UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 13
Fall 2023

1. **Orthogonal LLSE**

   a. Consider zero-mean random variables X, Y, Z with Y, Z orthogonal. Show that

   $$\mathbb{L}(X \mid Y, Z) = \mathbb{L}(X \mid Y) + \mathbb{L}(X \mid Z).$$

   b. Now, for *any* zero-mean random variables X, Y, Z, explain why it holds that

   $$\mathbb{L}(X \mid Y, Z) = \mathbb{L}(X \mid Y) + \mathbb{L}[X \mid (Z - \mathbb{L}(Z \mid Y))].$$

Solution:

a. Let $f(Y) := \mathbb{L}(X \mid Y)$ and $g(Z) := \mathbb{L}(X \mid Z)$. We observe that X, $f(Y)$, and $g(Z)$ are all zero-mean, with Y orthogonal to $g(Z) \in \text{span}\{1, Z\}$. ($Y$ is orthogonal to 1 because it is zero-mean.) A similar argument establishes that $f(Y)$ and Z are orthogonal. Now,

$$\begin{aligned}
\mathbb{E}(X - f(Y) - g(Z)) &= 0 \\
\mathbb{E}((X - f(Y) - g(Z))Y) &= \mathbb{E}(g(Z)Y) = 0 \\
\mathbb{E}((X - f(Y) - g(Z))Z) &= \mathbb{E}(f(Y)Z) = 0,
\end{aligned}$$

where $X - f(Y) \perp Y$ and $X - g(Z) \perp Z$. As $X - (f(Y) + g(Z))$ is orthogonal to any linear function of $1, Y, Z$, by the orthogonality principle,

$$\begin{aligned}
\mathbb{L}(X \mid Y, Z) &= f(Y) + g(Z) \\
&= \mathbb{L}(X \mid Y) + \mathbb{L}(X \mid Z).
\end{aligned}$$

b. $W := Z - \mathbb{L}(Z \mid Y)$ is orthogonal to Y, so $\mathbb{L}(X \mid Y, W) = \mathbb{L}(X \mid Y) + \mathbb{L}(X \mid W)$ by part a. Then, $\mathbb{L}(X \mid Y, W) = \mathbb{L}(X \mid Y, Z)$ as $\text{span}\{1, Y, W\} = \text{span}\{1, Y, Z\}$.

2. **Hypothesis Testing for Bernoulli Random Variables**

   Suppose that

   - The null hypothesis is $X = 0$: $Y \sim \text{Bernoulli}(\frac{1}{4})$, and
   - The alternative hypothesis is $X = 1$: $Y \sim \text{Bernoulli}(\frac{3}{4})$.

   Using the Neyman–Pearson formulation of hypothesis testing, find the optimal randomized decision rule $\hat{X}$ with respect to the criterion

$$\begin{aligned} \min \ \ & \mathbb{P}(\hat{X} = 0 \mid X = 1) \\ \text{s.t.} \ \ & \mathbb{P}(\hat{X} = 1 \mid X = 0) \leq \beta, \end{aligned}$$

   where $\beta \in [0, 1]$ is a given upper bound on the probability of false alarm (PFA).

   (Note that the Neyman–Pearson decision rule may change depending on the value of β. In particular, consider the two separate cases of $\beta \leq \frac{1}{4}$ and $\beta > \frac{1}{4}$.)

   Solution: The likelihood ratio is the discrete function

$$L(y) = \frac{f_{Y|X}(y \mid 1)}{f_{Y|X}(y \mid 0)} = \begin{cases} 3 & \text{if } y = 1 \\ \frac{1}{3} & \text{if } y = 0. \end{cases}$$

   By Neyman–Pearson, the optimal decision rule with randomization is given by

   - If $\mathbb{P}(Y = 1 \mid X = 0) = \frac{1}{4} \geq \beta$, then

$$\hat{X} = \begin{cases} 0 & \text{if } Y = 0 \\ \text{Bernoulli}(\gamma) & \text{with } \gamma = \beta / \frac{1}{4} \text{ if } Y = 1. \end{cases}$$

   - Otherwise, the threshold is $Y = 0$, and

$$\hat{X} = \begin{cases} \text{Bernoulli}(\gamma) & \text{with } \gamma = \frac{4}{3}\beta - \frac{1}{3} \text{ if } Y = 0 \\ 1 & \text{if } Y = 1. \end{cases}$$

     The value of γ above is chosen to make

$$\mathsf{PFA} = \mathbb{P}(Y = 1 \mid X = 0) + \gamma \cdot \mathbb{P}(Y = 0 \mid X = 0) = \frac{1}{4} + \frac{3}{4}\gamma = \beta.$$

3. **Gaussian LLSE**

Let X, Y, Z be i.i.d. $\mathcal{N}(0,1)$.

a. Find $\mathbb{L}(X^2 + Y^2 \mid X + Y)$.

b. Find $\mathbb{L}(X + 2Y \mid X + 3Y + 4Z)$.

c. Find $\mathbb{L}((X + Y)^2 \mid X - Y)$.

Solution:

a. We note that

$$\operatorname{cov}(X^2 + Y^2, X + Y) = \mathbb{E}((X^2 + Y^2)(X + Y)) = \mathbb{E}(X^3 + X^2 Y + XY^2 + Y^3) = 0.$$

Thus, $\mathbb{L}(X^2 + Y^2 \mid X + Y) = \mathbb{E}(X^2 + Y^2) = 2$.

b. We find that

$$\begin{aligned} \operatorname{cov}(X + 2Y, X + 3Y + 4Z) &= \mathbb{E}[(X + 2Y)(X + 3Y + 4Z)] = \mathbb{E}(X^2) + 6\,\mathbb{E}(Y^2) = 7 \\ \operatorname{var}(X + 3Y + 4Z) &= \operatorname{var}(X) + 9\operatorname{var}(Y) + 16\operatorname{var}(Z) = 26 \\ \mathbb{L}(X + 2Y \mid X + 3Y + 4Z) &= \frac{7}{26}(X + 3Y + 4Z). \end{aligned}$$

c. We observe that $\operatorname{cov}(X + Y, X - Y) = 0$, so that the jointly Gaussian $X + Y$ and $X - Y$ are independent. Hence,

$$\mathbb{L}((X + Y)^2 \mid X - Y) = \mathbb{E}((X + Y)^2) = \operatorname{var}(X + Y) = 2.$$