UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 10

Fall 2023

1. **Poisson Process Arrival Times**

   Consider a Poisson process $(N_t)_{t \geq 0}$ with rate 1. Let T_k be the time of the kth arrival, $k \geq 1$.

   a. Find $\mathbb{E}(T_3 \mid N_1 = 2)$.

   b. Given $T_3 = s$, where $s > 0$, find the joint distribution of T_1 and T_2.

   c. Find $\mathbb{E}(T_2 \mid T_3 = s)$.

   Solution:

   a. By the memoryless property, $\mathbb{E}(T_3 \mid N_1 = 2) = 1 + \mathbb{E}(T_1) = 2$.

   b. The distribution of the sum of k i.i.d. Exponential random variables is Erlang:

   $$f_{T_k}(s) = \frac{s^{k-1}e^{-s}}{(k-1)!} \mathbb{1}_{s \geq 0}\,.$$

   Then, by Bayes' rule and the memorylessness of Exponential distributions,

   $$\begin{aligned} f_{T_1,T_2|T_3}(s_1, s_2 \mid s) &= \frac{f_{T_1,T_2,T_3}(s_1, s_2, s)}{f_{T_3}(s)} \\ &= \frac{e^{-s_1}e^{-(s_2-s_1)}e^{-(s-s_2)}}{s^2 e^{-s}/2!} \mathbb{1}_{\{0 \leq s_1 \leq s_2 \leq s\}} \\ &= \frac{2}{s^2} \mathbb{1}_{\{0 \leq s_1 \leq s_2 \leq s\}}\,. \end{aligned}$$

   In other words, T_1 and T_2 are uniformly distributed on the feasible region $\{0 \leq s_1 \leq s_2 \leq s\}$. In particular, the joint distribution is precisely that of the order statistics of 2 i.i.d. Uniform($[0, s]$) random variables.

   c. By part b, T_2 is the maximum of 2 Uniform($[0, s]$) random variables. Thus, for $0 \leq x \leq s$,

   $$\begin{aligned} F_{T_2|T_3}(x \mid s) &= \mathbb{P}(T_2 \leq x \mid T_3 = s) = \left(\frac{x}{s}\right)^2 \\ f_{T_2|T_3}(x \mid s) &= \frac{2x}{s^2} \\ \mathbb{E}(T_2 \mid T_3 = s) &= \int_0^s \frac{2x^2}{s^2}\,dx = \frac{2s}{3}. \end{aligned}$$

2. **Poisson Process Practice**

Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ. Let T_k, $k \geq 1$ denote the time of the kth arrival. Given $0 \leq s < t$, we write $N(s,t) \coloneqq N(t) - N(s)$. Compute the following:

a. $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.

b. $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.

c. $\mathbb{E}(T_2 \mid N(2) = 1)$.

Solution:

a. The event $\{N(1) + N(2,4) + N(3,5) = 0\}$ is the same as the intersection of $\{N(1) = 0\}$ and $\{N(2,5) = 0\}$, which are independent with probabilities $e^{-\lambda}$ and $e^{-3\lambda}$. Hence

$$\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0) = e^{-4\lambda}.$$

b. $N(1,3) = N(1,2) + N(2,3)$, with $N(2,3)$ independent of $N(1,2)$, so $\mathbb{E}(N(1,3) \mid N(1,2) = 3) = 3 + \lambda$.

c. Since $N(2) = 1$, the second interarrival time T_2 has not yet lapsed at $t = 2$. From the memoryless property of the Exponential distribution,

$$\mathbb{E}(T_2 - 2 \mid N(2) = 1) = \frac{1}{\lambda}.$$

Hence the answer is $2 + \lambda^{-1}$.

3. **Poisson Process Warmup**

Give an interpretation of the following fact in terms of a Poisson process with rate λ. If N is Geometric with parameter p and $(X_k)_{k\in\mathbb{N}}$ are i.i.d. Exponential(λ), then $X_1 + \cdots + X_N$ has an Exponential distribution with parameter λp.

Solution: Consider a Poisson process with rate λ, and split the process by keeping each arrival independently with probability p. In the original process, the interarrival times are i.i.d. Exponential(λ), and $X_1 + \cdots + X_N$ represents the amount of time until the first arrival we keep. By Poisson splitting, we know that the split process is a Poisson process with rate λp, so the time until its first arrival is an Exponential random variable with parameter λp.