Advanced Computer Vision Methods - Homework 1.

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1 Problem

Take derivative of the following function: $f(x) = (x + x^3)^4 - x^3$

1. Introduce an additional variable: $u = x + x^3$
2. Solve with chain rule: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} - \frac{\partial (x^3)}{\partial x}$
3. Result: $\frac{\partial f}{\partial x} = 4(x + x^3)^3 \cdot (1 + 3x^2) - 3x^2$

2 Problem

Linearize the following function $f(x) = 2x^2 + x^3$ at $x_0 = 4$.

1. Derive the form: $f(x_0 + \delta) = f(x_0) + \Delta \approx f(x_0) + \frac{\partial f}{\partial x}|_{x_0} \cdot \delta$
2. Calculate: $f(x_0) = 2 \cdot 4^2 + 4^3 = 2 \cdot 16 + 64 = 96$
3. Calculate: $\frac{\partial f}{\partial x}|_{x_0} = (4x + 3x^2)|_{x_0} = 4 \cdot 4 + 3 \cdot 4^2 = 16 + 48 = 64$
4. Result: $f(x_0 + \delta) = 96 + 64 \cdot \delta$, where $\delta = x_i - x_0 = x_i - 4$

Visualization of function $f(x)$ and its linear approximation at x_0 is shown in figure 1.

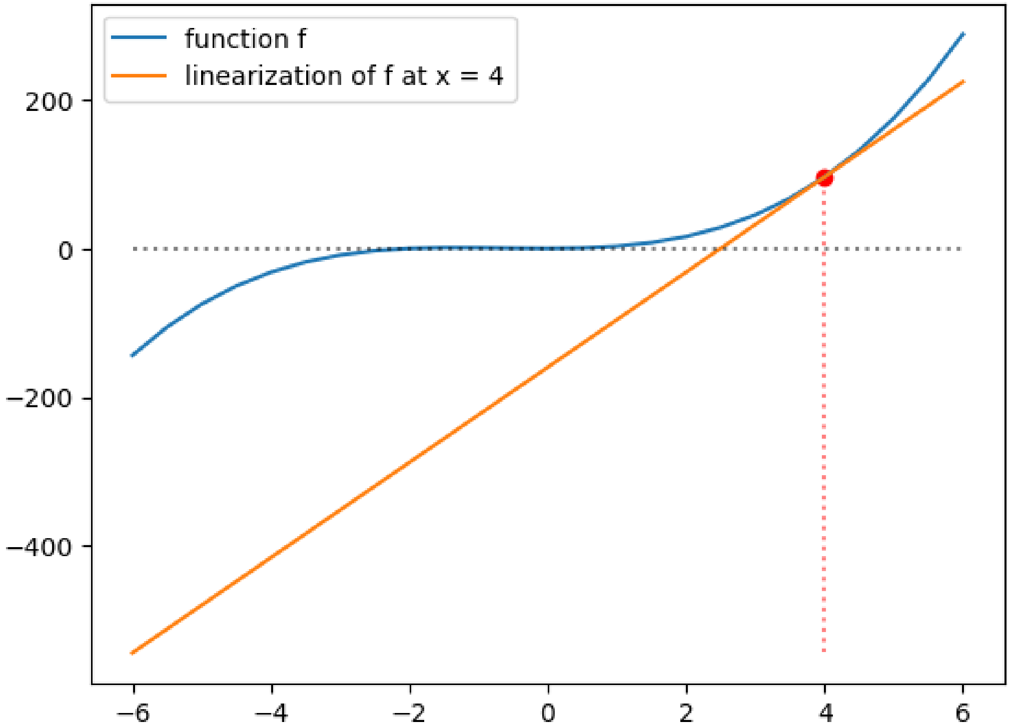

Figure 1: Linearization of function $f(x)$ at $x_0 = 4$.

3 Problem

Linearize the following function $f(x_1(\vec{p}), x_2(\vec{p}))$ for small changes in parameters $\vec{p}$, i.e., $\vec{p_0} + \delta$ using Jacobian: $f(x_1(\vec{p}), x_2(\vec{p})) = x_1^2(\vec{p}) + x_2^3(\vec{p})$ with parameter vector defined as: $\vec{p} = [p_1, p_2]^T$ and with parameters of $f(x_1, x_2)$ defined as: $x_1(\vec{p}) = 2p_2$ and $x_2(\vec{p}) = p_1 + p_2^2$.

1. Calculate Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 2p_2 \end{bmatrix}$$

2. Calculate gradient ∇f:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1(\vec{p}) \\ 3x_2^2(\vec{p}) \end{bmatrix}$$

3. Everything together:

$$f(\vec{X}(\vec{p_0}+\delta)) \approx f(\vec{X}(\vec{p_0}))+\nabla f^T \cdot J \cdot \vec{\delta} = f(\vec{X}(\vec{p_0}))+(\begin{bmatrix} 2x_1(\vec{p}) & 3x_2^2(\vec{p}) \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 1 & 2p_2 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix})|_{\vec{p_0}}$$

4 Problem

Assume a vector $\vec{x} \epsilon R^d$ and a symmetric positive definite matrix $A \epsilon R^{dXd}$. Compute gradient $\nabla f(\vec{x}) = \frac{\partial f}{\partial \vec{x}}$ of function $f(\vec{x}) = 20\vec{x}^T A\vec{x} + 10$.

1. Simply use equation (81) from the matrix cookbook:
   $\nabla f(\vec{x}) = \frac{\partial f}{\partial \vec{x}} = 20(A + A^T)\vec{x}$

2. $A+A^T = 2A$, since A is **symmetrical**, meaning we can further simplify the above into: $\nabla f(\vec{x}) = 40A\vec{x}$