

Learning Composable Chains-of-Thought

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Abstract

A common approach for teaching large language models (LLMs) to reason is to train on chain-of-thought (CoT) traces of in-distribution reasoning problems, but such annotated data is costly to obtain for every problem of interest. We want reasoning models to generalize beyond their training distribution, and ideally to generalize compositionally: combine atomic reasoning skills to solve harder, unseen reasoning tasks. We take a step towards compositional generalization of reasoning skills when addressing a target compositional task that has no labeled CoT data. We find that simply training models on CoT data of atomic tasks leads to limited generalization, but minimally modifying CoT formats of constituent atomic tasks to be **composable** can lead to improvements. We can train “atomic CoT” models on the atomic tasks with Composable CoT data and combine them with multitask learning or model merging for better zero-shot performance on the target compositional task. Such a combined model can be further bootstrapped on a small amount of compositional data using rejection sampling fine-tuning (RFT). Results on string operations and natural language skill compositions show that training LLMs on Composable CoT outperforms multitask learning and continued fine-tuning baselines within a given training data budget.¹

1 Introduction

Large language models (LLMs) have succeeded at addressing many problems largely by virtue of the massive amounts of data they are trained on. Many problems that defied earlier approaches have become easy because they are now in-distribution for models that have seen similar data during pre-training or instruction-tuning. However, these models still fail at challenging reasoning tasks and it is impossible to scale training data to cover all possible tasks of interest. Ideally, we want models that can *generalize* to new settings, and particularly, can apply basic “skills” learned during training in novel combinations to solve problems at inference time. How to empower LLMs with this capability, also called compositional generalization [1, 2, 3, 4], remains an open question. For instance, large reasoning models [5, 6], built on pre-trained LLMs, are typically trained on a large amount of data annotated with chain-of-thought (CoT) traces, but demonstrating how to generalize from easy problems to harder ones would break the dependence on the scale of training data [7, 8], and would facilitate more efficient and robust reasoning with LLMs.

We explore the setting of compositional reasoning where pre-trained LLMs are fine-tuned on CoT data of simple reasoning tasks (atomic tasks) and then evaluated on the combinations of them (compositional tasks) with limited compositional supervision. We find that models trained with atomic CoT data of the standard CoT format demonstrate limited generalization: they typically memorize and reproduce the atomic CoT patterns rather than successfully *composing* them. We propose a simple modification of the CoT format of the atomic task training data, which we call

¹Code and data are available at: https://github.com/fc2869/composable_cot.

While CoT reasoning can be elicited through zero-shot prompting [9, 10], recent works show that fine-tuning pre-trained LLMs on CoT traces leads to strong reasoning models [11, 6]. We define a dataset for a reasoning task \mathcal{T} as a set of (prompt, answer) pairs: $D_{\mathcal{T}} = \{(\mathbf{q}, a)\}$. A dataset with CoT traces is then $D_{\mathcal{T}}^{\text{CoT}} = \{(\mathbf{q}, \mathbf{t}, a)\}$. To fine-tune M parametrized as θ on $D_{\mathcal{T}}^{\text{CoT}}$ with supervised fine-tuning, we minimize the following supervised learning loss objective: $\mathcal{L}_{D_{\mathcal{T}}^{\text{CoT}}}(\theta) = -\frac{1}{N} \sum^N (\log p_{\theta}(\mathbf{t} \mid \mathbf{q}) + \log p_{\theta}(a \mid \mathbf{q}, \mathbf{t}))$ where $|D_{\mathcal{T}}^{\text{CoT}}| = N$.

Atomic and compositional tasks Consider a set of k tasks that represent basic reasoning skills $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k$, which we call **atomic** tasks. We define **compositional** tasks $\mathcal{T}_{(i,j)}$ where $i, j \in [1, \dots, k]$ and $i \neq j$, as those tasks that can be expressed as a composition of two atomic tasks. That is, there exist \mathcal{T}_i and \mathcal{T}_j such that $\mathcal{T}_{(i,j)} = g(\mathcal{T}_i, \mathcal{T}_j)$ where g is some function to combine the two atomic tasks. We discuss more details for g in Appendix A.

We define a collection of reasoning skills as *basic* if each skill cannot be reduced into a combination of others; i.e., each task cannot be efficiently solved by a rule-based transformation of the solution of some other combination of tasks.

Compositional reasoning from atomic CoT For a pair of atomic tasks \mathcal{T}_i and \mathcal{T}_j sampled from the k tasks, we assume access to atomic CoT data $D_{\mathcal{T}_i}^{\text{CoT}}$ and $D_{\mathcal{T}_j}^{\text{CoT}}$ with dataset sizes of N_i and N_j . We define models fine-tuned on atomic CoT data as **atomic CoT models**.

We assume for their composition $\mathcal{T}_{(i,j)}$, we only have access to a training dataset $D_{\mathcal{T}_{(i,j)}}$ of size $N_{(i,j)}$. We make two assumptions about this data which follow from practical considerations about how our compositional CoTs would work in practice. First, we assume that this data only contains the direct answer and *not* a labeled example of a CoT. This reflects that high-quality annotated CoT supervision may be harder to obtain in practice than correct answers. Second, we assume that $N_{(i,j)}$ is small. We may be able to collect a small amount of data for each new compositional task of interest, but these compositional tasks are too numerous to undertake large-scale data collection on. We are interested in the following question: *Can we achieve good compositional performance by fine-tuning with the atomic CoT data and a limited amount of compositional direct answer data?*

3 Learning Composable Chains-of-Thought

3.1 Constructing Composable CoT Training Data

Consider an atomic CoT dataset $D_{\mathcal{T}}^{\text{CoT}} = \{(\mathbf{q}, \mathbf{t}, a)\}$ for $\mathcal{T} \in \{\mathcal{T}_i, \mathcal{T}_j\}$ and we call it **standard CoT** data.

Given a pair of atomic tasks, we assume the CoT traces in each atomic task data follow a certain distribution distinct to that dataset. A pre-trained LLM M_0 fine-tuned on the standard CoT data is only optimized to generate CoT traces that can replicate patterns in those two distributions. It is unclear whether a fine-tuned model can produce compositional CoTs for a prompt drawn from the compositional task distribution, as this will generally be out-of-distribution from the perspective of each of the two datasets. Without additional supervision signals, such fine-tuned models typically only replicate one of the learned atomic reasoning patterns in the generated CoT; we show the empirical evidence for this in Section 5.3.

In order to generate two atomic CoTs in one sequence $\mathbf{t}_i \mathbf{t}_j$, the model must allocate substantial probability $p(\mathbf{t}_i \mid \mathbf{q} \mathbf{t}_j)$, despite these not being in the training distribution. Our goal is to make this as in-distribution as possible even for a model that does not train on explicit compositional examples.

Construction We define a set of *chain-of-thought tags* $\mathcal{P} = \{p_1, \dots, p_n\}$ for $k \in \{1, \dots, n\}$.

q
t₁
...
t_k
a

Question: Multiply the ASCII value of "a" by 2.
<tag 1> aaksebnab zldjxhl ... </tag 1>
... Additional proxy prefix CoTs
<tag k> The ASCII value of the letter a is 97, and [...] </tag k>
Answer: 194

Figure 2: Construction of Composable CoT data with k chain-of-thought tags. We insert $k - 1$ proxy prefixes at the end of the prompt, before the generation of \mathbf{t}_k .

For each training example $D_{\mathcal{T}}^{\text{CoT}} = (\mathbf{q}, \mathbf{t}, a)$ in the atomic CoT dataset, we randomly determine whether to augment it with one of the n tags. If an example is assigned tag p_k , then we want to treat it as the k -th step in a notional compositional reasoning process. To achieve this, we append $k - 1$ proxy prefix CoTs ($\mathbf{t}'_1, \dots, \mathbf{t}'_{k-1}$) to the end of the prompt: $D_{\mathcal{T}}^{\text{suf}} = (\mathbf{q}\mathbf{t}'_1 \dots \mathbf{t}'_{k-1}, \mathbf{t}_k)$ where $\mathbf{t}_k = \langle \text{suffix} \rangle \mathbf{t} a \langle / \text{suffix} \rangle$. The procedure is illustrated in Figure 2. We discuss different design choices for forming the proxy prefix CoTs in Appendix B. Here we present a simple yet effective approach where the proxy prefix CoT is a sequence of randomly sampled letters of a random length. Such a design aims at teaching models to generate robust continuation following an arbitrary prefix CoT. While the method applies to $n > 2$, we focus on the case where $n = 2$ for simplicity, i.e., we only have a **prefix CoT** tag and a **suffix CoT** tag. Concretely, this gives us an implementation consisting of a **prefix CoT** \mathbf{t}_{pre} , which is a CoT conditioned only on the prompt \mathbf{q} , and a suffix CoT \mathbf{t}_{suf} is conditioned on the prompt and a proxy prefix CoT. The prefix CoT dataset is constructed by the following: $D_{\mathcal{T}}^{\text{pre}} = (\mathbf{q}, \mathbf{t}_{\text{pre}})$ where $\mathbf{t}_{\text{pre}} = \langle \text{prefix} \rangle \mathbf{t} a \langle / \text{prefix} \rangle$.

Training objective: Given our construction, we fine-tune M_0 with the augmented dataset $D_{\mathcal{T}}^{\text{aug}} = D_{\mathcal{T}}^{\text{pre}} + D_{\mathcal{T}}^{\text{suf}}$ with a multitask learning objective to minimize the sum of the negative log likelihood of generating n prefix CoTs and m suffix CoTs, where $N_{\text{pre}} + N_{\text{suf}} = N$:

$$\mathcal{L}_{D_{\mathcal{T}}^{\text{aug}}}(\theta) = \mathcal{L}_{D_{\mathcal{T}}^{\text{pre}}}(\theta) + \mathcal{L}_{D_{\mathcal{T}}^{\text{suf}}}(\theta) = -\frac{1}{N_{\text{pre}}} \sum \log p_{\theta}(\mathbf{t}_{\text{pre}} | \mathbf{q}) - \frac{1}{N_{\text{suf}}} \sum \log p_{\theta}(\mathbf{t}_{\text{suf}} | \mathbf{q}, \mathbf{t}'_{\text{pre}})$$

3.2 Combining Atomic CoT Models

ComposableCoT-MTL We apply multitask learning (MTL) to fine-tune M_0 on the combined dataset of $D_{\mathcal{T}_i}^{\text{aug}} + D_{\mathcal{T}_j}^{\text{aug}}$ and obtain a single MTL model M_{comb} that can generate prefix and suffix CoTs for both atomic tasks.

ComposableCoT-Merge Model merging is another way to combine multiple models into a single multi-task model [12, 13, 14]. Starting from M_0 , we fine-tune two models M_i and M_j (parametrized by θ_i and θ_j) on $D_{\mathcal{T}_i}^{\text{aug}}$ and $D_{\mathcal{T}_j}^{\text{aug}}$ respectively to optimize for atomic task performance. Then we use Task Arithmetic [12] to merge the two models into a single model M_{comb} parametrized by θ_{comb} as a linear combination of the differences between the two fine-tuned parameters and the base model parameter: $\theta_{\text{comb}} = \alpha(\theta_i - \theta_0) + \beta(\theta_j - \theta_0) + \theta_0$ where α and β are tunable scaling factors.

Inference Both variants of M_{comb} can be used for **zero-shot** evaluation on the compositional task $\mathcal{T}_{(i,j)}$. At inference time, we sample a response from M_{comb} , append $\langle \text{suffix} \rangle$ to the end of the generated response when it stops generation, and continue generation until the model stops again.

3.3 Improving Composition with Rejection Sampling Fine-tuning

M_{comb} can be further improved with self-taught reasoning [15] by rejection sampling fine-tuning (RFT) [16, 17] on the limited compositional data. Recall that for the compositional task, we only have the direct answer labels instead of CoT traces. M_{comb} can serve as a starting point for RFT where we fine-tune M_{comb} with its own, correct CoT responses on the compositional task.

Algorithm 1 shows the algorithm. Concretely, we sample responses from M_{comb} for each example in the compositional training data. Using the direct answer labels to verify the sampled responses, we can collect a supervised fine-tuning dataset D_{RFT} to continued fine-tune M_{comb} . Such a process can be repeated for multiple iterations. Note that some tasks do not require a single correct answer to a given question (e.g. open-ended generation), and it would be hard to verify the correctness of sampled outputs only based on direct answer labels. For those tasks, we follow [15, 18] to perform rationalization: we first append the direct answer label to the end of the prompt and sample post-hoc explanations for the given answer from the model; because M_{comb} is optimized to generate an answer following a CoT, we extract the generated answer following the generated explanation and filter out explanations whose following answer is not the same as the provided gold answer; finally, we use the accepted explanations as surrogates for CoT to form the RFT data.

Algorithm 1 Bootstrapping Atomic CoT Models Trained on Composable CoT

Input: The combined model M_{comb} ; dataset $D_{\mathcal{T}_{(i,j)}} = \{(\mathbf{q}_v, a_v)\}_{v=1}^{N_{(i,j)}}$; the number of iterations c .
Output:

```
1:  $M_0 \leftarrow M_{\text{comb}}$  ▷ Initialization
2: for  $w$  in  $1 \dots c$  do
3:   if use rationalization then
4:      $(\tilde{\mathbf{t}}_v, \tilde{a}_v) \leftarrow M_{w-1}(q_v a_v) \forall v \in \{1, \dots, N_{(i,j)}\}$  ▷ Performance rationalization
5:   else
6:      $(\tilde{\mathbf{t}}_v, \tilde{a}_v) \leftarrow M_{w-1}(q_v) \forall v \in \{1, \dots, N_{(i,j)}\}$ 
7:   end if
8:    $D_{\text{RFT}} \leftarrow \{(\mathbf{q}_v, \tilde{\mathbf{t}}_v, a_v) \text{ s.t. } v \in \{1, \dots, N_{(i,j)}\} \text{ and } \tilde{a}_v = a_v\}$  ▷ CoTs with correct answers
9:    $M_w \leftarrow \text{SFT}(M_{\text{comb}}, D_{\text{RFT}})$  ▷ Fine-tune the combined model on the accepted CoT data
10: end for
```

4 Experimental Setup

We evaluate on two sets of tasks: a set of **string operation** tasks and tasks derived from the **Skill-Mix** [19] dataset. Each setting involves atomic tasks and compositional tasks. We ensure that all atomic tasks are learnable through supervised fine-tuning with a small amount of training data ($N_i, N_j \leq 500$); the single task learning performance can be found in Appendix E

String operation tasks We consider the following atomic tasks that involve string operations. (1) **Last letter in alphabet**: Determine the next letter in the alphabet following the last letter in a sequence of letters. (2) **Letter concentration**: Adapted from [9, 20, 4], this task prompts the LLM to concatenate the first, second, second-to-last, or last letter of each word in a given sequence of words. (3) **ASCII multiplication**: Perform multiplicative operations of the ASCII value of a given letter.

We consider the following compositions of the atomic tasks.

1. Last letter + multiplication: Given a sequence of letters, find the next letter in the alphabet following the last letter, determine its ASCII value, and then perform multiplication with a given constant.
2. Concatenation + last letter: Given a sequence of words, concatenate the first, second, or second-to-last letter of each word and then find the next letter in the alphabet following the last letter of the concatenated sequence.
3. Concatenation + multiplication: Given a sequence of words, concatenate the first, second, or second-to-last letter of each word, find the ASCII value of the last letter of the concatenated sequence, and then perform multiplication.

Data and CoT traces of these tasks are automatically generated using fixed templates. The data generation procedure and the examples can be found in Appendix C.

Skill-Mix Given the definition and an example of a language skill (e.g. hyperbole), the model needs to write a sentence to demonstrate the skill about a given topic. Because each language skill in Skill-Mix dataset consists of only a few examples, we consider an atomic task to be handling skills over a *category* of skills, and we evaluate on two categories that are mainly mutually exclusive: literary devices (*Skill-Mix-Literary*) and rhetorical devices (*Skill-Mix-Rhetorical*). Atomic CoT traces for Skill-Mix are distilled from GPT-4o [21], following [22]. Examples and details can be found in Appendix D. The composition tasks we consider combine **literary** and **rhetorical** skills: generate a sentence to demonstrate two provided skills, each of which is sampled from one of the categories.

Evaluation Metrics All string operation tasks are evaluated using exact match accuracy and a regex-based answer extractor is used to extract the answer from the generated response. For Skill-Mix tasks, we use the metrics from [19] to measure the quality of the generated sentence based on a rubric (namely, *Full Marks* and *Skill Fraction*), and use GPT-4o-mini for automatic generation. Details can be found in Appendix D.2.

Zero-shot/Few-shot Baselines Figure 3 summarizes the high-order variables of the configurations we evaluate. For zero-shot compositional generalization, we include the following baselines: (1)

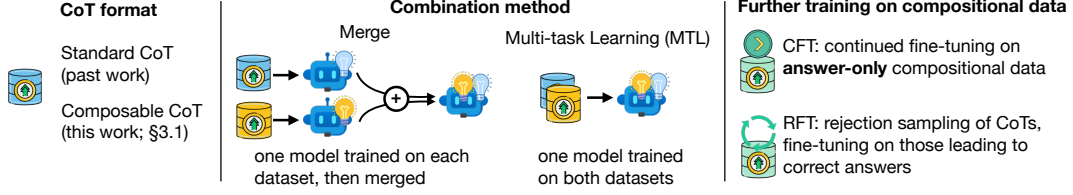


Figure 3: Summary of settings for methods evaluated. Names in the results table reference configurations described in this figure; e.g., ComposableCoT-Merge uses ComposableCoTs with model merging, and in the zero-shot setting does not use further tuning.

Few-shot direct answer prompting: we prompt M_0 with 5-shot demonstrations drawn from the compositional data; (2) Few-shot CoT prompting: we prompt M_0 with 5-shot CoT demonstrations drawn from the *atomic* data; (3) Model merging of atomic CoT models (*StandardCoT-Merge*): we fine-tune two models M_i and M_j based on M_0 with $D_{\mathcal{T}_i}^{\text{CoT}}$ and $D_{\mathcal{T}_j}^{\text{CoT}}$ respectively and merge them into M_{comb} with Task Arithmetic; (4) Multitask learning of atomic CoTs (*StandardCoT-MTL*): we fine-tune M_0 to be a single multitask learning model $M_{\text{SCoT-MTL}}$ on $D_{\mathcal{T}_i}^{\text{CoT}} + D_{\mathcal{T}_j}^{\text{CoT}}$.

Baselines with Compositional Supervision With the *same* compositional training dataset with only the answer label $D_{\mathcal{T}_{(i,j)}}$, we compare bootstrapping Composable CoT with the following baselines. (1) Continued fine-tuning (CFT) the multitask model of atomic CoTs (*CFT on StandardCoT-MTL*): we continue fine-tune the multitask model $M_{\text{SCoT-MTL}}$ on $D_{\mathcal{T}_{(i,j)}}$; (2) Continued fine-tuning the merged model of atomic CoTs (*CFT on StandardCoT-Merge*): we continue fine-tune the merged model of the two atomic CoT models M_{comb} on $D_{\mathcal{T}_{(i,j)}}$; (3) Multitask learning of atomic CoTs and compositional answers (*StandardCoT + Comp Answer*): we fine-tune a single multitask learning model based on M_0 on the combined dataset of $D_{\mathcal{T}_i}^{\text{CoT}} + D_{\mathcal{T}_j}^{\text{CoT}} + D_{\mathcal{T}_{(i,j)}}$. We also include supervised learning baselines (SFT) where M_0 is fine-tuned on the same compositional answer data $D_{\mathcal{T}_{(i,j)}}$.

The differences of methods we evaluate for each setting are summarized in Table 10.

Models and Training We use Llama 2 7B-base [23] and Qwen2.5 7B-base [24] for experiments. We use LoRA [25] for supervised fine-tuning experiments. For rejection sampling, we sample 10 responses for each prompt and use temperature $\tau = 0.9$ for inference; for the other methods, we use greedy decoding. For Skill-Mix tasks, we perform rationalization for RFT because it is an open-ended generation task (see Section 3.3). Configuration and hyperparameters are in Appendix F.

5 Results

5.1 Zero-shot Generalization

We evaluate the compositional generalization of the proposed method *without compositional supervision*, including ComposableCoT-Merge and ComposableCoT-MTL. For all methods that we compare with, we control the amount of training data to be the same as N_i and N_j . For reference, we also include the supervised fine-tuning baseline by fine-tuning M_0 with $N_{(i,j)}$ compositional answer data. Details of the training data for each task can be found in Appendix G.

Learning ComposableCoT achieves better zero-shot generalization. Table 1 shows that ComposableCoT variants outperform all baselines on a range of settings for both models. Moreover, while having seen no compositional training data, our method achieves comparable or even better performance than supervised fine-tuning baselines *with* compositional supervision (e.g., last letter + multiplication). We also find that combining atomic CoT models trained on Composable CoT is better than combining models trained on standard CoT: ComposableCoT-Merge is better than StandardCoT-Merge in most settings, and so is ComposableCoT-MTL to StandardCoT-MTL. These indicate that the Composable CoT format leads to better "composability" at inference time. We note one error case of ComposableCoT-Merge on concatenation + last letter on Qwen 2.5-7B; we discuss it in details in Section 5.3.

Table 1: Zero-shot compositional generalization of ComposableCoT with different combination approaches vs. baselines. *Without any compositional supervision*, using model merging or multitask learning to combine atomic CoT models trained on Composable CoT data outperforms baselines across settings and models, and is sometimes comparable to SFT with compositional supervision.

Methods	Last Letter + Mult EM	Concat + Last Letter EM	Concat + Mult EM	Skill-Mix Literary + Rhetorical Full Marks	Skill Fraction
Llama 2-7B					
<i>SFT on Base Model with Compositional Supervision</i>	3.1	5.0	9.0	35.5	60.1
Few-shot Answer	1.0	0.0	0.0	4.1	16.4
Few-shot CoT	2.0	3.0	1.0	7.3	23.1
StandardCoT-Merge	2.0	12.5	2.3	11.0	31.6
ComposableCoT-Merge (Ours)	16.0	19.1	3.0	19.6	37.1
StandardCoT-MTL	5.0	0.0	0.0	17.6	38.7
ComposableCoT-MTL (Ours)	18.7	6.5	3.1	22.9	49.9
Qwen 2.5-7B					
<i>SFT on Base Model with Compositional Supervision</i>	4.6	31.9	2.0	35.5	60.3
Few-shot Answer	2.4	0.0	2.7	34.7	56.0
Few-shot CoT	2.0	0.0	21.3	31.8	41.6
StandardCoT-Merge	70.4	54.8	77.0	29.8	48.0
ComposableCoT-Merge (Ours)	95.4	19.2	75.4	39.6	62.1
StandardCoT-MTL	3.6	60.9	72.1	42.0	58.2
ComposableCoT-MTL (Ours)	96.3	63.3	74.3	49.0	66.7

The optimal method to combine atomic CoT models differs for different settings. Regardless of using StandardCoT or ComposableCoT, combining atomic CoT models with model merging is generally better than with multitask learning for concatenation + multiplication, and multitask learning is better for Skill-Mix literary + rhetorical. We hypothesize that such differences are caused by the level of conflicts in the model parameter space between different pairs of atomic tasks.

5.2 Compositional Performance with Limited Supervision

We evaluate the performance of Composable CoT models after being further improved with one iteration of RFT using the limited compositional supervision. We compare it with multitask learning and continued fine-tuning baselines given the same compositional answer dataset $D_{\mathcal{T}_{(i,j)}}$ of size $N_{(i,j)} \leq 500$. For reference, we include the baseline of fine-tuning M_0 on the same compositional answer data. Details of the data condition can be found in Appendix G.

Table 2 shows that within the same budget of compositional training data, **using RFT on top of ComposableCoT-MTL and ComposableCoT-Merge achieves the best compositional task performance**, outperforming multitask learning and continued fine-tuning baselines across settings.

We further investigate if the performance is mainly driven by RFT or by learning Composable CoT format. We compare RFT upon StandardCoT-Merge with RFT upon ComposableCoT-Merge for Llama 2-7B, and StandardCoT-MTL with ComposableCoT-MTL for Qwen 2.5-7B.² Table 2 shows that RFT is a better way to improve the compositional task performance of StandardCoT models with compositional data than MTL and SFT. One explanation is that SFT or MTL based on the atomic

²To choose the ablation baselines to compare with, we take the StandardCoT model that wins on more categories and consider a model going from zero performance to nonzero as a “double win”. On Llama 2-7B, StandardCoT-Merge achieves nonzero accuracy on two tasks that StandardCoT-MTL achieves zero accuracy on, making it a better starting point for further fine-tuning. On Qwen 2.5-7B, both StandardCoT models have nonzero performance, and we choose StandardCoT-MTL as its performance is generally higher.

Table 2: Compositional task performance of rejection sampling fine-tuning (RFT) upon merged Composable atomic CoT models and other baselines. *Mult* stands for ASCII multiplication and *concat* stands for letter concatenation. *SFT* stands for supervised fine-tuning with the compositional answer data; *CFT* stands for continued fine-tuning; *MTL* stands for multitask learning method. Results on last letter + mult are omitted because the zero-shot performance already saturates. RFT on ComposableCoT variants achieves the best compositional task performance using the same amount of compositional answer data.

Category	Method	Last Letter + Mult EM	Concat + Last Letter EM	Concat + Mult EM	Skill-Mix Literary + Rhetorical Full Marks	Skill Fraction
Llama 2-7B						
SFT	SFT on Base Model	3.1	5.0	9.0	35.5	60.1
	CFT on StandardCoT-Merge	2.0	16.0	14.0	44.1	65.1
	CFT on StandardCoT-MTL	3.0	26.0	11.0	38.0	62.1
MTL	StandardCoT + Comp Answer	5.0	46.0	13.3	22.9	45.5
RFT	StandardCoT-Merge	0.0	23.0	29.7	26.1	52.0
	ComposableCoT-Merge (Ours)	72.0	46.0	40.0	45.3	66.6
Qwen 2.5-7B						
SFT	SFT on Base Model	-	31.9	2.0	35.5	60.3
	CFT on StandardCoT-Merge	-	41.1	9.3	51.0	71.4
	CFT on StandardCoT-MTL	-	60.3	12.7	34.7	56.3
MTL	StandardCoT + Comp Answer	-	65.1	7.1	41.2	55.3
RFT	StandardCoT-MTL	-	82.1	89.0	44.9	63.4
	ComposableCoT-MTL (Ours)	-	86.9	88.4	57.6	71.5

CoT models using compositional answer data can lead to a distribution shift in the output space (from generating a CoT sequence to generating a direct answer), and mitigating such distribution shift requires more advanced techniques, usually called CoT internalization [26] or latent CoT [27], beyond the focus of this work; meanwhile, RFT does not have this problem. Moreover, **RFT upon ComposableCoT models is generally better than RFT upon StandardCoT models.**³

5.3 Intrinsic Evaluation of Generated CoTs

To understand differences in models trained with different CoT formats, we conduct intrinsic evaluations on CoTs generated by ComposableCoT and StandardCoT models for zero-shot composition. For the string operation tasks, we extract template-based patterns of each atomic CoT from the generated outputs of models evaluated on the compositional task. For Skill-Mix, we consider the CoT pattern of an atomic task to be used if the generated response explicitly mentions the skill corresponding to that atomic skill category. Table 3 shows results with models trained from Qwen 2.5-7B. Using the same combination method (model merging or MTL), combining ComposableCoT leads to consistently higher presence of both atomic CoT patterns in the generated responses compared to StandardCoT. Atomic models trained with the Composable CoT format therefore leverage the combination of learned skills in some form more frequently than StandardCoT. We note the exception of ComposableCoT-Merge on Concat + Multi which seldomly leverages both atomic CoT in the outputs, explaining to its low compositional performance, as mentioned in Section 5.1. We thus hypothesize that in some cases, model merging can still be an unstable combination method. Examples of error cases in the generated CoTs can be found in Appendix H.

³Note that RFT requires a reasonably good model to start with: RFT fails for StandardCoT-Merge on Llama 2-7B because it is unable to sample enough correct responses for training from this StandardCoT-Merge model.

Table 3: Intrinsic evaluation of the generated CoTs from atomic CoT models evaluated on the compositional task in the zero-shot setting. “% \mathcal{T}_1 CoT” denotes the percentage of generated responses that use the CoT format of the first atomic task of the composition, and likewise for the second. † denotes that the ComposableCoT method has a significantly higher “% Both CoT” than the StandardCoT counterpart at the 0.01 level using a paired bootstrap test. Combined Composable CoT models generate responses including both atomic CoT patterns more frequently than combined atomic CoT models.

	Method	Performance	% \mathcal{T}_1 CoT	% \mathcal{T}_2 CoT	% Both CoT
Last Letter + Mult	StandardCoT-Merge	70.4	85.3	95.1	85.3
	ComposableCoT-Merge	95.4	100.0	100.0	† 100.0
	StandardCoT-MTL	3.6	0.0	100.0	0.0
	ComposableCoT-MTL	96.3	98.9	100.0	† 98.9
Concat + Last Letter	StandardCoT-Merge	77.0	90.3	98.7	90.0
	ComposableCoT-Merge	75.4	91.6	100.0	91.6
	StandardCoT-MTL	72.1	99.7	32.1	32.1
	ComposableCoT-MTL	74.3	100.0	83.1	† 81.3
Concat + Mult	StandardCoT-Merge	54.8	100.0	99.4	99.4
	ComposableCoT-Merge	19.2	44.6	60.5	17.7
	StandardCoT-MTL	60.9	100.0	66.7	66.7
	ComposableCoT-MTL	63.3	100.0	85.9	† 85.0
Skill-Mix	StandardCoT-Merge	29.8	60.0	59.2	35.9
Literary	ComposableCoT-Merge	39.6	64.1	66.9	† 43.3
+ Rhetorical	StandardCoT-MTL	42.0	65.3	58.0	37.6
	ComposableCoT-MTL	49.0	64.5	65.7	† 42.0

6 Related Work

As an important cognitive capability of humans [1, 2], compositional generalization has been considered a core capability for human-level reasoning models [28, 29]. As defined by [28], compositional generalization has three components: systematicity, productivity, and primitive application. This work mainly focuses on *systematicity*, the capability of applying known components in unseen combinations. Systematic generalization has been considered difficult for LLMs because of the limitations of the transformer architecture and autoregressive generation [30, 4], and has been explored extensively in the context of parsing [31, 32, 33, 34].

Recent theoretical analyses show that the compositional reasoning capability of LLMs can be improved by generating CoT [35, 36], but empirical results show that non-trivial effort needs to be put through prompt engineering [37, 38] or data selection [39, 20, 40, 41] to observe such improvements with CoT [42], particularly in domains where compositional solutions to problems are crucial [43, 44]. Prior work has explored more principled approaches, but they usually rely on heuristics to determine data quality [8, 45] or involve computationally intensive methods [7, 3].

We are inspired by a line of work on efficient methods for combining models of different capabilities. Past work on model merging [13, 46, 47] has shown that trained models can be merged to retain the union of their skills, including reasoning [48, 49, 50], but only limited prior work has shown any kind of skill composition [51]. Our work is the first to use model merging for compositional generalization with CoT. We believe that other approaches for combining models could be used, such as learning methods that make modular updates [52, 53].

7 Conclusion

We propose Composable Chain-of-Thought, a data augmentation scheme to convert CoT data of atomic reasoning skills into a format that facilitates inference-time compositional generalization. Training atomic CoT models with Composable CoT and combining them with model merging or multitask learning leads to better zero-shot compositional reasoning performance than building models with the standard CoT format. Such a combined model can be further improved by a limited amount of compositional data with rejection sampling fine-tuning. Learning to reason with

composable CoT shows a promising approach to improve compositional reasoning in LLMs, and could be extended to build more efficient and robust large reasoning models.

Limitations: Our experiments focus on pairwise compositional tasks where two atomic reasoning skills are composed and do not cover compositions of more than two atomic tasks. There is a lack of high-quality datasets that support n -way compositional tasks, and it is a non-trivial effort to collect an appropriate evaluation suite for the purpose. We note that our framework of Composable CoT generalizes to compositional tasks that consist of more than two atomic skills as discussed in Section 3.1. In addition, we only focus on small-scale datasets where both the atomic skills and compositional skills can be learned with a small amount of training data. Our focus here is on conducting controlled experiments, but we believe our methods and their principles can be scaled up to more complex settings.

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Table 4: Performance of atomic CoT models fine-tuned on different variants of proxy prefix on Llama 2-7B. Using random letters as the proxy prefix achieves the best out-of-domain performance when evaluated with an unseen prefix at inference time.

Type of Proxy Prefix	Exact Match Accuracy	
	In Domain Prefix	Out-of-Domain Prefix
Random Letters	83.0	90.0
Random from the Prompt	86.4	82.5
Random Text	90.6	70.0

A A Note on Composing Tasks

There exist various possible ways to combine atomic tasks into a compositional task with the combination function g . We simplify g into two types: (1) composite: the output of one atomic task is used as part of the input of another task, $g(\mathcal{T}_i, \mathcal{T}_j) = \mathcal{T}_i \circ \mathcal{T}_j$ or $g(\mathcal{T}_i, \mathcal{T}_j) = \mathcal{T}_j \circ \mathcal{T}_i$; (2) concatenation: the outputs of the two atomic tasks are concatenated using the same input, $g(\mathcal{T}_i, \mathcal{T}_j) = \mathcal{T}_i \oplus \mathcal{T}_j$ or $g(\mathcal{T}_i, \mathcal{T}_j) = \mathcal{T}_j \oplus \mathcal{T}_i$. Among tasks evaluated in Section 4, the string operation tasks need to be solved by a composite function, while the Skill-Mix task can be solved by either a composite function or a concatenation function.

B Design Choices for Constructing Composable CoT Data

When designing the proxy prefix CoT, we would like to consider the following constraints. (1) We do not assume any prior knowledge about what would possibly be put in the prefix CoT at inference time; (2) We do not assume strong relevance between the proxy prefix CoT and the suffix, i.e., not all the information in the proxy prefix CoT is useful for predicting the suffix CoT and the final answer. Based on these considerations, we experiment with the following variants:

- **Random letters:** We sample random letters from the alphabet to form a sequence of random lengths to simulate an *arbitrary* prefix CoT.
- **Random from the prompt:** We sample random letters and words from the prompt \mathbf{q} to form a sequence of random lengths to simulate a prefix CoT in a similar distribution as the input distribution.
- **Random text:** We sample random sentences from OpenWebText [54] to simulate a prefix CoT drawn from the pre-training data distribution.

We evaluate these variants by fine-tuning models on Composable CoT datasets that **only have suffix CoT examples**. Note that this is different from the multitask learning setting discussed in Section 3.1 where the Composable CoT dataset consists of both the prefix CoT examples and the suffix CoT examples. This experiment mainly aims at stress testing the model’s capability of learning a single atomic task with a given proxy prefix CoT variant. We use the same hyperparameter configurations for all proxy prefix variants for a given task.

We evaluate the fine-tuned models on the in-domain task in two settings: (1) In-domain prefix: we append the same type of prefix as we have used for training to the end of the prompt of the in-domain test example and evaluate the model on it; (2) Out-of-domain prefix: we randomly sample a prefix from the other two variants and append it to the end of the prompt of the in-domain test example and evaluate the model on it. We run experiments on the three string operation tasks and report the average performance. Table 4 shows that while using random letters as the proxy prefix leads to the worst in-domain performance, it generalizes the best to out-of-domain prefixes, which is a more desirable behavior.

C Details of String Operation Tasks

Last letter in alphabet We synthetically generate data for Last letter in alphabet. We randomly sample letters from the English alphabet of a random length and concatenate them into a sequence.

Then we extract the last letter from the sequence and derive the next letter following it in the alphabet. An example can be found in Example C.2. We automatically generate a chain-of-thought for each generated problem, using a fixed template shown in Example C.2.

ASCII multiplication Similarly, we randomly sample letters from the English alphabet of a random length and concatenate them into a sequence. Then, we randomly sample another letter s and randomly sample an integer $a \in \{1, \dots, 9\}$. We find the ASCII value of s as $f(s)$ and compute the product $a \cdot f(s)$ as the gold answer. An example can be found in Example C.3. We automatically generate a chain-of-thought for each generated problem, using a fixed template shown in Example C.3.

Letter concatenation We follow [9] to generate the dataset by randomly sampling from the most popular first and last names in the United States and the United Kingdom from <https://namecensus.com> and randomly concatenating them into a sequence of names. While the original task in [9] only requires concatenating the last letter of each name together, we raise the difficulty level by randomly asking for concatenations of the first, second, second-to-last, or the last letter. An example can be found in Example C.1. The CoT template is also shown in Example C.1.

Compositional tasks We synthetically construct the compositional tasks of the string operation tasks in similar procedures as used to generate the atomic data. An example of last letter + ASCII multiplication can be found in Example C.4, concatenation + last letter in Example C.5, and concatenation + multiplication in Example C.6. We made a design decision to exclude one variant of concatenation + last letter that concatenates the last letter of each word and finds the next letter following the last letter in the concatenated sequence; this variant can be solved by the reasoning shortcut of only applying Last letter in alphabet rather than a composition of both.

C.1 Atomic Task Example: Letter Concatenation Example

[Instruction]

Take the second-to-the-last letter of each word in the sequence and concatenate them in lower case: Tequan Monjur Khia Jodi-leigh answer

[Chain-of-Thought + Answer String]

The second-to-the-last letter of the 1st word is a. The second-to-the-last letter of the 2nd word is u. The second-to-the-last letter of the 3rd word is i. The second-to-the-last letter of the 4th word is g. So the answer is auig.

[Answer String]

auig

C.2 Atomic Task Example: Last letter in alphabet

[Instruction]

Find the Last letter in alphabet following the last letter in the sequence: wqsisibnnicdlpwqbnoicdcxcxrfoilpcbnixucbsssejxuzods answer:

[Chain-of-Thought + Answer String]

The last letter is s, and the letter following it in alphabet is t. So the answer is t.

[Answer String]

t

C.3 Atomic Task Example: ASCII Multiplication

[Instruction]

Find the ASCII value of the letter after '<letter>' and multiply the ASCII value by 2: byaxaxcpoteznwnwseselyjlrctxtxcbfvmfezbycplymfotjbfv

jlhotzjbjcpycbtzhorepyjckofj <letter> d answer:

[Chain-of-Thought + Answer String]

The ASCII value of the letter d is 100, and multiplying the ASCII value by 2 gives us 200. So the answer is 200.

[Answer String]

200

C.4 Compositional Task Example: Last letter + ASCII Multiplication

[Instruction]

Find the ASCII value of the Last letter in alphabet following the last letter in the sequence and multiply the ASCII value by 5: knnxqsvshqugxfuqljumsbihgxvqihnxuufuknxvumuupkpkshljqsbkiz answer:

[Answer String]

485

C.5 Compositional Task Example: Concatenation + Last Letter

[Instruction]

Take the second-to-the-last letter of each word in the sequence, concatenate them in lower case, and find the Last letter in alphabet following the last letter in the sequence of the concatenated letters: Tyjai Ahijah Denzil Amine answer:

[Answer String]

o

C.6 Compositional Task Example: Concatenation + Multiplication

[Instruction]

Take the second-to-the-last letter of each word in the sequence, concatenate them in lower case, then find the ASCII value of the last letter in the sequence of the concatenated letters, and multiply the ASCII value by 3: Zarriah Amylee Li Javarie answer:

[Answer String]

315

D Details of Skill-Mix Tasks

D.1 Modifications of Skill-Mix

We adapt the Skill-Mix dataset from [19]. For each example, the model is given a natural language skill, its definition, an example of the skill, and a topic to focus on, and the model needs to write a grammatical sentence to demonstrate the skill on the topic. Because we mainly focus on pairwise composition, we only use the $k = 2$ and $k = 1$ composition sets of the Skill-Mix data. We apply the following modifications to the dataset to fit our setting of compositional reasoning.

1. Filtering the categories of skills: We keep examples with skills of the rhetorical and literary categories out of the five categories from the original dataset. This is because the rhetorical and literary skills have the least overlap while other categories have more (e.g. the logical and rhetorical skills have a large body of overlaps).
2. Removing the requirements of post-hoc explanation and refinement from the prompt. The original dataset evaluates models by prompting the models to first write a sentence, provide an explanation for the written sentence, and then do another round of refinement based on

feedback from the grader (an LLM-as-a-judge). To fit into our setting of chain-of-thought reasoning and direct answering, we remove these irrelevant elements in the prompt.

3. Using a public test set: The original evaluation of [19] hides the test set from the public and models can only be evaluated based on API calls to the hidden test set. To ensure reproducibility of our results, we use a public test set collected by [22].

As an open-ended generation task, Skill-Mix does not have a single ground truth sentence. [22] collect a silver-labeled dataset of sentences and their post-hoc explanations generated by GPT-4o [21] and graded as correct by the grader. We further prompt GPT-4o to transform the (sentence, explanation) pairs into (CoT, sentence) pairs to resolve potential coreference issues. Examples of the prompt, CoT traces, and sentence can be found in Examples D.1, D.2, and D.3.

D.1 Atomic Task Example: Skill-Mix-Literary

[Instruction]

Greetings! I am interested in natural language processing and I was wondering if you could help me generate an example of text that illustrates multiple skills in semantics or syntax. The example should be a minimal natural piece of text with up to a few lines in the context of Vikings that illustrates all of the following skills: oxymoron. Please keep the text as short as possible, and make sure the concepts can be found fully from the text.

For reference, here are the definitions and examples for the concepts:
****oxymoron****: A figure of speech in which apparently contradictory terms appear in conjunction. For example, "The hall was filled with a deafening silence." Here, the phrase "deafening silence" is an oxymoron.

Please start the minimal natural piece of text with 'Answer:'.

Thanks very much!

[Chain-of-Thought + Answer String]

Explanation: The goal is to construct a sentence that showcases the use of an oxymoron while remaining relevant to the Viking theme. To achieve this, we focus on creating a phrase that contains seemingly contradictory terms. We begin with the context of Vikings enduring harsh weather conditions. The word "fiery" conveys an intense heat, while "frost" indicates severe cold. When these terms are combined into "fiery frost," they form an oxymoron as they depict both extreme temperature sensations simultaneously. This juxtaposition effectively creates a semantic paradox, leading to vivid imagery and emphasizing the harsh conditions faced by Vikings.

Answer: "In the bitter cold, the Viking felt a fiery frost."

[Answer String]

Answer: "In the bitter cold, the Viking felt a fiery frost."

D.2 Atomic Task Example: Skill-Mix-Rhetorical

[Instruction]

Greetings! I am interested in natural language processing and I was wondering if you could help me generate an example of text that illustrates multiple skills in semantics or syntax. The example should be a minimal natural piece of text with up to a few lines in the context of Hiking that illustrates all of the following skills: begging the question or assuming the conclusion. Please keep the text as short as

possible, and make sure the concepts can be found fully from the text.

For reference, here are the definitions and examples for the concepts:
****begging the question or assuming the conclusion****: A circular argument that paraphrases the question. For example, "Of course the earth is round; people have known this for hundreds of years."

Please start the minimal natural piece of text with 'Answer:'.

Thanks very much!

[Chain-of-Thought + Answer String]

Explanation: To illustrate the concept of begging the question or assuming the conclusion within the context of hiking, we start by identifying a statement that relies on its own conclusion as a premise. First, recognize that the claim "hiking is beneficial" needs supporting evidence to be considered valid. However, if we use the phrase "because it's good for your health" as justification, we notice that it merely reiterates the initial claim using slightly different wording. This is because declaring something "beneficial" inherently implies a positive impact, such as being "good for your health." Thus, the reasoning becomes circular, as it depends on the same assumption it seeks to prove.

Answer: "Hiking is beneficial because it's good for your health."

[Answer String]

Answer: "Hiking is beneficial because it's good for your health."

D.3 Compositional Task Example: Skill-Mix Literary + Rhetorical

[Instruction]

Greetings! I am interested in natural language processing and I was wondering if you could help me generate an example of text that illustrates multiple skills in semantics or syntax. The example should be a minimal natural piece of text with up to a few lines in the context of Vikings that illustrates all of the following skills: anaphora resolution, begging the question or assuming the conclusion. Please keep the text as short as possible, and make sure the concepts can be found fully from the text.

For reference, here are the definitions and examples for the concepts:
****anaphora resolution****: Resolving the antecedent of a pronoun or noun phrase. For example, "The car is falling apart, but it still works." Here, "it" is the anaphor and "car" is the antecedent.
*****begging the question or assuming the conclusion****: A circular argument that paraphrases the question. For example, "Of course the earth is round; people have known this for hundreds of years."

Please start the minimal natural piece of text with 'Answer:'.

Thanks very much!

[Answer String]

Answer:
The Viking chief, undefeated thanks to his ship, asserted, "It remains unconquered because it's the 'Indomitable'."

Table 5: Single-task learning performance by evaluating the atomic CoT models on the in-domain atomic tasks.

CoT Format	Last Letter EM	ASCII Mult EM	Concat EM	Skill-Mix Literary Full Marks	Skill-Mix Rhetorical Full Marks	Skill Fraction
Llama 2-7B						
StandardCoT	100.0	85.7	83.0	63.5	63.5	53.3
ComposableCoT	95.0	86.0	77.0	71.4	71.4	72.4
Qwen 2.5-7B						
StandardCoT	90.0	99.0	77.4	77.4	77.6	70.5
ComposableCoT	99.4	99.7	77.3	77.4	77.6	76.7
						81.9

D.2 Evaluation Metrics

We use GPT-4o-mini as the LLM-as-a-judge to grade the generated sentence using the exact same grading rubric as provided by [19]; the grader judges the quality of the sentence based on if: (1) All skills are used; (2) The sentence makes sense; (3) The sentence attaches to the given topic; (4) The sentence is short. We use the evaluation metrics for each generated sentence in [19], including the following:

1. **Full Marks:** 1 if the generated sentence satisfies all four criteria above and 0 otherwise.
2. **Skill Fraction:** The fraction of skills being demonstrated if all the other three criteria are satisfied; 0 otherwise

We aggregate these metrics by averaging over all generated responses. In general, full marks evaluate the model’s capability of writing a perfect sentence for the task, while skill fraction evaluates how good the model is at handling skills given that it is good at the other stylistic capabilities. We use Curator [55] for an efficient implementation of the evaluation pipeline.

E Single-Task Learning Performance

We report the single-task learning performance of the atomic CoT models by evaluating them on the in-domain atomic tasks. We would like the atomic tasks to be easy to learn to reflect the practical settings where we train models on basic, easy-to-learn skills and generalize to harder, unseen tasks. The training data conditions and hyperparameters for training can be found in Appendix F. Table 5 shows that all atomic tasks we evaluate are learnable within a small amount of training data ($N_i, N_j \leq 500$).

In addition, we observe that training on ComposableCoT or StandardCoT does not lead to consistent differences in atomic CoT performance, while the exception is on Skill-Mix-Rhetorical for Llama 2-7B where fine-tuning on ComposableCoT outperforms fine-tuning on StandardCoT by a large margin.

F Training Configurations

F.1 General Configurations

We conduct all fine-tuning experiments with LoRA[25] using the following set of hyperparameters: we use a rank of 8, $\alpha = 16$, and a dropout rate of 0.2 to prevent overfitting. We apply LoRA adapters to all linear modules, including the attention matrices Q , K , V and MLP matrices of all layers. We use bfloat16 precision for training and we use the efficient implementation of LoRA by LlamaFactory [56]. We use a training batch size of 4 and train for 5 epochs for all experiments that share the same number of training data; for methods that use a potentially smaller amount of training data (e.g. RFT methods usually get fewer data examples than the number of compositional training data provided, depending on how many correct responses we can sample from the model), we adjust the batch size to match the number of steps.

Table 6: Optimal learning rate for each method in the experiments with compositional supervision.

Category	Method	Last Letter + Mult	Concat + Last Letter	Concat + Mult	Skill-Mix Literary + Rhetorical
Llama 2-7B					
SFT	SFT on Base Model	1e-3	1e-3	5e-4	5e-4
	CFT on StandardCoT-Merge	1e-3	5e-4	1e-4	1e-4
	CFT on StandardCoT-MTL	1e-4	1e-4	1e-4	1e-3
MTL	StandardCoT + Comp Answer	1e-3	5e-4	1e-3	5e-4
RFT	StandardCoT-Merge	-	1e-3	1e-3	5e-4
	ComposableCoT-Merge (Ours)	1e-4	1e-4	1e-3	1e-3
Qwen 2.5-7B					
SFT	SFT on Base Model	-	1e-3	1e-3	5e-4
	CFT on StandardCoT-Merge	-	5e-4	5e-4	1e-4
	CFT on StandardCoT-MTL	-	1e-3	1e-3	1e-3
MTL	StandardCoT + Comp Answer	-	5e-4	5e-4	1e-3
RFT	StandardCoT-MTL	-	1e-3	1e-4	5e-4
	ComposableCoT-MTL (Ours)	-	1e-3	1e-3	5e-4

F.2 Configuration for Rejection Sampling Fine-tuning

In addition to the sampling parameters (see Section 4), we consider the following configuration of RFT for sampling the correct responses: if the model generates multiple correct responses for a given question, we only randomly select *one* of them to be added into the RFT dataset D_{RFT} . In this way we ensure the diversity of examples in D_{RFT} so that the dataset will not be filled with samples from a small set of examples where the model is good at.

F.3 Hyperparameters: Learning Rate

We find in preliminary experiments that learning rate is the most important hyperparameter for the fine-tuning experiments of our interest. We perform hyperparameter sweeps for learning rate over the space of $\{5e-3, 1e-3, 5e-4, 1e-4, 5e-5\}$ on a validation set for each experiment. The optimal learning rate for each method for the experiments with compositional supervision in Table 6.

F.4 Hyperparameters: Model Merging

For methods that use model merging as the combination, we use Task Arithmetic [47] to combine the atomic CoT models. We perform a hyperparameter sweep for the scalars α and β over the space of $\alpha \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$ and $\beta = 1 - \alpha$ on a validation set for each task.

G Data Statistics

G.1 General Data Conditions for Experiments

Table 7 summarizes the number of training data and test data used in the evaluations in Sections 5.1 and 5.2. Note that for letter concatenation + multiplication we have two sizes of the compositional training data for Llama 2-7B and Qwen 2.5-7B: this is because all methods on Llama 2-7B perform poorly on zero-shot evaluation for this task and we need a slightly larger amount of compositional training data so that different methods can start to show distinguishable compositional task performance from each other. Regardless, we still consider 500 to be a reasonably small amount of training data, satisfying our ideal data conditions defined earlier.

Table 7: Data conditions for each task used for our evaluation.

		# Train	# Test
Atomic Tasks	Last Letter	100	700
	ASCII Mult	100	700
	Concat	500	700
	Skill-Mix Literary	100	126
	Skill-Mix Rhetorical	100	105
Compositional Tasks	Last Letter + Mult	100	700
	Concat + Last Letter	100	504
	Concat + Mult (Llama 2-7B)	500	700
	Concat + Mult (Qwen 2.5-7B)	100	700
	Skill-Mix Literary + Rhetorical	100	245

Table 8: The detailed breakdown of the number of training data used by each method in the zero-shot setting. N_i and N_j denotes the number of training data from the atomic tasks \mathcal{T}_i and \mathcal{T}_j seen by the method during training.

	Method	N_i	N_j
Last Letter + Mult; Skill-Mix Literary + Rhetorical	StandardCoT-Merge	0	0
	ComposableCoT-Merge	100	100
	StandardCoT-MTL	100	100
	ComposableCoT-MTL	100	100
Concat + Last Letter; Concat + Mult	StandardCoT-Merge	500	100
	ComposableCoT-Merge	500	100
	StandardCoT-MTL	500	100
	ComposableCoT-MTL	500	100

G.2 Training Data Used by Each Method

We show a detailed breakdown in Table 8 of the number of training data used by each zero-shot method for both models and in Table 9 for Qwen 2.5-7B by each method with compositional answer data in the experiments in Section 5.2. Note that the statistics for Llama 2-7B in the setting with compositional supervision are mostly the same except $N_{(i,j)} = 500$ for concat + last letter and concat + mult.

H Error Analyses

In addition to not being able to perform the individual atomic task correctly, we show three types of common errors made by ComposableCoT variants in the zero-shot compositional evaluation setting.

1. Example H.1 shows an example where the generated CoT is only able to replicate CoT of one atomic CoT and repeat the same CoT in the prefix and suffix.
2. Example H.2 shows an example where the combined model fails to continue generation after generating the prefix CoT. This is a common error for Composable models combined with model merging.
3. Example H.3 shows an example where the combined model uses the wrong atomic CoT in the prefix that should have been used in the suffix.

H.1 Error Case: Replicating One Atomic CoT Pattern

[Instruction]

Take the first letter of each word in the sequence, concatenate them in lower case, and find the next letter in alphabet following the

Table 9: The detailed breakdown of the number of training data used by each method with compositional supervision for Qwen 2.5-7B. N_i and N_j denotes the number of training data from the atomic tasks \mathcal{T}_i and \mathcal{T}_j seen by the method during training. $N_{(i,j)}$ denotes the number of compositional answer data seen during training.

	Method	N_i	N_j	$N_{(i,j)}$
Last Letter + Mult; Skill-Mix Literary + Rhetorical	SFT on Base Model	0	0	100
	CFT on StandardCoT-Merge	100	100	100
	CFT on StandardCoT-MTL	100	100	100
	MTL on StandardCoT + Comp Answer	100	100	100
	RFT on StandardCoT-Merge	100	100	100
	RFT on ComposableCoT-Merge	100	100	100
	RFT on StandardCoT-MTL	100	100	100
	RFT on ComposableCoT-MTL	100	100	100
Concat + Last Letter; Concat + Mult	SFT on Base Model	0	0	100
	CFT on StandardCoT-Merge	500	100	100
	CFT on StandardCoT-MTL	500	100	100
	MTL on StandardCoT + Comp Answer	500	100	100
	RFT on StandardCoT-Merge	500	100	100
	RFT on ComposableCoT-Merge	500	100	100
	RFT on StandardCoT-MTL	500	100	100
	RFT on ComposableCoT-MTL	500	100	100

Table 10: Summary of methods evaluated in the zero-shot compositional evaluation and the composition with limited compositional answer data. “Merge” stands for model merging; “MTL” stands for multitask learning; “CFT” stands for continued fine-tuning; “RFT” stands for rejection sampling fine-tuning. “-” means the property is not applicable to the method (e.g. *MTL on Standard + Comp Answer* mixes Standard CoT data with compositional answer data, and trains a single MTL model from the pretrained model, so there is no atomic CoT model trained or combined.)

Method	# Atomic CoT Models Trained	Atomic CoT Format	Combination Method	Model trained on Compositional Data	How is Compositional Data Used
<i>Zero-shot Evaluation</i>					
StandardCoT-Merge	2	Standard	Merge	-	-
ComposableCoT-Merge (Ours)	2	Composable	Merge	-	-
StandardCoT-MTL	1	Standard	MTL	-	-
ComposableCoT-MTL (Ours)	1	Composable	MTL	-	-
<i>Evaluation with Limited Compositional Answer Data</i>					
CFT on StandardCoT-Merge	2	Standard	Merge	StandardCoT-Merge	CFT
CFT on StandardCoT-MTL	1	Standard	MTL	StandardCoT-MTL	CFT
MTL on StandardCoT + Comp Answer	-	Standard	-	Pretrained Model	Mix with Atomic CoT data and MTL
RFT on StandardCoT-Merge	2	Standard	Merge	StandardCoT-Merge	RFT
RFT on ComposableCoT-Merge (Ours)	2	Composable	Merge	ComposableCoT-Merge	RFT
RFT on StandardCoT-MTL	1	Standard	MTL	StandardCoT-MTL	RFT
RFT on ComposableCoT-MTL (Ours)	1	Composable	MTL	ComposableCoT-MTL	RFT


```

last letter in the sequence of the concatenated letters:  Jessika Berk
Hamilton Stephanie answer:
[Answer String]
t
[Model Output]
<prefix> The last letter is e, and the letter following it in alphabet is
f.  So the answer is f.</prefix> <suffix> The last letter is e, and the
letter following it in alphabet is f.  So the answer is f.</suffix>
[Predicted Answer]
f

```

H.2 Error Case: Failing to Continue Generation after Prefix CoT

```

[Instruction]
Take the last letter of each word in the sequence, concatenate them in
lower case, then find the ASCII value of the last letter in the sequence
of the concatenated letters, and multiply the ASCII value by 6:  Ayat
Amaris Heavenly Mariela answer:
[Answer String]
582
[Model Output]
<prefix> The last letter of the 1st word is t.  The last letter of the
2nd word is s.  The last letter of the 3rd word is y.  The last letter
of the 4th word is a.  So the answer is tasy, and the ASCII value of the
last letter in the sequence of the concatenated letters is 121, so the
answer is 726.</prefix> <suffix>
[Predicted Answer]
726

```

H.3 Error Case: Use the Wrong Order of Atomic CoTs

```

[Instruction]
Take the second letter of each word in the sequence, concatenate them
in lower case, and find the next letter in alphabet following the last
letter in the sequence of the concatenated letters:  D'shaun Takia Hiroki
Arjun answer:
[Answer String]
s
[Model Output]
<prefix> the last letter is n, and the letter following it in alphabet
is o.  so the answer is o.</prefix> <suffix> the second letter of d is d,
the second letter of t is a, the second letter of h is i, and the second
letter of a is r.  so the answer is dair.</suffix>
[Predicted Answer]
dair

```

I Computation resources

We conducted all our experiments primarily on a server with NVIDIA RTX A6000 48GB GPU, and a server with GPU of NVIDIA GH200 120GB. Each individual experiment can be run on a single A6000 of 48 GB within 1 hour.

J Licensing

We use the following publicly available datasets from prior works with open licenses.

Skill-Mix [19] uses the Apache-2.0 license and data is available at: <https://huggingface.co/spaces/dingliyu/skillmix>.

Letter concatenation The dataset uses the Apache-2.0 license and the data is available at: <https://huggingface.co/datasets/ChilleD/LastLetterConcat>