UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 02
Fall 2023

1. **Choosing from Any Jar Makes No Difference**

   Each of k jars contains w white and b black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $w/(w+b)$.

   Solution: We derive a recursion for the probability p_i that a white ball is chosen from the ith jar. We have, using the total probability theorem,

$$p_{i+1} = \frac{w+1}{w+b+1}p_i + \frac{w}{w+b+1}(1-p_i) = \frac{1}{w+b+1}p_i + \frac{w}{w+b+1},$$

   starting with the initial condition $p_1 = w/(w+b)$. Thus, we have

$$p_2 = \frac{1}{w+b+1} \cdot \frac{w}{w+b} + \frac{w}{w+b+1} = \frac{w}{w+b}.$$

   More generally, this calculation shows that if $p_{i-1} = w/(w+b)$, then $p_i = w/(w+b)$. Thus, we obtain $p_i = w/(w+b)$ for all i.

2. **Borel–Cantelli Lemma**

If $A_1, A_2, \ldots$ is a sequence of events with $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$, then

$$\mathbb{P}(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) = 0.$$

Remark: later we will see how Borel–Cantelli may be used to show some laws of large numbers.

Solution: If infinitely many of $A_1, A_2, \ldots$ occur, then at least one of $A_n, A_{n+1}, \ldots$ occurs for any $n \in \mathbb{Z}_{>0}$. So,

$$\mathbb{P}(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) \leq \Pr\left(\bigcup_{m=n}^{\infty} A_m\right) \leq \sum_{m=n}^{\infty} \mathbb{P}(A_m) \xrightarrow[n\to\infty]{} 0$$

because $\sum_{i=1}^{\infty} \mathbb{P}(A_i)$ converges. In more detail,

$$\sum_{m=n}^{\infty} \mathbb{P}(A_m) = \sum_{m=1}^{\infty} \mathbb{P}(A_m) - \sum_{m=1}^{n-1} \mathbb{P}(A_m),$$

and as $n \to \infty$, the second term converges to $\sum_{m=1}^{\infty} \mathbb{P}(A_m)$, so $\sum_{m=n}^{\infty} \mathbb{P}(A_m)$ converges to 0 as $n \to \infty$.

Note: This result is incredibly useful for proving convergence results.

3. **Middle School**

A middle school is composed of 40% sixth graders, 40% seventh graders and 20% eighth graders. The average height of students in these grades are 4, 4.5, and 5 ft. respectively. The variance of heights within each grade are 1, $\frac{1}{2}$, and $\frac{1}{2}$ sq. ft. respectively. Suppose you pick a student at random. Let X denote their grade, and Y denote their height.

What is $\mathbb{E}(Y)$?

Solution:

a. From the definition of expectation,

$$\mathbb{E}(Y) = \frac{2}{5} \cdot 4 + \frac{2}{5} \cdot \frac{9}{2} + \frac{1}{5} \cdot 5 = \frac{22}{5} = 4.4.$$

b. Y "depends on" X, so it may be hard to find $\text{var}(Y)$ directly. Instead, since we can find $\mathbb{E}(Y \mid X)$ and $\text{var}(Y \mid X)$, we can use the law of total variance:

$$\begin{aligned}
\text{var}(Y) &= \mathbb{E}(\text{var}(Y \mid X)) + \text{var}(\mathbb{E}(Y \mid X)) \\
&= \left(\frac{2}{5} \cdot 1 + \frac{2}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2}\right) + \left(\frac{2}{5}(4 - 4.4)^2 + \frac{2}{5}(4.5 - 4.4)^2 + \frac{1}{5}(5 - 4.4)^2\right) \\
&= 0.4 + 0.2 + 0.1 + 0.064 + 0.004 + 0.072 \\
&= 0.84.
\end{aligned}$$

4. **General Tail-Sum Formula**

Suppose Y is a nonnegative random variable and p is a positive integer. Show that

$$\mathbb{E}(Y^p) = \int_0^\infty p y^{p-1}\, \mathbb{P}(Y > y)\, dy.$$

Hint: In this problem, you may swap integrals with expectations.

Solution: As noted, the key step lies in exchanging integration and expectation:

$$\begin{aligned}
\int_0^\infty p y^{p-1}\, \mathbb{P}(Y > y)\, dy &= \int_0^\infty \mathbb{E}(p y^{p-1}\, \mathbb{1}_{Y>y})\, dy \\
&= \mathbb{E}\left(\int_0^\infty p y^{p-1}\, \mathbb{1}_{Y>y}\ dy\right) \\
&= \mathbb{E}\left(\int_0^Y p y^{p-1}\, dy\right) \\
&= \mathbb{E}(Y^p).
\end{aligned}$$

5. **Compact Arrays**

Consider an array of $n \geq 1$ entries, where each entry is chosen uniformly randomly from $\{0, \ldots, 9\}$. We want to make the array more compact by moving all the zeros to the end of the array. For example, if we take the array

$$\begin{bmatrix} 6 & 4 & 0 & 0 & 5 & 3 & 0 & 5 & 1 & 3 \end{bmatrix}$$

and make it compact, we now have

$$\begin{bmatrix} 6 & 4 & 5 & 3 & 5 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Let i be a fixed positive integer in $\{1, \ldots, n\}$. Suppose that the ith entry of the array is nonzero. (The array is indexed starting from 1.) Let X_i be the random variable equal to the index that the ith entry has been moved to after making the array compact. Calculate $\mathbb{E}(X_i)$.

Solution: Let Y_j, $j = 1, \ldots, i-1$, be the indicator that the jth entry of the original array is 0. Then the ith entry is moved backwards $\sum_{j=1}^{i-1} Y_j$ positions, so

$$\mathbb{E}(X_i) = i - \sum_{j=1}^{i-1} \mathbb{E}(Y_j) = i - \frac{i-1}{10} = \frac{9i+1}{10}.$$

The variance is also straightforward to compute by the independence of the indicators Y_j. We note that $\operatorname{var}(Y_j) = \frac{1}{10} \cdot \frac{9}{10} = \frac{9}{100}$, so

$$\operatorname{var}(X_i) = \operatorname{var}\left(i - \sum_{j=1}^{i-1} Y_j\right) = \sum_{j=1}^{i-1} \operatorname{var}(Y_j) = \frac{9(i-1)}{100}.$$

6. **Expected Sorting Distance**

Let $(a_1, \ldots, a_n)$ be a random permutation of $\{1, \ldots, n\}$, so that it is equally likely to be any possible permutation. When sorting the list $(a_1, \ldots, a_n)$, the element a_i must move a distance of $|a_i - i|$ places from its current position to reach the position in the sorted order. Find the expected total distance that the elements will have to be moved,

$$\mathbb{E}\left(\sum_{i=1}^{n} |a_i - i|\right)$$

Note: To simplify your answer, you can use the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: By the linearity of expectation, we have that

$$\mathbb{E}\left(\sum_{i=1}^{n} |a_i - i|\right) = \sum_{i=1}^{n} \mathbb{E}(|a_i - i|).$$

Because all of the permuations are equally likely, a_i is equally likely to be any number from 1 to n. Thus

$$\begin{aligned}
\mathbb{E}(|a_i - i|) &= \sum_{k=1}^{n} \frac{1}{n} |k - i| \\
&= \frac{1}{n} \sum_{k=1}^{n-i} k + \frac{1}{n} \sum_{k=1}^{i-1} k \\
&= \frac{(n-i)(n-i+1) + (i-1)i}{2n}.
\end{aligned}$$

Putting it all together, and using the closed-form formula for $\sum_{k=1}^{n} k^2$, we obtain

$$\mathbb{E}\left(\sum_{i=1}^{n} |a_i - i|\right) = \frac{n^2 - 1}{3}.$$