205IQ-bootstrap

B-MAT-400

Previously: Probability density function

- The probability density function f of a continuous random variable X is a positive and integrable function on $\mathbb{R}$ verifying:

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$$

Previously: Cumulative distribution function

- The cumulative distribution function is the probability that X takes a value less than or equal to x:

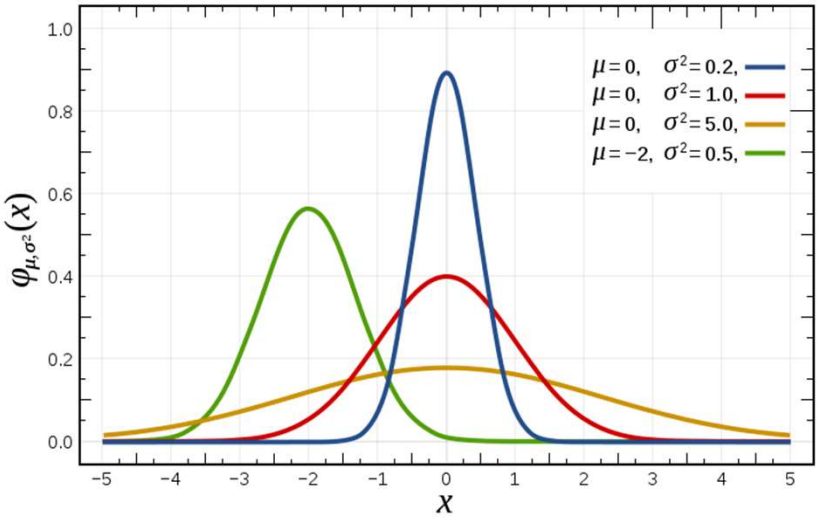
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to +\infty} F(x) = 1$
- F is continuous, derivable and increasing on $\mathbb{R}$
- $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$

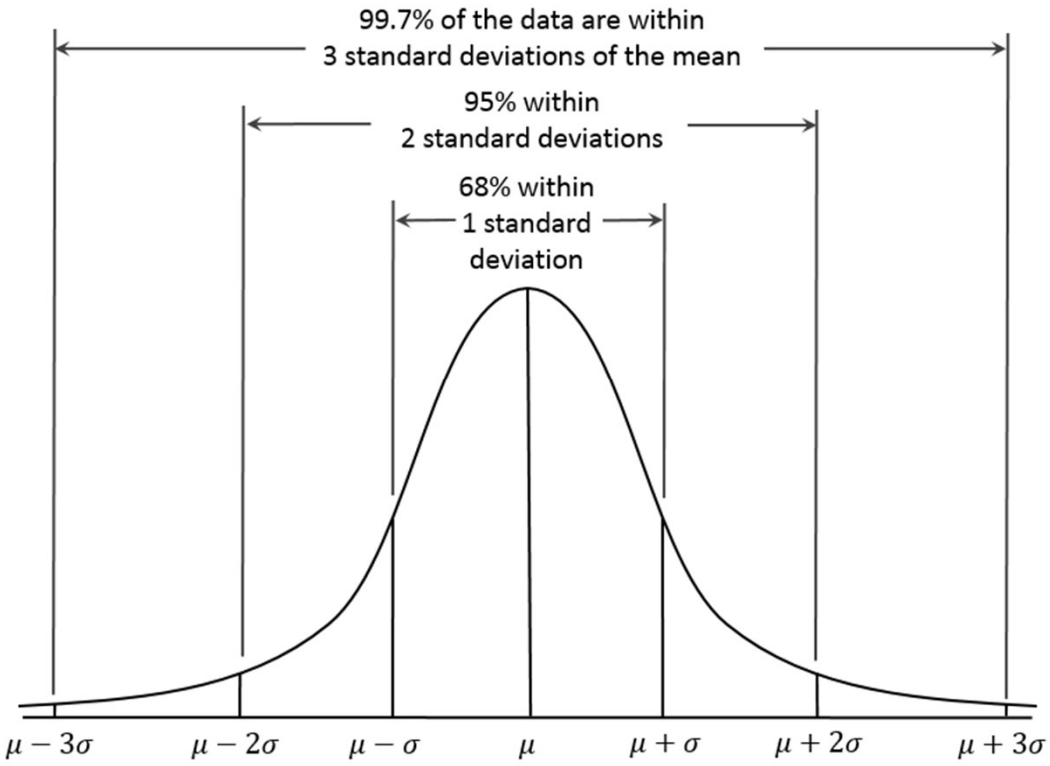
Normal distribution

- The normal distribution, also called Gaussian distribution, is a very common continuous probability distribution
- It is defined by its mean μ and its standard deviation σ:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

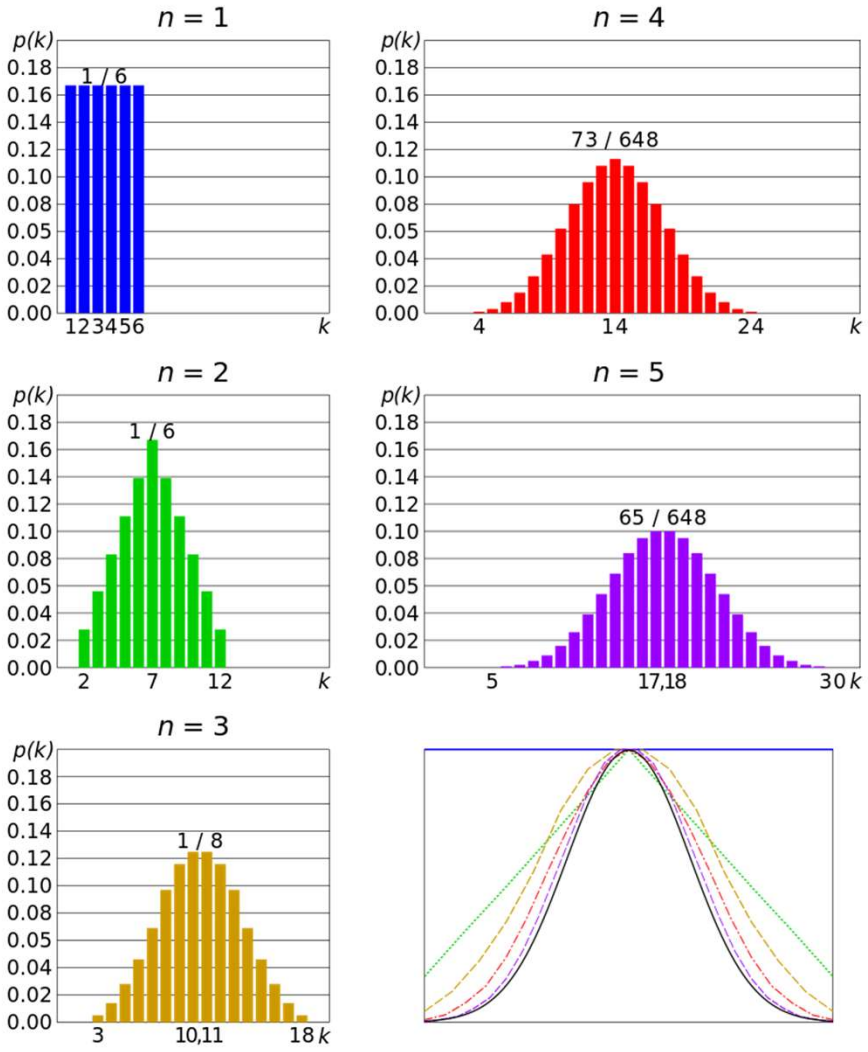


Normal distribution



Central limit theorem

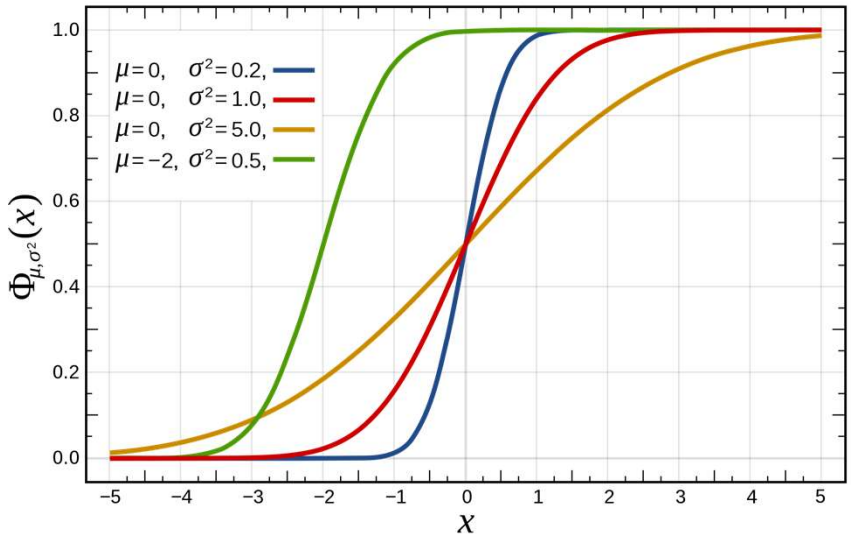
- Under fairly common conditions, the sum of many random variables will have an approximately normal distribution
- Example: the sum of many dice rolls
- Binomial (if n is large and p not close to 1 or 0) and Poisson (if λ is large) distributions can be approximated by the normal distribution



Cumulative distribution function

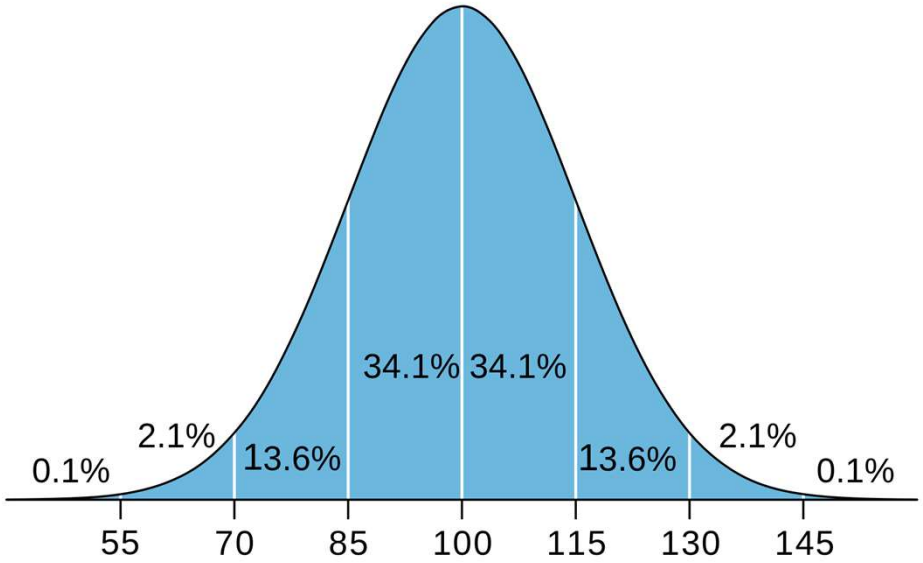
- The cumulative distribution function of a normal distribution is:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$



IQ tests

- Historically, score obtained by dividing the “mental age” of a child by its actual age and multiplying by 100
- Current tests are calibrated to follow a normal distribution with a mean of 100 and a standard deviation of 15



205IQ

- Goal: Compute normal distributions to calibrate IQ tests
- Inputs: μ σ [IQ1] [IQ2]
  - μ: mean
  - σ: standard deviation
  - IQx: zero, one or two IQ values
- Outputs:
  - Zero IQ value: values of the density function for every integer from 0 to 200
  - One IQ value: percentage of people with an IQ less than the given value
  - Two IQ values: percentage of people with an IQ between the given values

Points of attention

- **Beware of precision!**
- Be smart when choosing your integration interval
- Efficient method to compute integrals
  - Cf. 110borwein and 204ducks

Exercises

- Create a function that plots the normal distribution density function given its mean and standard deviation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Create a function that computes the cumulative distribution function of a normal distribution given its mean, standard deviation and a value of x

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$