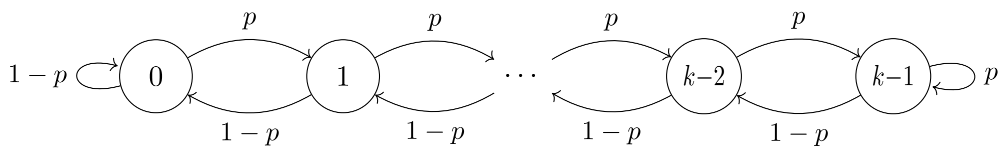
UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 9
Fall 2023

1. **Finite Random Walk**

Let $0 < p < 1$, and consider the following finite *random walk* with bias p on $\mathcal{X} = \{0, \ldots, k-1\}$, also known as the finite *birth-death chain*.



a. Find the stationary distribution π.
Hint: Write $q = 1 - p$ and define $r \coloneqq \frac{p}{q}$. Be careful when $r = 1$.

b. Find the limit of $\pi(0)$ and $\pi(k-1)$, as functions of k, as $k \to \infty$.

Solution:

a. Let us solve the detailed balance equations:

$$p \cdot \pi(i-1) = q \cdot \pi(i) \quad \text{for all } i = 1, \ldots, k-1,$$

or $\pi(i) = r\pi(i-1)$. Iterating this recurrence relation, we have $\pi(i) = r^i \pi(0)$, so

$$\sum_{i=0}^{k-1} \pi(i) = \pi(0) \sum_{i=0}^{k-1} r^i = \pi(0) \frac{1-r^k}{1-r} = 1.$$

We can then solve for $\pi(0) = \frac{1-r}{1-r^k}$ and $\pi(i) = r^i \frac{1-r}{1-r^k}$ in general. However, this formula is undefined when $r = 1$, or $p = \frac{1}{2}$. Instead, we find that $\pi(i) \equiv \frac{1}{k}$ for all $i = 0, \ldots, k-1$. In short, the stationary distribution is given by

$$\pi(i) = \begin{cases} r^i \frac{1-r}{1-r^k} & \text{if } p \neq \frac{1}{2} \\ \frac{1}{k} & \text{if } p = \frac{1}{2}. \end{cases}$$

b. First, the limit of $\pi(0)$ is $\lim_{k\to\infty} \frac{1}{k} = 0$ if $r = 1$ and $(1-r)\lim_{k\to\infty} \frac{1}{1-r^k}$ if $r \neq 1$. Now, $1/(1-r^k) \to 1$ if $r < 1$ and $\to 0$ if $r > 1$. Therefore

$$\lim_{k\to\infty} \pi(0) = \begin{cases} 1-r & \text{if } r < 1 \\ 0 & \text{if } r \geq 1. \end{cases}$$

For $\pi(k-1)$, we know that the limit is also 0 if $r = 1$, so let $r \neq 1$. Then

$$\lim_{k\to\infty} \pi(k-1) = \lim_{k\to\infty} \frac{r^{k-1} - r^k}{1-r^k} = \lim_{k\to\infty} \frac{1-r}{\frac{1}{r^{k-1}} - r} = \begin{cases} 0 & \text{if } r < 1 \\ \frac{r-1}{r} & \text{if } r > 1. \end{cases}$$

Therefore the limit is

$$\lim_{k\to\infty} \pi(k-1) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 - \frac{1}{r} & \text{if } r > 1. \end{cases}$$

2. **Moving Books Around**

You have N books labelled $1, \ldots, N$ on your shelf. At each time step, you pick a book i with probability $\frac{1}{N}$, place it on the left of all others on the shelf, then repeat this process, each step independent of any other step. Construct a suitable Markov chain which takes values in the set of all $N!$ permutations of the books.

a. Find the transition probabilities of the Markov chain.

b. Find its stationary distribution.

   Hint: You can guess the stationary distribution before computing it.

Solution:

a. The state space consists of all $N!$ permutations on N books. The transition probabilities are then

$$P((\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \ldots, \sigma_N), (\sigma_i, \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \sigma_N)) = \frac{1}{N}$$

for $i = 1, \ldots, N$, and 0 otherwise.

b. By symmetry, every state $\sigma \in S_N$ should have the same stationary probability,

$$\pi(\sigma) = \frac{1}{N!}.$$

We can verify that this probability distribution satisfies the balance equations. Let $\sigma^{(1)} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \ldots, \sigma_N)$ be a permutation, and for $i = 2, \ldots, n$, let $\sigma^{(i)}$ be the permutation with σ_1 in the ith position, $(\sigma_2, \ldots, \sigma_{i-1}, \sigma_1, \sigma_i, \ldots, \sigma_N)$. With this notation,

$$\pi(\sigma^{(1)}) = \sum_{i=1}^{N} \pi(\sigma^{(i)}) P(\sigma^{(i)}, \sigma^{(1)}) = \sum_{i=1}^{N} \frac{1}{N!} \cdot \frac{1}{N} = \frac{1}{N!}.$$

3. **Product of Rolls of a Die**

A fair die with labels 1 through 6 is rolled until the product of the last two rolls is 12. What is the expected number of rolls?

Hint: You can model this process as a Markov chain with 3 states, choosing your states according to the outcome of last roll. For example, assign one state if its outcome was 1 or 5, which is useless if you want the product to be 12. If the outcome was 2, 3, 4 or 6, it's useful and can be assigned to another state. Assign a third state to the case when the product of the last two outcomes was 12.

Solution: Taking the hint, we model this process as a Markov chain with 3 states, where the states correspond to the outcome of the last roll. Let s_1 be the state where the last outcome is 1 or 5; s_2 where the last outcome is 2, 3, 4, or 6; and s_3 where the product of the last two rolls is 12, so the transition probability matrix is

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 \end{bmatrix}.$$

Let T_i be the expected number of rolls needed to get to state s_3 starting from state s_i for $i = 1, 2$. Then we have the first-step equations

$$T_1 = 1 + \frac{1}{3}T_1 + \frac{2}{3}T_2$$

$$T_2 = 1 + \frac{1}{3}T_1 + \frac{1}{2}T_2.$$

Solving the equations, we get $T_1 = 10.5$ and $T_2 = 9$, so the expected number of rolls is

$$T = 1 + \frac{1}{3}T_1 + \frac{2}{3}T_2 = 10.5.$$