UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 05
Fall 2023

1. **Convergence in Probability**

   Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequences $(Y_n)_{n\in\mathbb{N}}$ converge in probability to some limit.

   a. $Y_n = \prod_{i=1}^n X_i$.
   b. $Y_n = \max\{X_1, \dots, X_n\}$.
   c. $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

   Solution:

   a. By the independence of the random variables,

   $$\mathbb{E}(Y_n) = \mathbb{E}(X_1)\cdots\mathbb{E}(X_n) = 0$$
   $$\mathrm{var}(Y_n) = \mathbb{E}(Y_n^2) = (\mathrm{var}(X_1))^n = \left(\frac{1}{3}\right)^n.$$

   Since $\mathrm{var}(Y_n) \to 0$ as $n \to \infty$, by Chebyshev's inequality, the sequence converges to its mean 0 in probability.

   b. Consider $\varepsilon \in (0, 1]$. We see that

   $$\begin{aligned}\mathbb{P}(|Y_n - 1| \geq \varepsilon) &= \mathbb{P}(\max\{X_1, \dots, X_n\} \leq 1 - \varepsilon) \\ &= \mathbb{P}(X_1 \leq 1 - \varepsilon, \dots, X_n \leq 1 - \varepsilon) \\ &= \mathbb{P}(X_1 \leq 1 - \varepsilon)^n \\ &= \left(1 - \frac{\varepsilon}{2}\right)^n,\end{aligned}$$

   so $\mathbb{P}(|Y_n - 1| \geq \varepsilon) \to 0$ as $n \to \infty$, and we are done.

   c. We can find the expectation, then bound the variance:

   $$\mathbb{E}(Y_n) = \frac{1}{n} \cdot n\,\mathbb{E}(X_1^2) = \frac{1}{3},$$
   $$\mathrm{var}(Y_n) = \frac{1}{n}\,\mathrm{var}(X_1^2) \leq \frac{1}{n} \to 0 \qquad \text{as } n \to \infty,$$

   since $X_1^2 \leq 1$. Hence, we see that $Y_n \to \frac{1}{3}$ in probability as $n \to \infty$.

2. **Bernoulli Convergence**

Consider an independent sequence of random variables $X_n \sim \text{Bernoulli}(\frac{1}{n})$.

a. Show that X_n converges to 0 in probability.

b. Argue that

$$\mathbb{P}\left(\left\{\lim_{n\to\infty} X_n = 0\right\}\right) = \mathbb{P}\left(\bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}\right).$$

c. Using part b, show that X_n does **not** converge almost surely to 0.
Hint: Consider applying the union bound and the independence of the X_n.

Solution:

a. We want to show that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - 0| > \varepsilon) = 0.$$

Because each X_n can only be 0 or 1, if $\varepsilon \geq 1$, then $\mathbb{P}(|X_n - 0| > \varepsilon) = 0$, so the limit is also zero. If $0 < \varepsilon < 1$, then

$$\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{P}(X_n = 1) = \frac{1}{n} \to 0.$$

b. Since each X_n can only take on the values 0 or 1, the limit of X_n is 0 iff the sequence $X_1, X_2, \ldots$ is eventually always 0. In other words, $\{\lim_{n\to\infty} X_n = 0\}$ occurs if and only if there exists an N such that for all $n \geq N$, $X_n = 0$. Thus

$$\left\{\lim_{n\to\infty} X_n = 0\right\} = \bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}.$$

c. Applying the union bound to the equality in part b,

$$\begin{aligned}
\mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right) &\leq \sum_{N=1}^{\infty} \mathbb{P}(X_n = 0 \text{ for all } n \geq N) \\
&= \sum_{N=1}^{\infty} \mathbb{P}\left(\bigcap_{n=N}^{\infty} \{X_n = 0\}\right)
\end{aligned}$$

Because the X_n are independent, this equals

$$\begin{aligned}
&= \sum_{N=1}^{\infty} \prod_{n=N}^{\infty} \mathbb{P}(X_n = 0) \\
&= \sum_{N=1}^{\infty} \frac{N-1}{N} \cdot \frac{N}{N+1} \cdot \frac{N+1}{N+2} \cdots
\end{aligned}$$

By telescoping, this infinite product is zero for any value of N, so we have

$$= \sum_{N=1}^{\infty} 0 = 0.$$

Since this probability is not 1, X_n does not converge almost surely to 0. In fact, since this probability is 0, X_n *almost surely does not converge* to 0. A related result is Kolmogorov's 0–1 law, which states that a sequence of independent random variables either converges or does not converge with probability 1.

3. **Mean Square Convergence**

   A sequnce of random variables $\{X_n\}_{n\geq 0}$, each satisfying $\mathbb{E}[X_n^2] < \infty$, is said to converge in *mean square* to a random variable X if

$$\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0.$$

   a. Show that convergence in mean square implies convergence in probability.
   b. Consider the sequence of random variables $\{X_n\}_{n\geq 1}$, where each $X_n \sim \text{Bernoulli}(1/n)$. Show that this sequence converges to 0 in mean square.
   c. Does it converge almost surely?

Solution:

a. Assume that $\mathbb{E}[(X_n - X)^2] \to 0$, as $n \to \infty$. For any $\epsilon > 0$,

$$\begin{aligned}\Pr(|X_n - X| > \epsilon) &= \Pr\big((X_n - X)^2 > \epsilon^2\big) \\ &\leq \frac{\mathbb{E}[(X_n - X)^2]}{\epsilon^2} \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

b. $\mathbb{E}[X_n^2] = \mathbb{E}[X_n] = \frac{1}{n} \to 0$, as $n \to \infty$.

c. It does not converge almost surely though, since for any $\epsilon \in (0,1)$ and any m we have that

$$\begin{aligned}\Pr(|X_n - 0| < \epsilon, \text{ for all } n \geq m) &= \lim_{n\to\infty} \prod_{i=m}^{n} \Big(1 - \frac{1}{i}\Big) = \lim_{n\to\infty} \prod_{i=m}^{n} \Big(\frac{i-1}{i}\Big) \\ &= \lim_{n\to\infty} \frac{m-1}{m} \frac{m}{m+1} \cdots \frac{n-1}{n} \\ &= \lim_{n\to\infty} \frac{m-1}{n} = 0.\end{aligned}$$

   Now take the union over m.

$$\begin{aligned}&\Pr(\exists m \text{ such that } |X_n| < \epsilon \text{ for all } n \geq m) \\ &= \Pr\Big(\bigcup_{m=1}^{\infty} \{|X_n| < \epsilon \text{ for all } n \geq m\}\Big) \\ &\leq \sum_{m=1}^{\infty} \Pr(|X_n| < \epsilon \text{ for all } n \geq m) = 0.\end{aligned}$$

   This implies that $\Pr(\lim_{n\to\infty} X_n = 0) = 0$, which means $(X_n)_{n=1}^{\infty}$ does not converge a.s. to 0. Since (a) and (b) imply that $X_n \to 0$ in probability as $n \to \infty$, if $(X_n)_{n=1}^{\infty}$ were to converge to a random variable X a.s., then X would have to be 0 (because a.s. convergence implies convergence in probability), but we have seen that $(X_n)_{n=1}^{\infty}$ does *not* converge to 0 a.s., which means $(X_n)_{n=1}^{\infty}$ does not converge a.s. to anything.