UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 01
Fall 2023

1. **Deriving Facts from the Axioms**

   a. Let $n \in \mathbb{Z}_{>0}$ and $A_1, \ldots, A_n$ be any events. Prove the **union bound**: $\Pr(\bigcup_{i=1}^n A_i) \le \sum_{i=1}^n \Pr(A_i)$.

   b. Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of increasing events. Prove that $\lim_{n\to\infty} \Pr(A_n) = \Pr(\bigcup_{i=1}^\infty A_i)$.
   [This can be viewed as a **continuity** property for probability measures.]

   c. Let $A_1, A_2, \ldots$ be a sequence of events. Prove that the union bound holds for countably many events: $\Pr(\bigcup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty \Pr(A_i)$.

Solution:

a. From inclusion-exclusion, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \le \Pr(A) + \Pr(B)$. The result now follows from induction. Formally, the case of $n = 1$ is trivial and the case of $n = 2$ was proven above; let $n \ge 3$.

$$\Pr\left(\bigcup_{i=1}^n A_i\right) = \Pr\left(A_1 \cup \bigcup_{i=2}^n A_i\right) \le \Pr(A_1) + \Pr\left(\bigcup_{i=2}^n A_i\right)$$
$$\le \Pr(A_1) + \sum_{i=2}^n \Pr(A_i) = \sum_{i=1}^n \Pr(A_i).$$

b. Write $A_1' = A_1$ and $A_i' = A_i \setminus \bigcup_{j=1}^{i-1} A_j$ for $i \in \mathbb{N}$, $i \ge 2$. Now, the A_i' for $i \in \mathbb{Z}_{>0}$ are disjoint, and $\bigcup_{i=1}^n A_i' = \bigcup_{i=1}^n A_i = A_n$, so $\Pr(A_n) = \Pr(\bigcup_{i=1}^n A_i') = \sum_{i=1}^n \Pr(A_i')$. Hence,

$$\lim_{n\to\infty} \Pr(A_n) = \lim_{n\to\infty} \sum_{i=1}^n \Pr(A_i') = \sum_{i=1}^\infty \Pr(A_i') = \Pr\left(\bigcup_{i=1}^\infty A_i'\right) = \Pr\left(\bigcup_{i=1}^\infty A_i\right),$$

by countable additivity.

Note: A function $f : \mathbb{R} \to \mathbb{R}$ is called **continuous** if for every $x \in \mathbb{R}$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, we have $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$. If we view $\bigcup_{i=1}^\infty A_i$ as "$\lim_{i\to\infty} A_i$", then the continuity property of probability says that $\lim_{n\to\infty} \Pr(A_n) = \Pr(\lim_{n\to\infty} A_n)$, which explains the name.

Note 2: Some students noticed that by inclusion-exclusion, $\Pr(\bigcup_{i=1}^\infty A_i) = \Pr(A_1) + \Pr(\bigcup_{i=2}^\infty A_i) - \Pr(A_1 \cap (\bigcup_{i=2}^\infty A_i))$, but $A_1 \cap (\bigcup_{i=2}^\infty A_i) = A_1$, so $\Pr(\bigcup_{i=1}^\infty A_i) = \Pr(\bigcup_{i=2}^\infty A_i)$; indeed, one can prove by induction that for any $n \in \mathbb{Z}_{>0}$, $\Pr(\bigcup_{i=1}^\infty A_i) = \Pr(\bigcup_{i=n}^\infty A_i)$. This is *not* sufficient to prove the problem, but it can be explained by noticing that the LHS, $\lim_{n\to\infty} \Pr(A_n)$, does not depend on finitely many events (since we are taking a limit), so "throwing away" the events $A_1, \ldots, A_n$ does not change the probability.

c. As in the previous part, define $A'_1 = A_1$ and $A'_i = A_i \setminus \bigcup_{j=1}^{i-1} A_i$ for $i \in \mathbb{N}$, $i \geq 2$. Now, $\Pr(\bigcup_{i=1}^{\infty} A_i) = \Pr(\bigcup_{i=1}^{\infty} A'_i) = \sum_{i=1}^{\infty} \Pr(A'_i)$, and for all $i \in \mathbb{Z}_{>0}$ we have $\Pr(A'_i) \leq \Pr(A_i)$ since $A'_i \subseteq A_i$, so $\Pr(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \Pr(A_i)$.

Note: The fact we used above is that if $B \subseteq A$, then $\Pr(B) \leq \Pr(A)$; this follows because $A = B \cup (A \setminus B)$ is a disjoint union, so $\Pr(A) = \Pr(B) + \Pr(A \setminus B) \geq \Pr(B)$.

2. **Superhero Basketball**

Superman and Captain America are playing a game of basketball. At the end of the game, Captain America scored n points and Superman scored m points, where $n > m$ are positive integers. Supposing that each basket counts for exactly one point, what is the probability that after the start of the game (when they are initially tied), Captain America was always *strictly* ahead of Superman?

(Assume that all sequences of baskets which result in the final score of n baskets for Captain America and m baskets for Superman are equally likely.)

Hint: Think about symmetry. Is it more likely that Superman scored the first point and there was a tie at some point in the game, or Captain America scored the first point and there was a tie at some point in the game?

Solution: Let T be the event that Captain America and Superman were tied at least once after the first point, let C be the event that Captain America scores the first point, and let S be the event that Superman scores the first point. Then we wish to find

$$\mathbb{P}(T^c) = 1 - \mathbb{P}(T) = 1 - (\mathbb{P}(C \cap T) + \mathbb{P}(S \cap T)).$$

First, we observe that $\mathbb{P}(C \cap T) = \mathbb{P}(S \cap T)$ by symmetry. Given any sequence of baskets where the first point is scored by Captain America and there is a tie, flipping all of the baskets until the first tie yields a sequence of baskets where the first point is scored by Superman and there is a tie, and the same is true vice versa. Thus, the outcomes in $C \cap T$ are in bijection with the outcomes in $S \cap T$.

Moreover, since Captain America won the game, $\mathbb{P}(S \cap T) = \mathbb{P}(S)$. If Superman scored the first point, then there must have been a point when Captain America caught up. Now it suffices to find $\mathbb{P}(S)$, which is $\frac{m}{m+n}$. Therefore

$$\mathbb{P}(T^c) = 1 - 2\,\mathbb{P}(S) = 1 - \frac{2m}{m+n} = \frac{n-m}{n+m}.$$

Note: This is yet another famous problem known as the *ballot problem*.

3. **Independence and Pairwise Independence**

   A collection of events $\{A_i\}_{i\in I}$ is said to be *pairwise independent* if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\cdot\mathbb{P}(A_j)$ for all distinct indices $i \neq j$.

   You flip a fair coin 99 times, where the result of each flip is independent of all other flips. For $i = 1, \ldots, 99$, let A_i be the event that the ith flip comes up heads. Let B be the event that in total, an *odd* number of heads are seen. Show that the events $A_1, \ldots, A_{99}, B$ are pairwise independent but *not* independent.

   Solution: A good choice of sample space is $\Omega = \{H, T\}^{99}$, the set of all 2^{99} possible configurations of 99 coin flips, or equivalently $\{0,1\}^{99}$, the set of all 99-bit binary strings. We first check that $A_1, \ldots, A_{99}, B$ are pairwise independent: for $1 \leq i < j \leq 99$,

$$\mathbb{P}(A_i \cap A_j) = \frac{2^{97}}{2^{99}} = \frac{2^{98}}{2^{99}} \cdot \frac{2^{98}}{2^{99}} = \mathbb{P}(A_i)\cdot\mathbb{P}(A_j).$$

   We also see that $\mathbb{P}(B) = \frac{1}{2}$ by symmetry: there is a unique correspondence between outcomes with an odd number of heads and outcomes with an even number of heads, found by simply switching the result of the first flip. $\mathbb{P}(A_i \cap B) = \frac{1}{4}$ by the same argument, now applied to the outcomes in A_i. So, for $1 \leq i \leq 99$,

$$\mathbb{P}(A_i \cap B) = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2} = \mathbb{P}(A_i)\cdot\mathbb{P}(B).$$

   This shows pairwise independence. However, $A_1, \ldots, A_{99}, B$ are not independent:

$$\mathbb{P}\left(\bigcap_{i=1}^{99} A_i \cap B\right) = \frac{1}{2^{99}} \neq \left(\frac{1}{2}\right)^{100} = \prod_{i=1}^{99}\mathbb{P}(A_i)\cdot\mathbb{P}(B).$$

   This formalizes the idea that B is "determined by" the events $A_1, \ldots, A_{99}$.

4. **The Probabilistic Method**

We introduce a proof technique — the *probabilistic method*. If we wish to show that there exists an element with property A in a set $\mathcal{X}$, it suffices to show that there exists a probability distribution p over $\mathcal{X}$ such that under p, the probability assigned to elements with property A is greater than 0.

(Why does this work? If there is no element with property A, then there cannot possibly exist a p that assigns a positive probability to elements with property A, because we require that $p(\varnothing) = 0$.) Such a proof method is nonconstructive, meaning that it doesn't provide a method for finding such an element, yet it demonstrates the element exists.

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest colored red. Show that no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

Hint: If we sample an inscribed cube uniformly at random among all possible inscribed cubes, what is (an upper bound on) the probability that the sampled cube has at least one blue vertex?

Solution: Pick an inscribed cube uniformly at random, enumerate its vertices $1, \ldots, 8$, and let B_i be the event that vertex i is blue. Note that

$$\mathbb{P}\left(\bigcup_{i=1}^{8} B_i\right) \leq \sum_{i=1}^{8} \mathbb{P}(B_i) = \sum_{i=1}^{8} \frac{1}{10} = \frac{8}{10} < 1.$$

In other words, the probability of at least one vertex being blue is strictly less than 1, so the probability that no vertex is blue (every vertex is red) is strictly greater than 0. Because a *randomly sampled* inscribed cube has nonzero probability of having all vertices red, there must exist at least one inscribed cube with all vertices red by the probabilistic method.

5. **Joint Occurrence**

You know that at least one of the events A_i, $i = 1, \ldots, n$, is certain to occur, but certainly no more than two occur. n is an integer ≥ 2. Show that if the probability of occurrence of any single event is p, and the probability of joint occurrence of any two distinct events is q, we have $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n(n-1)}$.

Solution: By the union bound, since

$$1 = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i) = np,$$

we see that $p \geq \frac{1}{n}$. Now we observe that the events $A_i \cap A_j$ for $i < j$, $i, j \in \{1, \ldots, n\}$, are pairwise disjoint, so by finite additivity,

$$1 \geq \mathbb{P}\left(\bigcup_{i<j} A_i \cap A_j\right) = \sum_{i<j} \mathbb{P}(A_i \cap A_j) = \binom{n}{2} q,$$

so $q \leq \frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)}$.

6. **Suspicious Game**

You are playing a card game with your friend in which you take turns picking a card from a deck. (Assume that you never run out of cards.) If you draw one of the special *bullet* cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that $\frac{1}{3}$ of the deck is filled with bullet cards. However, you don't fully trust your friend: you believe he is lying with probability $\frac{1}{4}$. Assume that if your friend is lying, then the opposite is true: $\frac{2}{3}$ of the deck is filled with bullet cards!

What is the probability that you win the game if you go first?

Solution: Let p denote the probability of randomly selecting a bullet card; p stays the same since you never run out of cards. Because the game ends when the first bullet card is drawn, the number of turns N before the game ends is a Geometric random variable with parameter p. The probability that you win is the probability that N is even, so we have

$$\begin{aligned}\mathbb{P}(\text{win} \mid p) = \mathbb{P}(N \text{ is even} \mid p) &= \sum_{\substack{k=1 \\ k \text{ is even}}}^{\infty} p(1-p)^{k-1} = p(1-p)\sum_{i=0}^{\infty}(1-p)^{2i} \\ &= \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.\end{aligned}$$

Now, we don't know whether $p = \frac{1}{3}$ or $\frac{2}{3}$, so we can use the law of total probability:

$$\begin{aligned}\mathbb{P}(\text{win}) &= \mathbb{P}\left(\text{win} \mid p = \frac{1}{3}\right) \cdot \mathbb{P}\left(p = \frac{1}{3}\right) + \mathbb{P}\left(\text{win} \mid p = \frac{2}{3}\right) \cdot \mathbb{P}\left(p = \frac{2}{3}\right) \\ &= \frac{2}{5} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} \\ &= 0.3625.\end{aligned}$$

Note. It may be tempting to first calculate p as

$$\mathbb{P}(B) = \mathbb{P}(B \mid L) \cdot \mathbb{P}(L) + \mathbb{P}(B \mid L^c) \cdot \mathbb{P}(L^c) = \frac{5}{12},$$

where B is the event of drawing a bullet card and L is the event that your friend is lying. Then, one would plug in $p = \frac{5}{12}$ to find the probability of winning as $\frac{7}{19} \approx 0.3684$. However, this does not work as it is not the case that N is a Geometric random variable with parameter $\frac{5}{12}$: the order of conditioning is reversed in this case.

Remark. The reason why the game is suspicious is because $\frac{1-p}{2-p} \leq \frac{1}{2}$ for $p \in [0,1]$, so your chances of winning are always unfavorable!