MEIJER G-FUNCTIONS

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Definition 0.1. We write W for the Weyl algebra generated over $\mathbb{C}$ by z and ∂ subject to the relation $[\partial, z] = 1$. We give this a grading with $|z| = 1$ and $|\partial| = -1$. We put $\Delta = z\,\partial \in W_0$.

Remark 0.2. It is not hard to see that $\{\Delta^k \mid k \geq 0\}$ and $\{z^k\partial^k \mid k \geq 0\}$ are both bases for W_0 over $\mathbb{C}$.

Definition 0.3. We put $M = W_1 + W_0$, which is a bimodule for W_0. We call the elements of M *Meijer operators*. Any such operator can be written in the form $L = z\,f(\Delta) - g(\Delta)$ for some polynomials f and g. The *bidegree* of L is the pair $(\deg(f), \deg(g))$ (with the convention $\deg(0) = -\infty$).

We will study the sets $\operatorname{ann}(L, U) = \{u \in U \mid Lu = 0\}$ for various W-modules U:

Definition 0.4.

(a) We write H for the space of holomorphic functions $u(z)$ on $\mathbb{C}^\times$, with ∂ acting as differentiation and z acting as multiplication by the identity function. We call this the *holomorphic module*.

(b) We write S for the space of doubly infinite sequences $(a_k)_{k\in\mathbb{Z}}$ that are rapidly decreasing in the sense that $|k^N a_k| \to 0$ as $|k| \to \infty$ for all $N \geq 0$. This can be regarded as a W-module by the rules $(\partial a)_k = (k+1)a_{k+1}$ and $(za)_k = a_{k-1}$ (so $(\Delta a)_k = k\,a_k$). We also write S_0 for the subset of sequences where $a_k = 0$ for $|k| \gg 0$. We call this the *series module*.

(c) We write E for the space of holomorphic functions $m(t)$ on $\mathbb{C}$, with ∂ acting as $e^{-t}\frac{d}{dt}$ and z as multiplication by e^t. We call this the *exponential module*.

(d) We write F for the space of meromorphic functions $v(s)$ on $\mathbb{C}$, with Δ acting as multiplication by s, and z acting as the shift operator $(zv)(s) = v(s-1)$. We call this the *Mellin module*.

Remark 0.5. The exponential module is useful because $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is a universal cover and it turns out that this is sufficient to handle all monodromy issues for operators of bidegree (p, q) with $p \neq q$. If $p = q$ then certain relevant functions will have a pole at $z = 1$ as well as $z \in \{0, \infty\}$ so we need to consider the universal cover of $\mathbb{C} \setminus \{0, 1\}$ by the elliptic modular function instead. We will return to this later.

Definition 0.6. We define $\tau\colon E \to E$ by $(\tau m)(t) = m(t + 2\pi i)$. For $\alpha \in \mathbb{C}^\times$ we put

$$E_\alpha = \ker(\tau - \alpha) = \{m \in E \mid m(t + 2\pi i) = \alpha\, m(t) \text{ for all } t\}$$
$$\widetilde{E}_\alpha = \bigcup_{n\geq 0} \ker((\tau-\alpha)^n) = \mathbb{C}[t].E_\alpha$$

We can consider various homomorphisms between the above modules.

(a) Taylor expansion gives an injective homomorphism $\tau\colon H \to S$. In fact, it is well-known that the Fourier transform gives an isomorphism from S to the space of smooth functions on the circle, and this converts τ to the obvious restriction map.

(b) Identifying z with e^t gives an isomorphism between H and $E_0 < E$.

(c) Given a function $v(s) \in F$ we can choose a contour C in the Riemann sphere and attempt to define $u(z) = \oint_C v(s)\, z^s\, ds$, but this can fail in various ways to be well-defined. This construction should give a homomorphism between certain groups related to H and F, called the *Mellin transform*. However, I am not yet sure of the best formulation for this.

Proposition 0.7. *If L has bidegree (p, q) with $p < q$ then* $\operatorname{ann}(L, E)$ *has dimension q over* $\mathbb{C}$.

Proof. We have $L = zF - G$, where $F \sum_{k=0}^p a_k z^k \partial^k$ and $G = \sum_{k=0}^q b_k z^k \partial^k$ say with $a_p, b_q \neq 0$. This means that L acts on E as the operator

$$e^t \sum_{k=0}^p a_k \frac{d^k}{dt^k} - \sum_{k=0}^q b_k \frac{d^k}{dt^k}.$$

This is $-b_q$ times a monic polynomial of degree q in $\frac{d}{dt}$, with holomorphic coefficients. The standard Frobenius method now shows that for any t_0, the kernel of L on holomorphic germs at t_0 has dimension q. The spaces of local solutions form a vector bundle with flat connection over the simply connected space $\mathbb{C}$, so the evident map from global solutions to germs at 0 is an isomorphism. $\square$

Corollary 0.8. *If L has bidegree (p,q) with $p > q$ then $\operatorname{ann}(L,E)$ has dimension p over $\mathbb{C}$.*

Proof. If $L = zf(\Delta) - g(\Delta)$, put $L^* = zg(-\Delta) - f(-\Delta)$. The proposition shows that $\operatorname{ann}(L^*,E)$ has dimension p, and one can check that composition with $t \mapsto -t$ gives an isomorphism $\operatorname{ann}(L,E) \simeq \operatorname{ann}(L^*,E)$. $\square$

Corollary 0.9. *If L has bidegree (p,q) with $p \neq q$ then $\operatorname{ann}(L,E) = \bigoplus_{\alpha\neq 0}(\operatorname{ann}(L,\widetilde{E}_\alpha))$.*

Proof. It is not hard to see that $\operatorname{ann}(L,E)$ is preserved by τ. As $\operatorname{ann}(L,E)$ is also finite-dimensional, it must split as a direct sum of its generalised eigenspaces. Note also that τ is invertible, so all eigenvalues are nonzero. The claim is clear from this. $\square$

We now study the spaces $K = \operatorname{ann}(L,S)$ and $K_0 = K \cap S_0$, where again $L = zf(\Delta) - g(\Delta)$ has bidegree (p,q). Put $P = \{n \in \mathbb{Z} \mid f(n) = 0\}$ and $Q = \{n \in \mathbb{Z} \mid g(n) = 0\}$ (so $|P| \leq p$ and $|Q| \leq q$, and often P and Q will be empty). Suppose for the moment that $p < q$. If $P = \emptyset$ we will show that $K = K_0 = 0$. If $P \neq \emptyset$ then the most common situation is that $\dim(K) = 1$ and $\dim(K_0) = 0$, but it will take a little work to formulate a precise statement. We put

$$R = \{i \in \mathbb{Z} \mid \exists j \in P \text{ with } j > i \text{ and } \{i, i+1, \ldots, j-1\} \cap Q = \emptyset\}.$$

Proposition 0.10.

(a) *The restriction map $K \to \operatorname{Map}(R,\mathbb{C})$ is zero.*
(b) *The restriction map $K \to \operatorname{Map}(P \setminus R,\mathbb{C})$ is an isomorphism.*
(c) *We have $\dim(K) = |P \setminus R| \leq \min(|P|, |Q|+1)$.*
(d) *If $\max(P) \leq \max(Q)$ then $K = K_0$. Otherwise there is a unique element $b \in K$ with $b_{\max(P)} = 1$ and $b_i = 0$ for $i < \max(P)$, and we have $K = K_0 \oplus \mathbb{C}b$.*

Proof. First note that K is just the space of rapidly decreasing sequences a satisfying $f(k-1)a_{k-1} = g(k)a_k$ for all k.

Suppose that $a \in K$ and $i \in R$, so there exists $j > i$ with $g(j) = 0$ and $f(k) \neq 0$ for $i \leq k < j$. The recurrence relation gives

$$f(i)f(i+1)\cdots f(j-1)a_i = g(i+1)g(i+2)\cdots g(j)a_j,$$

from which we deduce that $a_i = 0$. This proves (a).

Next, note that for $k \ll 0$ we will have $f(k-1), g(k) \neq 0$ so we can write the recurrence relation as $a_{k-1} = a_k\, g(k)/f(k-1)$. As $p < q$ we have $|g(k)/f(k-1)| \to \infty$ as $k \to -\infty$. Thus, the only way the sequence can be rapidly decreasing is if $a_k = 0$ for $k \ll 0$. Now suppose that $a_{k-1} = 0$; we claim that a_k is also zero. If $k \in R$ then this holds by part (a), if $k \in P \setminus R$ then it holds by assumption, and if $k \notin P$ then it follows from the relation $f(k-1)a_{k-1} = g(k)a_k$. It now follows by induction that $a = 0$, so the restriction $K \to \operatorname{Map}(P \setminus R,\mathbb{C})$ is injective.

Now suppose we have $i \in P \setminus R$. If i is maximal in P, we put

$$b_{ik} = \begin{cases} 0 & \text{if } k < i \\ \prod_{j=i+1}^{k} \frac{f(j-1)}{g(j)} & \text{if } k \geq i. \end{cases}$$

This gives an element $b_i \in K$, which lies in K_0 iff $\max(Q) \geq i = \max(P)$. We are using the standard convention that the empty product is one, so $b_{ii} = 1$, but $b_{ij} = 0$ for all $j \in P \setminus \{i\}$.

Suppose instead that i is not maximal in P, and let j be the smallest element in P with $j > i$. As $i \notin R$ the set $\{i, i+1, \ldots, j-1\} \cap Q$ must be nonempty; let m be the smallest element. Put

$$b_{ik} = \begin{cases} 0 & \text{if } k < i \text{ or } k > m \\ \prod_{j=i+1}^{k} \frac{f(j-1)}{g(j)} & \text{if } i \leq k \leq m. \end{cases}$$

Again we have $b_i \in K$ with $b_{ii} = 1$ and $b_{ij} = 0$ for $j \in P \setminus \{i\}$.

All claims are now clear except for the fact that $|P \setminus R| \leq |Q| + 1$. This holds because every element of $P \setminus R$ is either maximal in P or dominated by an element of Q. $\square$

Remark 0.11. Suppose that $L = z\, f(\Delta) - g(\Delta)$ and $L^* = z\, g(-\Delta) - f(-\Delta)$. We find that the map $(a_n)_{n\in\mathbb{Z}} \to (a_{-n})_{n\in\mathbb{Z}}$ gives an isomorphism $\operatorname{ann}(L, S) \simeq \operatorname{ann}(L^*, S)$. Using this we can understand $\operatorname{ann}(L, S)$ in the case where $p > q$. The case where $p = q$ will require a slightly different approach.

Definition 0.12. Suppose that $L = z\, f(\Delta) - g(\Delta)$, where

$$f(t) = \alpha \prod_{j=1}^{p} (t - a_j)$$

$$g(t) = \beta \prod_{j=1}^{q} (t - b_j).$$

Suppose that $m \in \mathbb{C}$ is such that $\exp(m) = (-1)^p \alpha/\beta$. We then put

$$v(s) = v_{L,m}(s) = e^{ms} \prod_{j=1}^{p} \Gamma(a_i - s + 1)^{-1} \prod_{j=1}^{q} \Gamma(s + 1 - b_i)^{-1}.$$

Recall that the Gamma function has poles but no zeros, so $v(s)$ is holomorphic.

Proposition 0.13. *The map $w(z) \mapsto w(e^{2\pi i s})v(s)$ gives an isomorphism from the space of meromorphic functions on $\mathbb{C}^\times$ to $\operatorname{ann}(L, F)$.*

Proof. It is not hard to see that a nonzero meromorphic function $u \in F$ satisfies $Lu = 0$ if and only if $u(s)/u(s-1) = f(s-1)/g(s)$. Using the functional equation $x\,\Gamma(x) = \Gamma(x+1)$ one can check that $v(s)$ has the above property, so $u \in \operatorname{ann}(L, F)$. If v is another element of $\operatorname{ann}(L, F)$ then we have $(u/v)(s) = (u/v)(s-1)$, so $(u/v)(s) = w(e^{2\pi i s})$ for some holomorphic function on $\mathbb{C}^\times$, as claimed. $\square$

To understand the nature of $v_{L,m}(s)$ and related functions, we need to know about the asymptotics of the Gamma function.

References