Short Conclusion on Combinatorial Proof

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1 Pascal's Recurrence Relation

Pascal's Formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

LHS: Choose k out of n objects.
RHS: If element#1 is already been chosen, choose remaining k-1 from n-1 objects; If element#1 is not chosen, choose k from n.

Stirling Number of the Second Kind

$$S(p, k) = kS(p-1, k) + S(p-1, k-1)$$

LHS: Arrange p students into k groups.
RHS: If student#1 forms a group himself, then arrange p-1 into k-1; If student#1 joins with others, then arrange p-1 into k, and he can choose k groups to join.

Stirling Number of the First Kind

$$s(p, k) = (p-1)s(p-1, k) + s(p-1, k-1)$$

LHS: Arrange p people in k round tables.
RHS: If the king sit alone, then arrange p-1 into k-1 tables; If the king sit with others, then arrange p-1 into k tables, and the king have p-1 positions to sit.

Pascal's relation is a common recurrence relation. Usually, when solving problems using recurrence, we consider the ① first/final step or ② the previous step

2 Other Recurrence Relations

Ramsey Number

$$r(m, n) \leq r(m-1, n) + r(m, n-1)$$

This conclusion (Ramsey Theory) can be proved by mathematical induction.
Induction hypothesis: $r(m, n-1)$ and $r(m-1, n)$ both exists.
Suppose in $K_{r(m-1,n)+r(m,n-1)}$, one of the vertices x is incident to $|R_x|$ red edges and $|B_x|$ blue edges. Then

$$|R_x| + |B_x| = r(m-1, n) + r(m, n-1) + 1$$

By the Pigeonhole Principle, either $|R_x| \geq r(m-1, n)$ or $|B_x| \geq r(m, n-1)$.
If $|R_x| > r(m-1, n)$, consider the $K_{r(m-1,n)}$ formed by the $|R_x|$ vertices. If it provides a red K_{m-1}, then by adding x, we have a red K_m. If it provides a blue K_n, then we are done.

Similar proof can be made when $|B_x| \geq r(m, n-1)$.

$\rightarrow$ Complement

$$r(m,n) \leq \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$$

Suppose $f(m,n) = \binom{m+n-2}{m-1}$, by Pascal's formula, we have

$$f(m,n) = f(m-1,n) + f(m,n-1)$$

Derangements

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

RHS: Since 1 is not in its natural position, there are (n-1) ways to choose the first element. Suppose k is in the first position, if 1 is in k's natural position, then D_{n-2} ways to arrange others; if not, since 1 cannot be in k's natural position, there are D_{n-1} ways to arrange them.

$$D_{n+1} = P_1^n D_{n-1} + P_2^n D_{n-2} + \cdots P_{n-1^n} D_1 + P_n^n D_0$$

RHS: Count by cyclic permutation groups containing 1.

Catalan Number

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0$$

RHS: In triangulation, choose a base edge. Discuss on which triangle contains this base.

Large Schröder Number

$$R_n = R_{n-1} + \sum_{k=1}^{n} R_{k-1} R_{n-k}$$

LHS: Number of subdiagonal HVD-lattice paths (Schröder path) from (0,0) to (n,n).
RHS: If the first step is D, then R_{n-1}. If the first step is H, suppose it touches the line $y = x$ at (k,k) for the first time.

3 Summation

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

LHS: Count subsets containing different number of elements.
RHS: For each elements, it can either be in the subset or not.

$$\binom{r+k+1}{k} = \binom{r}{0} + \binom{r+1}{1} + \cdots + \binom{r+k}{k}$$

$$\binom{n+k}{k+1} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k}$$

LHS: Directly choose.
RHS: Choose using a decision tree. Each time take the "Chosen" branch (first) or the "Not chosen" branch.
The two identities are the recursive form of Pascal's formula.

$$H_k^{n+1} = \sum_{i=0}^{k} H_{i^n}$$

Counting the integer solutions to $x_1 + x_2 + \cdots + x_k \leq n$
LHS: Plug in x_{k+1}.
RHS: Discuss on the exact value of their sum.

4 Concerning Leader

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

LHS: Choose k people out of n, and then choose a leader among them.
RHS: Choose the leader first then the remaining k-1 people.

$$n2^{n-1} = 1\binom{m}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$$

LHS: Choose the leader first, then a group with no constraints.
RHS: Discuss on how many people are in the group, then choose a leader among them.

$$n(n-1)2^{n-2} = \sum_{k=1}^{n} k^2\binom{n}{k}$$

Choose two leaders (not necessarily distinct).

5 Different Groups

Vandermonde's identity

$$\binom{n_1+n_2}{m} = \sum_{k=0}^{m}\binom{n_1}{k}\binom{n_2}{m-k}$$

LHS: From two groups each containing n_1 and n_2 people, choose m in total.
RHS: Discuss on how many of them are chosen from the first group.

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$$

A special case of Vandermonde's identity when $n_1 = n_2 = n$.

6 Special Member

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

In a set containing three distinct element a, b and c, count the number of k subset containing at least one of them.
LHS: All subsets subtract those do not contain a, b and c.
RHS: If a is chosen, then choose k-1 from n-1; If a is not chosen and b is chosen, then choose k-1 from n-2; If a and b are not chosen and c is chosen, then choose k-1 from n-3.
This is an incomplete form of $\binom{n+k}{k+1} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k}$, *when we stop at the third branch.*

$$\binom{n+3}{k} = \binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}$$

LHS: From a group of n+3 people containing a, b and c, choose k people.
RHS: Choose none of a, b and c; one of them; two of them; three of them.
This is a special case of Vandermonde's identity when $n_2 = 3$.

$$H_{b-2}^{a} = H_b^a - H_b^{a-1} - H_{b-1}^{a-1}$$

LHS: From a dishes choose b of them, where a particular dish A is chosen at least 2 times.
RHS: From general cases subtracts those choosing exactly 0 or 1 of A.

7 Special Position

$$\binom{n+2}{3} = 1 \cdot n + 2 \cdot (n-1) + \cdots + (n-1) \cdot 2 + n \cdot 1$$

LHS: Choose 3 out of n.
RHS: Discuss on the middle number, choose one less than it and one greater than it respectively.

8 One-to-one Correspondence

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}$$

Each odd combination can be one-one corresponded to an even combination by deleting or adding a particular element.
Or think using probability: when consider whether to choose the n-th element, it has already been decided since the oddness (evenness) has to be preserved.

Catalan Number

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$

LHS: Number of rectangular lattice paths from (0,0) to (n,n) which never get above the line $y = x$.
RHS: Number of paths which touch the line $y = x + 1$ at least once ⇔ Number of paths from (-1,1) to (n,n)

Partition Number

$$p_n^s = p_n^t$$

Use Ferrers diagram,

$$\text{self-conjugate} \Leftrightarrow \text{distinct odd}$$

$$p_n^o = p_n^d$$

⇒: Merge the same numbers until distinct
⇐: Divide by 2 until odd

$$\text{odd} \Leftrightarrow \text{distinct}$$

9 Inequality

Sperner's Theorem

The maximal antichian contains exactly $\binom{n}{\lfloor 1/2 \rfloor}$ elements. Suppose $\mathcal{A}$ is an antichian. Use two different methods to count the number of (A, C), where A is in $\mathcal{A}$, and $\mathcal{C}$ is the maximal chain containing A.
Lemma 1: Each maximal chian of S has exactly n+1 elements; There are n! maximal chians in total.
Lemma 2: The intersection of an antichian and a chain can have at most 1 element.
Method 1: Choose $\mathcal{C}$ first. There are n! ways to do this. After $\mathcal{C}$ is chosen, there are at most 1 way to choose A.
Method 2: Fix A first. If $|A| = k$, there are $k!(n-k)!$ ways to choose $\mathcal{C}$. Hence, there are

10 Higher Dimensions

General Pascal's Formula

$$\binom{n}{m_1 \cdots m_p} = \sum_{k=0}^{p} \binom{n}{m_1 \cdots m_k - 1 \cdots m_p}$$

LHS: Partition n objects into p different groups, each containing m_i objects.
RHS: Discuss on which group the particular object goes into.

$$p^n = \sum_{k_1 + \cdots + k_p = n} \binom{n}{k_1 \cdots k_p}$$

LHS: Partition n objects into p different groups, where each element can go to one of the groups.
RHS: Discuss on how many each group have.
Compare with the general Pascal's formula.

General Vandermonde's Identity

$$\binom{n_1 + \cdots + n_p}{m} = \sum_{k_1 + \cdots + k_p = m} \binom{n_1}{k_1}\binom{n_2}{k_2} \cdots \binom{n_p}{k_p}$$

LHS: From p different groups each containing n_k people, choose m.
RHS: Discuss on how many are chosen from each groups.

$$k_1 k_2 \cdots k_p \binom{n}{k_1 \cdots k_p} = n(n-1)\cdots(n-p+1)\binom{n}{k_1 - 1 \cdots k_p - 1}$$

LHS: Partition n into p groups, and from each group choose a leader.
RHS: Choose the leader first then form the group.