MAS290 METHODS FOR DIFFERENTIAL EQUATIONS — EXAMPLES

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Example 1. Consider a system as follows (where a, b, c, d, p, q are constants with $a, d, ad - bc \neq 0$).

$$\begin{aligned} \dot{x} &= f(x, y) = x(ax + by - p) \\ \dot{y} &= g(x, y) = y(cx + dy - q). \end{aligned}$$

For an equilibrium point, we must have either $x = 0$ or $ax + by = p$, and also $y = 0$ or $cx + dy = q$. If $x = 0$ then the equation $cx + dy = q$ becomes $y = q/d$, and if $y = 0$ then the equation $ax + by = p$ becomes $x = p/a$. If x and y are both nonzero then we must have $ax + by = p$ and $cx + dy = q$; these equations can be solved in the usual way to give $x = \frac{dp-bq}{ad-bc}$ and $y = \frac{aq-cp}{ad-bc}$. We thus have four equilibrium points:

$$u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ q/d \end{bmatrix} \qquad u_3 = \begin{bmatrix} p/a \\ 0 \end{bmatrix} \qquad u_4 = \frac{1}{ad - bc} \begin{bmatrix} dp - bq \\ aq - cp \end{bmatrix}.$$

The Jacobian matrix is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2ax + by - p & bx \\ cy & cx + 2dy - q \end{bmatrix}.$$

Evaluating this at the equilibrium points gives

$$J_1 = \begin{bmatrix} -p & 0 \\ 0 & -q \end{bmatrix} \qquad J_2 = \begin{bmatrix} -(dp - bq)/d & 0 \\ cq/d & q \end{bmatrix} \qquad J_3 = \begin{bmatrix} p & bp/a \\ 0 & -(cp - aq)/a \end{bmatrix}$$

and

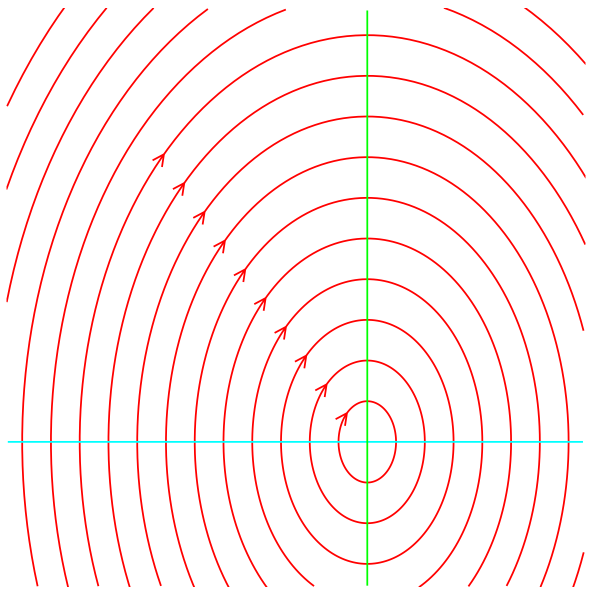
$$J_4 = \frac{1}{ad - bc} \begin{bmatrix} a(dp - bq) & b(dp - bq) \\ c(aq - cp) & d(aq - cp) \end{bmatrix}.$$

The first three of these are triangular so one can just read off the trace, determinant and eigenvalues. For J_4, the standard formulae for the eigenvalues do not simplify in any useful way.

Example 2. Consider the system

$$\dot{x} = f(x, y) = 1 + y$$
$$\dot{y} = g(x, y) = 1 - 2x.$$

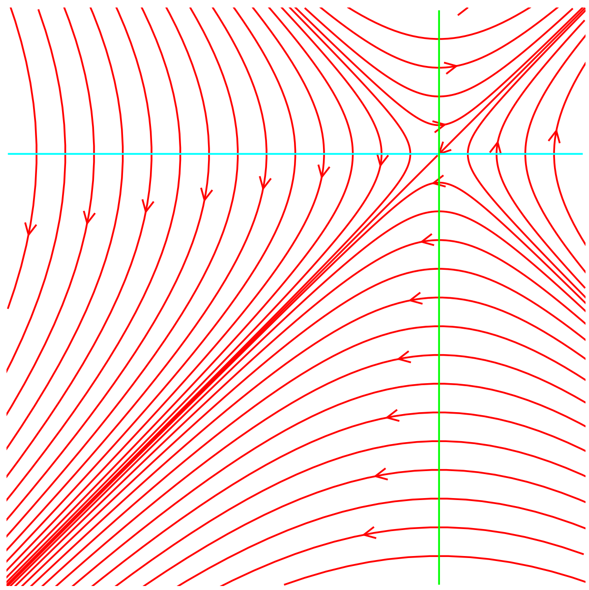
For an equilibrium point, we need $1 + y = 0$ and $1 - 2x = 0$, so $x = 1/2$ and $y = -1$. Thus, there is a unique equilibrium point at $(1/2, -1)$. The Jacobian is $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, which has trace $\tau = 0$ and determinant $\delta = 2$. As $\tau = 0$ and $\delta > 0$, this is a centre. The bottom left entry in J is $-2 < 0$, so the rotation is clockwise. In this case the functions f and g are just linear + constant, so there is no error in linearization, and the whole phase diagram is exactly the same as the usual phase diagram for a centre, except that it has been shifted away from the origin. The picture is as follows:



Example 3. Consider the system

$$\begin{aligned}\dot{x} &= f(x,y) = y-1\\ \dot{y} &= g(x,y) = x-1.\end{aligned}$$

For an equilibrium point, we need $y-1 = x-1 = 0$. Thus, there is a unique equilibrium point at $(1,1)$. The Jacobian is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has trace $\tau = 0$ and determinant $\delta = -1$. As $\delta < 0$, this is a saddle. In this case the functions f and g are just linear + constant, so there is no error in linearization, and the whole phase diagram is exactly the same as the usual phase diagram for a saddle, except that it has been shifted away from the origin. The picture is as follows:



Example 4. Consider the system

$$\dot{x} = x - x^3$$
$$\dot{y} = y - y^3.$$

The x-nullcline is given by $x - x^3 = 0$, which factors as $x(1+x)(1-x) = 0$, so $x = 0$ or $x = \pm 1$. Similarly, the y-nullcline is given by $y - y^3 = 0$, so $y = 0$ or $y = \pm 1$. This means that there are nine equilibrium points (n, m) with $n, m \in \{-1, 0, 1\}$. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 1 - 3x^2 & 0 \\ 0 & 1 - 3y^2 \end{bmatrix}.$$

If x and y are both zero then J is the identity matrix, corresponding to an improper unstable node. If $x = 0$ and $y = \pm 1$ (or *vice versa*) then the eigenvalues are 1 and -2, one positive and one negative, so we have a saddle. If both x and y are ± 1 then $J = -2I$, corresponding to an improper stable node.

The above is enough for a reasonably good sketch of the phase diagram. However, in this case it is possible to just give a formula for the solution:

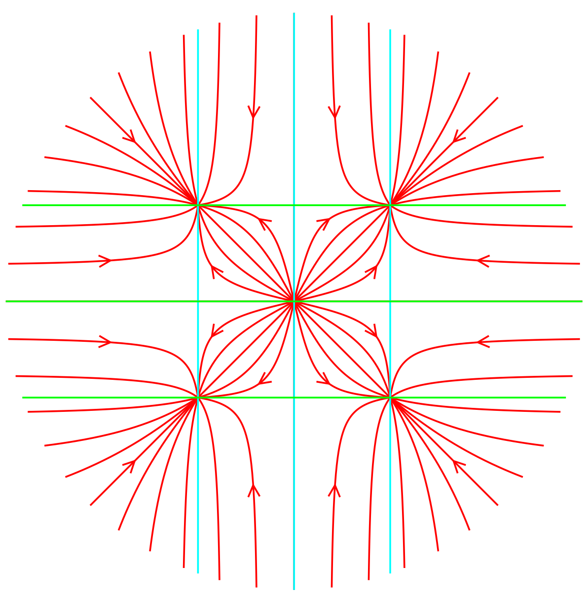
$$x = (1 + (x_0^{-2} - 1)e^{-2t})^{-1/2}$$
$$y = (1 + (y_0^{-2} - 1)e^{-2t})^{-1/2}.$$

To check that this works, it will be convenient to put $C = x_0^{-2} - 1$, so $x = (1 + Ce^{-2t})^{-1/2}$. This gives

$$\dot{x} = -\tfrac{1}{2}(1 + Ce^{-2t})^{-3/2} \times (-2) \times Ce^{-2t} = (1 + Ce^{-2t})^{-3/2} Ce^{-2t}$$
$$x - x^3 = x^3(x^{-2} - 1) = (1 + Ce^{-2t})^{-3/2}(1 + Ce^{-2t} - 1) = (1 + Ce^{-2t})^{-3/2} Ce^{-2t} = \dot{x},$$

as required. Essentially the same argument gives $\dot{y} = y - y^3$.

The phase diagram is as follows:



Example 5. Consider the system

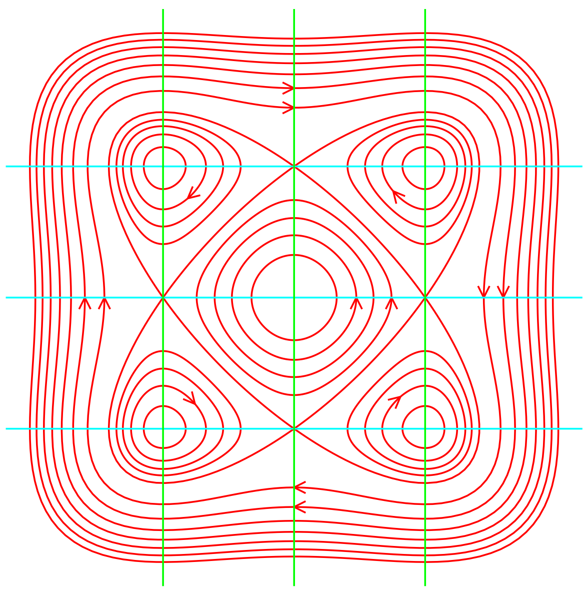
$$\dot{x} = y^3 - y = y(y+1)(y-1)$$
$$\dot{y} = x - x^3 = x(1+x)(1-x).$$

The x-nullcline consists of three horizontal lines, with equations $y = 0$, $y = 1$ and $y = -1$. Similarly, the y-nullcline consists of three vertical lines, with equations $x = 0$, $x = 1$ and $x = -1$. This means that there are nine equilibrium points (n, m) with $n, m \in \{-1, 0, 1\}$. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 3y^2 - 1 \\ 1 - 3x^2 & 0 \end{bmatrix},$$

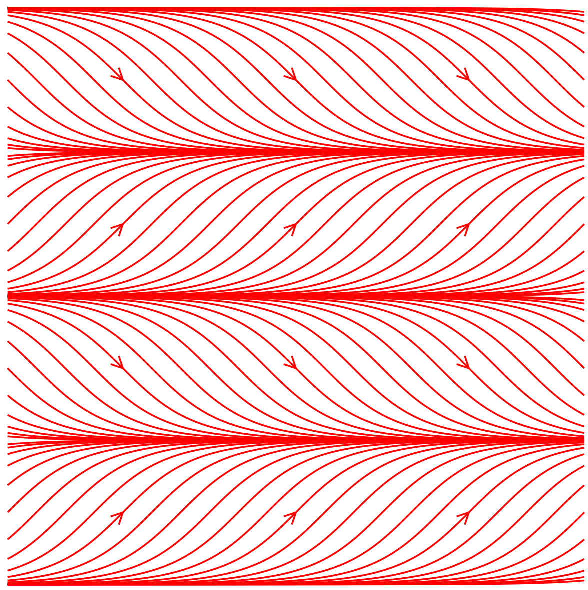
so the trace is $\tau = 0$ and the determinant is $\delta = (1 - 3x^2)(1 - 3y^2)$. As $\tau = 0$ we see that the equilibrium points are centres if $\delta > 0$, and saddles if $\delta < 0$. If x and y are both zero then $\delta = 1$, corresponding to a cycle. The bottom left entry in J is $1 - 3x^2 = 1 > 0$, so the rotation is anticlockwise. If $x = 0$ and $y = \pm 1$ then $\delta = -2 < 0$, corresponding to a saddle. The same applies if $x = \pm 1$ and $y = 0$. Finally, if both x and y are ± 1 then $\delta > 0$, which means we have another cycle. The bottom left entry is $-2 < 0$, so the rotation is clockwise.

The phase portrait is as follows:



Check direction of rotation.

Example 6. Consider the equations $\dot{x} = 1$ and $\dot{y} = \sin(\pi y)$. As $\dot{x}$ is never zero, there are no equilibrium points. It is clear that $x = x_0 + t$, but the behaviour of y is less obvious. For any integer n we have a solution $(x, y) = (t, n)$ (which works because $\dot{y} = 0$ and also $\sin(\pi y) = \sin(n\pi) = 0$). If $0 < y < 1$ then $\dot{y} = \sin(\pi y) > 0$ so y increases, but solutions never cross so (x, y) must stay below $y = 1$. In fact we find that y increases asymptotically, tending towards the limit $y = 1$ but never reaching it. Similarly, if $1 < y < 2$ then y decreases asymptotically towards $y = 1$. The phase diagram is as follows:



Example 7. The *Duffing oscillator* is the system $\dot{x} = y$ and $\dot{y} = 2x - x^3$; it is used to model various kinds of oscillations in electrical engineering. The x-nullcline is the line $y = 0$. The y-nullcline is given by $2x - x^3 = 0$, which means that $x = 0$ or $x = \pm\sqrt{2}$. Thus, there are three equilibrium points:

$$u_1 = (0, 0) \qquad u_2 = (\sqrt{2}, 0) \qquad u_3 = (-\sqrt{2}, 0).$$

The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 - 3x^2 & 0 \end{bmatrix}.$$

At u_1 this becomes $J = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, which has $\tau = 0$ and $\delta = -2 < 0$, indicating a saddle. It is easy to see that the vectors $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$ are eigenvectors with eigenvalues $\pm\sqrt{2}$.

At u_2 or u_3 we have $J = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$, which has trace $\tau = 0$ and determinant $\delta = 4 > 0$. This means that the linearization has a centre, and so suggests (but does not prove) that the original system has a centre. Now consider the function

$$V = 2y^2 + x^4 - 4x^2 + 4 = 2y^2 + (x^2 - 2)^2.$$

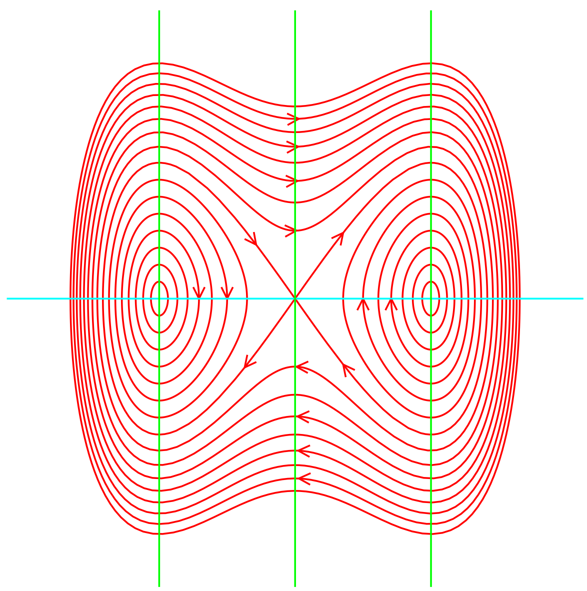
We have

$$\dot{V} = 4y\dot{y} - 4x^3\dot{x} - 8x\dot{x} = 4y(2x - x^3) - 4x^3y - 8xy = 0,$$

so V is a conserved quantity. Consider a point $(x, y) = (\sqrt{2} + a, b)$ close to u_2, so a and b are small. We then have

$$V = 8a^2 + 2b^2 + \text{terms of higher order}.$$

Using this, we see that u_2 is indeed a centre, and essentially the same argument works for u_3 as well.



Example 8. The *damped Duffing oscillator* is given by $\dot{x} = y$ and $\dot{y} = 2x - x^3 - 0.1y$. The x-nullcline is the line $y = 0$. The y-nullcline is given by $2x - x^3 = 0.1y$. When x is small we can neglect the term x^3 so the y-nullcline equation becomes $y = 2x/0.1 = 20x$, which describes a steep line through the origin. This should be compared with the vertical line $x = 0$ which is part of the y-nullcline for the undamped Duffing oscillator. The y-nullcline for the damped oscillator also includes curves through $(\pm\sqrt{2}, 0)$ that can again be described approximately as steeply sloping straight lines. The equilibrium points are again

$$u_1 = (0,0) \qquad u_2 = (\sqrt{2}, 0) \qquad u_3 = (-\sqrt{2}, 0),$$

exactly the same as in the undamped case. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 - 3x^2 & -0.1 \end{bmatrix},$$

so $\tau = -0.1$ and $\delta = 3x^2 - 2$. At u_1 this becomes $(\tau, \delta) = (-0.1, -2)$, so in particular $\delta < 0$, so we have a saddle. At u_2 we have $(\tau, \delta) = (-0.1, 4)$, so $\tau^2 - 4\delta < 0$; we therefore have a stable focus. There is also a stable focus at u_3, for the same reason.

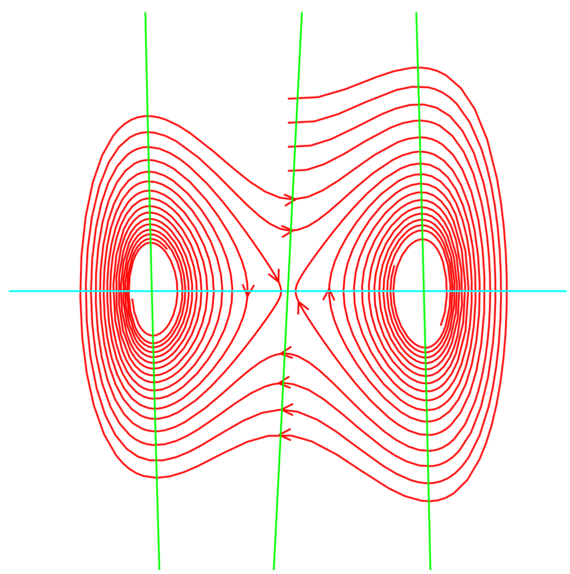
We can again consider the function

$$V = 2y^2 + x^4 - 4x^2 + 4,$$

but it is no longer conserved. Instead we have

$$\dot{V} = 4y\dot{y} - 4x^3\dot{x} - 8x\dot{x} = 4y(2x - x^3 - 0.1y) - 4x^3y - 8xy = -0.4y^2.$$

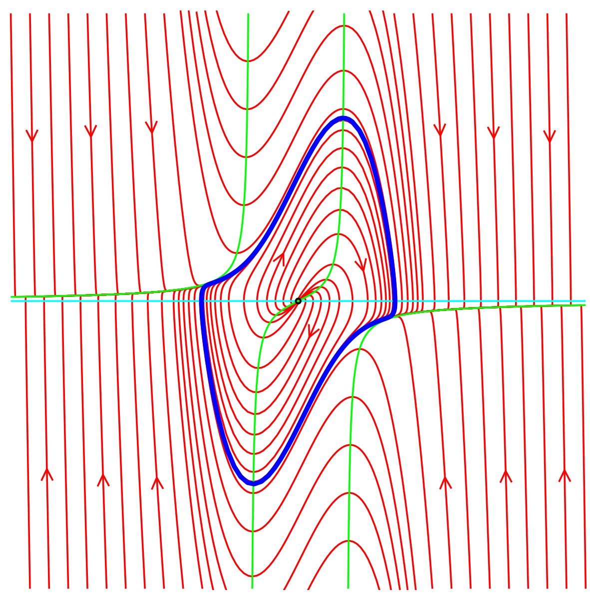
This shows that $\dot{V} \leq 0$, and that $\dot{V}$ can only be equal to 0 if $y = 0$. Using the description $V = 2y^2 + (x^2 - 2)^2$ we also see that V can only be zero at u_2 and u_3. This means that V almost satisfies the conditions for a Lyapunov function, but not quite.



Example 9. The *van der Pol oscillator* has equations $\dot{x} = y$ and $\dot{y} = 2(1-x^2)y - x$. The x-nullcline is the line $y = 0$. The y-nullcline is given by $2(1-x^2)y - x = 0$, or equivalently $y = \frac{x}{2(1-x^2)}$. (More precisely, the y-nullcline contains the points $\left(x, \frac{x}{2(1-x^2)}\right)$ for all $x \neq \pm 1$, but it does not contain any points (x, y) with $x = \pm 1$.) For an equilibrium point we must have $y = 0$ and also $2(1-x^2)y - x = 0$ which gives $x = 0$. Thus, the only equilibrium point is therefore at the origin. The Jacobian is

$$J = \begin{bmatrix} 0 & 1 \\ -4xy - 1 & 2 - 2x^2 \end{bmatrix},$$

which becomes $J = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ at the origin. This has $\tau = 2$ and $\delta = 1$ and $\tau^2 - 4\delta = 0$, so the eigenvalues are both equal to one. However, J is not diagonalizable, so we have a degenerate unstable node at the origin. This means that solutions starting near the origin will be quickly pushed away from the origin. However, it turns out that they do not escape to infinity. Instead, the system has a new feature called a *limit cycle*. This is a strangely shaped closed curve of finite size that wraps around the origin. All solutions that start inside the limit cycle spiral outwards to approach the curve asymptotically as $t \to \infty$. Solutions that start outside the limit cycle move very quickly in a nearly vertical direction until they are close to the x-axis, at which point they turn sharply sideways and move much more slowly to approach the limit cycle and spiral in to it from the outside.



Example 10. Consider a system of the form

$$\begin{aligned}\dot{x} &= f(x,y) = a(x^2-1) + b(x^2-1)\\ \dot{y} &= g(x,y) = c(x^2-1) + d(x^2-1),\end{aligned}$$

where a, b, c and d are nonzero constants with $ad - bc \neq 0$. The x-nullcline is given by $a(x^2-1)+b(x^2-1)$. If a and b have the same sign, we find that the x-nullcline is an ellipse, with equation

$$(x,y) = (\sqrt{1+b/a}\cos(t),\ \sqrt{1+a/b}\sin(t)).$$

Now consider a case where a and b have opposite signs, say $a > 0$ and $b < 0$ with $a > |b|$. In this case we find that the x-nullcline is a hyperbola, with formula

$$(x,y) = (\sqrt{1-|b|/a}\cosh(t),\ \sqrt{a/|b|-1}\sinh(t)).$$

There are similar formulae for other combinations of signs. By the same argument, the y-nullcline is also either an ellipse or a hyperbola, depending on the signs and relative sizes of c and d.

For an equilibrium point we must have $f = g = 0$, or in matrix form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x^2-1 \\ y^2-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have assumed that $ad - bc \neq 0$ so the matrix above has an inverse and it follows that the only solution is $x^2 - 1 = y^2 - 1 = 0$. This in turn means that $x = \pm 1$ and $y = \pm 1$, so there are precisely four equilibrium points:

$$u_1 = (1,1) \qquad u_2 = (-1,-1) \qquad u_3 = (1,-1) \qquad u_4 = (-1,1).$$

The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2ax & 2by \\ 2cx & 2dy \end{bmatrix},$$

so the trace is $\tau = 2(ax + dy)$ and the determinant is $\delta = 4(ad - bc)xy$. We therefore have the following table:

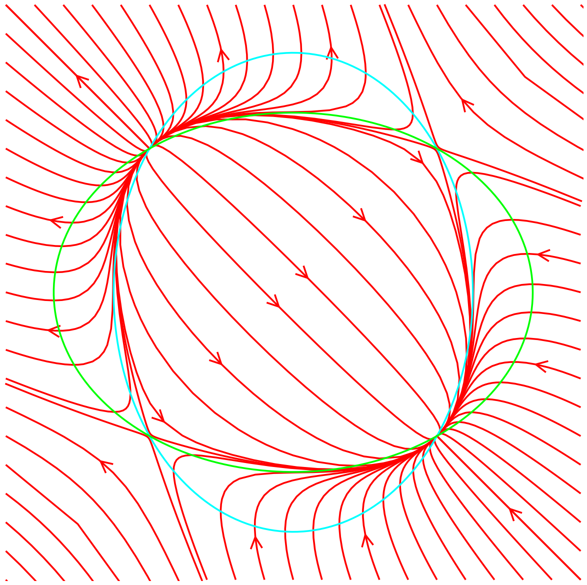
	u_1	u_2	u_3	u_4
τ	$2(a+d)$	$-2(a+d)$	$2(a-d)$	$-2(a-d)$
δ	$4(ad-bc)$	$4(ad-bc)$	$-4(ad-bc)$	$-4(ad-bc)$
$\tau^2-4\delta$	$4(a+d)^2-16(ad-bc)$	$4(a+d)^2-16(ad-bc)$	$4(a-d)^2+16(ad-bc)$	$4(a-d)^2+16(ad-bc)$

This is about as far as we can get without choosing specific numerical values for a, b, c and d.

We now consider the case where $(a,b,c,d) = (-16,-9,9,16)$, so the equations are $\dot{x} = -16x^2 - 9y^2 + 25$ and $\dot{y} = 9x^2 + 16y^2 - 25$. The table becomes

	u_1	u_2	u_3	u_4
τ	0	0	-64	64
δ	-700	-700	700	700
$\tau^2-4\delta$	2800	2800	1296	1296

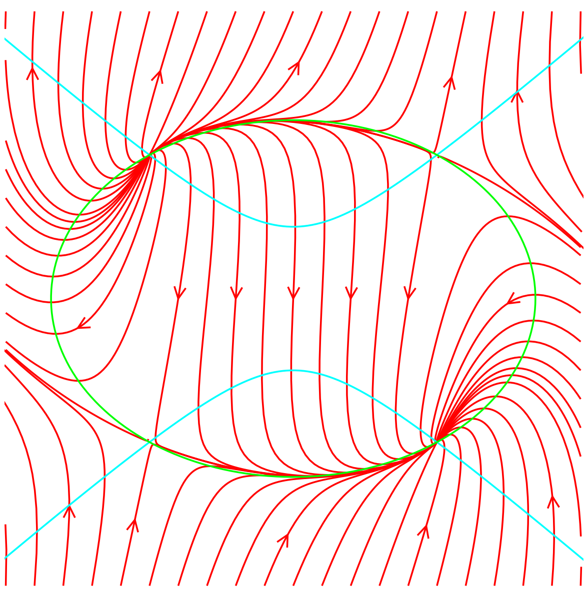
From this we see that u_1 and u_2 are saddles, and u_3 is a stable node, and u_4 is an unstable node. The x-nullcline is an ellipse with equation $(x,y) = (\frac{5}{4}\cos(t), \frac{5}{3}\sin(t))$. The y-nullcline is another ellipse with equation $(x,y) = (\frac{5}{3}\cos(t), \frac{5}{4}\sin(t))$. The picture is as follows:



Now consider instead the case where $(a,b,c,d) = (-9,12,25,46)$, so $\dot{x} = 12y^2 - 9x^2 - 3$ and $\dot{y} = 25x^2 + 46y^2 - 71$. The table becomes

	u_1	u_2	u_3	u_4
τ	74	-74	-110	110
δ	-2856	-2856	2856	2856
$\tau^2 - 4\delta$	16900	16900	676	676

Again we see that u_1 and u_2 are saddles, and u_3 is a stable node, and u_4 is an unstable node. However, the nullclines are different and so the overall picture is different. The y-nullcline is again an ellipse, but the x-nullcline is now a hyperbola.



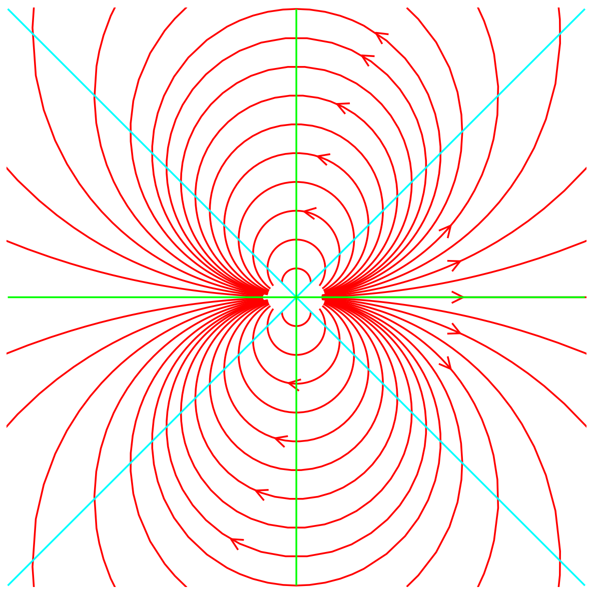
Example 11. Consider the system

$$\dot{x} = f(x,y) = x^2 - y^2$$
$$\dot{y} = g(x,y) = 2xy.$$

The x-nullcline is given by $x^2 - y^2 = 0$, or $(x-y)(x+y) = 0$, so $x = y$ or $x = -y$. The y-nullcline is given by $2xy = 0$, so $x = 0$ or $y = 0$. The only point lying on both nullclines is the origin, so the origin is the only equilibrium point. The Jacobian is $J = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$, which is zero at the origin. Thus, the linearization at the origin is the constant system, which is not structurally stable. We therefore cannot deduce very much about the behaviour of the solution from this linearization. However, we can make a reasonable sketch using the nullclines. Moreover, it happens that this system has an explicit solution:

$$x = \frac{x_0 - t(x_0^2 + y_0^2)}{(1 - tx_0)^2 + t^2 y_0^2} \qquad\qquad y = \frac{y_0}{(1 - tx_0)^2 + t^2 y_0^2}$$

The phase diagram is as follows:



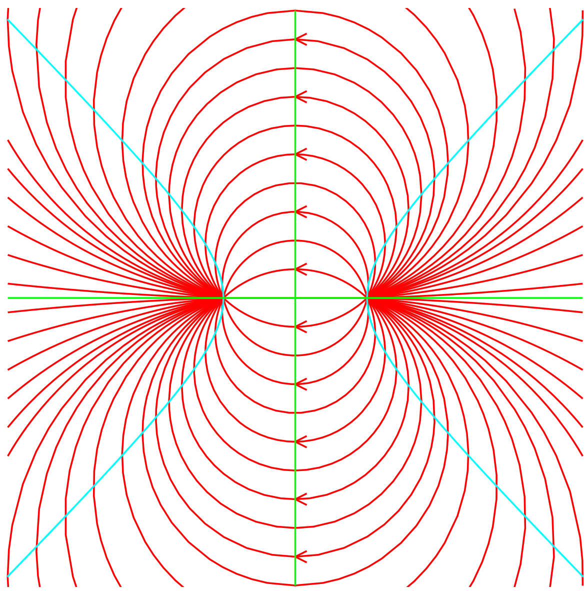
Example 12. Consider the system

$$\dot{x} = f(x,y) = x^2 - y^2 - 1/4$$
$$\dot{y} = g(x,y) = 2xy.$$

The x-nullcline is given by $x^2 - y^2 - 1/4 = 0$, or $x = \pm\sqrt{y^2 + 1/4}$. The y-nullcline is given by $2xy = 0$, so $x = 0$ or $y = 0$. If $x = 0$ then the y-nullcline equation becomes $-y^2 - 1/4 = 0$ which is impossible (as y is assumed to be real). If $y = 0$ then the y-nullcline equation becomes $x^2 - 1/4 = 0$, so $x = \pm 1/2$. Thus, the equilibrium points are $u_1 = (-1/2, 0)$ and $u_2 = (1/2, 0)$.

The Jacobian is $J = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$, which is $-2I$ at u_1 and $+2I$ at u_2. This means that u_1 is an improper stable node and u_2 is an improper unstable node.

The phase diagram is as follows:



Example 13. Consider the system

$$\dot{x} = f(x,y) = x^3 - 3xy^2 - 1$$
$$\dot{y} = g(x,y) = 3x^2y - y^3.$$

The y-nullcline is given by $3x^2y - y^3 = 0$, but this can be factored as $y((\sqrt{3}x)^2 - y^2) = 0$ or $y(y-\sqrt{3}x)(y+\sqrt{3}x) = 0$. This means that $y = 0$ or $y = \pm\sqrt{3}x$, so we have three straight lines through the origin.

For the x-nullcline, we must have $x^3 - 3xy^2 - 1$, which rearranges to give $y = \pm\sqrt{\frac{x^3-1}{3x}}$. Note that when $x < 0$, both $x^3 - 1$ and $3x$ are negative and so $(x^3-1)/(3x)$ is positive, so the above square root makes sense. Similarly, when $x > 1$, both $x^3 - 1$ and $3x$ are positive, so $(x^3-1)/(3x)$ is also positive, so the square root again makes sense. However, when $0 < x < 1$ we see that $(x^3-1)/(3x)$ is negative; it follows that the x-nullcline does not pass through this region.

Any equilibrium point must lie on the y-nullcline, so $y = 0$ or $y = \pm\sqrt{3}x$. If $y = 0$ then the x-nullcline equation $x^3 - 3xy^2 - 1 = 0$ becomes $x^3 = 1$, so $x = 1$. If $y = \pm\sqrt{3}x$ then the x-nullcline equation becomes $x^3 - 3x \times 3x^2 = 1$, so $x^3 = -1/8$, so $x = -1/2$. Thus, we have three equilibrium points:

$$u_1 = (1,0) \qquad u_2 = (-1/2, +\sqrt{3}/2) \qquad u_3 = (-1/2, -\sqrt{3}/2).$$

The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}.$$

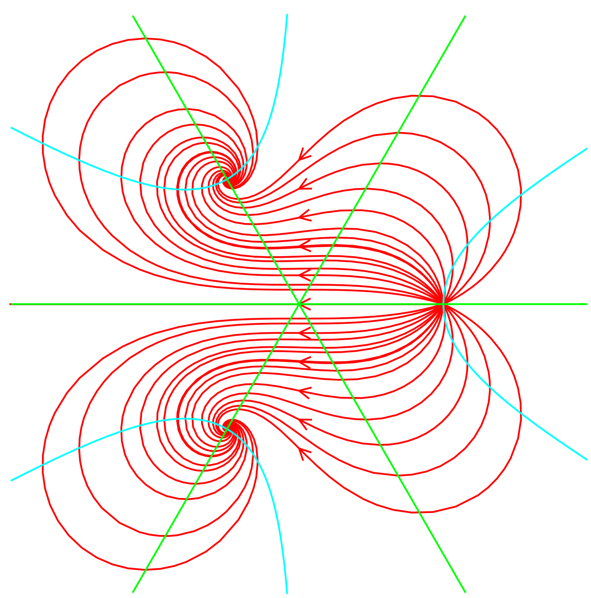
The trace is $\tau = 6(x^2 - y^2)$, and the determinant is

$$\begin{aligned} \delta &= (3(x^2-y^2))^2 - (-6xy) \times 6xy = 9(x^4 - 2x^2y^2 + y^4) + 36x^2y^2 \\ &= 9(x^4 + 2x^2y^2 + y^4) = 9(x^2+y^2)^2. \end{aligned}$$

At each of the equilibrium points we have $x^2 + y^2 = 1$ and so $\delta = 9 > 0$. At u_1 we have $\tau = 6$, so $\tau^2 - 4\delta = 0$. The eigenvalues of the Jacobian are $\lambda_1, \lambda_2 = (\tau \pm \sqrt{\tau^2 - 4\delta})/2$, which in this case means that $\lambda_1 = \lambda_2 = 3$. This is an improper unstable node.

At u_2 and u_3 we have $\tau = -3$ and so $\tau^2 - 4\delta = -27 < 0$. As $\tau^2 - 4\delta < 0$ these points are foci, and as $\tau < 0$ they are stable. The bottom left entry in J is $6xy$. At u_2 this is negative, so the rotation is clockwise; at u_3 it is positive, so the rotation is anticlockwise.

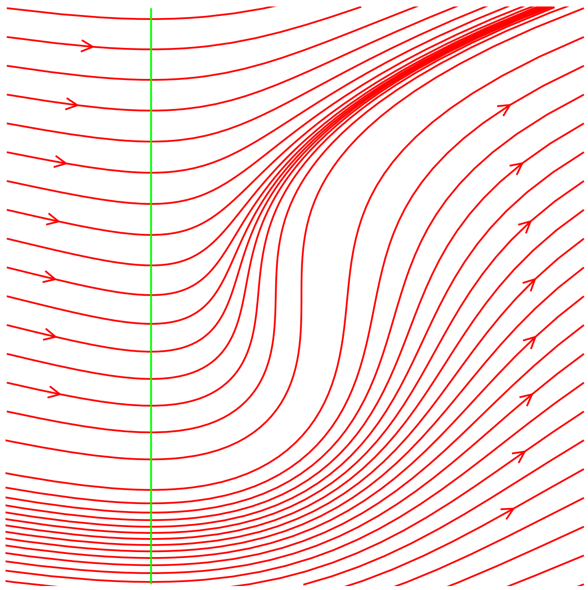
The phase diagram is as follows:



Example 14. Consider the system

$$\dot{x} = f(x, y) = x^2 + y^2$$
$$\dot{y} = g(x, y) = x + 1.$$

The y-nullcline is the line $x = -1$, but the x-nullcline is just the origin. No points lie on both nullclines, so there are no equilibria. The function $\dot{x} = x^2 + y^2$ is always nonnegative, so all trajectories move to the right. To the left of the line $x = -1$ they move downwards, and to the right of that line they move upwards.



Example 15. Consider the system

$$\dot{x} = f(x, y) = 9y^2 - 1$$
$$\dot{y} = g(x, y) = 9x^2 - 1.$$

The x-nullcline is given by $9y^2 - 1 = 0$ or equivalently $y = \pm 1/3$. The y-nullcline is given by $9x^2 - 1 = 0$ or equivalently $x = \pm 1/3$. It follows that there are four equilibrium points:

$$u_1 = (1/3, 1/3) \qquad u_2 = (-1/3, -1/3) \qquad u_3 = (1/3, -1/3) \qquad u_4 = (-1/3, 1/3).$$

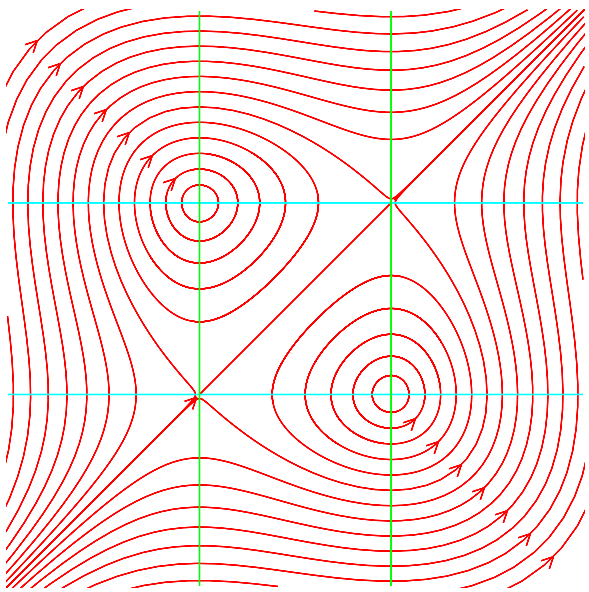
The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 18y \\ 18x & 0 \end{bmatrix},$$

so the trace is $\tau = 0$ and the determinant is $-324xy$.

At u_1 and u_2 we have $\delta = -36 < 0$ so there is a saddle. The Jacobian is $\pm 6 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so it is easy to see that the eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

At u_3 and u_4 we have $\delta = 36 > 0$ and $\tau = 0$ so there is a centre. The bottom left entry in J is $18x$. At u_3 this is positive so the rotation is anticlockwise, and at u_4 it is negative so the rotation is clockwise.



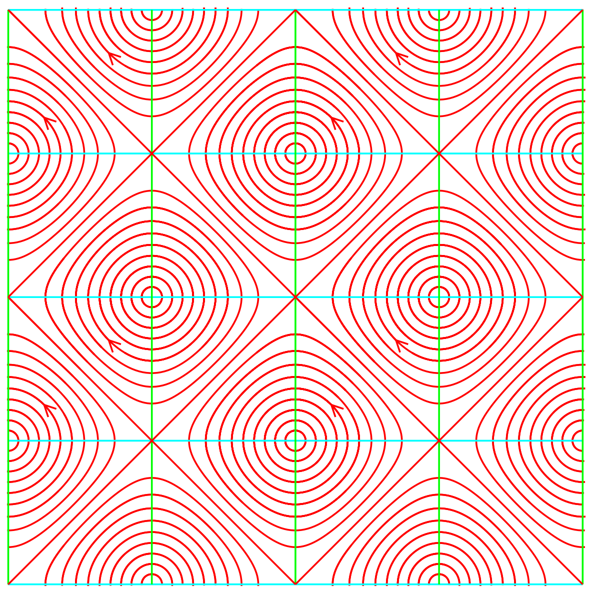
Example 16. Consider the system

$$\dot{x} = f(x, y) = \sin(\pi y)$$
$$\dot{y} = g(x, y) = \sin(\pi x).$$

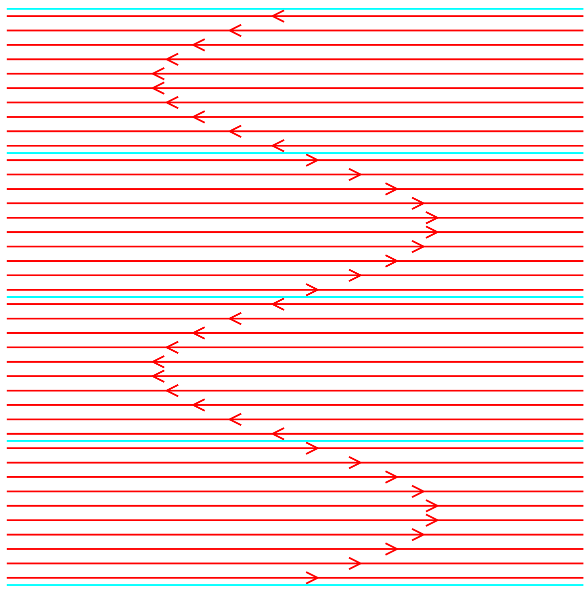
The x-nullcline is given by $\sin(\pi y) = 0$, which means that y must be in integer. Similarly, the y-nullcline is given by $\sin(\pi x) = 0$, which means that x must be in integer. Thus, the equilibrium points are of the form (n, m), where n and m are both integers. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & \pi\cos(\pi y) \\ \pi\cos(\pi x) & 0 \end{bmatrix},$$

so the trace is $\tau = 0$ and the determinant is $\delta = -\pi^2\cos(\pi x)\cos(\pi y)$. At an equilibrium point (n, m) we have $\cos(\pi x) = (-1)^n$ and $\cos(\pi y) = (-1)^m$, so $\delta = (-1)^{n+m+1}\pi^2$. If n and m are both odd, or both even, then we have $\delta = -\pi^2 < 0$, so (n, m) is a saddle. If n is odd and m is even then $\delta = \pi^2 > 0$ and $\tau = 0$ so we have a centre. The bottom left entry in J is $\cos(\pi x) = (-1)^n = -1$, so the rotation is clockwise. Similarly, if n is even and m is odd then we again have a centre, but in this case the rotation is anticlockwise.



Example 17. Consider the system where $\dot{x} = \sin(\pi y)$ and $\dot{y} = 0$. The solution is just $y = y_0$ (a constant) and $x = x_0 + \sin(\pi y_0)t$. The phase diagram just consists of horizontal lines. In all our other examples the equilibrium points are well-separated from each other. However, in this case we have a whole line of equilibrium points where $y = 0$, and another whole line of equilibrium points where $y = \pi$, and similarly for every multiple of π. The Jacobian is $J = \begin{bmatrix} 0 & \pi\cos(\pi y) \\ 0 & 0 \end{bmatrix}$, and if $y = n\pi$ this becomes $J = \begin{bmatrix} 0 & (-1)^n \\ 0 & 0 \end{bmatrix}$. This is an unusual kind of matrix with only one eigenvalue where it is not possible to find two linearly independent eigenvectors. The linearized system is called a *shear flow*.



Example 18. Consider the equations

$$\dot{x} = x^2 - y^2 + 2xy$$
$$\dot{y} = x^2 - y^2 - 2xy.$$

For an equilibrium point, both $\dot{x}$ and $\dot{y}$ must be zero. By adding and subtracting these equations, we see that $x^2 - y^2 = 0$ and $xy = 0$. As $xy = 0$ we must have $x = 0$ or $y = 0$. If $x = 0$ then the equation $x^2 - y^2 = 0$ gives $y = 0$, and if $y = 0$ then the same equation gives $x = 0$. Thus, the only equilibrium point is the origin.

The Jacobian is $J = \begin{bmatrix} 2x + 2y & 2x - 2y \\ 2x - 2y & -2x - 2y \end{bmatrix}$, which is zero at the origin. Thus, linearization does not tell us very much about the behaviour near the origin.

It is more useful to note that there are three solutions that we can write down explicitly. The first is $(x, y) = (1/(2t), -1/(2t))$. To check that this works, we have

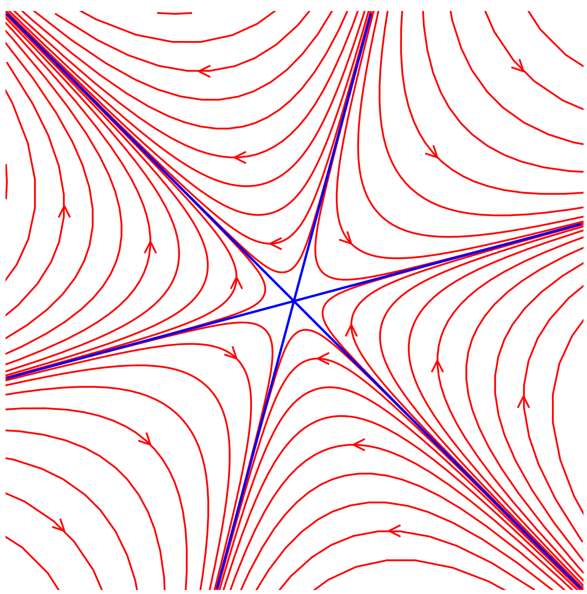
$$\dot{x} = \frac{d}{dt}\left(\frac{1}{2t}\right) = \frac{-1}{2t^2} \qquad\qquad x^2 - y^2 + 2xy = \frac{1}{4t^2}(1 - 1 - 2) = \frac{-1}{2t^2}$$
$$\dot{y} = \frac{d}{dt}\left(\frac{-1}{2t}\right) = \frac{1}{2t^2} \qquad\qquad x^2 - y^2 - 2xy = \frac{1}{4t^2}(1 - 1 + 2) = \frac{1}{2t^2}.$$

The other two solutions are

$$(x, y) = \left(\frac{-1 + \sqrt{3}}{4t}, \frac{1 + \sqrt{3}}{4t}\right) \qquad \text{and} \qquad (x, y) = \left(\frac{-1 - \sqrt{3}}{4t}, \frac{1 - \sqrt{3}}{4t}\right).$$

They can be checked in the same way. The first solution covers the line $y = -x$, the second covers the line $y = (2 + \sqrt{3})x$, and the third covers the line $y = (2 + \sqrt{3})x$.

The picture below shows the three special solutions in blue, and the remaining flow lines in red. The nullclines have not been shown.



Example 19. Consider the equations

$$\dot{x} = -x + (x^2 - y^2)/4$$
$$\dot{y} = y - (x^2 - y^2)/4.$$

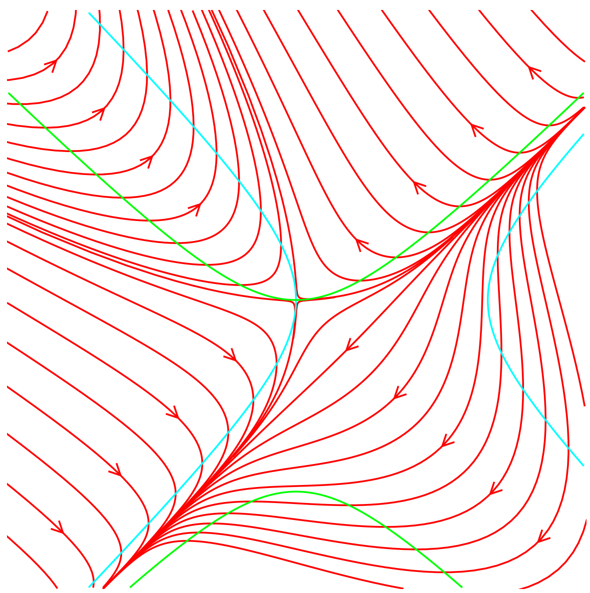
The x-nullcline is given by $-x + (x^2 - y^2)/4 = 0$, or equivalently $x^2 - 4x - y^2 = 0$. It is best to regard this as a quadratic equation for x, with solution $x = 2 \pm \sqrt{4 + y^2}$. (This makes sense for all possible values of y, because $4 + y^2$ is always positive.) Similarly, the y-nullcline is $y = -2 \pm \sqrt{x^2 + 4}$.

At an equilibrium point we must have $-x + (x^2 - y^2)/4 = 0$ and also $y - (x^2 - y^2)/4 = 0$. Adding these equations gives $y - x = 0$, so $x = y$, so $x^2 - y^2 = 0$. Substituting this back into our two equations gives $x = y = 0$. Thus, the only equilibrium point is at the origin. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} -1 + x/2 & -y/2 \\ -x/2 & 1 + y/2 \end{bmatrix},$$

which becomes $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ at the origin. The eigenvalues are ± 1, which means that the origin is a saddle.

Next, it is easy to see that the equations $x = 1 - t$ and $y = -1 - t$ give a solution to the equations, which covers the line $x - y = 2$. Together with the nullclines, this gives a reasonable sketch of the phase diagram. A detailed picture is as follows:



Example 20. Consider the system

$$\dot{x} = f(x,y) = x^2 + 2y^2 - y$$
$$\dot{y} = g(x,y) = 2x + 2y.$$

For an equilibrium point we must have $x^2+2y^2-y=0$ and $2x+2y=0$. The second equation gives $y=-x$, and substituting this into the first equation gives $3x^2+x=0$ or $x(3x+1)=0$. We thus have $x=0$ or $x=-1/3$, so the two equilibria are $u_1=(0,0)$ and $u_2=(-1/3,1/3)$. The Jacobian is

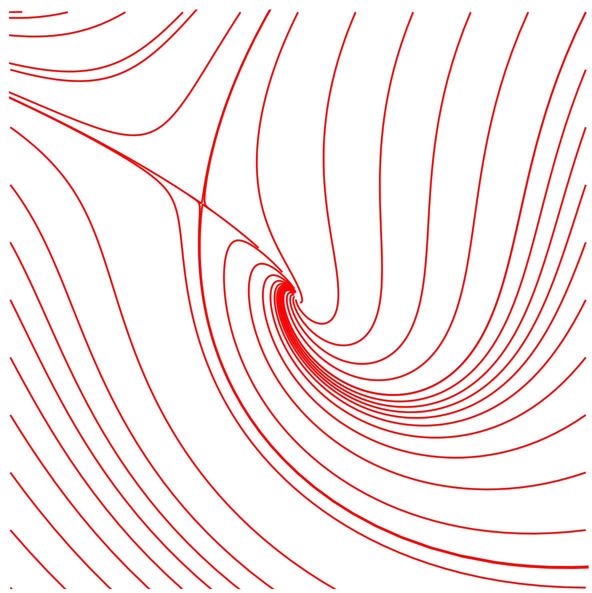
$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2x & 4y-1 \\ 2 & 2 \end{bmatrix}.$$

Evaluating this at the equilibrium points gives

$$J_1 = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} -2/3 & 5/3 \\ 2 & 2 \end{bmatrix}.$$

For J_1 we have $\tau=2$ and $\delta=2$ so $\tau^2-4\delta=-4<0$. As $\tau>0$ and $\tau^2-4\delta<0$ we see that the point u_1 is an unstable focus. As the bottom left entry in J is negative, it is a clockwise focus.

For J_2 we have $\tau=4/3$ and $\delta=-14/3$. As $\delta<0$, this is a saddle.



Example 21. Consider the system

$$\dot{x} = f(x, y) = x^2 - y^2$$
$$\dot{y} = g(x, y) = x + 1.$$

The y-nullcline is the line $x = -1$, and the x-nullcline is given by $y = \pm x$. Thus, the equilibrium points are $u_1 = (-1, 1)$ and $u_2 = (-1, -1)$. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 1 & 0 \end{bmatrix},$$

which has $\tau = 2x$ and $\delta = 2y$.

At u_1 we have $\tau = -2$ and $\delta = 2$, so $\tau^2 - 4\delta = -4$. As $\tau^2 - 4\delta < 0$ this is a focus, and as $\tau < 0$ it is stable. The bottom left entry in J is 1 which is positive, so the rotation is anticlockwise.

At u_2 we have $\delta = -2$ which is negative, so there is a saddle. The matrix J is $\begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix}$. The eigenvalues are

$$\lambda_1, \lambda_2 = (\tau \pm \sqrt{\tau^2 - 4\delta})/2 = (-2 \pm \sqrt{12})/2 = -1 \pm \sqrt{3} \simeq -2.73,\ 0.73.$$

Corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 + \sqrt{3} \\ -1 \end{bmatrix} \simeq \begin{bmatrix} 2.73 \\ -1 \end{bmatrix} \qquad\qquad v_2 = \begin{bmatrix} \sqrt{3} - 1 \\ 1 \end{bmatrix} \simeq \begin{bmatrix} 0.73 \\ 1 \end{bmatrix}.$$

These determine the angles of the flow lines approaching the saddle.