# 204ducks-bootstrap

B-MAT-400

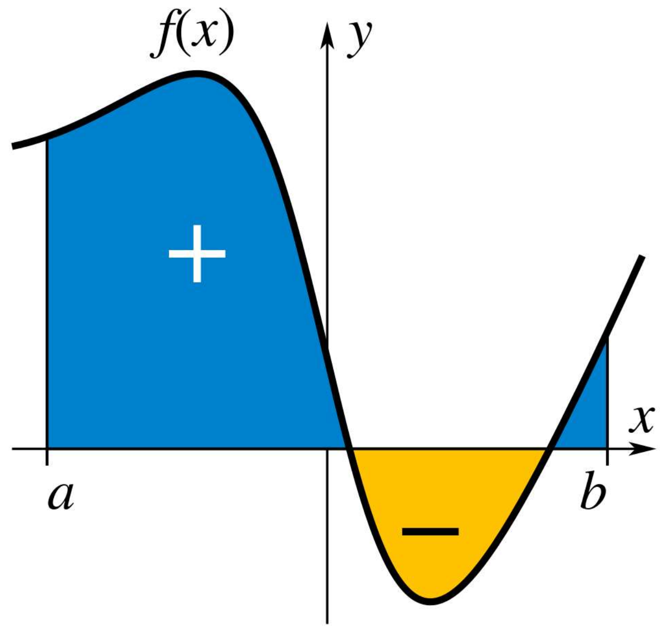
# Reminder: integrals

- The integral of a continuous function is the signed area defined by its graph, the $x$-axis and the boundaries

$$\int_a^b f(x)dx$$

- It is the inverse of the derivative:

$$F(x) = \int_a^x f(t)dt \Longrightarrow F'(x) = f(x)$$



# Previously: random variables and distributions

- Random variable: values that depends of the outcome of an experiment, defined as a function $X: \Omega \rightarrow E$
  - Example: sum of 2 dice rolls, $E = \{2,3,4,5,6,7,8,9,10,11,12\}$

- Probability distribution: function that maps every value of a random variable to its probability

$$p(x) = P(X = x), x \in E$$

# Continuous random variables

- A continuous random variable $X$ can take any real value between two boundaries
- Example: duration of a phone call
- $\mathbb{R}$ is uncountable, so we cannot enumerate the events $\{X = x\}$
- Instead, we consider the probability on intervals $[a, b]$:
$$\{a \leq X \leq b\}$$
- Example: phone calls with a duration between 1 and 2 minutes

# Cumulative distribution function

- If $X$ is a continuous random variable: $X\colon \Omega \longrightarrow \mathbb{R}$
- The cumulative distribution function is the probability that $X$ takes a value less than or equal to $x$:
$$F(x) = P(X \leq x)$$
- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to +\infty} F(x) = 1$
- $F$ is continuous, derivable and increasing on $\mathbb{R}$
- $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$
- $P(X = x) = P(x \leq X \leq x) = 0$

# Probability density function

- The probability density function $f$ of a continuous random variable $X$ is a positive and integrable function on $\mathbb{R}$ verifying:

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$$

- It is related to the cumulative distribution function:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

# Expected value and standard deviation

- Expected value:

$$E[X] = \mu = \int_{-\infty}^{+\infty} x f(x) dx$$

- Variance:

$$Var(X) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx$$

- Standard deviation:

$$\sigma = \sqrt{Var(X)} = \sqrt{\int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx}$$

# 204ducks

- Ducks flying away after a disturbance
- Chance that a duck comes back after $t$ minutes defined by the density

$$\begin{aligned} f: [0, +\infty[ &\longrightarrow \mathbb{R} \\ t &\longmapsto ae^{-t} + (4-3a)e^{-2} \ + (2a-4)e^{-4t} \end{aligned}$$

- Input: constant $a$ between $0$ and $2.5$
- Outputs:
  - Average time for a duck to come back
  - Standard deviation of the ducks' time to come back
  - Time after which 55% and 99% of ducks are back
  - Percentage of ducks that come back after 1 and 2 minutes

# Points of attention

- **Beware of precision!**
- $f$ is only defined on $[0, +\infty[$. What about $]-\infty, 0[$?
- Efficient method to compute integrals
  - Cf. 110borwein
- Efficient method to solve $f(x) = c$
  - Cf. 105torus

# Exercise: numerical integrals (cf. 110borwein)

- Create a function that computes (given $a$, $b$ and $f$):

$$\int_a^b f(x)dx$$

- Methods for numerical integration:
  - Rectangle rule
  - Trapezoidal rule
  - Simpson's rule…
- How can you go from $[a, b]$ to $]-\infty, +\infty[$?

# Exercise: solving equations (cf. 105torus)

- Create a function that solves $f(x) = c$, given $f$ and $c$.
- Root-finding algorithms:
  - Bisection
  - Newton
  - Secant