UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 12
Fall 2023

1. **Generating Erdős–Rényi Random Graphs**

   Let G_1 and G_2 be independent Erdős–Rényi random graphs on n vertices with probabilities p_1 and p_2 respectively. Let G be $G_1 \cup G_2$, that is, the graph generated by combining the edges in G_1 and G_2.

   a. Is G an Erdős–Rényi random graph on n vertices with probability $p_1 + p_2$?
   b. Is G an Erdős–Rényi random graph?

   Solution:

   a. No; an edge appears in G if it appears in G_1 or in G_2, which occurs with probability $p_1 + p_2 - p_1 p_2$ by inclusion-exclusion.
   b. Yes; each edge appears with probability $p = p_1 + p_2 - p_1 p_2$ independently.

2. **Voltage MAP**

You are trying to detect whether voltage V_1 or voltage V_2 was sent over a channel with independent Gaussian noise $Z \sim N(V_3, \sigma^2)$. Assume that both voltages are equally likely to be sent.

a. Derive the MAP detector for this channel.

b. Using the Gaussian Q-function, determine the average error probability for the MAP detector.

c. Suppose that the average transmit energy is $(V_1^2 + V_2^2)/2$ and that the average transmit energy is constrained such that it cannot be more than $E > 0$. What voltage levels V_1, V_2 should you choose to meet this energy constraint but still minimize the average error probability?

Solution:

a. Note that since both outputs are equiprobable, the MAP rule is equivalent to the ML rule. Note that the likelihood ratio is:

$$L(y) = \frac{f(y \mid x = V_1)}{f(y \mid x = V_2)}$$

$$\hat{x} = \begin{cases} V_1, & \text{if } L(y) > 1 \\ V_2, & \text{if } L(y) \leq 1 \end{cases}$$

Thus, we have:

$$f(y \mid x = V_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \varepsilon^{-(y-V_1-V_3)^2/2\sigma^2}$$

$$f(y \mid x = V_2) = \frac{1}{\sqrt{2\pi\sigma^2}} \varepsilon^{-(y-V_2-V_3)^2/2\sigma^2}$$

so:

$$L(y) = \varepsilon^{(V_1-V_2)(y-(V_1+V_2)/2-V_3)/\sigma^2}$$

WLOG, we may assume $V_1 > V_2$, so the ML rule is:

$$\hat{x} = \begin{cases} V_1, & \text{if } y > \dfrac{(V_1+V_2)}{2} + V_3 \\ V_2, & \text{if } y \leq \dfrac{(V_1+V_2)}{2} + V_3 \end{cases}$$

b. We have:

$$\begin{aligned}
\Pr(\text{error}) &= \Pr(\hat{x} = V_2 \mid x = V_1)\Pr(x = V_1) + \Pr(\hat{x} = V_1 \mid x = V_2)\Pr(x = V_2) \\
&= \frac{1}{2}\Pr\Big(y < \frac{V_1+V_2}{2} + V_3 \,\Big|\, x = V_1\Big) \\
&\quad + \frac{1}{2}\Pr\Big(y > \frac{V_1+V_2}{2} + A_3 \,\Big|\, x = V_2\Big) \\
&= \frac{1}{2}\Big[1 - Q\Big(\frac{V_2-V_1}{2\sigma}\Big)\Big] + \frac{1}{2}Q\Big(\frac{V_1-V_2}{2\sigma}\Big) \\
&= Q\Big(\frac{V_1-V_2}{2\sigma}\Big).
\end{aligned}$$

c. We would like to

$$\text{minimize } Q\Big(\frac{V_1 - V_2}{2\sigma}\Big) \text{ subject to } \frac{V_1^2 + V_2^2}{2} \leq E.$$

Note that this is equivalent to maximizing $V_1 - V_2$ which is equivalent to maximizing $(V_1 - V_2)^2$, subject to the same constraint. Now, we have:

$$(V_1 - V_2)^2 \leq (|V_1| + |V_2|)^2 \leq 4E$$

where we have equality iff $V_1 = -V_2$, so the optimal choice is $V_1 = \sqrt{E}$, $V_2 = -\sqrt{E}$.

3. **Poisson Process MAP**

Customers arrive to a store according to a Poisson process with rate 1. The store manager learns of a rumor that one of the employees is sending every other customer to the rival store, so that *deterministically*, every odd-numbered customer $1, 3, 5, \ldots$ is sent away.

Let $X = 1$ be the hypothesis that the rumor is true and $X = 0$ the rumor is false, assuming that both hypotheses are equally likely. Suppose a customer arrives to the store at time 0. After that, the manager observes $T_1, \ldots, T_n$, where T_i is the time of the ith subsequent sale, $i = 1, \ldots, n$. Derive the MAP rule to determine whether the rumor was true or not.

Solution: Note that both hypotheses are a priori equally likely, so the MAP rule is equivalent to the MLE rule. Also note that the interarrival times τ_i are independent whether conditioned on $X = 1$ or on $X = 0$. The density of an interarrival interval given $X = 1$ is Erlang of order 2, so for $0 \le t_1 < \cdots < t_n$,

$$f_{T_1,\ldots,T_n|X}(t_1, \ldots, t_n \mid 1) = \prod_{i=1}^{n} (t_i - t_{i-1}) e^{-(t_i - t_{i-1})} = e^{-t_n} \prod_{i=1}^{n} (t_i - t_{i-1}).$$

The density of an interarrival interval given $X = 0$ is Exponential, so

$$f_{T_1,\ldots,T_n|X}(t_1, \ldots, t_n \mid 0) = e^{-t_n}.$$

Taking the logarithm of both expressions, we see that the MAP is to declare $X = 1$ whenever

$$\sum_{i=1}^{n} \ln(T_i - T_{i-1}) \ge 0.$$