23. We will show later that for each $\delta > 0$ there exists $\epsilon > 0$ such that $U_\epsilon \subseteq S^*_\delta$. Every point $u \in U_\epsilon$ can be connected to e by the path $\gamma(t) = u + t(e - u)$, which goes in a straight line from u to e. This path stays inside U_ϵ, and $U_\epsilon \subseteq S^*_\delta$, so γ stays inside S^*_δ. This shows that u actually lies in S_δ, not just S^*_δ. We conclude that $U_\epsilon \subseteq S_\delta$.

Example 24. Suppose that $u \in S^*_\delta \subset R$, and that $v = \phi(t, u)$ for some $t > 0$. Suppose also that $\phi(s, u) \in R$ for all s with $0 \leq s \leq t$. In other words, there is a flow line running from u to v, and it stays in R. By Remark 19, we have $V(v) \leq V(u) \leq \delta$, so $v \in S^*_\delta$.

Now suppose that u is in S_δ, so there is a path γ from e to u in S^*_δ. The flow line gives a path from u to v in S^*_δ, and we can join these two paths together to get a path from e to v in S^*_δ. This shows that v actually lies in S_δ.

Proposition 25. *For all $\delta > 0$, there exists ϵ with $0 < \epsilon < \epsilon_0$ such that $U_\epsilon \subseteq S^*_\delta$.*

Proof. We will suppose that there is no such ϵ, and deduce a contradiction.

Let n_0 be a large integer, big enough that $1/n_0 < \epsilon_0$. By assumption, for $n \geq n_0$ we have $U_{1/n} \not\subseteq S^*_\delta$. We can therefore choose $u_n \in U_{1/n} \subseteq R$ with $u_n \notin S^*_\delta$. As $u_n \notin S^*_\delta$, we must have $V(u_n) > \delta$. However, as $u_n \in U_{1/n}$ we have $\|u - e\| < 1/n$, so $u_n \to e$. As V is continuous on R, it follows that $V(u_n) \to V(e) = 0$. As $V(u_n) > \delta$ for all n, this is impossible. We therefore have a contradiction, so the Proposition must be true after all. $\square$

Corollary 26. *For all $\delta > 0$, there exists ϵ with $0 < \epsilon < \epsilon_0$ such that $U_\epsilon \subseteq S_\delta$.*

Proof. This follows from the Proposition by Example 23. $\square$

Proposition 27. *For all ϵ with $0 < \epsilon < \epsilon_0$, there exists $\delta > 0$ such that $S_\delta \subseteq U_\epsilon$.*

Proof. We will suppose that there is no such δ, and deduce a contradiction.

By assumption, for all $n > 0$, the set $S_{1/n}$ is not contained in U_ϵ. We can therefore choose a point $u_n \in S_{1/n}$ such that $u_n \notin U_\epsilon$. As $u_n \notin U_\epsilon$, we have $\|u - e\| \geq \epsilon$. As $u_n \in S_\delta$, we can choose a continuous map $\gamma\colon [0,1] \to S^*_\delta$ with $\gamma(0) = e$ and $\gamma(1) = u_n$. Now $\|\gamma(t) - e\|$ is a continuous function of t that is 0 when $t = 0$, and is greater than or equal to ϵ when $t = 1$. By the Intermediate Value Theorem, there exists $t_0 \in [0,1]$ with $\|\gamma(t_0) - e\| = \epsilon$. We put $v_n = \gamma(t_0)$, so $\|v_n - e\| = \epsilon$. Moreover, the path γ joins e to v_n in S^*_δ, so $v_n \in S_\delta$. Because $\|v_n - e\| = \epsilon$, we have $\|v_n\| \leq \|e\| + \epsilon$ for all n, so the sequence (v_n) is bounded. Thus, the Bolzano-Weierstrass theorem says that we can find a subsequence (v_{n_k}) and a point $w \in \mathbb{R}^2$ such that $v_{n_k} \to w$. It follows that $\|v_{n_k} - e\| \to \|w - e\|$, but $\|v_{n_k} - e\| = \epsilon$, so $\|w - e\| = \epsilon$. In particular, we have $\|w - e\| < \epsilon_0$, so $w \in U_{\epsilon_0} \subseteq R$. As V is continuous on R, it follows that $V(v_{n_k}) \to V(w)$. However, $v_n \in S_{1/n}$, so $0 \leq V(v_n) \leq 1/n$, so $V(v_n) \to 0$. This means that $w \in R$ with $V(w) = 0$, but V is positive definite, so w must be equal to e. This means that $v_{n_k} \to e$, but that is impossible because $\|v_{n_k} - e\| = \epsilon > 0$ for all k.

Thus, the Proposition must be true after all. $\square$

Proposition 28. *Suppose that $0 < \epsilon < \epsilon_0$, and that $S_\delta \subseteq U_\epsilon$. Then any flow line that starts in S_δ stays in S_δ for all t.*

Proof. Consider a flow line $u(t) = (x(t), y(t))$ such that $u(0) \in S_\delta$. Put

$$t_0 = \sup\{t \mid u([0,t]) \subseteq S_\delta\}.$$

We want to prove that $t_0 = \infty$. We will suppose that $t_0 < \infty$, and deduce a contradiction. For $0 \leq t < t_0$ we have $u(t) \in S_\delta \subseteq U_\epsilon$, so $\|u(t) - e\| < \epsilon$. It follows that $\|u(t_0) - e\| \leq \epsilon < \epsilon_0$. By continuity, there exists $t_1 > t_0$ such that $\|u(t) - e\| < \epsilon_0$ for $0 \leq t \leq t_1$. This means that the flow line stays within $U_{\epsilon_0} \subseteq R$ from $t = 0$ to $t = t_1$. Thus, Remark 19 tells us that $V(u(t)) \leq V(u(0)) \leq \delta$ for $0 \leq t \leq t_1$, so these points $u(t)$

all lie in S_δ^*. Moreover, all these points are connected together in S_δ^* by the flow line, so they lie in S_δ by Example 24. Thus, $u([0,t_1]) \subseteq S_\delta$. This is a contradiction, because $t_1 > t_0$, and t_0 was defined to be the supremum of all numbers t such that $u([0,t]) \subseteq S_\delta$.

We must therefore have $t_0 = \infty$ after all, which means that $u(t) \in S_\delta$ for all $t \geq 0$. □

Theorem 29. *The point e is asymptotically stable.*

Proof. Suppose given $\epsilon > 0$. Put $\epsilon_1 = \min(\epsilon_0/2, \epsilon)$, so $0 < \epsilon_1 < \epsilon_0$. Proposition 27 says that we can choose δ such that $S_\delta \subseteq U_{\epsilon_1}$. Corollary 26 says that we can choose ϵ_2 such that $U_{\epsilon_2} \subseteq S_\delta$. Consider a flow line u that starts at $u(0) \in U_{\epsilon_2} \subseteq S_\delta$. By Proposition 28, we have $u(t) \in S_\delta \subseteq U_{\epsilon_1} \subseteq B_\epsilon(e) \subseteq R$ for all $t \geq 0$. In other words, the flow line is completely contained in R, so the function $V(u(t))$ is defined for all t and is weakly decreasing by Remark 19. Any nonnegative, weakly decreasing function converges to a limit, so $V(u(t))$ converges to some number V_0 as $t \to \infty$. As $0 \leq V(u(t)) \leq \delta$ for $t \geq 0$, we also have $0 \leq V_0 \leq \delta$.

Next, as $u(t) \in U_{\epsilon_1}$ we have $\|u(t)\| \leq \|e\| + \epsilon_1$ for all $t \geq 0$. This means that the sequence $(u(n))$ is bounded, so by the Bolzano-Weierstrass theorem there is a subsequence $(u(n_k))$ that converges to some point $v \in \mathbb{R}^2$. Here $\|u(n_k) - e\| < \epsilon_1$ for all k, so $\|v - e\| \leq \epsilon_1 < \epsilon_0$, so $v \in R$. As V is continuous on R we have $V(u(n_k)) \to V(v)$, but also $V(u(t)) \to V_0$, so $V(v) = V_0$. In particular, this means that $v \in S_\delta$.

Next, consider $\phi(s,v)$ for some $s > 0$. These points lie on a flow line starting at the point $v \in S_\delta$ when $s = 0$, so they are still in S_δ by Proposition 28. Note also that $\phi(s, u(t)) = u(s+t)$, so

$$\phi(s,v) = \lim_{k\to\infty} \phi(s, u(n_k)) = \lim_{k\to\infty} u(s+n_k),$$

so

$$V(\phi(s,v)) = \lim_{k\to\infty} V(u(s+n_k)),$$

but $V(u(t)) \to V_0$, so $V(\phi(s,v)) = V_0$, which is independent of s. This means that $W(\phi(s,v)) = \frac{d}{ds}V(\phi(s,v)) = 0$, but W is negative definite, so $\phi(s,v) = e$. In particular, we can take $s = 0$, and we get $v = e$, so $V_0 = V(v) = V(e) = 0$. This means that $V(u(t)) \to 0$ as $t \to \infty$.

Finally, suppose we have another small number $\epsilon' > 0$. By Proposition 27, there exists $\delta' > 0$ such that $S_{\delta'} \subseteq U_{\epsilon'}$. As $V(u(t)) \to 0$, we can find T such that $V(u(t)) < \delta'$ for $t \geq T$. This means that when $t \geq T$ we have $u(t) \in S_{\delta'} \subseteq U_{\epsilon'}$, so $\|u(t) - e\| < \epsilon'$. This proves that $u(t) \to e$, as required. Thus, e is asymptotically stable. □