Some simple mathematical models

July 1, 2011

The birth of modern science

"Philosophy is written in this grand book the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and to read the alphabet in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."

Galileo's falling stone

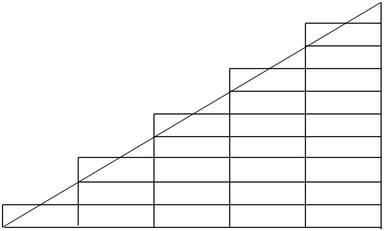
Galileo's assumption/hypothesis

"When ... I observe a stone initially at rest falling from an elevated position and continually acquiring new increments of speed, why should I not believe that such increases take place in a manner which is exceedingly simple? ...
A motion is said to be uniformly accelerated when starting from rest, it acquires, during equal increments of time, equal increments of speed. That is the concept of accelerated motion that is most simple and easy, and it is confirmed to be that actually occurring in Nature, by exact correspondence with experimental results based upon it."

Galileo's falling stone

Galileo's deduction/prediction:

Distances are to each other as the squares of the times.



Galileo's falling stone

Galileo's experiment:

The ball and the inclined plane ...

time t	1	2	3	4	5
distance y	1	4	9	16	25

Galileo's falling stone

Galileo's other assumption:

"In order to handle this matter in a scientific way, it is necessary to cut loose from such irksome difficulties [as friction or air resistance], otherwise it is not possible to give any exact description. Once we have discovered our theorems about the stone, assuming there is no resistance, we can correct them or apply them with limitations as experience will teach. I can show by experiments that these external and incidental resistances are scarcely observable for bodies travelling over short distances."

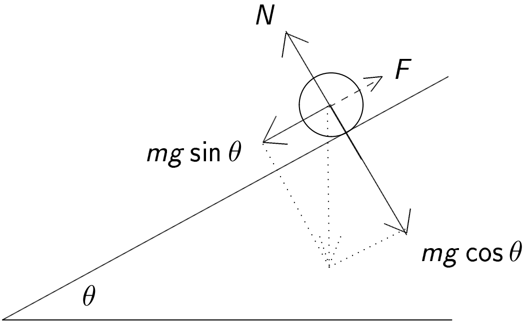
Galileo's falling stone

Our modern version of Galileo's simple mathematical model:

$$\begin{aligned}
\text{Acceleration} \qquad a(t) &= \frac{d^2y}{dt^2} = g, \\
\text{so velocity} \qquad v(t) &= \frac{dy}{dt} = \int g\,dt = gt, \\
\text{hence distance} \qquad y(t) &= \int gt\,dt = \frac{1}{2}gt^2.
\end{aligned}$$

Galileo's falling stone

Ball rolling on an inclined plane:



Galileo's falling stone

Falling stone with air resistance:

Taking into account the air resistance $-kmv$ should improve our real-life modelling.

$$\begin{aligned} \text{mass} \times \text{accel.} &= \text{force}, \\ \text{so} \quad ma = m\frac{d^2y}{d^2t} &= mg - km\frac{dy}{dt}, \\ \text{hence} \quad ma = m\frac{dv}{dt} &= mg - kmv, \\ \text{and therefore} \quad a(t) = \frac{dv}{dt} &= g - kv. \end{aligned}$$

Galileo's falling stone

Separating variables and integrating:

$$
\begin{aligned}
\int \frac{dv}{g - kv} &= \int dt, \\
\text{hence} \qquad -\frac{1}{k}\ln(g - kv) &= t + c_1, \\
\text{so that} \qquad \ln(g - kv) &= -kt + c_2. \\
\text{Taking exponentials,} \qquad g - kv &= c_3 e^{-kt}, \\
\text{so} \qquad v(t) &= \frac{g}{k} - c_4 e^{-kt}, \\
&\to \frac{g}{k} \quad \text{as } t \to \infty.
\end{aligned}
$$

This equation tells us that in the long run the velocity will approach a constant limiting value g/k, called the *terminal velocity*. But we could have found the terminal velocity much more easily. How?

Galileo's falling stone

How can we test our model?

What were our assumptions?

When might it be necessary to improve our model?

How could we do this?

Galileo's and Huygens' simple pendulum

Galileo's first observation

"One must observe that each pendulum has its own time of vibration so definite and determinate that it is not possible to make it move with any other period than that which nature has given it. For let any one take in his hand the cord to which the weight is attached and try, as much as he pleases, to increase or decrease the frequency of its vibrations; it will be time wasted."

Galileo's and Huygens' simple pendulum

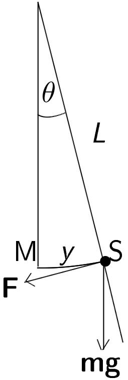
Galileo's assumptions:

That the angle of swing is small, and that friction/resistance may be ignored.

Galileo's experimental conclusion:

"As to the times of vibration of bodies suspended by threads of different lengths, they bear to each other the same proportion as the square roots of the lengths of the thread; or one might say that the lengths are to each other as the squares of the times...."

Galileo's and Huygens' simple pendulum



Our modern modelling of the pendulum:

$$
\begin{aligned}
\text{From the geometry,} \quad y &\approx arc\ MS = L\theta. \\
\text{Force towards centre} \quad F &= mg\sin\theta \approx mg\theta \approx \frac{mg}{L}y
\end{aligned}
$$

Galileo's and Huygens' simple pendulum

$$\begin{aligned} \text{Force} &= \text{mass} \times \text{acceleration}, \\ \text{so that} \qquad -\frac{mg}{L}y &= F = m\frac{d^2y}{dt^2}, \\ \text{giving} \qquad \frac{d^2y}{dt^2} &= -\frac{g}{L}y. \end{aligned}$$

General solution: $y(t) = A\sin\left(\sqrt{\frac{g}{L}}\, t + \alpha\right)$,

where A is the "amplitude", α is the "phase", $\sqrt{\frac{g}{L}}$ is the "frequency".

The "period" of this function is $2\pi\sqrt{\frac{L}{g}}$, in harmony with Galileo's experimental conclusion.

Galileo's and Huygens' simple pendulum

How can we test our model?

What assumptions have we made in this model?

When might it fail to model vibrations accurately?

How could we improve it?

Fibonacci's rabbits

Liber Abbaci (Book of Counting), 1202, by Leonardo of Pisa, also called Leonardo Fibonacci – son of Bonacci.

Problem: *A certain person places one pair of rabbits in a certain place surrounded on all sides by a wall. We want to know how many pairs can be bred from that pair in one year, assuming that it is in their nature that each month they give birth to another pair, and in the second month after birth each new pair can also breed.*

$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \cdots$

Fibonacci's answer is 377 pairs of rabbits, and his sequence of numbers is now known as the **Fibonacci sequence**.

Fibonacci's rabbits

Leonardo's solution:

"After the first month there will be two pairs, after the second, three. In the third month, two pairs will produce, so at the end of that month there will be five pairs. In the fourth month, three pairs will produce, so there will be eight pairs. Continuing thus, in the sixth month there will be five plus eight equals thirteen, in the seventh month, eight plus thirteen equals twenty-one, etc. There will be 377 pairs at the end of the twelfth month. For the sequence of numbers is as follows, where each is the sum of the two predecessors, and thus you can do it in order for an infinite number of months."

Fibonacci's rabbits

$$
\begin{tikzcd}
B \arrow[d] & & & & \\
Y \arrow[d] & & & & \\
A \arrow[r] \arrow[d] & B \arrow[dr] & & & \\
A \arrow[r] \arrow[d] & B \arrow[dr] & Y \arrow[dr] & & \\
A \arrow[r] & B & Y & A \arrow[r] & B
\end{tikzcd}
$$

(B, Y, A)	Total
$(1, 0, 0)$	1
$(0, 1, 0)$	1
$(1, 0, 1)$	2
$(1, 1, 1)$	3
$(2, 1, 2)$	5
$(3, 2, 3)$	8

Fibonacci's rabbits

After n months, let there be (B_n, Y_n, A_n) pairs of babies, young rabbits and adults, respectively. Then

$$\begin{aligned} A_{n+1} &= A_n + Y_n \\ B_{n+1} &= A_n + Y_n \\ Y_{n+1} &= B_n \\ \text{so that} \quad A_{n+1} &= B_{n+1} = Y_{n+2}. \end{aligned}$$

Let the total number of (pairs of) rabbits at n months be given by

$$F_n = A_n + B_n + Y_n.$$

Then we deduce that

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for } n \geq 0, \quad F_0 = F_1 = 1.$$

Fibonacci's rabbits

$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377, \cdots$

The number of pairs of rabbits after Fibonacci's 12th month is F_{13}, as he starts with F_1, not F_0, with a pair of young rabbits, not babies. We have $F_{13} = 377$. In the next month after that,
$F_{11} = 233$ pairs will produce;
and in the 100th month, F_{98} pairs will produce.

Leonardo got there without much fuss. But our notation and technique is very powerful and can be applied to a host of other problems.

Is there a quick way of arriving at this number F_{13}, or higher Fibonacci numbers like F_{100}?

Fibonacci's rabbits

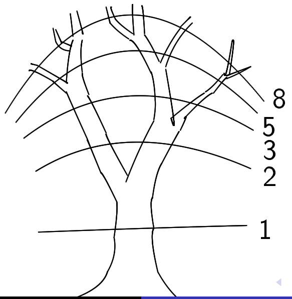
$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}, \quad \text{for} \quad n \geq 0.$$

Hence

$$\begin{aligned}
\begin{pmatrix} F_2 \\ F_1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
\begin{pmatrix} F_3 \\ F_2 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
\begin{pmatrix} F_{13} \\ F_{12} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{12} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 377 \\ 233 \end{pmatrix}.
\end{aligned}$$

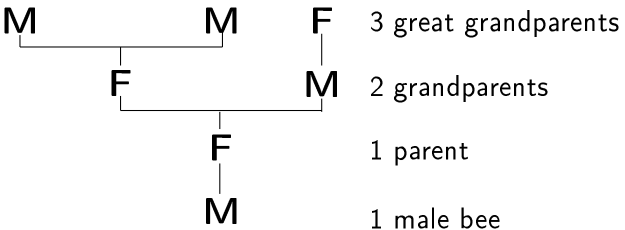
Fibonacci's rabbits

This same model for population growth can be applied to other situations. Assuming that each branch of a tree gives rise to a new branch, but only after skipping a season's maturation period, we obtain a visually very plausible model:



Fibonacci's rabbits

In certain species of bee, the forebears follow a Fibonacci law. Female bees are born from the mating of male and female, male bees are born asexually to single females.

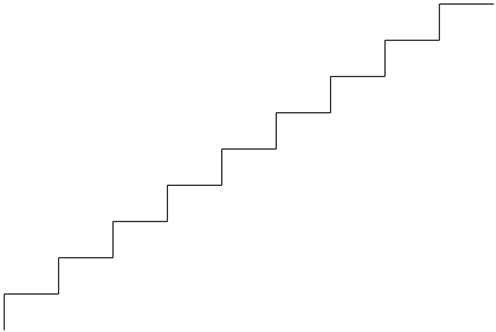


Exercise: Prove that the number B_n of bee ancestors in the nth generation backward, satisfies the Fibonacci generating equation. (Let $B_n = M_n + F_n$, the sum of the number of males and females, and write down the equations connecting M_n, F_n, M_{n+1}, F_{n+1}.)

Fibonacci's rabbits

Fibonacci numbers also arise from the following problem. Why?

Problem: *Taking either one or two steps at a time, how many different ways are there to climb n steps?*



Fibonacci's rabbits

How accurate is the Fibonacci model for the biological situations we are modelling?

What were the assumptions we made?

What could be done to improve the model?

Red blood cells

These give your blood its red colour.
Their function is to carry oxygen around the body.
When there are too few in your body you suffer from *anaemia*.
They are manufactured in your bone marrow, and each cell lives for about 4 months before dying, probably in your kidney or spleen.

You should normally have about 2×10^{13} red blood cells in your body at any moment, unless you live in Lesotho. Why?

Your bone marrow produces about 2×10^{6} red blood cells per second, or 1728×10^{8} per day.

Red blood cells

Assume that a fixed **number** b of cells are produced every day, and that a fixed **proportion** m of existing cells die each day.

Problem 1: What percentage of cells normally survive each day in the equilibrium situation?

Problem 2: Is this equilibrium stable? On our assumptions above, what will happen if the red blood cell count is suddenly different from normal?

Red blood cells

Let C_n be the number of red blood cells present at day n.
Let m be the proportion of red blood cells that die each day.
Let b be the number of red blood cells produced each day in the bone marrow.
Then

$$C_{n+1} = (1 - m)C_n + b.$$

For equilibrium, $C_n = C$, say, for all n. Substituting in the equation we have

$$C = (1 - m)C + b, \quad \text{hence} \quad mC = b,$$

$$m = \frac{b}{C} = \frac{1728 \times 10^8}{2 \times 10^{13}} = 0.00864.$$

Thus, the percentage of red blood cells that die each day is 0.864%, so that 99.136% of them survive each day.

Red blood cells

Next, suppose $C_n \neq C$, and let $D_n = C_n - C$, so D_n is the deviation from the normal. Then

$$
\begin{aligned}
D_{n+1} &= C_{n+1} - C \\
&= (1-m)C_n + b - C \\
&= (1-m)C_n + mC - C \\
&= (1-m)(C_n - C) \\
&= (1-m)D_n. \\
\text{Hence} \qquad D_n &= (1-m)^n D_0 \to 0, \quad \text{so} \quad C_n \to C.
\end{aligned}
$$

The equilibrium situation is a **stable** one: after a small deviation the system will return to the equilibrium.

Red blood cells

How can we test our model?

What assumptions have we made in this model?

When might it fail to model blood cell count accurately?

How could we improve it?

Crop yields and bacteria cultures

In the Chinese classic *Jiu zhang suan shu (The Nine Chapters of the Mathematical Art)*, written over 2000 years ago, the following problem is solved on the Chinese counting board:

Problem: Now there are 3 classes of paddy: top, medium and low grade. Given 3 bundles of top grade paddy, 2 bundles of medium grade paddy and 1 bundle of low grade paddy, the yield is 39 *dou*. For 2 bundles of top grade, 3 bundles of medium grade and 1 bundle of low grade, the yield is 34 *dou*. And for 1 bundle of top grade, 2 bundles of medium grade and 3 bundles of low grade, the yield is 26 *dou*. How much does one bundle of each grade yield?
Answer: Top grade paddy yields nine and a quarter *dou* per bundle; medium grade paddy four and a quarter *dou*; and low grade paddy two and three quarters *dou*.

Crop yields and bacteria cultures

high grade	1	2	3
medium grade	2	3	2
low grade	3	1	1
yield	26	34	39

$$\begin{aligned} 3x+2y+z &= 39 \qquad (1)\\ 2x+3y+z &= 34 \qquad (2)\\ x+2y+3z &= 26 \qquad (3) \end{aligned}$$

Exercise: Solve the Chinese problem by doing row operations on the matrix of coefficients.

Crop yields and bacteria cultures

In the 21st century, three species of bacteria $B_1,\ B_2,\ B_3$ are mixed together in the same culture in a laboratory. It is known that each requires three types of food, $F_1,\ F_2,\ F_3$, and:

$$\begin{aligned}
&B_1 \text{ consumes } F_1,\ F_2,\ F_3 \text{ at the rates } (1,2,2);\\
&B_2 \text{ consumes } F_1,\ F_2,\ F_3 \text{ at the rates } (3,2,1);\\
&B_3 \text{ consumes } F_1,\ F_2,\ F_3 \text{ at the rates } (8,4,1).
\end{aligned}$$

It is found experimentally that when 2750 units of F_1, 1500 units of F_1, and 500 units of F_3 are supplied per day, a constant situation is reached in which all food is consumed.

Problem 1: Find the amounts of bacteria present.
Problem 2: Which set of constant states is possible, and which is not, regardless of the food supply?

Crop yields and bacteria cultures

Solution: Suppose that there are amounts X_1, X_2, X_3, respectively, of the three bacteria present. Then

$$\begin{pmatrix} 1 & 3 & 8 \\ 2 & 2 & 4 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2750 \\ 1500 \\ 500 \end{pmatrix}, \qquad \text{or} \qquad AX = F.$$

Since $det(A) = 0$, so there are many possible solutions:

$$(X_1, X_2, X_3) = (t - 250,\ 1000 - 3t,\ t), \quad t \in \mathbb{R}.$$

Notice that for positive values of the X_i we must have $250 \le t \le \frac{1000}{3}$.

Crop yields and bacteria cultures

For the second question, consider the map

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad \text{defined by} \;\; f(X) = AX.$$

The rank is easily found to be 2, so the kernel of f has dimension 1 (it's a line) and the range is the 2-dimensional plane through the origin, spanned by the two image vectors:

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Taking the vector product of these two, we get a vector in direction $(-2, 5, -4)$ perpendicular to the required plane, hence the equation of the plane is

$$2x - 5y + 4z = 0.$$

Crop yields and bacteria cultures

The range of f is the plane

$$2x - 5y + 4z = 0.$$

One point on this plane is that experimentally discovered equilibrium state $(2750,\ 1500,\ 500)$.

But all possible equilibrium states must lie on this plane!

Age-stratified populations & a butterfly's life cycle

Consider a population of individuals each of which gives birth, lives for a while, then dies. Assume there is a maximum life span, and divide the population into k age-groups, numbering $a_1(n), a_2(n), \ldots, a_k(n)$ individuals at time n, measured in years or breeding seasons. Suppose that

1. everyone in stage k dies at the end of the year/season;
2. the probability that an individual in class j survives another year/season is p_j;
3. the probability that an individual in class j gives birth to exactly one individual is f_j (the fertility rate).

Age-stratified populations & a butterfly's life cycle

For example, take $k = 4$. The numbers in the next season are given by the *Leslie matrix*:

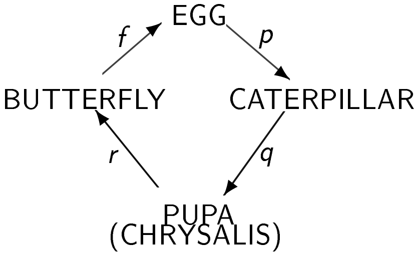
$$\begin{pmatrix} a_1(n+1) \\ a_2(n+1) \\ a_3(n+1) \\ a_4(n+1) \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \end{pmatrix} \begin{pmatrix} a_1(n) \\ a_2(n) \\ a_3(n) \\ a_4(n) \end{pmatrix}.$$

After N years we will have

$$\begin{pmatrix} a_1(N) \\ a_2(N) \\ a_3(N) \\ a_4(N) \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \end{pmatrix}^N \begin{pmatrix} a_1(0) \\ a_2(0) \\ a_3(0) \\ a_4(0) \end{pmatrix}.$$

A butterfly's life cycle

Consider the four stages of a butterfly's life cycle: egg, caterpillar, pupa (or chrysalis) and adult butterfly.



Clearly $f_1 = f_2 = f_3 = 0$. Assume that a butterfly succeeds in surviving to lay eggs with probability f, and then always lays N eggs. Then f_4 must be replaced by fN. Let the respective probabilities of an egg, caterpillar and pupa surviving to move into the next class be p, q, r.

A butterfly's life cycle

Let $E_n,\ C_n,\ P_n,\ B_n$ be the numbers of individual eggs, caterpillars, pupa, and butterflies, in a population after the nth season. Then the numbers for the next season will be

$$\begin{pmatrix} E_{n+1} \\ C_{n+1} \\ P_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} fNB_n \\ pE_n \\ qC_n \\ rP_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & fN \\ p & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & r & 0 \end{pmatrix} \begin{pmatrix} E_n \\ C_n \\ P_n \\ B_n \end{pmatrix}.$$

A butterfly's life cycle

The parameters f, p, q, r could be estimated from experimental data. The long term fate of the population will depend on their values, and that of N. To see how, we look at powers of the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & fN \\ p & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & r & 0 \end{pmatrix}^4 = fNpqr \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} E_{n+4} \\ C_{n+4} \\ P_{n+4} \\ B_{n+4} \end{pmatrix} = fNpqr \begin{pmatrix} E_n \\ C_n \\ P_n \\ B_n \end{pmatrix}$$

A butterfly's life cycle

It's easy to check that matrix equation, but how do we arrive at it in the first place? When powers of a matrix are in view, it is natural to find the eigenvalues.

The Cayley-Hamilton theorem says that the matrix will satisfy its characteristic equation, and the eigenvalues of this Leslie matrix are easily seen to be the solutions of the cubic

$$\lambda^3 = fNpqr$$

A butterfly's life cycle

Thus we conclude:

$$\begin{array}{llcll} \text{if} & fNpqr & < & 1, & \text{ultimate extinction occurs;} \\ \text{if} & fNpqr & = & 1, & \text{there is unstable equilibrium;} \\ \text{if} & fNpqr & > & 1, & \text{there is unsustainable expansion.} \end{array}$$

How realistic is this model?

What were our assumptions?

How can we test the model?

How can we improve it?

What can we learn from it anyway?