UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 09
Fall 2023

1. **System Shocks**

   For a positive integer n, let $X_1, \dots, X_n$ be independent Exponentially distributed random variables, each with mean 1. Let $\gamma > 0$. A system experiences shocks at times $k = 1, \dots, n$, and the size of the shock at time k is X_k.

   a. Suppose that the system fails if any shock exceeds the value γ. What is the probability of system failure?

   b. Suppose instead that the effect of the shocks is cumulative, i.e. the system fails when the total amount of shock received exceeds γ. What is the probability of system failure?

   Solution:

   a. The system fails if $\max\{X_1, \dots, X_n\} > \gamma$, so

   $$\begin{aligned}\mathbb{P}(\max\{X_1, \dots, X_n\} > \gamma) &= 1 - \mathbb{P}(\max\{X_1, \dots, X_n\} \le \gamma) \\ &= 1 - \prod_{k=1}^{n} \mathbb{P}(X_k \le \gamma) = 1 - (1 - e^{-\gamma})^n.\end{aligned}$$

   b. $\mathbb{P}(X_1 + \cdots + X_n > \gamma) = \mathbb{P}(N_\gamma < n)$, where $(N_t)_{t \ge 0}$ is a Poisson process with rate 1, so

   $$\mathbb{P}(X_1 + \cdots + X_n > \gamma) = \sum_{k=0}^{n-1} \frac{\gamma^k}{k!} e^{-\gamma}.$$

2. **Basketball II**

Captain America and Superman are playing an untimed basketball game in which the two players score points according to independent Poisson processes with rates λ_C and λ_S respectively. The game is over when one player has scored k more points than the other.

a. Suppose $\lambda_C = \lambda_S$, and suppose Captain America has a head start of $m < k$ points. Find the probability that Captain America wins.
Hint: if $\alpha_i = \frac{1}{2}\alpha_{i-1} + \frac{1}{2}\alpha_{i+1}$, then $\alpha_{i+1} - \alpha_i = \alpha_i - \alpha_{i-1}$.

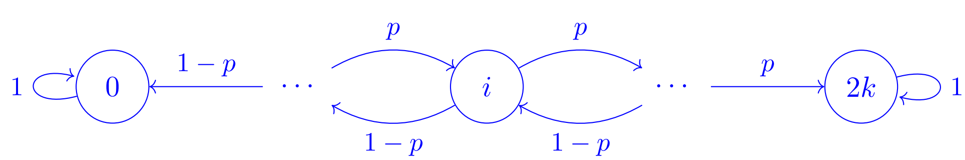
b. Keeping the assumptions, find the expected time $\mathbb{E}(T)$ it will take for the game to end.
Hint: consider the telescoping sum $\beta_j = \beta_0 + (\beta_1 - \beta_0) + \cdots + (\beta_j - \beta_{j-1})$.

Solution:

a. Consider the merged process with rate $\lambda_C + \lambda_S$. We see that each point is one for Captain America with probability $p := \frac{\lambda_C}{\lambda_C + \lambda_S}$ and one for Superman with probability $1 - p$. Then, the Markov chain whose state is the number of additional points Superman needs to score to win has transition probabilities

$$\begin{aligned}
P(0,0) &= 1 \\
P(i, i+1) &= p, \text{ where } 0 < i < 2k \\
P(i, i-1) &= 1 - p, \text{ where } 0 < i < 2k \\
P(2k, 2k) &= 1.
\end{aligned}$$

As $\lambda_C = \lambda_S$, i.e. $p = \frac{1}{2}$, this is also known as the *symmetric gambler's ruin* problem for $n = 2k$, which has the following transition diagram:



Let α_i be the probability of eventually reaching the absorbing state $2k$ starting from i. The system of first-step equations and boundary conditions are

$$\alpha_i = \frac{1}{2}\alpha_{i-1} + \frac{1}{2}\alpha_{i+1}, \quad \alpha_0 = 0, \quad \alpha_{2k} = 1.$$

We see that the values $\alpha_0, \ldots, \alpha_{2k}$ are in fact evenly spaced out on the number line $[0, 1]$, with each α_i being the midpoint of $[\alpha_{i-1}, \alpha_{i+1}]$. Thus α_i is directly proportional to i, the "distance" of state i from 0, and we find the final answer of

$$\mathbb{P}(\text{Captain America wins}) = \alpha_{m+k} = \frac{m+k}{2k}.$$

b. In the CTMC above, the holding time τ_n for each jump is i.i.d. Exponential(2λ), where $\lambda = \lambda_C = \lambda_S$. If N_i is the number of jumps made until the game ends, starting from i, then by the law of total expectation with independence,

$$\mathbb{E}(T) = \mathbb{E}\left(\sum_{n=1}^{N_j} \tau_n\right) = \mathbb{E}(N_j \cdot \mathbb{E}(\tau_1)) = \mathbb{E}(N_j) \cdot \mathbb{E}(\tau_1) = \frac{\mathbb{E}(N_j)}{2\lambda}.$$

To compute $\beta_i \coloneqq \mathbb{E}(N_i)$, let $\Delta_i \coloneqq \mathbb{E}(N_{i+1}) - \mathbb{E}(N_i)$. The first-step equations are

$$\beta_i = 1 + \frac{1}{2}\beta_{i-1} + \frac{1}{2}\beta_{i+1}, \quad \beta_0 = \beta_{2k} = 0,$$

which we can rewrite as $\Delta_i = \Delta_{i-1} - 2$. In particular, we have $\Delta_{2k-1} = \Delta_0 - 2(2k-1)$, and therefore

$$-\beta_{2k-1} = \Delta_{2k-1} = \Delta_0 - 2(2k-1) = \beta_1 - 2(2k-1).$$

But $\beta_{2k-1} = \beta_1$ by symmetry, so $\beta_1 = 2k - 1 = \Delta_0$, and the previous recurrence gives us $\Delta_i = \Delta_0 - 2i = 2k - 1 - 2i$. To calculate β_j, we use a telescoping sum:

$$\beta_j = \beta_0 + \sum_{i=0}^{j-1}(\beta_{i+1} - \beta_i) = \sum_{i=0}^{j-1}(2k - 1 - 2i) = j(2k - j).$$

As $j = m + k$ was our starting state, we have $\mathbb{E}(N_{m+k}) = (k+m)(k-m)$, and thus

$$\mathbb{E}(T) = \frac{(k+m)(k-m)}{2\lambda}.$$

3. **Illegal U-Turns**

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal, and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for time $\tau > 0$. Let N be the number of police cars you see before you make a U-turn.

a. Find $\mathbb{E}(N)$.
b. Let $n \geq 2$. Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.
c. Find the expected time that you wait until you make a U-turn.

Solution:

a. We note that N is equal to the number of successive interarrival intervals that are smaller than τ, where these intervals are independent and each shorter than τ with probability $1 - e^{-\lambda\tau} := 1 - p$. Thus

$$\mathbb{P}(N = k) = e^{-\lambda\tau}(1 - e^{-\lambda\tau})^k = p(1-p)^k,$$

so N is a shifted Geometric random variable with parameter p, i.e. $N+1 \sim \text{Geometric}(p)$, and $\mathbb{E}(N) = \frac{1}{p} - 1 = e^{\lambda\tau} - 1$.

b. Let S_n be the nth interarrival time. The event $\{N \geq n\}$ indicates that the time between cars $n-1$ and n is at most τ, so we want to compute

$$\mathbb{E}(S_n \mid S_n < \tau) = \frac{\int_0^\tau t\lambda e^{-\lambda t}\,dt}{\int_0^\tau \lambda e^{-\lambda t}\,dt}.$$

Using integration by parts in the numerator, we find that the answer is

$$= \frac{\lambda^{-1} - (\tau + \lambda^{-1})e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.$$

c. You make the U-turn at time $T = S_1 + \cdots + S_N + \tau$, with $S_i \leq \tau$ for $i = 1, \ldots, N$, so

$$\begin{aligned}
\mathbb{E}(T) &= \tau + \mathbb{E}(S_1 + \cdots + S_N) \\
&= \tau + \sum_{n=0}^{\infty} \mathbb{P}(N = n) \cdot \mathbb{E}(S_1 + \cdots + S_N \mid N = n) \\
&= \tau + \sum_{n=0}^{\infty} \mathbb{P}(N = n) \cdot n \cdot \mathbb{E}(S_1 \mid S_1 \leq \tau) \\
&= \tau + (e^{\lambda\tau} - 1) \cdot \frac{\lambda^{-1} - (\tau + \lambda^{-1})e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.
\end{aligned}$$