Assignment 2 - Probability

Kannan N S

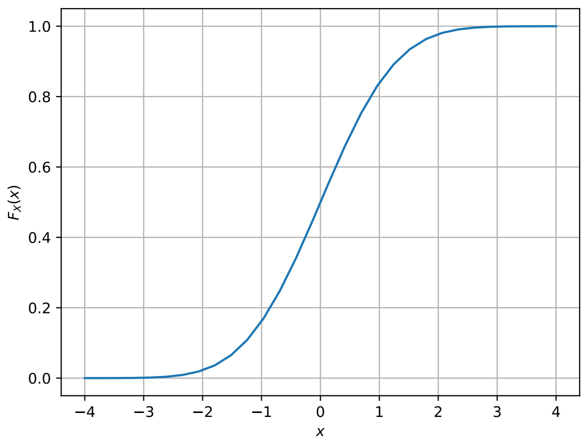

Fig. 1.2. The CDF of X

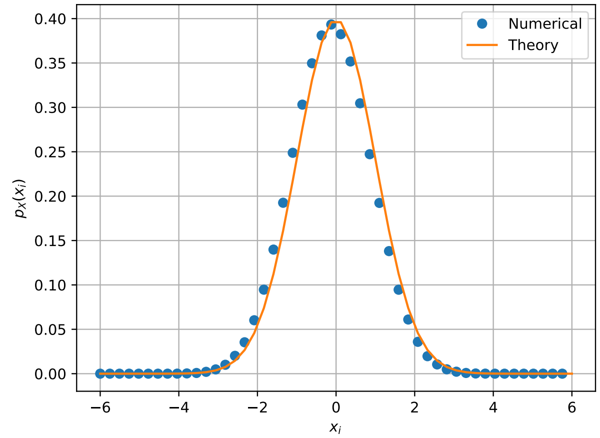

Fig. 1.3. The PDF of X

Contents

1 Central Limit Theorem

1.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{1.1.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
Code/myexrand.c
Code/mycoeffs.h
```

1.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 1.2

1.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{1.3.1}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 1.3 using the code below

```
code/pdf_plot.py
```

1.4 Find the mean and variance of X by writing a C program.

Solution:

```
Code/myexrand.c
Code/mycoeffs.h
```

- Mean of uniform distribution = 0.000326.
- Variance of uniform distribution = 1.000907.

1.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{1.5.1}$$

repeat the above exercise theoretically.
Solution: Since the area under the curve is given by:

$$\int_{-\infty}^{\infty} p_x(x)(dx)$$

The mean of $p_x(x)(dx)$ is given by:

$$= \int_{-\infty}^{\infty} x.p_x(x)(dx)$$
$$= \int_{-\infty}^{0} x.p_x(x)(dx) + \int_{0}^{\infty} x.p_x(x)(dx)$$

Since $x.p_x(x)$ is an odd function, integrating it from minus infinity to positive infinity will give zero.
So mean of Gauss distribution is Zero.
Var(x) $= E[X - E[X]]^2$
$= E[X^2 - 2XE[X] + E[X]^2]$
$= E[X^2] - E[X]^2$
We know that $E[X]$ is zero so $E[X]^2$ is also zero.

$$Var(x) = E[X^2]$$
$$= \int_{-\infty}^{\infty} (x)^2.p_x(x)(dx)$$
$$= \int_{-\infty}^{\infty} \frac{(x)^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)(dx)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x)^2 \exp\left(-\frac{x^2}{2}\right)(dx)$$

We can rewrite the above integral as,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) x(dx)$$

Let us assume y $= \frac{1}{2} * x^2$

$$\frac{dy}{dx} = \frac{2x}{2}$$
$$dy = xdx$$
$$x = \sqrt{2y}$$

Let us substitute,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2y} \exp(-y)\, dy$$
$$= \frac{\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{y} \exp(-y)\, dy$$

Since it is an even function we know $\int_{-\infty}^{\infty} F(x)(dx) = 2\int_{0}^{\infty} F(x)(dx)$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \sqrt{y} \exp(-y)\, dy$$

We can see that, $\int_{0}^{\infty} \sqrt{y} \exp(-y)\, dy$ is a Gamma function of the form

$$\Gamma(a) = \int_{0}^{\infty} y^{a-1} \exp(-y)\, dy$$
$$E[X^2] = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} y^{(\frac{1}{2}+1)-1} \exp(-y)\, dy$$
$$E[X^2] = \frac{2}{\sqrt{\pi}} \Gamma(\frac{1}{2} + 1)$$

We know that, $\Gamma(\zeta + 1) = \zeta\Gamma(\zeta)$

$$= \frac{2}{\sqrt{\pi}} * \frac{1}{2}\Gamma(\frac{1}{2})$$
$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$
$$E[X^2] = \frac{2}{\sqrt{\pi}} * \frac{1}{2} * \sqrt{\pi}$$
$$E[X^2] = 1.$$

So, the Variance of Gauss distribution is equal to 1.