MAS290 METHODS FOR DIFFERENTIAL EQUATIONS

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1. Introduction

2. First order autonomous ODEs

In this section we discuss systems of differential equations like these.

Example 2.1. Suppose that the number of fish in the sea is F, and the number of sharks is S. The sharks breed, the fish breed, and the sharks eat the fish. The rates of change of F and S are given by

$$\frac{dF}{dt} = (\alpha - \beta S)F \qquad\qquad \frac{dS}{dt} = -(\gamma - \delta F)S,$$

for some constants α, β, γ and δ. These are called the *Lotka Volterra equations.* We will discuss later why they are a good model.

Example 2.2. Some aspects of the weather are described by the *Lorenz equations*:

$$\frac{dx}{dt} = 10(y-x) \qquad\qquad \frac{dy}{dt} = (28-z)x - y \qquad\qquad \frac{dz}{dt} = xy - \tfrac{8}{3}z.$$

Example 2.3. Perhaps explain a simplified version of the Hodgkin-Huxley model.

Example 2.4. The one-sector growth model is a complex system of differential equations that relates capital investment, labour costs, interest rates, savings rates, profits and similar variables.

- All of these are *ordinary* differential equations; there are derivatives with respect to time, but not with respect to other variables. This is different from the equations that govern the flow of heat along an iron bar, for example. Heat will only flow if some parts of the bar are hotter than others, so the equations involve the derivatives of temperature with respect to position as well as with respect to time. Equations like this are called *partial differential equations*; you will study them in later courses.
- All the above equations are of *first order*; on the left hand side we have only the first derivative dx/dt, not higher derivatives like d^2x/dt^2. This is different from most of the equations of physics, which usually involve acceleration, which is a second derivative. However, there is a way to convert second order equations to first order systems, which we will discuss later.
- All the above equations are *deterministic*; there are no random effects. The theory of *stochastic differential equations* includes random effects, which are important in economics and finance. You will be able to study these in your final year in Sheffield.

- All the above equations are *autonomous*; they do not include external effects. For example, a more realistic description of populations of sharks and fish would include the external effect of humans catching fish.

3. Review of differential equations in one variable

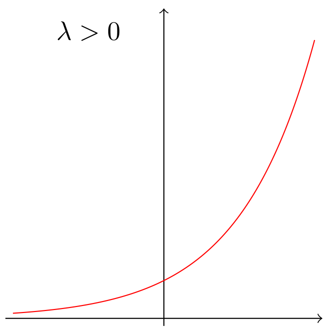
Suppose that we have a variable x that depends on a variable t, and we write $\dot{x}$ for dx/dt. If we know that $x(t)$ satisfies a differential equation like $\dot{x}(t) = f(x(t))$, then we may be able to deduce a formula for $x(t)$.

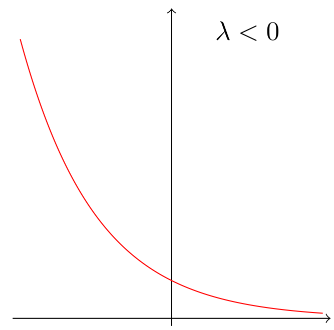
Example 3.1. Suppose we know that $\dot{x} = \lambda x$ for some constant λ, and that $x = x_0$ when $t = 0$. It is then well-known that $x = e^{\lambda t} x_0$ for all t. To check this, put $y = e^{-\lambda t} x$. Using the product rule and the equation $\dot{x} = \lambda x$ we get

$$\dot{y} = \frac{d}{dt}(e^{-\lambda t} x) = x \frac{d}{dt} e^{-\lambda t} + e^{-\lambda t} \frac{dx}{dt} = -\lambda x e^{-\lambda t} + e^{-\lambda t} \lambda x = 0.$$

As $\dot{y} = 0$, we see that y is constant. At $t = 0$ we have $y = e^0 x_0 = x_0$, so we must have $y = x_0$ for all time. As $e^{-\lambda t} x = y = x_0$, it follows that $x = e^{\lambda t} x_0$ as claimed.

We will need to recall the shape of the graph of $e^{\lambda t}$. If $\lambda > 0$, then it looks like the picture on the left below. It is always positive, and never crosses the x-axis. As $t \to +\infty$, the function $e^{\lambda t}$ grows large very quickly.





If $\lambda < 0$ then the graph of $e^{\lambda t}$ looks like the picture on the right above. Again, it is always positive, and never crosses the x-axis. It approaches zero as $t \to +\infty$.

Example 3.2. If $\dot{x} = \sqrt{x}$ then it works out that $x = (t + c)^2/4$ for some constant c. To see this, put $y = 2\sqrt{x} = 2x^{1/2}$, so $x = y^2/4$. We then have

$$\dot{y} = 2 \times \tfrac{1}{2} \times x^{\frac{1}{2}-1} \dot{x} = x^{-1/2} \dot{x} = x^{-1/2} x^{1/2} = 1.$$

This means that $y = t + c$ for some constant c, so $x = (t + c)^2/4$.

Remark 3.3. The above two examples ($\dot{x} = \lambda x$ and $\dot{x} = \sqrt{x}$) are first order differential equations in one variable. *First order* means that only the first derivative $\dot{x} = dx/dt$ is involved, not the second derivative $\ddot{x} = d^2x/dt^2$ or any higher derivatives. An example of a second order equation is $\ddot{x} + \omega^2 x = 0$, with solution $x = A\cos(\omega t) + B\sin(\omega t)$. In the first half of this course we will mostly study first order equations.

Remark 3.4. The equations $\dot{x} = \lambda x$ and $\dot{x} = \sqrt{x}$ are also *autonomous*, which means that the variable t does not appear explicitly. An example of a non-autonomous equation is $\dot{x} = x + t$, with solution $x = Ae^t - 1 - t$. In this course we will mainly study autonomous equations.

Definition 3.5. Consider a second order, autonomous differential equation $\ddot{x} + b\dot{x} + cx = 0$. The *auxiliary polynomial* is $p(\lambda) = \lambda^2 + b\lambda + c = 0$. The roots of the equation $p(\lambda) = 0$ are $(-b \pm \sqrt{b^2 - 4c})/2$.

(a) If $b^2 \geq 4c$ then the roots $\lambda_1 = (-b - \sqrt{b^2 - 4c})/2$ and $\lambda_2 = (-b + \sqrt{b^2 - 4c})/2$ are both real numbers.
(b) If $b^2 < 4c$ then the numbers $\lambda = -b/2$ and $\omega = \sqrt{4c - b^2}/2$ are real, and the two roots of the auxiliary equation are $\lambda - i\omega$ and $\lambda + i\omega$. We call λ the *growth rate*, and ω the *angular frequency*.

Proposition 3.6.

(a) *If* $b^2 \geq 4c$ *then the solutions of the equation* $\ddot{x} + b\dot{x} + cx = 0$ *are of the form* $x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ *(where* A_1 *and* A_2 *are real constants).*

(b) *If* $b^2 < 4c$ *then the solutions are of the form* $(B\cos(\omega t) + C\sin(\omega t))e^{\lambda t}$ *(where* B *and* C *are real constants).*

Proof. The first thing is to prove that the above formulae do in fact satisfy the equation $\ddot{x} + b\dot{x} + cx = 0$. In case (a) we have

$$\begin{aligned} x &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \\ \dot{x} &= A_1 \lambda_1 e^{\lambda_1 t} + A_2 \lambda_2 e^{\lambda_2 t} \\ \ddot{x} &= A_1 \lambda_1^2 e^{\lambda_1 t} + A_2 \lambda_2^2 e^{\lambda_2 t} \\ \ddot{x} + b\dot{x} + cx &= A_1(\lambda_1^2 + b\lambda_1 + c)e^{\lambda_1 t} + A_2(\lambda_2^2 + b\lambda_2 + c)e^{\lambda_2 t} \\ &= A_1 p(\lambda_1) e^{\lambda_1 t} + A_2 p(\lambda_2) e^{\lambda_2 t} = 0, \end{aligned}$$

as required. In case (b), we recall that

$$\begin{aligned} \cos(\omega t) &= (e^{i\omega t} + e^{-i\omega t})/2 \\ \sin(\omega t) &= -i(e^{i\omega t} - e^{-i\omega t})/2, \end{aligned}$$

so

$$\begin{aligned} x &= (B\cos(\omega t) + C\sin(\omega t))e^{\lambda t} \\ &= (Be^{i\omega t} + Be^{-i\omega t} - iCe^{i\omega t} + iCe^{-i\omega t})e^{\lambda t}/2 \\ &= \frac{B - iC}{2} e^{(\lambda + i\omega)t} + \frac{B + iC}{2} e^{(\lambda - i\omega)t}. \end{aligned}$$

In other words, if we put $A_1 = (B - iC)/2$ and $A_2 = (B + iC)/2$ and $\lambda_1 = \lambda + i\omega$ and $\lambda_2 = \lambda - i\omega$, then λ_1 and λ_2 are the roots of the auxiliary polynomial, and $x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$. Thus, we have $\ddot{x} + b\dot{x} + cx = 0$ just as in case (a); the only difference is that some of the numbers involved are complex, but that does not matter.

The more difficult problem is to prove that *every* solution has the form described above. Suppose that $x(t)$ is a function of t such that $\ddot{x} + b\dot{x} + cx = 0$. Let λ_1 and λ_2 be the roots of the auxiliary polynomial, so

$$t^2 + bt + c = (t - \lambda_1)(t - \lambda_2) = t^2 - (\lambda_1 + \lambda_2)t + \lambda_1 \lambda_2,$$

so $\lambda_1 + \lambda_2 = -b$ and $\lambda_1 \lambda_2 = c$. Put

$$\begin{aligned} P_1(t) &= e^{-\lambda_1 t}(\dot{x}(t) - \lambda_2 x(t)) \\ P_2(t) &= e^{-\lambda_2 t}(\dot{x}(t) - \lambda_1 x(t)). \end{aligned}$$

We then have

$$\begin{aligned} \dot{P}_1 &= -\lambda_1 e^{-\lambda_1 t}(\dot{x} - \lambda_2 x) + e^{-\lambda_1 t}(\ddot{x} - \lambda_2 \dot{x}) \\ &= e^{-\lambda_1 t}(\ddot{x} - (\lambda_1 + \lambda_2)\dot{x} + \lambda_1 \lambda_2 x) = e^{-\lambda_1 t}(\ddot{x} + b\dot{x} + cx) = 0. \end{aligned}$$

Thus, P_1 is actually constant. By a similar calculation, P_2 is also constant. We can also rearrange the definition of P_1 and P_2 to get

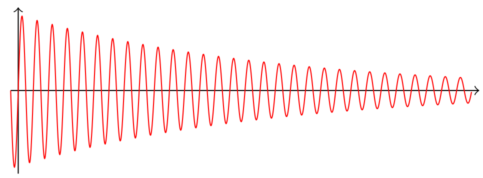
$$\begin{aligned} \dot{x} - \lambda_2 x &= P_1 e^{\lambda_1 t} \\ \dot{x} - \lambda_1 x &= P_2 e^{\lambda_2 t}. \end{aligned}$$

Subtracting these equations and rearranging gives

$$x = \frac{P_1}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{P_2}{\lambda_2 - \lambda_1} e^{\lambda_2 t}.$$

In other words, if we put $A_1 = P_1/(\lambda_1 - \lambda_2)$ and $A_2 = P_2/(\lambda_2 - \lambda_1)$ then $x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ as required. All this works perfectly well even in case (b) where the numbers λ_k, P_k and A_k may be complex. $\square$

Remark 3.7. In many applications, it works out that $\lambda < 0$ and ω is much larger than $|\lambda|$. In that case, the graph of the function $e^{\lambda t}\sin(\omega t)$ is like this:



4. Phase portraits

Consider a differential system

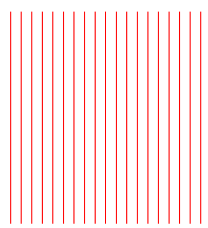
$$\dot{x}(t) = f(x(t), y(t)) \qquad\qquad \dot{y}(t) = g(x(t), y(t)).$$

If we are given initial conditions $x(0) = x_0$ and $y(0) = y_0$, we can solve the equations to find $x(t)$ and $y(t)$ for all t (or at least, for some range of values of t). The points $(x(t), y(t))$ will then trace out a curve in the plane, passing through the initial point (x_0, y_0). If we change the initial point then we will usually get a different curve. The collection of all solution curves (which cover the whole plane) is called the *phase portrait* of the system.

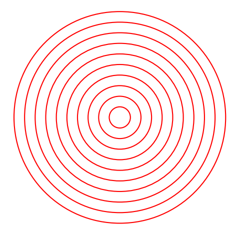
Example 4.1. Phase portraits for three very simple systems are shown below.

$\dot{x} = 1,\ \dot{y} = 0$

$\dot{x} = 0,\ \dot{y} = 1$



$\dot{x} = -y,\ \dot{y} = x$



For the system $\dot{x} = 1, \dot{y} = 0$ the solution is just $x = x_0 + t$ and $y = y_0$, which gives a horizontal line. Similarly, for the system $\dot{x} = 0, \dot{y} = 1$ the phase portrait just consists of vertical lines. For the third system, we can choose any $r \geq 0$ and then the functions $x = r\cos(t)$ and $y = r\sin(t)$ give a solution. The corresponding curve is a circle of radius r centred at the origin. All these circles together form the phase portrait.

Often we can sketch the phase portrait and deduce useful information about the behaviour of the differential system, even if we cannot find a formula for the solutions.

5. Critical points

To do

In this section we consider differential equations of the form $\dot{x} = u(x)$, where $x\,\mathbb{R} \to \mathbb{R}^n$ and $u\,\mathbb{R}^n \to \mathbb{R}^n$. The main topic is the behaviour near equilibrium points $a \in \mathbb{R}^n$ where $u(a) = 0$. We can reduce to the case where $a = 0$, so $u(x) \simeq Ax + O(\|x\|^2)$ for some matrix A. If $x(0) = c \simeq 0$ then for small t we have $\dot{x} \simeq Ax$ so $x \simeq e^{At}c$. This means that 0 is a stable equilibrium if all the eigenvalues of A lie in the left half plane.

For more global stability analysis we seek Lyapunov functions $V(x)$ such that $V^{-1}\{0\} = 0$ and $u.\nabla V < 0$ away from the origin, so that $(V \circ x)' \leq 0$ for all solutions to our ODE.

We also need to cover other material about sketching trajectories.

6. Second order linear ODEs

Here we need to cover power series methods, classification of singular points, Sturm-Liouville theory.