UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 11
Fall 2023

1. **CTMC Introduction**

   Consider the continuous-time Markov chain defined on the state space $\{1, 2, 3, 4\}$ which has transition rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

   a. Find the stationary distribution π of this chain.

   b. Find the stationary distribution μ of the jump chain, the DTMC which only keeps track of the jumps. Formally, if $(X_t)_{t\geq 0}$ transitions at times $T_1, T_2, \ldots$, then its jump chain is $(Y_n)_{n=1}^{\infty}$, where $Y_n \coloneqq X_{T_n}$.

   c. Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

   d. From state 1, what is the expected amount of time until the chain is in state 4?

Solution:

a. Solving $\pi Q = 0$ with $\sum_i \pi(i) = 1$, we find that the stationary distribution is

$$\pi = \begin{bmatrix} \frac{3}{38} & \frac{7}{38} & \frac{9}{38} & \frac{1}{2} \end{bmatrix}.$$

b. Recall that μ is given by

$$\mu(i) = \frac{q(i)\pi(i)}{\sum_{j=1}^{4} q(j)\pi(j)}.$$

   Using part a, we have

$$\mu = \begin{bmatrix} \frac{9}{85} & \frac{21}{85} & \frac{36}{85} & \frac{19}{85} \end{bmatrix}.$$

c. The *holding time*, the time the chain remains in state 1 before jumping, has an Exponential distribution with rate 3, so the expected amount of time it stays in state 1 is $\frac{1}{3}$.

d. We can compute the expected hitting times using first-step equations. Let $\beta(i)$ be the mean time needed to reach state 4 from i, so that $\beta(4) = 0$. For $i = 1, 2, 3$, we have

$$\beta(i) = \frac{1}{q(i)} + \sum_{j\neq i} p(i,j)\beta(j) = \frac{1}{q(i)} + \sum_{j\neq i} \frac{q(i,j)}{q(i)}\beta(j).$$

   If we delete row and column 4 of the rate matrix Q and consider the submatrix corresponding to the remaining rows and columns, we get

$$Q' = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -4 \end{bmatrix}.$$

Let $\beta' = \begin{bmatrix}\beta(1) & \beta(2) & \beta(3)\end{bmatrix}^{\mathsf{T}}$. Then we can rewrite the first-step equations as

$$Q'\beta' = -\begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\mathsf{T}},$$

which yields the solution $\beta' = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\mathsf{T}}$. Thus the expected time until the chain is in state 4, starting from state 1, is 1.

2. **Frogs**

   Three frogs are playing near a pond. When they are in the sun, they get too hot and jump in the lake at rate 1. When they are in the lake, they get too cold and jump onto land at rate 2. The rates here refer to those of the Exponential distribution. Let X_t be the number of frogs in the sun at time $t \geq 0$.

   a. Find the stationary distribution of $(X_t)_{t\geq 0}$.

   b. Find the answer to part a again, this time using the observation that the three frogs are independent two-state Markov chains.

   Solution:

   a. Let the states $S = \{0, 1, 2, 3\}$ be the number of frogs in the sun. The Markov chain has $\lambda_0 = 6$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\mu_3 = 3$, $\mu_2 = 2$, and $\mu_1 = 1$, where λ_i and μ_i are the rates of jumping forwards and backwards respectively from state i. Using detailed balance, we compute the stationary distribution to be

   $$\pi = \frac{1}{27}\begin{bmatrix} 1 & 6 & 12 & 8 \end{bmatrix}.$$

   b. The individual frogs follow independent Markov chains, each with stationary distribution

   $$\pi = \frac{1}{3}\begin{bmatrix} 1 & 2 \end{bmatrix}.$$

   The stationary probability of being in state $i \in S$ is therefore

   $$\mathbb{P}(X_t = i) = \binom{3}{i}\left(\frac{1}{3}\right)^{3-i}\left(\frac{2}{3}\right)^{i}.$$

3. **Lazy Server**

Customers arrive at a queue at the times of a Poisson process with rate λ. The queue is in a service facility with infinite capacity, in which there is an infinitely powerful but lazy server who visits the facility at the times of a Poisson process with rate μ. These two processes are independent. When the server visits the facility, it instantaneously serves all the customers in the queue, then immediately leaves. In other words, at any time, the only customers waiting in the queue are those who arrived after the server's most recent visit.

a. Model the queue length as a CTMC, and find its stationary distribution.
b. Supposing that the CTMC is at stationarity, find the mean number of customers waiting in the queue at any given time.

Solution:

a. We can model the queue length as a continuous-time Markov chain on the state space $S = \mathbb{N}$. The rate at at which a customer arrives is λ, and the rate at which the server arrives is μ, so the rates are $q(i, i+1) = \lambda$ for $i \in \mathbb{N}$ and $q(i, 0) = \mu$ for $i \in \mathbb{Z}^+$. Now, the balance equation for state $i \in \mathbb{Z}^+$ reads $\lambda \cdot \pi(i-1) = (\lambda + \mu) \cdot \pi(i)$, a recurrence relation whose base case we can find by

$$\sum_{i \in \mathbb{N}} \pi(i) = \sum_{i \in \mathbb{N}} \left(\frac{\lambda}{\lambda + \mu} \right)^i \pi(0) = \frac{1}{1 - \frac{\lambda}{\lambda + \mu}} \pi(0) = \frac{\lambda + \mu}{\mu} \pi(0) = 1.$$

With $\pi(0) = \frac{\mu}{\lambda + \mu}$, the stationary distribution is given by

$$\pi(i) = \frac{\mu}{\lambda + \mu} \left(\frac{\lambda}{\lambda + \mu} \right)^i.$$

b. If X is a random variable with $\mathbb{P}(X = i) = \pi(i)$ for all $i \in S$, then we see that

$$X + 1 \sim \text{Geometric}\left(\frac{\mu}{\lambda + \mu} \right).$$

Thus $\mathbb{E}(X) = \frac{\lambda + \mu}{\mu} - 1 = \frac{\lambda}{\mu}$. One possible interpretation of this fact is that $\frac{1}{\mu}$ is the mean amount of time a customer spends in the system.