UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 1
Fall 2023

1. **Independence**

   Events $A, B \in \mathcal{F}$ are said to be **independent** if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

   a. Show that if events A, B are independent, then the probability exactly one of the events occurs is
   $$\mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A)\,\mathbb{P}(B).$$

   b. Show that if the event A is independent of itself, then $\mathbb{P}(A) = 0$ or 1.

   Solution:

   a. The probability of the event that exactly one of A and B occurs is
   $$\begin{aligned} &\mathbb{P}(A \cap B^c) + \mathbb{P}(A^c \cap B) \\ &= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A)\,\mathbb{P}(B). \end{aligned}$$

   b. $\mathbb{P}(A \cap A) = \mathbb{P}(A) \cdot \mathbb{P}(A)$, so $\mathbb{P}(A) = \mathbb{P}(A)^2$. This implies that $\mathbb{P}(A) \in \{0, 1\}$.

2. **Balls and Bins**

Suppose n bins are arranged from left to right. You sequentially throw n balls; each ball lands in a bin chosen uniformly at random, independent of all other balls.

a. Formulate an appropriate probability space for modelling the outcome of this experiment.

b. Let A_i denote the event that exactly i bins are empty, $i = 0, \ldots, n$. Compute the probability of the event

$$\{\text{all empty bins sit to the left of all bins containing at least one ball}\}$$

in terms of the $\mathbb{P}(A_i)$'s.

c. Practice your CS70 skills by computing $\mathbb{P}(A_1)$.

Solution:

a. There is some flexibility in how we do this. For example, we can take the order of throws into account in our model, or just the final configuration. We'll do the former, since it makes the third part easier due to the uniformity that $\mathbb{P}$ enjoys.

Hence, any outcome of our experiment is the sequence of bins where the balls land, i.e. we take $\Omega = \{1, \ldots, n\}^n$. Accordingly, if $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$, then ω_i denotes the bin that the ith toss lands in.

Since everything is discrete, it is natural to choose $\mathcal{F} = 2^\Omega$, and since each ball is equally likely to land in any bin, we have $\mathbb{P}$ uniform over the sample space, i.e.

$$\mathbb{P}(\{\omega\}) = n^{-n} \quad \forall \omega \in \Omega.$$

b. Call our event of interest B. By the law of total probability, we have

$$\mathbb{P}(B) = \sum_{i=0}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=0}^{n} \frac{1}{\binom{n}{i}} \mathbb{P}(A_i).$$

The last identity follows because if we have exactly i empty bins, there are $\binom{n}{i}$ ways to choose which bins are empty, and only one of those ways will put all empty bins on the left. Since all such configurations of the empty bins are equally likely, the identity follows.

c. There are n choices for the empty bin, then $n-1$ choices for the bin that will have two balls, followed by $\binom{n}{2}$ choices of said two balls, and lastly $(n-2)!$ ways to throw the remaining $n-2$ balls into the $n-2$ remaining bins, so that each bin has exactly one ball. Therefore the desired probability is

$$\frac{n(n-1)(n-2)!\binom{n}{2}}{n^n} = \frac{n!\binom{n}{2}}{n^n}.$$

3. **Coin Flipping and Symmetry**

Alice and Bob have $2n+1$ fair coins, $n \geq 1$. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. A fair coin lands on heads with probability $\frac{1}{2}$; assume that coin tosses are independent.

a. Formulate this scenario in terms of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Describe explicitly what the outcomes are and what the probability measure of any event is defined to be.

b. Show that the probability that after all coins have been tossed, Bob will have gotten more heads than Alice is $\frac{1}{2}$.

Hint: Consider the event $A = \{\text{more heads in the first } n+1 \text{ tosses than the last } n \text{ tosses}\}$.

Solution:

a. We can let the sample space be $\Omega = \{H, T\}^{2n+1}$, the set of all possible configurations of $2n+1$ coin tosses. We take $\mathcal{F} = 2^{\Omega}$, and the probability measure $\mathbb{P}$ is *uniform*, such that $\mathbb{P}(\{\omega\}) = 2^{-(2n+1)}$ for every outcome $\omega \in \Omega$. (Thus, $\mathbb{P}(A) = 2^{-(2n+1)}|A|$ for all $A \in \mathcal{F}$.)

b. Define the events

$$A = \{\text{there are more heads in the first } n+1 \text{ tosses than the last } n \text{ tosses}\},$$
$$B = \{\text{there are more tails in the first } n+1 \text{ tosses than the last } n \text{ tosses}\}.$$

$\mathbb{P}(A) = \mathbb{P}(B)$ by symmetry, as every outcome in A can be uniquely identified with an outcome in B by swapping heads and tails, and vice versa. We also note that $A \cap B = \varnothing$, as it is impossible for the first $n+1$ tosses to have more heads *and* more tails than the last n tosses, and $A \cup B = \Omega$. Thus $\mathbb{P}(A) + \mathbb{P}(B) = 1$, and $\mathbb{P}(A) = \frac{1}{2}$.

Alternatively, if the probability that Bob has more heads than Alice in the first n tosses is p, then the probability that Bob has fewer heads than Alice in the first n tosses is also p by symmetry, and the probability that they are tied after n tosses is $1 - 2p$.

So, the probability that Bob wins is $p + (\frac{1}{2})(1-2p) = \frac{1}{2}$. Bob can win either by having more heads than Alice in the first n tosses, or by having the same number of heads as Alice in the first n tosses, then flipping heads on the last toss.