UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 3
Fall 2023

1. **Uncorrelatedness and Independence**

   a. Show that if $X_1, \dots, X_n$ are pairwise uncorrelated, then

   $$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{var}(X_i).$$

   b. Find an example where a pair of random variables are uncorrelated but not independent.

Solution:

a. By linearity of expectation, pairwise uncorrelatedness of $X_1, \dots, X_n$ implies uncorrelatedness of $X_1 + \cdots + X_k$ and X_{k+1} for $k = 1, 2, \dots, n-1$ (you should verify this yourself). Then, since $\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y)$ for uncorrelated X and Y, we have

$$\begin{aligned}
&\operatorname{var}(X_1 + \cdots + X_n) \\
&= \operatorname{var}(X_1 + \cdots + X_{n-1}) + \operatorname{var}(X_n) \\
&= \operatorname{var}(X_1 + \cdots + X_{n-2}) + \operatorname{var}(X_{n-1}) + \operatorname{var}(X_n) \\
&\quad \vdots \\
&= \operatorname{var}(X_1) + \cdots + \operatorname{var}(X_n).
\end{aligned}$$

b. Consider $X \sim \text{Uniform}\{-1, 0, 1\}$, $Z \sim \text{Uniform}\{-1, 1\}$, independent of each other (Z is called a *Rademacher* random variable). Let $Y = XZ$. Then,

$$\begin{aligned}
\operatorname{cov}(X, Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] \\
&= \mathbb{E}[X^2 Z] - 0 \cdot 0 \\
&= \mathbb{E}[X^2]\,\mathbb{E}[Z] \\
&= \tfrac{2}{3} \cdot 0 \\
&= 0.
\end{aligned}$$

However, X and Y are not independent since

$$\mathbb{P}(X = 0, Y = 0) = \mathbb{P}(X = 0) = \tfrac{1}{3},$$

$$\mathbb{P}(X = 0)\,\mathbb{P}(Y = 0) = \tfrac{1}{3} \cdot \tfrac{1}{3} \neq \mathbb{P}(X = 0, Y = 0).$$

2. **Galton–Watson Branching Process**

Consider a population of N individuals for some positive integer N. Let ξ be a random variable taking values in $\mathbb{N}$ with $\mathbb{E}(\xi) = \mu$ and $\text{var}(\xi) = \sigma^2$. At the end of each year, each individual, independently of all other individuals and generations, leaves behind a number of offspring which has the same distribution as ξ. For each $n \in \mathbb{N}$, let X_n denote the size of the population at the end of the nth year.

a. Compute $\mathbb{E}(X_n)$.

b. Compute $\text{var}(X_n|X_{n-1})$. Then, write $\text{var}(X_n)$ in terms of $\text{var}(X_{n-1})$.

Solution:

a. We first note that $X_0 = N$, so $\mathbb{E}(X_0) = N$ and $\text{var}(X_0) = 0$. Then, conditioned on the number of people in the previous year X_{n-1}, we have

$$\begin{aligned}
\mathbb{E}(X_n) = \mathbb{E}(\mathbb{E}(X_n \mid X_{n-1})) &= \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{X_{n-1}} \xi_i \mid X_{n-1}\right)\right) \\
&= \mathbb{E}\left(X_{n-1}\,\mathbb{E}(\xi)\right) \\
&= \mu\,\mathbb{E}(X_{n-1}).
\end{aligned}$$

By recursion, we find that $\mathbb{E}(X_n) = \mu^n N$.

b. As we computed above, $\mathbb{E}(X_n \mid X_{n-1}) = \mu X_{n-1}$. The conditional variance is $\text{var}(X_n \mid X_{n-1}) = \sigma^2 X_{n-1}$. Then, we have

$$\text{var}\, X_n = \mathbb{E}[\sigma^2 X_{n-1}] + \text{var}(\mu X_{n-1}) = \sigma^2 \mu^{n-1} N + \mu^2\, \text{var}\, X_{n-1}.$$

First, suppose that $\mu = 1$. Then, the recurrence simplifies to $\text{var}\, X_n = \sigma^2 N + \text{var}\, X_{n-1}$, which means that the variance increases linearly:

$$\text{var}(X_n) = \sigma^2 N n.$$

For $\mu \neq 1$, the solution to the recurrence is obtained by finding a pattern after a few iterations:

$$\begin{aligned}
\text{var}(X_n) = \sigma^2\mu^{n-1}N + \mu^2\,\text{var}\,X_{n-1} &= \sigma^2\mu^{n-1}N + \sigma^2\mu^n N + \mu^4\,\text{var}\,X_{n-2} \\
= \cdots = \sigma^2\mu^{n-1}N\sum_{k=0}^{n-1}\mu^k &= \sigma^2\mu^{n-1}N\frac{1-\mu^n}{1-\mu}
\end{aligned}$$

We have used the formula for a finite geometric series.

3. **Minimum and Maximum of Exponentials**

Let $\lambda_1, \lambda_2 > 0$, and $X_1 \sim \text{Exponential}(\lambda_1)$, $X_2 \sim \text{Exponential}(\lambda_2)$ are independent. Also, define $U := \min(X_1, X_2)$ and $V := \max(X_1, X_2)$. Show that U and $V - U$ are independent.

Solution: For $u, w > 0$,

$$\begin{aligned}
\Pr(U \le u, V-U \le w, X_1 < X_2) &= \Pr(X_1 \le u, X_1 < X_2 \le X_1 + w) \\
&= \int_0^u \int_{x_1}^{x_1+w} \lambda_2 \exp(-\lambda_2 x_2)\, dx_2\, \lambda_1 \exp(-\lambda_1 x_1)\, dx_1 \\
&= \int_0^u \big\{\exp(-\lambda_2 x_1) - \exp\big(-\lambda_2(x_1+w)\big)\big\} \lambda_1 \exp(-\lambda_1 x_1)\, dx_1 \\
&= \big(1 - \exp(-\lambda_2 w)\big) \int_0^u \lambda_1 \exp\big(-(\lambda_1+\lambda_2)x_1\big)\, dx_1 \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}\big(1 - \exp\{-(\lambda_1+\lambda_2)u\}\big)\big(1 - \exp(-\lambda_2 w)\big).
\end{aligned}$$

By symmetry, interchanging the roles of 1 and 2 yields

$$\begin{aligned}
&\Pr(U \le u, V-U \le w, X_2 < X_1) \\
&\qquad = \frac{\lambda_2}{\lambda_1+\lambda_2}\big(1 - \exp\{-(\lambda_1+\lambda_2)u\}\big)\big(1 - \exp(-\lambda_1 w)\big).
\end{aligned}$$

Adding these two expressions yields

$$\Pr(U \le u, V-U \le w) = \big(1 - \exp\{-(\lambda_1+\lambda_2)u\}\big) p_w, \qquad \text{where}$$

$$p_w := \frac{\lambda_1}{\lambda_1+\lambda_2}\big(1 - \exp(-\lambda_2 w)\big) + \frac{\lambda_2}{\lambda_1+\lambda_2}\big(1 - \exp(-\lambda_1 w)\big).$$

The joint CDF splits into a product of factors $\Pr(U \le u) \Pr(V-U \le w)$ which proves independence. To interpret the second term, observe that $\lambda_1/(\lambda_1+\lambda_2)$ is the probability of the event $\{X_1 < X_2\}$; and conditioned on this event, $V - U \sim \text{Exponential}(\lambda_2)$ by the memoryless property.