SIMP

The main theory of SIMP:

$$E_{ijkl}(\rho) = \rho^p E^0_{ijkl} \tag{1}$$

where:

E^0_{ijkl} : stiffness tensor of the solid material

p : penalization

Strain Energy Density : $E_{ijkl}\epsilon_{ij}\epsilon_{kl}$

Sensitivity of the Strain Energy:

$$\frac{\partial}{\partial \rho}(\rho^p E^0_{ijkl}) = p\rho^{p-1}E_{ijkl}\epsilon_{ij}\epsilon kl$$

This derivative tells us how much the strain energy density will change with a change in density.

$$\begin{aligned}
&\min_{u\in U,\rho} \; l(u) \\
&\text{s.t.} : a_E(u,v) = l(v), \quad \text{for all } v \in U \ , \\
&\qquad E_{ijkl}(x) = \rho(x)^p E^0_{ijkl} \ , \\
&\qquad \int_\Omega \rho(x)\mathrm{d}\Omega \le V; \quad 0 < \rho_{\min} \le \rho \le 1 \ .
\end{aligned}$$

Lagrangian of the SIMP:

$$L = l(u) - \{a_e(u,\bar{u}) - l(\bar{u})\} + \Lambda\left(\int rho(x)d\Omega - V\right) + \int \lambda^+(\rho(x)-1)d\Omega + \int \lambda^-(\rho_{min} - \rho(x))d\Omega$$

where

$a(u,v)$: Energy Bilinear Form

$$a(u,v) = \int_\Omega E_{ijkl}(x)\epsilon_{ij}(u)\epsilon_{kl}(v)d\Omega$$

$l(u)$: Load Linear Form

$$l(u) = \int_\Omega fud\Omega + \int tuds$$

Breaking down the Lagrangian into components:

- $a_E(u, \bar{u}) - l(\bar{u})$: ensure the equilibrium constraint $a_E(u, v) = l(v)$ where $\bar{u}$ is the Lagrangian multiplier.
- $\Lambda(\int_\Omega \rho(x)d\Omega - V)$: term which ensures that $\int_\Omega \rho(x)d\Omega$ does not exceed V where Λ is the Lagrange multiplier.
- $\int_\Omega \lambda^+(x)(\rho(x) - 1)d\Omega + \int_\Omega \lambda^-(x)(\rho_{min} - \rho(x))d\Omega$: enforces the upper & lower bounds on the material density $\rho(x)$ where $\rho_{min} \leq \rho(x) \leq 1$ where $\lambda^+(x)$ & $\lambda^-(x)$ are the Lagrange multiplier.

Karush-Kuhn-Tucker (KKT) Conditions:

set of requirements that must be satisfied for a solution to be optimal in a nonlinear problem

Problem Form:

minimize f(x),

subject to

$g_i(x) \leq 0 \ \ for \ \ i = 1, ..., m$

$h_j(x) = 0 \ \ for \ \ j = 1, ..., p$

Stationarity:

Gradient of the objective function should equal to the sum of the gradients of the constraints times their Lagrange multiplier

Primal Feasibility:

The solution must satisfy all of the inequality and equality constraints

Dual Feasibility:

All Lagrange multipliers associated with the inequality constraints must be non-negative

Complementary Slackness:

The product of each Lagrange multiplier and its corresponding inequality constraint must be zero.

$$\lambda_i(x)g(x) = 0 \ \ i = 1, ..., m$$

This implies that:

if $\lambda_i > 0$; $g_i(x)$ is active meaning ($g_i(x) = 0$)

if $g_i(x) < 0$; $g_i(x)$ is not active meaning $\lambda_i = 0$

Stationarity of the Lagrangian:

Definition of the stationarity condition :

state where 1st derivative of the objective function with respect to design variable is zero. This condition is used to find the optimal solution where the objective function is neither increasing, nor decreasing, impliying local minimum, maximum or saddle point

$$\frac{\partial L}{\partial \rho} = \Lambda + \frac{\partial}{\partial \rho}(-a_E(u,\bar{u}) + l(\bar{u})) + \lambda^+ + \lambda^- = 0 \tag{2}$$

The derivative can be simplified as follows:

$$\begin{aligned} -\frac{\partial\ a_E(u,\bar{u})}{\partial \rho} + \Lambda + \lambda^+ + \lambda^- = 0 \quad (3) \\ where \quad \frac{\partial a_E(u,\bar{u})}{\partial \rho} = \frac{\partial E_{ijkl}}{\partial \rho}\epsilon_{ij}\epsilon kl \quad (4) \end{aligned} \tag{5}$$

Therefore, the stationarity condition can be written as:

$$p\rho^{p-1}E^0_{ijkl} = \Lambda + \lambda^+ + \lambda^- \tag{6}$$

- $\lambda^- \geq 0 \quad , \quad \lambda^+ \geq 0$: Dual Feasibility
- $\lambda^-(\rho_{min} - \rho(x)) = 0.$: Complementary Slackness
- $\lambda^+(\rho(x) - 1) = 0$: Complementary Slackness

If the complementary slackness conditions are applied, for intermediate densities $\rho_{min} < \rho < 1,\ \lambda^+ = \lambda^- = 0$.

Therefore, the stationarity condition becomes:

$$p\rho^{p-1}E^0_{ijkl}\epsilon_{ij}\epsilon_{kl} = \Lambda$$