MAS290 METHODS FOR DIFFERENTIAL EQUATIONS — FORMULAE

You should learn and remember all the formulae on this sheet.

Matrices

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the trace is $\tau = a + d$ and the determinant is $\delta = ad - bc$. The eigenvalues are $\lambda_1 = (\tau - \sqrt{\tau^2 - 4\delta})/2$ and $\lambda_2 = (\tau + \sqrt{\tau^2 - 4\delta})/2$, and we also have $\lambda_1 + \lambda_2 = \tau$ and $\lambda_1\lambda_2 = \delta$. The corresponding linear system can be classified as follows.

- If $\delta < 0$ then λ_1 and λ_2 are real with $\lambda_1 < 0 < \lambda_2$, and we have a saddle.
- If $\delta > 0$ and $\tau > 0$ and $\tau^2 - 4\delta > 0$ then λ_1 and λ_2 are real with $0 < \lambda_1 < \lambda_2$, and we have an unstable node.
- If $\delta > 0$ and $\tau > 0$ and $\tau^2 - 4\delta < 0$ then λ_1 and λ_2 are complex with $0 < \mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)$, and we have an unstable focus. The rotation is anticlockwise if $b < 0 < c$, and clockwise if $c < 0 < b$.
- If $\delta > 0$ and $\tau = 0$ then we have a centre. The eigenvalues are $\pm i\omega$, where $\omega = \sqrt{|\delta|}$. The rotation is anticlockwise if $b < 0 < c$, and clockwise if $c < 0 < b$.
- If $\delta > 0$ and $\tau < 0$ and $\tau^2 - 4\delta < 0$ then λ_1 and λ_2 are complex with $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) <$, and we have a stable focus. The rotation is anticlockwise if $b < 0 < c$, and clockwise if $c < 0 < b$.
- If $\delta > 0$ and $\tau < 0$ and $\tau^2 - 4\delta > 0$ then λ_1 and λ_2 are real with $\lambda_1 < \lambda_2 < 0$, and we have a stable node.
- Cases where $\delta = 0$ or $\tau^2 - 4\delta = 0$ will not be discussed here.

Fundamental solutions

The fundamental solution for a matrix A is a matrix P depending on t with $\dot{P} = AP$ and $P = I$ when $t = 0$.

- Suppose that there are eigenvalues λ_1 and λ_2 with corresponding eigenvectors v_1 and v_2 that are linearly independent. Put
$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}.$$
Then $A = VDV^{-1}$ and $P = VEV^{-1}$.
- If $\lambda_1 \neq \lambda_2$ then we also have
$$P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A)$$
- If A has complex eigenvalues $\lambda \pm i\omega$ (with $\omega \neq 0$) then the above formula can also be written as
$$P = e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I))$$
- If A has only one eigenvalue λ, then we instead have
$$P = e^{\lambda t}(I + t(A - \lambda I)).$$
- In all cases we have $\det(P) = e^{\tau t}$, where $\tau = \mathrm{trace}(A) = \lambda_1 + \lambda_2$.

Definiteness of quadratic functions

Consider a quadratic function $Q = ax^2 + 2bxy + cy^2$.

- If $ac - b^2 > 0$ and $a, c > 0$ then Q is positive definite.
- If $ac - b^2 > 0$ and $a, c < 0$ then Q is negative definite.
- If $ac - b^2 \leq 0$ then Q is neither positive definite nor negative definite.

Constant coefficients

Consider an equation $Ay'' + By' + Cy = 0$, where A, B and C are constant with $A \neq 0$. Let λ_1 and λ_2 be the roots of the auxiliary polynomial $At^2 + Bt + C$.

- If $\lambda_1 \neq \lambda_2$ then the general solution is $y = Pe^{\lambda_1 x} + Qe^{\lambda_2 x}$ with P and Q constant.
- If $\lambda_1, \lambda_2 = \lambda \pm i\omega$ (with $\omega \neq 0$) then the general solution can also be given in the form $y = e^{\lambda x}(M\cos(\omega x) + N\sin(\omega x))$ with M and N constant.
- If there is only one root λ, then the general solution is $y = e^{\lambda x}(P + Qx)$.

Reduction of order

Suppose that y satisfies $Ay'' + By' + Cy = 0$. Put $v = \int B/A\,dx$ and $u = \int y^{-2}e^{-v}\,dx$ and $z = uy$. Then we also have $Az'' + Bz' + Cz = 0$.

Sturm-Liouville form

Consider an operator $L(y) = Ay'' + By' + Cy$. Then we also have $L(y) = ((py')' + qy)/r$, where

$$v = \int B/A\,dx \qquad p = e^v \qquad q = pC/A \qquad r = p/A.$$

Normal form

Consider an operator $L(y) = y'' + Py' + Qy$. Put

$$v = \int P\,dx \qquad m = e^{-v/2} \qquad R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2 \qquad z = y/m.$$

Then $y'' + Py' + Qy = 0$ if and only if $z'' + Rz = 0$.