UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 4
Fall 2023

1. **Covariance Matrix**

For a random vector $X = [X_1, X_2, \ldots, X_n]^\mathsf{T}$, its covariance matrix Σ is defined with entries $\Sigma_{ij} = \mathrm{Cov}(X_i, X_j)$. Suppose that $\mathbb{E}[X] = 0$.

a. Show that Σ is positive semi-definite, i.e. for all $v \in \mathbb{R}^n$, we have $v^\mathsf{T}\Sigma v \geq 0$.

b. Show that if the X_i's are *pairwise* independent, then Σ is diagonal.

c. Give an example of two random variables X_1, X_2 with a diagonal covariance matrix, but such that X_1, X_2 are not independent.

Solution:

a. Note that we can write $\Sigma = \mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^\mathsf{T}]$. Therefore for any $v \in \mathbb{R}^n$, we have

$$v^\mathsf{T}\Sigma v = v^\mathsf{T}\,\mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^\mathsf{T}]v = \mathbb{E}[v^\mathsf{T}XX^\mathsf{T}v] = \mathbb{E}[(v^\mathsf{T}X)(v^\mathsf{T}X)^\mathsf{T}] \geq 0.$$

b. If X_i's are pairwise independent, then $\mathrm{Cov}(X_i, X_j) = 0$ for all $i \neq j$, and so Σ is diagonal.

c. Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 = ZX_1$, where Z is uniform on $\{-1, 1\}$.