UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 6
Fall 2023

1. **Chernoff Bound**

   Prove that

   - $P(X > t) \leq \frac{M_X(\lambda)}{e^{\lambda t}}$ $\forall \lambda > 0$, and $P(X < t) \leq \frac{M_X(\lambda)}{e^{\lambda t}}$ $\forall \lambda < 0$.
   - For $X \sim \mathcal{N}(\mu, \sigma^2)$, upper bound the probability of deviation from mean $P(|X - \mu| > t)$.

   Solution: See Chernoff Bound references (e.g. Wikipedia).

2. **Exponential Bounds**

   Let $X \sim \text{Exponential}(\lambda)$. For $x > \lambda^{-1}$, find bounds on $\mathbb{P}(X \geq x)$ using Markov's inequality, Chebyshev's inequality, and the Chernoff bound.

   Solution: Since $\mathbb{E}(X) = \lambda^{-1}$, Markov's inequality gives

$$\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}(X)}{x} = \frac{1}{\lambda x},$$

   and from $\text{var}(X) = \lambda^{-2}$, Chebyshev's inequality gives

$$\begin{aligned}\mathbb{P}(X \geq x) = \mathbb{P}(X - \lambda^{-1} \geq x - \lambda^{-1}) &\leq \mathbb{P}(|X - \lambda^{-1}| \geq x - \lambda^{-1}) \\ &\leq \frac{\text{var}(X)}{(x - \lambda^{-1})^2} = \frac{1}{(\lambda x - 1)^2}.\end{aligned}$$

   By the Chernoff bound, for any $s > 0$,

$$\mathbb{P}(X \geq x) = \mathbb{P}\left(\exp(sX) \geq \exp(sx)\right) \leq \frac{M_X(s)}{\exp(sx)} = \frac{\lambda}{(\lambda - s)\exp(sx)}.$$

   We wish to optimize this bound over $s > 0$; we note that it suffices to maximize the denominator $(\lambda - s)\exp(sx)$. Differentiating,

$$-\exp(sx) + x(\lambda - s)\exp(sx) = 0,$$

   so $1 = x(\lambda - s)$, that is, $s = \lambda - x^{-1}$. Thus

$$\begin{aligned}\mathbb{P}(X \geq x) &\leq \frac{\lambda}{(\lambda - (\lambda - x^{-1}))\exp((\lambda - x^{-1})x)} = \frac{\lambda}{x^{-1}\exp(\lambda x - 1)} \\ &= \lambda x \exp\left(-(\lambda x - 1)\right).\end{aligned}$$

   Observe that the Chernoff bound is the only one which decreases exponentially with x, which is the true behavior: $\mathbb{P}(X \geq x) = \exp(-\lambda x)$.

3. **Almost Sure Convergence Implies Convergence in Probability**

   For random variables $X_1, X_2, \ldots$ and X on a common probability space $(\Omega, \mathcal{F}, P)$, prove that if $X_n \overset{\text{a.s.}}{\to} X$ then $X_n \overset{\text{p}}{\to} X$.

   Solution: The Wikipedia page contains good proofs on the topic.