CV_Assignment2

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1 Solve

If the infinity point of a straight line is required, it can be obtained by intersecting the straight line with the infinite straight line.

Homogeneous equation of the infinity line:

$$l_1 \quad : \quad 0x + 0y + Kz = 0 \qquad K \neq 0$$

According to the question:

$$l_2 \quad : \quad x - 3y + 4 = 0$$

Homogeneous form:

$$l_2 \quad : \quad \frac{x}{z} - 3\frac{y}{z} + 4 = 0 \quad \Rightarrow \quad x - 3y + 4z = 0$$

According to Theorem : On the projective plane, the intersection of two lines l, l' is the point $x = l \times l'$, we get:

$$\begin{aligned} x &= (0, 0, k)^T \times (1, -3, 4)^T \\ &= \begin{vmatrix} i & j & k \\ 0 & 0 & K \\ 1 & -3 & 4 \end{vmatrix} \\ &= (3K, K, 0) \end{aligned}$$

Since the infinite line parameter K can be any non-zero number, usually we take $K = 1$, so we get the homogeneous coordinate of the infinity point of $l_2(x - 3y + 4 = 0)$:

$$x = (3, 1, 0)$$

2 Prove

For these four points $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3), D(x_4, y_4, z_4)$

coplanar $\Leftrightarrow \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$

First:

$$\begin{aligned}
\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} &= \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 & 0 \end{vmatrix} \\
&= x_1 \begin{vmatrix} y_2 - y_1 & z_2 - z_1 & 0 \\ y_3 - y_1 & z_3 - z_1 & 0 \\ y_4 - y_1 & z_4 - z_1 & 0 \end{vmatrix} - y_1 \begin{vmatrix} x_2 - x_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & z_3 - z_1 & 0 \\ x_4 - x_1 & z_4 - z_1 & 0 \end{vmatrix} \\
&+ z_1 \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \\ x_4 - x_1 & y_4 - y_1 & 0 \end{vmatrix} - \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} \\
&= - \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} \\
&= \begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \end{vmatrix}
\end{aligned}$$

Three points must be coplanar. When four points are coplanar, the product of any point and the other three points must be 0.

So we get:

$$\begin{aligned}
\overrightarrow{AB} &= (x_1 - x_2, y_1 - y_2, z_1 - z_2) \\
\overrightarrow{AC} &= (x_1 - x_3, y_1 - y_3, z_1 - z_3) \\
\overrightarrow{AD} &= (x_1 - x_4, y_1 - y_4, z_1 - z_4)
\end{aligned}$$

$$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = 0$$

Then

$$\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \end{vmatrix} = 0$$

Finally

Four points $A(x_1,y_1,z_1),B(x_2,y_2,z_2),C(x_3,y_3,z_3),D(x_4,y_4,z_4)$ **coplanar** $\Leftrightarrow$

$$\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Prove completed

3 Solve

Obtained from the question: $\mathbf{p}_n = (x,y)^T \qquad \mathbf{p}_d = (x_d,y_d)^T$

$$\begin{cases} x_d = x(1+k_1r^2+k_2r^4)+2\rho_1xy+\rho_2(r^2+2x^2)+xk_3r^6 \\ y_d = y(1+k_1r^2+k_2r^4)+2\rho_2xy+\rho_1(r^2+2y^2)+yk_3r^6 \end{cases} \text{where} \quad r^2=x^2+y^2$$

Thus:

$$\begin{aligned} x_d &= x(1+k_1(x^2+y^2)+k_2(x^2+y^2)^2)+2\rho_1xy+\rho_2(x^2+y^2+2x^2)+xk_3(x^2+y^2)^3 \\ &= x(1+k_1(x^2+y^2)+k_2(x^4+y^4+2x^2y^2))+2\rho_1xy+\rho_2(3x^2+y^2)+xk_3(x^6+y^6+3x^4y^2+3x^2y^4) \\ &= x+2\rho_1xy+\rho_2(3x^2+y^2)+k_1(x^3+xy^2)+k_2(x^5+xy^4+2x^3y^2)+k_3(x^7+xy^6+3x^5y^2+3x^3y^4) \end{aligned}$$

$$\begin{aligned} y_d &= y(1+k_1(x^2+y^2)+k_2(x^2+y^2)^2)+2\rho_2xy+\rho_1(x^2+y^2+2y^2)+yk_3(x^2+y^2)^3 \\ &= y(1+k_1(x^2+y^2)+k_2(x^4+y^4+2x^2y^2))+2\rho_2xy+\rho_1(x^2+3y^2)+yk_3(x^6+y^6+3x^4y^2+3x^2y^4) \\ &= y+2\rho_2xy+\rho_1(x^2+3y^2)+k_1(x^2y+y^3)+k_2(x^4y+y^5+2x^2y^3)+k_3(x^6y+y^7+3x^4y^3+3x^2y^5) \end{aligned}$$

So we get:

$$\frac{\mathrm{d}\mathbf{p}_d}{\mathrm{d}\mathbf{p}_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix}$$

$$\begin{aligned} \frac{\partial x_d}{\partial x} &= 1+2\rho_1y+6\rho_2x+k_1(3x^2+y^2)+k_2(5x^4+y^4+6x^2y^2)+k_3(7x^6+y^6+15x^4y^2+9x^2y^4) \\ &= (1+2\rho_1y+k_1y^2+k_2y^4+k_3y^6)+6\rho_2x+(3k_1+6k_2y^2+9k_3y^4)x^2+(5k_2+15k_3y^2)x^4+7k_3x^6 \\ \frac{\partial x_d}{\partial y} &= 2\rho_1x+2\rho_2y+2k_1xy+k_2(4xy^3+4x^3y)+k_3(6xy^5+6x^5y+12x^3y^3) \\ &= 2\rho_1x+(2\rho_2+2k_1x+4k_2x^3+6k_3x^5)y+(4k_2x+12k_3x^3)y^3+6k_3xy^5 \\ \frac{\partial y_d}{\partial x} &= 2\rho_2y+2\rho_1x+2k_1xy+k_2(4x^3y+4xy^3)+k_3(6x^5y+12x^3y^3+6xy^5) \\ &= 2\rho_2y+(2\rho_1+2k_1y+4k_2y^3+6k_3y^5)x+(4k_2y+12k_3y^3)x^3+6k_3yx^5 \\ \frac{\partial y_d}{\partial y} &= 1+2\rho_2x+6\rho_1y+k_1(x^2+3y^2)+k_2(x^4+5y^4+6x^2y^2)+k_3(x^6+7y^6+9x^4y^2+15x^2y^4) \\ &= (1+2\rho_2x+k_1x^2+k_2x^4+k_3x^6)+6\rho_1y+(3k_1+6k_2x^2+9k_3x^4)y^2+(5k_2+15k_3x^2)y^4+7k_3y^6 \end{aligned}$$

4 Solve

Obtained from the question:

$$\mathbf{r} = \theta\mathbf{n} \in \mathbb{R}^{3\times1} \qquad \alpha \triangleq \sin\theta \qquad \beta \triangleq \cos\theta \qquad \gamma \triangleq 1-\cos\theta$$

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad \mathbf{n}\mathbf{n}^T = \begin{bmatrix} n_1^2 & n_1n_2 & n_1n_3 \\ n_1n_2 & n_2^2 & n_2n_3 \\ n_1n_3 & n_2n_3 & n_3^2 \end{bmatrix} \qquad \mathbf{n}^\wedge = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \in \mathbb{R}^{3\times3}$$

$$\mathbf{R} = \cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{n}\mathbf{n}^T + \sin\theta\mathbf{n}^\wedge$$

$$\mathbf{R} = \beta\mathbf{I} + \gamma\mathbf{n}\mathbf{n}^T + \alpha\mathbf{n}^\wedge$$

$$\mathbf{u} \triangleq (R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}, R_{33})^T \in \mathbb{R}^{9\times1}$$

Thus:

$$\begin{aligned}
\mathbf{R} &= \beta\mathbf{I} + \gamma\mathbf{n}\mathbf{n}^T + \alpha\mathbf{n}^\wedge \\
&= \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} + \begin{bmatrix} \gamma n_1^2 & \gamma n_1n_2 & \gamma n_1n_3 \\ \gamma n_1n_2 & \gamma n_2^2 & \gamma n_2n_3 \\ \gamma n_1n_3 & \gamma n_2n_3 & \gamma n_3^2 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha n_3 & \alpha n_2 \\ \alpha n_3 & 0 & -\alpha n_1 \\ -\alpha n_2 & \alpha n_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \beta + \gamma n_1^2 & \gamma n_1n_2 - \alpha n_3 & \gamma n_1n_3 + \alpha n_2 \\ \gamma n_1n_2 + \alpha n_3 & \beta + \gamma n_2^2 & \gamma n_2n_3 - \alpha n_1 \\ \gamma n_1n_3 - \alpha n_2 & \gamma n_2n_3 + \alpha n_1 & \beta + \gamma n_3^2 \end{bmatrix}
\end{aligned}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{r}^T} = \begin{bmatrix} \frac{\mathrm{d}R_{11}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{12}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{13}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{21}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{22}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{23}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{31}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{32}}{\mathrm{d}\mathbf{r}^T} \\ \frac{\mathrm{d}R_{33}}{\mathrm{d}\mathbf{r}^T} \end{bmatrix} \qquad \begin{cases} \frac{\mathrm{d}R_{11}}{\mathrm{d}\mathbf{r}^T} = \left[\frac{2\gamma n_1}{\theta}, 0, 0\right] \\ \frac{\mathrm{d}R_{12}}{\mathrm{d}\mathbf{r}^T} = \left[\frac{\gamma n_2}{\theta}, \frac{\gamma n_1}{\theta}, \frac{-\alpha}{\theta}\right] \\ \frac{\mathrm{d}R_{13}}{\mathrm{d}\mathbf{r}^T} = \left[\frac{\gamma n_3}{\theta}, \frac{\alpha}{\theta}, \frac{\gamma n_1}{\theta}\right] \\ \frac{\mathrm{d}R_{21}}{\mathrm{d}\mathbf{r}^T} = \left[\frac{\gamma n_2}{\theta}, \frac{\gamma n_1}{\theta}, \frac{\alpha}{\theta}\right] \\ \frac{\mathrm{d}R_{22}}{\mathrm{d}\mathbf{r}^T} = \left[0, \frac{2\gamma n_2}{\theta}, 0\right] \\ \frac{\mathrm{d}R_{23}}{\mathrm{d}\mathbf{r}^T} = \left[\frac{-\alpha}{\theta}, \frac{\gamma n_3}{\theta}, \frac{\gamma n_2}{\theta}\right] \\ \frac{\mathrm{d}R_{31}}{\mathrm{d}\mathbf{r}^T} = \left[\frac{\gamma n_3}{\theta}, \frac{-\alpha}{\theta}, \frac{\gamma n_1}{\theta}\right] \\ \frac{\mathrm{d}R_{32}}{\mathrm{d}\mathbf{r}^T} = \left[\frac{\alpha}{\theta}, \frac{\gamma n_3}{\theta}, \frac{\gamma n_2}{\theta}\right] \\ \frac{\mathrm{d}R_{33}}{\mathrm{d}\mathbf{r}^T} = \left[0, 0, \frac{2\gamma n_3}{\theta}\right] \end{cases}$$

So we get:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{r}^T} = \begin{bmatrix} \frac{2\gamma n_1}{\theta} & 0 & 0 \\ \frac{\gamma n_2}{\theta} & \frac{\gamma n_1}{\theta} & \frac{-\alpha}{\theta} \\ \frac{\gamma n_3}{\theta} & \frac{\alpha}{\theta} & \frac{\gamma n_1}{\theta} \\ \frac{\gamma n_2}{\theta} & \frac{\gamma n_1}{\theta} & \frac{\alpha}{\theta} \\ 0 & \frac{2\gamma n_2}{\theta} & 0 \\ \frac{-\alpha}{\theta} & \frac{\gamma n_3}{\theta} & \frac{\gamma n_2}{\theta} \\ \frac{\gamma n_3}{\theta} & \frac{-\alpha}{\theta} & \frac{\gamma n_1}{\theta} \\ \frac{\alpha}{\theta} & \frac{\gamma n_3}{\theta} & \frac{\gamma n_2}{\theta} \\ 0 & 0 & \frac{2\gamma n_3}{\theta} \end{bmatrix} = \frac{1}{\theta} \begin{bmatrix} 2\gamma n_1 & 0 & 0 \\ \gamma n_2 & \gamma n_1 & -\alpha \\ \gamma n_3 & \alpha & \gamma n_1 \\ \gamma n_2 & \gamma n_1 & \alpha \\ 0 & 2\gamma n_2 & 0 \\ -\alpha & \gamma n_3 & \gamma n_2 \\ \gamma n_3 & -\alpha & \gamma n_1 \\ \alpha & \gamma n_3 & \gamma n_2 \\ 0 & 0 & 2\gamma n_3 \end{bmatrix}$$

5 Programming

View details in *RANSAC.py* **and** *RANSAC.md*