CV_Assignment1

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1 Prove

$\mathbf{M_i} = \begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0}^T & \mathbf{1} \end{bmatrix} \in \mathbb{R}^{4\times4}, \mathbf{R_i} \in \mathbb{R}^{3\times3}$ is an orthonormal matrix , $det(\mathbf{R}_i) = 1$ and $\mathbf{t_i} \in \mathbb{R}^{3\times1}$ is a vector

To prove $< \{\mathbf{M_i}\}, \odot >$ is a group

1.1 Closure

$\forall \mathbf{M_i}, \mathbf{M_j} \in \{\mathbf{M_i}\}$

$$\mathbf{M_i} \odot \mathbf{M_j} = \begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \begin{bmatrix} \mathbf{R_j} & \mathbf{t_j} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \mathbf{R_i}\mathbf{R_j} & \mathbf{R_i}\mathbf{t_j} + \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{2}$$

$\mathbf{R_i}\mathbf{t_j} \in \mathbb{R}^{3\times1} \Rightarrow \mathbf{R_i}\mathbf{t_j} + \mathbf{t_i} \in \mathbb{R}^{3\times1}$

$\mathbf{R_i}\mathbf{R_j} \in \mathbb{R}^{3\times3}, \mathbf{R_i}, \mathbf{R_j}$ are orthonormal matrix and $det(\mathbf{R_i}) = det(\mathbf{R_j}) = 1$
Now we should prove $\mathbf{R_i}\mathbf{R_j}$ is an orthonormal matrix and $det(\mathbf{R_i}\mathbf{R_j}) = 1$

Prove:

$$\mathbf{R_i}\mathbf{R_i}^T = \mathbf{E} \quad and \quad \mathbf{R_j}\mathbf{R_j}^T = \mathbf{E} \tag{3}$$

$$\mathbf{R_i}\mathbf{R_j}(\mathbf{R_i}\mathbf{R_j})^T = \mathbf{R_i}\mathbf{R_j}\mathbf{R_j^T}\mathbf{R_i^T} \tag{4}$$

$$= \mathbf{R_i}\mathbf{E}\mathbf{R_i^T} \tag{5}$$

$$= \mathbf{E} \tag{6}$$

So we get $\mathbf{R_i}\mathbf{R_j}$ is an orthonormal matrix and $det(\mathbf{R_i}\mathbf{R_j}) = 1$

Proof of colsure is complete

1.2 Associativity

$\forall \mathbf{M_i}, \mathbf{M_j}, \mathbf{M_k} \in \{\mathbf{M_i}\}$

$$(\mathbf{M_i} \odot \mathbf{M_j}) \odot \mathbf{M_k} = \left(\begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \begin{bmatrix} \mathbf{R_j} & \mathbf{t_j} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix}\right) \odot \begin{bmatrix} \mathbf{R_k} & \mathbf{t_k} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \mathbf{R_i R_j} & \mathbf{R_i t_j + t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \begin{bmatrix} \mathbf{R_k} & \mathbf{t_k} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} \mathbf{R_i R_j R_k} & \mathbf{R_i R_j t_k + R_i t_j + t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{9}$$

$$\mathbf{M_i} \odot (\mathbf{M_j} \odot \mathbf{M_k}) = \begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \left(\begin{bmatrix} \mathbf{R_j} & \mathbf{t_j} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \begin{bmatrix} \mathbf{R_k} & \mathbf{t_k} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix}\right) \tag{10}$$

$$= \begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \begin{bmatrix} \mathbf{R_j R_k} & \mathbf{R_j t_k + t_j} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{11}$$

$$= \begin{bmatrix} \mathbf{R_i R_j R_k} & \mathbf{R_i R_j t_k + R_i t_j + t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{12}$$

According to Equation(9) **and Equation**(12) **, we get** $(\mathbf{M_i} \odot \mathbf{M_j}) \odot \mathbf{M_k} = \mathbf{M_i} \odot (\mathbf{M_j} \odot \mathbf{M_k})$

Proof of associativity is complete

1.3 Existence Unity

$\forall \mathbf{M_i} \in \{\mathbf{M_i}\}$ **,** $\exists \mathbf{E} \in \{\mathbf{M_i}\}$ **is an unity , which make** $\mathbf{M_i} \odot \mathbf{E} = \mathbf{E} \odot \mathbf{M_i} = \mathbf{M_i}$

$$\mathbf{E} = \begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = \mathbf{I_4} \tag{14}$$

$$where \quad \mathbf{R_i} = \mathbf{I_3} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \quad and \quad \mathbf{t_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

The existence of unitary proof is complete

1.4 Existence Inverse Unit

$\forall \mathbf{M_i} \in \{\mathbf{M_i}\}$, $\exists \mathbf{N} \in \{\mathbf{M_i}\}$ **is an inverse unit , which make** $\mathbf{M_i} \odot \mathbf{N} = \mathbf{N} \odot \mathbf{M_i} = \mathbf{I_4}$

$$\mathbf{N} = \begin{bmatrix} \mathbf{R_i^{-1}} & -\mathbf{R_i^{-1} t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \tag{16}$$

$$\begin{aligned} \mathbf{M_i} \odot \mathbf{N} &= \begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \begin{bmatrix} \mathbf{R_i^{-1}} & -\mathbf{R_i^{-1} t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} && (17) \\ &= \begin{bmatrix} \mathbf{I_3} & \mathbf{0_{3\times 1}} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} && (18) \\ &= \mathbf{I_4} && (19) \end{aligned}$$

$$\begin{aligned} \mathbf{N} \odot \mathbf{M_i} &= \begin{bmatrix} \mathbf{R_i^{-1}} & -\mathbf{R_i^{-1} t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} \odot \begin{bmatrix} \mathbf{R_i} & \mathbf{t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} && (20) \\ &= \begin{bmatrix} \mathbf{I_3} & \mathbf{0_{3\times 1}} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix} && (21) \\ &= \mathbf{I_4} && (22) \end{aligned}$$

The existence of inverse unitary proof is complete and the inverse unit is $\begin{bmatrix} \mathbf{R_i^{-1}} & -\mathbf{R_i^{-1} t_i} \\ \mathbf{0^T} & \mathbf{1} \end{bmatrix}$

1.5 Summary

So far,we can prove $< \{\mathbf{M_i}\}, \odot >$ **forms a group**

2 Prove

$$G(x, y; \sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{23}$$

$$\begin{aligned} LoG &= \sigma^2 \nabla^2 G && (24) \\ &= \sigma^2 (\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}) && (25) \\ &= \sigma^2 (\frac{x^2 - \sigma^2}{2\pi\sigma^6} e^{-\frac{x^2+y^2}{2\sigma^2}} + \frac{y^2 - \sigma^2}{2\pi\sigma^6} e^{-\frac{x^2+y^2}{2\sigma^2}}) && (26) \\ &= \sigma^2 (\frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^6} e^{-\frac{x^2+y^2}{2\sigma^2}}) && (27) \\ &= \frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}} && (28) \end{aligned}$$

$$\frac{\partial G}{\partial \sigma} = \lim_{\Delta\delta \to 0} \frac{G(x,y;\sigma+\Delta\delta) - G(x,y;\sigma)}{\Delta\delta} \tag{29}$$

$$\xlongequal{\Delta\delta=(k-1)\sigma} \lim_{k \to 1} \frac{G(x,y;k\sigma) - G(x,y;\sigma)}{k\sigma - \sigma} \tag{30}$$

$$\approx \frac{G(x,y;k\sigma) - G(x,y;\sigma)}{k\sigma - \sigma} \tag{31}$$

$$\approx \frac{DoG}{k\sigma - \sigma} \tag{32}$$

$$\frac{\partial G}{\partial \sigma} = \frac{\partial(\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2)}}}{\partial \sigma} \tag{33}$$

$$= \frac{x^2+y^2-2\sigma^2}{2\pi\sigma^5} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{34}$$

$$= \frac{LoG}{\sigma} \tag{35}$$

According Equation(32) **and Equation**(35)**:**

$$DoG \approx (k-1)LoG \tag{36}$$

$$k = \frac{\Delta\delta}{\sigma} + 1 \tag{37}$$

3 Prove

To prove A^TA is non-singular(invertible),we can prove that $|A^TA| \neq 0$

$$A \in \mathbb{R}^{m\times n} \qquad A^T \in \mathbb{R}^{n\times m} \qquad \mathbf{rank}(A) = n \tag{38}$$

$$A^TA \in \mathbb{R}^{n\times n} \tag{39}$$

So we just need to prove

$$\mathbf{rank}(A^TA) = n \tag{40}$$

Because the rank of the matrix is equal to the row rank and the column rank

$$\mathbf{rank}(A) = \mathbf{rowrank}(A) = \mathbf{colrank}(A) \tag{41}$$

$$\mathbf{rank}(A^T) = \mathbf{rowrank}(A^T) = \mathbf{colrank}(A^T) \tag{42}$$

$$\mathbf{rowrank}(A) = \mathbf{colrank}(A^T) \tag{43}$$

$$\mathbf{colrank}(A) = \mathbf{rowrank}(A^T) \tag{44}$$

So we get:

$$\mathbf{rank}(A) = \mathbf{rank}(A^T) = n \tag{45}$$

Then we will prove

$$\mathbf{rank}(AA^T) = \mathbf{rank}(A^TA) \tag{46}$$

By proving that $Ax = 0$ and $A^TAx = 0$ two n-ary homogeneous equations have the same solution,we can get rank$(A^TA) = $ rank(A)

$$Ax = 0 \Rightarrow A^TAx = 0 \tag{47}$$

$$A^TAx = 0 \Rightarrow x^TA^TAX = 0 \Rightarrow (AX)^TAX = 0 \Rightarrow AX = 0 \tag{48}$$

So the two n-ary homogeneous equations $Ax = 0$ and $A^TAx = 0$ have the same solution

Then the same:

$$\mathbf{rank}(A) = \mathbf{rank}(A^TA) \tag{49}$$

$$\mathbf{rank}(A^T) = \mathbf{rank}(AA^T) \tag{50}$$

According to Equation(45) , Equation(49) , Equtation(50) , we can get

$$\mathbf{rank}(A) = \mathbf{rank}(A^TA) = \mathbf{rank}(AA^T) = \mathbf{rank}(A^T) = n \tag{51}$$

So,we get rank$(A^TA) = n$, now we can prove A^TA is non-singular

4 Programming

See the source code and documentation under the Panorama_Stitching folder for details

5 Programming

See the source code and documentation under the Scale_Invariant_Point_Detect folder for details