1 Atmosphere Aerosol PSF

The cyl and somb functions are defined as follows:

$$\begin{aligned}
\mathrm{cyl}\left(\frac{r}{d}\right) &= 1 \;\text{ for } 0 \le r < d/2 \\
&= 0.5 \;\text{ for } r = d/2 \\
&= 0 \;\text{ for } r > d/2
\end{aligned}$$

$$\mathrm{somb}\left(\frac{r}{d}\right) = \frac{2J_1\left(\frac{\pi r}{d}\right)}{\left(\frac{\pi r}{d}\right)} \tag{1}$$

where J_1 is the first-order Bessel function of the first kind.

The Dirac-delta function $\delta(r)$ has infinite height, zero width and the area under the impulse response is one,

$$\int_0^\infty \frac{\delta(r)}{r\pi} dr = 1. \tag{2}$$

This is important when the psf is implemented in discrete, pixelated form.

Zero-order Hankel properties

$g(r) = 2\pi \int_0^\infty G(\rho')J_0(2\pi r\rho')\rho' d\rho'$	$G(\rho) = 2\pi \int_0^\infty g(r')J_0(2\pi \rho r')r' dr'$
$h(r)$	$H(\rho)$
$A\,f(r) + B\,h(r)$	$A\,F(\rho) + B\,H(\rho)$
$f(r)h(r)$	$F(\rho) \star\star H(\rho)$
$f(r) \star\star h(r)$	$F(\rho)H(\rho)$
$f\left(\frac{r}{b}\right)$	$\lvert b\rvert^2 F(b\rho)$

Elementary zero-order Hankel transform pairs

$g(r) = 2\pi \int_0^\infty G(\rho')J_0(2\pi r\rho')\rho' d\rho'$	$G(\rho) = 2\pi \int_0^\infty g(r')J_0(2\pi \rho r')r' dr'$
$\frac{\delta(r)}{r\pi}$	1
$\mathrm{somb}(r)$	$\frac{4}{\pi}\mathrm{cyl}(\rho)$
$\exp\left\{-\pi r^2/a\right\}$	$a\exp\left\{-\pi a\rho^2\right\}$

Manipulation of Zero-order Hankel transform pairs

$g(r) = 2\pi \int_0^\infty G(\rho')J_0(2\pi r\rho')\rho' d\rho'$	$G(\rho) = 2\pi \int_0^\infty g(r')J_0(2\pi \rho r')r' dr'$
$\frac{\pi}{4}\,\mathrm{somb}(r)$	$\mathrm{cyl}(\rho)$
$\lvert f_c\rvert^2 \frac{\pi}{4}\,\mathrm{somb}(f_c r)$	$\mathrm{cyl}\left(\frac{\rho}{f_c}\right)$
$\lvert f_c\rvert^2\, \exp\left\{-\pi (f_c r)^2/a\right\}$	$a \exp\left\{-\pi a \left(\frac{\rho}{f_c}\right)^2\right\}$
$\lvert f_c\rvert^2\, \exp\left\{-\pi^2 (f_c r)^2/\beta\right\}$	$\frac{\beta}{\pi} \exp\left\{-\beta \left(\frac{\rho}{f_c}\right)^2\right\}$
$\frac{\pi}{\beta}\lvert f_c\rvert^2\, \exp\left\{-\pi^2 (f_c r)^2/\beta\right\}$	$\exp\left\{-\beta \left(\frac{\rho}{f_c}\right)^2\right\}$

The original Kopeika aerosol MTF formulation is as follows:

$$\begin{aligned} M(\rho) &= \exp\left(-\beta\left(\frac{\rho}{f_c}\right)^2\right) \quad \text{for } \rho < f_c \\ &= \exp\left(-\beta\right) \quad \text{for } \rho \geq f_c \end{aligned} \tag{3}$$

This can be reformulated as

$$M(\rho) = \exp\left(-\beta\right) + \left[\exp\left\{-\beta\left(\frac{\rho}{f_c}\right)^2\right\} - \exp\left(-\beta\right)\right] \mathrm{cyl}\left(\frac{\rho}{f_c}\right). \tag{4}$$

Applying the Hankel transform

$$
\begin{aligned}
m(r) &= \exp(-\beta)\,\frac{\delta(r)}{r\pi} \\
&\quad + \left[\frac{\pi}{\beta}|f_c|^2\,\exp\left\{-\pi^2(f_cr)^2/\beta\right\} - \exp(-\beta)\,\frac{\delta(r)}{r\pi}\right] \\
&\quad \star\star\,|f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr) \\
&= \exp(-\beta)\,\frac{\delta(r)}{r\pi} \\
&\quad + \frac{\pi}{\beta}|f_c|^2\,\exp\left\{-\pi^2(f_cr)^2/\beta\right\}\star\star|f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr) \\
&\quad - \exp(-\beta)\,\frac{\delta(r)}{r\pi}\star\star|f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr) \\
&= \exp(-\beta)\,\frac{\delta(r)}{r\pi} \\
&\quad - \exp(-\beta)\,\frac{\delta(r)}{r\pi}\star\star|f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr) \\
&\quad + \frac{\pi}{\beta}|f_c|^2\,\exp\left\{-\pi^2(f_cr)^2/\beta\right\}\star\star|f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr).
\end{aligned}
$$

Which leads to the final formulation

$$
\begin{aligned}
m(r) &= \exp(-\beta)\left[\frac{\delta(r)}{r\pi} - |f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr)\right] \\
&\quad + \frac{\pi}{\beta}|f_c|^2\,\exp\left\{-\pi^2(f_cr)^2/\beta\right\}\star\star|f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr) \qquad (5)
\end{aligned}
$$

because

$$
\frac{\delta(r)}{r\pi}\star\star|f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr) = |f_c|^2\left(\frac{\pi}{4}\right)\mathrm{somb}(f_cr)
$$

where

$$
\mathrm{somb}\left(f_cr\right) = \frac{2J_1\left(\pi f_cr\right)}{\left(\pi f_cr\right)} \qquad (6)
$$