UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 10
Fall 2023

1. **Reversibility of CTMCs**

   We say that a CTMC with transition rate matrix Q and distribution π is *reversible* if π and Q satisfy the detailed balance equations

   $$\pi(i) \cdot q(i,j) = \pi(j) \cdot q(j,i) \quad \forall i, j \in S.$$

   Show that if π is a reversible distribution for a CTMC, then π is also a stationary distribution for the chain, and moreover the embedded jump chain is also reversible. *Remark*: the converse is also true — the CTMC is reversible if and only if the embedded chain is reversible.

   Solution: For any state $i \in S$, we have

   $$(\pi Q)_i = \sum_{j \in S} \pi_j q_{j,i} = \sum_{j \in S} \pi_i q_{i,j} = \pi_i \sum_{j \in S} q_{i,j} = 0,$$

   which shows that π is indeed a stationary distribution. Now recall the formula for finding the stationary distribution μ of the embedded jump chain from that of the CTMC:

   $$\mu_i = \frac{\pi_i q_i}{\sum_k \pi_k q_k}.$$

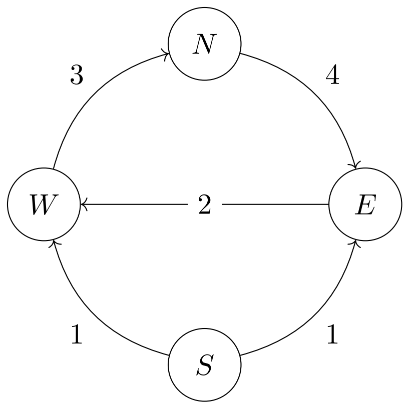
   We also know that $p_{i,j} = \frac{q_{i,j}}{q_i}$. By reversibility, it follows that

   $$\mu_i p_{i,j} = \frac{\pi_i q_{i,j}}{\sum_k \pi_k q_k} = \frac{\pi_j q_{j,i}}{\sum_k \pi_k q_k} = \mu_j p_{j,i}.$$

   Thus the embedded chain is also reversible. A similar substitution will show that the converse is true as well.

2. **Jump Chain Stationary Distribution**

Use properties of transient states and the jump chain to find the stationary distribution of this CTMC.



Solution: We see that state S is transient, so $\pi(S) = 0$. Now considering just the states N, E, and W, the jump chain has a uniform stationary distribution of $\frac{1}{3}$ in each state by symmetry. To convert this into the stationary distribution of the CTMC, we recall the following formula:

$$\pi_{\text{CTMC}}(i) = \frac{\frac{1}{q(i)}\pi_{\text{jump}}(i)}{\sum_j \frac{1}{q(j)}\pi_{\text{jump}}(j)}.$$

Intuitively, as the chain spends equal time at N, E, and W from a jump standpoint, the true fraction of time we spend in a state should be simply proportional to $\frac{1}{q(i)}$, the inverse of the sum of the rates leaving the state. Thus

$$\begin{bmatrix}\pi(N) & \pi(E) & \pi(W) & \pi(S)\end{bmatrix} = \frac{1}{\frac{1}{4}+\frac{1}{3}+\frac{1}{2}}\begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{3} & 0\end{bmatrix} = \begin{bmatrix}\frac{3}{13} & \frac{6}{13} & \frac{4}{13} & 0\end{bmatrix}.$$

3. $M/M/2$ **Queue**

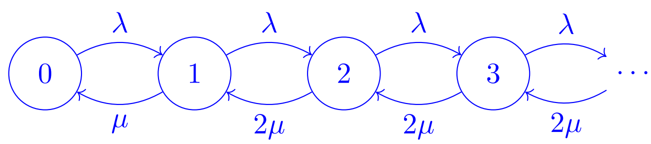
A queue has Poisson arrivals with rate λ and two servers with i.i.d. Exponential(μ) service times. The two servers work in parallel: when there are at least two customers in the queue, two are being served; when there is only one customer, only one server is active. Let X_t be the number of customers either in the queue or in service at time t.

a. Argue that the process $(X_t)_{t\geq 0}$ is a Markov process, and draw its state transition diagram.

b. Find the range of values of μ for which the Markov chain is positive recurrent. For this range of values, calculate the stationary distribution of the Markov chain.

Solution:

a. The queue length is a MC: customer arrivals are independent of the current number of customers in the queue, and departures only depend on the current number of customers being served. Also, even when $k = 1$ or 2 customers are being served, the completion of their services are independent of one another. Finally, when $k = 2$, even if one of the customers has been completely served, the other customer has the same service time distribution as before, because the Exponential distribution is memoryless.



b. It suffices to solve the detailed balance equations

$$\pi(1) = \frac{\lambda}{\mu}\pi(0)$$

$$\pi(i+1) = \frac{\lambda}{2\mu}\pi(i), \qquad i \in \mathbb{Z}^+.$$

Iterating these recurrences yields the following expression for the stationary distribution. We can find the base case $\pi(0)$ as the stationary distribution must normalize:

$$\sum_{i=0}^{\infty} \pi(i) = \pi(0) + \pi(0) \cdot \frac{\lambda}{\mu} \sum_{i=1}^{\infty} \left(\frac{\lambda}{2\mu}\right)^{i-1} = 1.$$

This series converges iff $\lambda < 2\mu$, in which case the Markov chain is positive recurrent. For μ in this range, we find the stationary distribution

$$\pi(0) = \frac{2\mu - \lambda}{2\mu + \lambda}$$

$$\pi(i) = \frac{2\mu - \lambda}{2\mu + \lambda}\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}, \qquad i \in \mathbb{Z}^+.$$