UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 5
Fall 2023

1. **On Almost Sure Convergence**

a. Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

b. Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

$$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \mid Y = y\big) = 1.$$

Does $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

c. Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \dots, 2^{k+1} - 1\}$, and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

d. Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$?

Solution:

a. Yes. If a sequence oscillates between two distinct values infinitely often, then it does not converge. Here, we have a sequence that oscillates infinitely often, with probability 1, which means that the sequence in fact **diverges** with probability 1.

The above may have been very cumbersome to read, which is why we often abbreviate "with probability 1" with "a.s." Then the above reads "$(X_n)_{n\in\mathbb{N}}$ oscillates between two values infinitely often a.s., so $(X_n)_{n\in\mathbb{N}}$ does not converge a.s."

b. Yes. Observe that when $Y = y \neq 0$, $(X_n)_{n\in\mathbb{N}}$ will converge to y^{-1}, but when $Y = 0$, $(X_n)_{n\in\mathbb{N}}$ does not converge. However, $\mathbb{P}(Y = 0) = 0$, since Y is a continuous random variable. In other words,

$$\mathbb{P}(X_n \text{ does not converge as } n \to \infty) = \mathbb{P}(Y = 0) = 0$$
$$\mathbb{P}(X_n \text{ converges as } n \to \infty) = \mathbb{P}(Y \neq 0) = 1,$$

so $(X_n)_{n\in\mathbb{N}}$ converges a.s.

c. No. The sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between 0 and successively higher powers of two infinitely often a.s., so it does not converge a.s.

d. Yes. Fix $\varepsilon > 0$. For $n \in \mathbb{Z}^+$, one has

$$\mathbb{P}(|X_n| > \varepsilon) = \frac{1}{2^k},$$

where $k = \lfloor \log_2 n \rfloor$. As $n \to \infty$, the above probability goes to 0, so $X_n \to 0$ in probability. Intuitively, $(X_n)_{n\in\mathbb{N}}$ has infinitely many oscillations, so it cannot converge a.s. However, the probability of each oscillation shrinks to 0, so $(X_n)_{n\in\mathbb{N}}$ converges in probability.

The expectations do not converge. For all n, one has $\mathbb{E}(X_n) = 1$, so it is not the case that $\mathbb{E}(X_n) \to 0$ as $n \to \infty$. Hence, convergence in probability is not sufficient to imply that the expectations converge. In fact, almost sure convergence is not sufficient either.

2. **Convergence in Probability**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequence $(Y_n)_{n\in\mathbb{N}}$ converges in probability to some limit where $Y_n = (X_n)^n$.

Solution: For any $\varepsilon > 0$, $\mathbb{P}(|Y_n| > \varepsilon) = \mathbb{P}(|X_n| > \varepsilon^{1/n}) = 1 - \varepsilon^{1/n} \to 0$ as $n \to \infty$. Thus, the sequence converges to 0 in probability.

3. **Convergence in L^p**

   Let $p \geq 1$. A sequence of random variables $(X_n)_{n\geq 1}$ is said to **converge in** L^p (norm) to a random variable X if

   $$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0.$$

   Prove that if $X_n \to X$ in L^p, then $X_n \to X$ in probability.

   Solution: Note that for $p \geq 1$, $x \mapsto x^p$ is a monotonic function. By Markov's inequality,

   $$\mathbb{P}(|X_n - X| \geq \varepsilon) = \mathbb{P}(|X_n - X|^p \geq \varepsilon^p) \leq \frac{\mathbb{E}(|X_n - X|^p)}{\varepsilon^p}.$$

   If $X_n \to X$ in L^p, i.e. $\mathbb{E}(|X_n - X|^p) \to 0$, then $\mathbb{P}(|X_n - X| \geq \varepsilon) \to 0$ for any $\varepsilon > 0$, which is precisely convergence in probability.