UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 08
Fall 2023

1. **Random Walk on an Undirected Graph**

   Consider a random walk on an undirected connected finite graph (that is, define a Markov chain where the state space is the set of vertices of the graph, and at each time step, transition to a vertex chosen uniformly at random out of the neighborhood of the current vertex). What is the stationary distribution π? Your answer may depend on $\deg(v)$ (i.e., the degree of a vertex v) for some v. *Hint*: assume first that the chain is *reversible*.

   Solution: Let $\mathcal{X}$ be the state space. The stationary distribution is

   $$\pi(v) = \frac{\deg(v)}{\sum_{v' \in \mathcal{X}} \deg(v')}, \qquad v \in \mathcal{X}.$$

   Clearly, π is a valid probability distribution. We check that the chain is reversible. Note that if u and v are neighbors, then

   $$\pi(u)P(u,v) = \frac{\deg(u)}{\sum_{v' \in \mathcal{X}} \deg(v')} \cdot \frac{1}{\deg(u)} = \frac{1}{\sum_{v' \in \mathcal{X}} \deg(v')}.$$

   Also,

   $$\pi(v)P(v,u) = \frac{\deg(v)}{\sum_{v' \in \mathcal{X}} \deg(v')} \cdot \frac{1}{\deg(v)} = \frac{1}{\sum_{v' \in \mathcal{X}} \deg(v')}.$$

   So, $\pi(u)P(u,v) = \pi(v)P(v,u)$ if u and v are neighbors. If u and v are not neighbors, then $P(u,v) = P(v,u) = 0$, so the equation holds in this case as well. The chain is reversible and so π is stationary.

2. **Recurrence and Transience of Random Walks**

"A drunk man will find his way home, but a drunk bird may get lost forever."
— Shizuo Kakutani

Consider the symmetric random walk $S_n = X_1 + \cdots + X_n$ in d dimensions, in which we start at the origin, and in each time step jump to an adjacent point on the d-dimensional lattice $\mathbb{Z}^d$ with uniform probability. That is,

$$X_i \sim_{\text{i.i.d.}} \text{Uniform}\{\pm e_1, \ldots, \pm e_d\},$$

where $\{e_1, \ldots, e_d\}$ are the unit vectors in $\mathbb{Z}^d$.

a. Show that if $\sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) = \infty$, then the random walk is recurrent.
Hint: Let N be the number of times the random walk visits the origin. It may help to notice that $\mathbb{E}(N) = \infty$ is equivalent to recurrence of the random walk.

b. Use part a to show that the random walk for $d = 1$ is recurrent. You may use Stirling's approximation, where $f(n) \sim g(n)$ indicates $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

c. Use part b to show that the random walk for $d = 2$ is recurrent.
Hint: consider two independent 1-dimensional random walks in orthogonal directions.

d. **Optional**. Show that the random walk for $d = 3$ is transient.

e. Use part d to show that the random walk for any $d > 3$ is also transient.

Solution:

a. Defining N as in the hint, we have by the tail-sum formula that

$$\sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) = \mathbb{E}\left(\sum_{n=0}^{\infty} \mathbb{1}_{S_n=0}\right) = \mathbb{E}(N) = \sum_{k=1}^{\infty} \mathbb{P}(N \geq k)$$

Let τ_k be the time index at which the origin is visited for the kth time. Then

$$\sum_{k=1}^{\infty} \mathbb{P}(N \geq k) = \sum_{k=1}^{\infty} \mathbb{P}(\tau_k < \infty) = \sum_{k=1}^{\infty} \mathbb{P}(\tau_1 < \infty)^k,$$

where $\mathbb{P}(\tau_k < \infty) = \mathbb{P}(\tau_1 < \infty)^k$ uses the (strong) Markov property. Since this summation diverges, we must have $\mathbb{P}(\tau_1 < \infty) = 1$, so S_n is recurrent. In fact, the converse of this statement is true as well.

b. It suffices to show that $\sum_{n=0}^{\infty} \mathbb{P}(S_{2n} = 0)$ diverges. By counting the possible paths,

$$\mathbb{P}(S_{2n} = 0) = \binom{2n}{n}\frac{1}{2^{2n}} \sim \frac{\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}{n\left(\frac{n}{e}\right)^{2n}} \cdot \frac{1}{2^{2n}} = \sqrt{\frac{2}{n}}.$$

As $\sum_{n=2}^{\infty} n^{-1/2}$ diverges, we deduce from part a that S_n is reccurent in dimension 1.

c. We may do a similar combinatorial computation as we did in part b to deduce recurrence. However, for a nicer approach, consider two independent 1-dimensional random walks

$$R_n := (S_n^{(1)}, S_n^{(2)}).$$

Observe that by rotating the coordinate plane by 45 degrees, this is equivalent to a symmetric random walk S_n on $\mathbb{Z}^2$. In particular,

$$\mathbb{P}(S_{2n} = (0,0)) = \mathbb{P}\left(S_{2n}^{(1)} = 0\right) \cdot \mathbb{P}\left(S_{2n}^{(2)} = 0\right) \sim \left(\frac{1}{\sqrt{n}}\right)^2 = \frac{1}{n}$$

using part b. As $\sum_{n=2}^{\infty} n^{-1}$ is still divergent, S_n is recurrent for $d = 2$ as well.

d. **Optional**. Summing over all $0 \le j,k \le 2n$ for which $j + k \le 2n$, we have

$$\begin{aligned}
\mathbb{P}(S_{2n} = 0) &= 6^{-2n} \sum_{j,k} \frac{(2n)!}{(j!k!(n-j-k)!)^2} \\
&= 2^{-2n}\binom{2n}{n} \sum_{j,k} \left(3^{-n} \frac{n!}{j!k!(n-j-k)!}\right)^2 \\
&\sim \frac{1}{\sqrt{n}} \sum_{j,k} \left(3^{-n} \frac{n!}{j!k!(n-j-k)!}\right)^2.
\end{aligned}$$

At this point, we may use Stirling's approximation to further simplify the summation, and eventually we find that $\mathbb{P}(S_{2n} = 0) \sim O(n^{-3/2})$. Since $\sum_{n=2}^{\infty} n^{-3/2} < \infty$, we see that S_n is transient for $d = 3$.

e. Suppose $d > 3$. If we let $S_n^{(i)}$ denote the ith coordinate of the random walk S_n, then we can generate a 3-dimensional random walk as the process $(S_n^{(1)}, S_n^{(2)}, S_n^{(3)})$. As random walks in $d = 3$ are transient,

$$\mathbb{P}(S_n = 0 \text{ i.o.}) \le \mathbb{P}\left((S_n^{(1)}, S_n^{(2)}, S_n^{(3)}) = 0 \text{ i.o.}\right) < 1$$

shows that S_n is also transient for any $d > 3$.

3. **Customers in a Store**

Consider two independent Poisson processes with rates λ_1 and λ_2, which measure the number of customers arriving in store 1 and 2.

a. What is the probability that a customer arrives in store 1 before any arrives in store 2?

b. What is the probability that in the first hour, a total of exactly 6 customers arrive in the two stores?

c. Given that exactly 6 have arrived in total at the two stores, what is the probability that exactly 4 went to store 1?

Solution:

a. Consider the sum of the two processes, itself a Poisson process with rate $\lambda_1 + \lambda_2$ by Poisson merging. Each customer in this process is marked as 1 with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and 2 otherwise. The probability of the first customer going to store 1 is then the probability of marking the first customer as 1, which is

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Alternatively, the arrival times of the first customer of the two stores are respectively $X \sim \text{Exponential}(\lambda_1)$ and $Y \sim \text{Exponential}(\lambda_2)$. By the law of total probability,

$$\begin{aligned}\mathbb{P}(X < Y) &= \int_0^\infty f_Y(y) \cdot \mathbb{P}(X < Y \mid Y = y)\, \mathrm{d}y \\ &= \int_0^\infty \lambda_2 e^{-\lambda_2 y} \cdot (1 - e^{-\lambda_1 y})\, \mathrm{d}y \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2}.\end{aligned}$$

b. Considering the merged process, this probability is

$$\mathbb{P}(\text{Poisson}((\lambda_1 + \lambda_2) \cdot 1) = 6) = \frac{(\lambda_1 + \lambda_2)^6}{6!} e^{-(\lambda_1+\lambda_2)}.$$

c. Conditioned on the total number of arrivals, the number of arrivals in each split process has Binomial distribution, so this probability is

$$\binom{6}{4} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^4 \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^2.$$