Time Series Notes 1.1

This note is based on James D. Hamilton's Time Series Analysis Chapter 1. It's mainly about Difference Equations.

1 First-Order Difference Equations

1.1 General Expression

A linear first-order difference equation:

$$y_t = \phi y_{t-1} + w_t$$

1.2 Solve it by Recursive Substitution

Suppose the sequence starts at date 0, with respect to every date, exists an equation:

$$y_0 = \phi y_{-1} + w_0$$
$$y_1 = \phi y_0 + w_1$$

$$y_t = \phi y_{t-1} + w_t$$

Subtituting y_{t-1} with the already-known equation, like:

$$y_1 = \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1$$

Based on the value of y_{-1}, we can compute all the values.

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + ... + \phi w_{t-1} + w_t$$

1.3 Dynamic Multipliers

Changing w_0 with other parameters unchanged, the effect is:

$$\frac{\partial y_t}{\partial w_0} = \phi^t$$

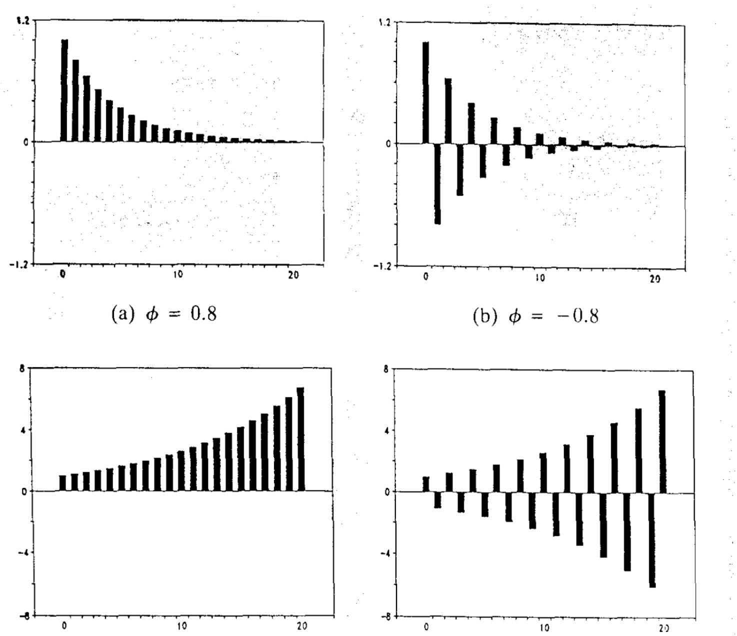
And the effect of w_t on y_{t+j} is given by:

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j$$

which is the dynamic multiplier. The multiplier only depends on j, w_t and y_{t+j}. It does not depend on t.

Different values of ϕ in first-order difference equation can produce a variety of dynamic responses of y to w.

- If $0 < \phi < 1$, the multiplier decays geometrically toward zero. See Figure1.1 a.
- If $-1 < \phi < 0$, the multiplier will alternate in sign, and the absolute value of the effect decays geometrically toward zero. See Figure1.1 b.
- If $\phi > 1$, the dynamic multiplier increases exponentially over time. See figure1.1 c.
- If $\phi < -1$,the system exhibits explosive oscillation. See figure1.1 d.



(a) $\phi = 0.8$ (b) $\phi = -0.8$

(c) $\phi = 1.1$ (d) $\phi = -1.1$

Figure 1: Multiplier changes

To summarize, if $|\phi| < 1$, the system is stable, and if $|\phi| > 1$, the system is explosive.

For the case that $\phi = 1$, the solution becomes:

$$y_{t+j} = y_{t-1} + w_t + w_{t+1} + ... + w_{t+j-1} + w_{t+j}$$

The output variable is the sum of the historical inputs. A one-unit increase in w will cause a **permanent** one-unit increase in y. See more for impulse-response function on $page5$.

1.4 Permanent Changes

Suppose there's a permanent change in w, the effect on y_{t+j} is given by:

$$\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + ... + \frac{\partial y_{t+j}}{\partial w_{t+j}} = \phi^j + \phi^{j-1} + ... + \phi + 1$$

When $|\phi| < 1$ the limit of it goes to infinity and is described as kind of "long-run" effect on y.

2 pth-Order Difference Equations

2.1 General Equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + w_t$$

Or we can rewrite this as:

Define the $(p \times 1)$ vector

$$\xi_t = \begin{bmatrix} y_t \\ y_{t-1} \\ ... \\ y_{t-p+1} \end{bmatrix}$$

Define F($p \times p$) by:

$$F = \begin{bmatrix} \phi_1 & \phi_2 & ... & \phi_{p-1} & \phi_p \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ ... \\ 0 & 0 & ... & 1 & 0 \end{bmatrix}$$

Note that there's not a identity matrix below the ϕ array. Also define the $p \times$ 1vector v_t:

$$v_t = \begin{bmatrix} w_t \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}$$

Then the first-order vector difference equation can be expressed as:

$$\xi_t = F\xi_{t-1} + v_t$$

The first order vector system is simply an alternative representation of the pth-order scalar system, and we can also solve it in a recursive way:

$$\xi_t = F^{t+1}\xi_{-1} + F^t v_0 + F^{t-1} v_1 + ... + F v_{t-1} + v_t.$$

Let $f_{11}^{(t)}$ denote the $(1,1)$ element of $\boldsymbol{F}^t$. The first equation can be rewritten as:

$y_t = f_{11}^{t+1}y_{-1} + f_{12}^{t+1}y_{-2} + ... + f_{1p}^{t+1}y_{-p} + f_{11}^{(t)}w_0 + f_{11}^{t-1}w_1 + ... + f_{11}^{(1)}w_{t-1} + w_t$

It can be seen from the above equation that the dynamic multiplier is

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^j$$

Here $f_1 1^j$ denotes the $(1,1)$ element of $\boldsymbol{F}^j$, which is ϕ_1. Also, we can compute the $(1,1)$ element of $\boldsymbol{F}^2$, which is the dynamic multiplier when $j = 2$.

Proposition 1 *The eigenvalues of the matrix $\boldsymbol{F}$ are the values of λ that satisfy*

$$\lambda^p = \phi_1\lambda^{p-2} - ... - \phi_{p-1}\lambda - \phi_p = 0$$

The proposition above shows that once we know the eigenvalues, it is straightforward for us to characterize the dynamic behavior of the system.

3 General Solutions of pth-Order Difference Equation

3.1 Distinct Eigenvalues

Recall that if the eigenvalues are distinct, always exists a nonsingular matrix $\boldsymbol{T}$:

$$\boldsymbol{F} = \boldsymbol{T\Lambda T}^{-1}$$

Λ is a matrix with the eigenvalues along the principal diagonal.This enables us to characterize the dynamic multiplier. Thus,

$$\boldsymbol{F}^j = \boldsymbol{T\Lambda^j T^{-1}}$$

Do the above matrix computation, we will have the $(1,1)$ element:

$$f_{11}^j = [t_{11}t^{11}]\lambda_1^j + [t_{12}t^{21}]\lambda_2^j + ... + [t_{1p}t^{p1}]\lambda_p^j$$

Let c_j denotes $[t_{1j}t^{j1}]$. Notice that the sum of c_j represents the $(1,1)$ element of $\boldsymbol{T}T^{-1}$. So,

$$c_1 + c_2 + ... + c_p = 1$$

Recall that

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^j$$

Subtituting, and we get:

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j + c_2\lambda_2^j + ... + c_p\lambda_p^j$$

It reveals that the dynamic multiplier is a weighted average of the eigenvalues raised to the jth power.

Proposition 2 *If the eigenvalues* $(\lambda_1, \lambda_2, ...\lambda_p)$ *of the matrix* $\boldsymbol{F}$ *are distinct, then the magnitude* c_i *can be written*

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{k=1k\neq i}^{p}(\lambda_i - \lambda_k)}$$

The pth-order difference equation implies that

$$y_{t+j} = f_{11}^{j+1} y_{t-1} + f_{12}^{j+1} + ... + f_{1p}^{j+1} y_{t-p} + w_{t+j} + \phi_1 w_{t+j-1} + \phi_2 w_{t+j-2} + ... + \phi_{j-1} w_{t+1} + \phi_j w_t$$

3.2 Higher-order systems

For a second-order system, suppose that $\phi_1^2 + 4\phi_2 > 0$.

Furthermore, if all of the eigenvalues are less than 1 in absolute value, then the system is stable and its dynamics are represented as a weighted average of decaying exponentials or decaying exponentials oscillating in sign. We have proved it in the previous section.

If the eigenvalues are real but at least one is **greater** than unity in absolute value, the system is explosive. Let λ_1 be the largest eigenvalue in absolute value. The dynamic multiplier is eventually dominated by an exponential function:

$$\lim_{j\to\infty} \frac{\partial y_{t+j}}{\partial w_t} \times \frac{1}{\lambda_1^j} = c_1$$

Here we will just cover a little about complex eigenvalues.

If $\phi_1^2 + 4\phi_2 < 0$, thenthe eigenvalues are complex. And for (ϕ_1, ϕ_2) lies below the parabola indicated in Figure1.2, their mod satisfies:

$$R^2 = a^2 + b^2 = -\phi_2$$

Thus, a system with complex eigenvalues is explosive whenever $\phi_2 < -1$.

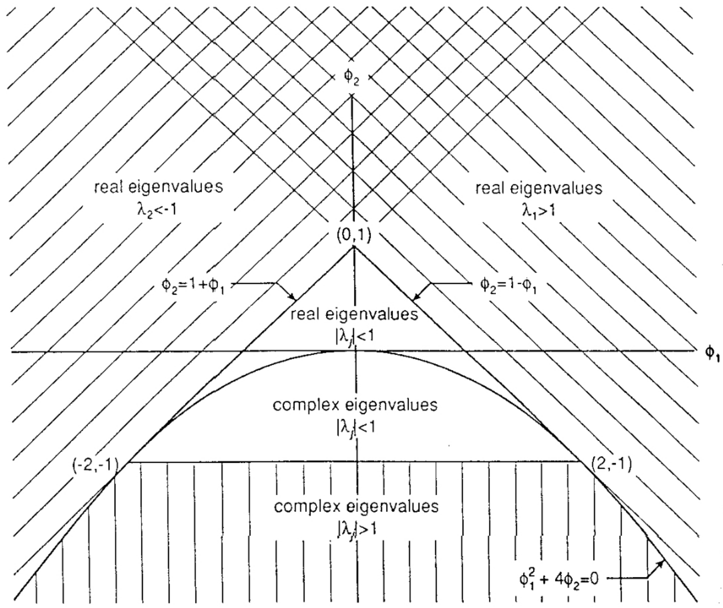


Figure 2: Summary

3.3 Repeated Eigenvalues

Now if $\boldsymbol{F}$ has repeated eigenvalues and $s < p$ linearly independent eigenvectors, using Jordan decomposition(See Matrix Analysis for more details),

$$\boldsymbol{F} = \boldsymbol{MJM^{-1}}$$

$$J = \begin{bmatrix} J_1 & 0 & 0 & ... & 0 \\ 0 & J_2 & 0 & ... & 0 \\ 0 & 0 & J_3 & ... & 0 \\ 0 & 0 & 0 & ... & J_s \end{bmatrix}$$

with

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda_i & 1 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & \lambda_i & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda_i \end{bmatrix}$$

Also, the same result exists here,

$$\boldsymbol{F^j} = \boldsymbol{MJ^jM^{-1}}$$

4 Long-Run and Present-Value Calculations

Proposition 3 *If the eigenvalues are all less than β^{-1}(β to be set)in mod, then the matrix $(\boldsymbol{I_p} - \beta\boldsymbol{F})^{-1}$ exists,and the effect of w on the present value of y is given as:*

$$1/(1 - \phi_1\beta - \phi_2\beta^2 - ... - \phi_{p-1}\beta^{p-1} - \phi_p\beta^p)$$

Setting $\beta = 1$,

$$\sum_{j=1}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} = 1/(1 - \phi_1 - \phi_2 - ... - \phi_p)$$

This shows the cumulative effect of a one-time change in w on y.

References