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DATS 6202-12
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Homework I

①

E.1: $S = (\{H,T\}, \{H,T\}, \{H,T\})$
$A = \{HT\}$
$B = \{H,T\}$

Outcome	$\{H,T\}$	$\{H,T\}$	$\{H,T\}$
Probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$P(A) = P(\{H,T\}) + P(\{H,T\}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$
$$P(B) = P(\{H,T\}) + P(\{H,T\}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$
$$P(A \cup B) = P(\{H,T\}) + P(\{H,T\}) + P(\{H,T\}) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$
$$P(A \cap B) = P(\{H,T\}) = \frac{1}{3}$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{3} + \frac{2}{3} - \frac{1}{3} = \frac{3}{3} = 1$$
$$P(A \cap B) = P(A) + P(B) - P(A \cap B) = \frac{2}{3} + \frac{2}{3} - \frac{1}{3} = \frac{3}{3} = 1$$

$\boxed{\therefore\ HHT = \frac{1}{3}\ (0.33\overline{3}\%)}$

E.2

												12 Picture		
		A	2	3	4	5	6	7	8	9	10	J	Q	K
52	26 Red	◇ 13												
		♡ 13												
	26 Black	♣ 13												
		♠ 13												

$S = \{52\}$
$A = 4$
$B = 26$

	A	B
Outcome	$\{Ace\}$	$\{Red\}$
Probability	$\frac{4}{52}$	$\frac{26}{52}$

$$P(A) = \frac{4}{52}$$
$$P(B) = \frac{26}{52}$$
$$P(A \cup B) = P(\{Ace\}) + P(\{Red\}) = \frac{4}{52} + \frac{26}{52} = \frac{30}{52}$$
$$P(A \cap B) = P(\{Ace\}) + P(\{Red\}) = \frac{4}{52} + \frac{26}{52} = \frac{30}{52}$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{26}{52} - \frac{30}{52}\ \frac{0}{52} = 0$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{4}{52} + \frac{26}{52} - \frac{0}{52} = \frac{30}{52}$$

$\therefore$ A and B are independent.

E.3:

i) $f(x) = \sin(6x-1)$
$f'(x) = \sin(5x)$
$f'(x) = 0.08x$

ii) $f(x) = x^8 + 30 + \frac{1}{x^4}$
$f'(x) = 8x^7 + 30 + \frac{1}{4x^3}$
$f'(x)\ 8x^7 + 30 + 4x^3$
$f'(x) = 42x^{10}$

iii) $f(x) = e^{(\frac{1}{x}) + (\frac{1}{x^2})}$
$f'(x) = e^{(\frac{1}{x} + \frac{1}{2x})}$
$f'(x) = e^{(\frac{2}{2x^2})}$
$f'(x) = e^{1x^2}$

iv) $f(x) = \sin^2(6x-1)$
$f(x) = \sin^2(5x)$
$f(x) = (0.007x)$

E4: $f(x) = 2x^3 + 24x^2 - 54x$

a) $f'(x) = 6x^2 + 48x - 54x = 0$
$= 6x(x + 48 - 54) = 0$
$6x = 0 \qquad x + 48 - 54 = 0$
$x = 0, \quad x = 6$

interval	x_0	$f'(x_0)$
$(-\infty, 0)$	-1	Positive
$(0, 6)$	1	negative
$6, \infty$	7	positive

$x = -1 \qquad -6(-1+48-54) = 42$
$x = 1 \qquad 6(1+48-54) = -30$
$x = 7 \qquad 42(7+48-54) = 42$

$-\infty \quad \text{I} \quad 0 \quad \text{II} \quad 6 \quad \text{III} \quad \infty$

$\therefore$ $f(x)$ is decreasing at $(0,6)$ and increasing at $(-\infty, 0) + (6, \infty)$

E.4 (cont'd):

b) $f''(x) = 12x + 48 - 54 \Rightarrow 12x - 6 \Rightarrow \frac{12}{12}x = \frac{6}{12} \Rightarrow x = \frac{1}{2}$

interval	x_0	$f''(x_0)$
$(-\infty, \frac{1}{2})$	-1	negative
$(\frac{1}{2}, \infty)$	1	positive

	−		+	
$-\infty$		$\frac{1}{2}$		∞ f''

$f''(x) = 12x - 6$

$f''(-1) = 12(-1) - 6$
$= -12 - 6$

$f''(1) = 12(1) - 6$
$= 12 - 6$
$= 6 > 0$

∴ $f''(x)$ concaves up at $(-\infty, \frac{1}{2})$ and concaves down at $(\frac{1}{2}, \infty)$

E.5:

CP $x = 0$; $x = 6$

$f(0.5) = 2(0.5)^3 + 24(0.5)^2 - 54(0.5)$
$= 2(0.125) + 24(0.25) - 54(0.5)$
$= 0.25 + 6 - 27$
$= -20.75$

∴ Points of Inflection $= (0.5, -20.75)$

-3	0	3

interval	x^0	$f'(x_0)$
$(-3, 0)$	-1	positive
$(0, 3)$	1	

$-\infty$	$\frac{1}{2}\infty$	0

Interval	X_0	f'
$(-\infty, \frac{1}{2}\infty)$		negative
$(\frac{1}{2}\infty, 0)$		negative

$f'(-1) = 2(-1)^3 + 24(-1)^2 - 54(-1)$
$= 2(-1) + 24(1) - 54(-1)$
$= -2 + 24 - (-54)$
$= 76$

Interval spans from $-\infty$ to zero, therefore any value would be negative

E.6:

i. $f(x,y) = x^2 + y^2$

$$\nabla f(x,y) = \begin{matrix} \dfrac{\partial f}{\partial x} = 2x + \cancel{y^2} \\ \dfrac{\partial f}{\partial y} = \cancel{x^2} + 2y \end{matrix} \qquad \nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

ii. $f(x,y) = x^2 + y^2$ at $(1, 2)$

$$\nabla f = \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$f(x,y) = x^2 + y^2$ at $(2, 1)$

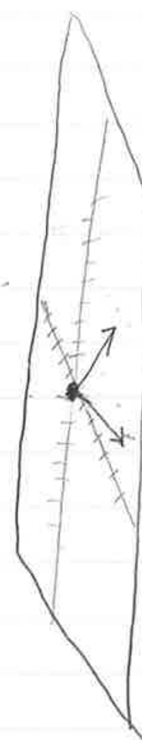
$$\nabla f = \begin{bmatrix} 2 \cdot 2 \\ 2 \cdot 1 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$f(x,y) = x^2 + y^2$ at $0, 0$

$$\nabla f = \begin{bmatrix} 2 \cdot 0 \\ 2 \cdot 0 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



E.7:

i. $f(x,y) = 2xy + x^2 + y$

$$\nabla f(x,y) = \frac{\partial f}{\partial x} = 2x^2 + y$$

$$\frac{\partial f}{\partial y} = 2x$$

$$\nabla f = \begin{bmatrix} 2x^2 + y \\ 2x \end{bmatrix}$$

ii: $f(x,y) = 2xy + x^2 + y$ at $(1, 1)$

$$\nabla f = \begin{bmatrix} 2(1)^2 + (2\cdot 1) \\ 2\cdot 1 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$f(x,y) = 2xy + x^2 + y$ at $(0, -1)$

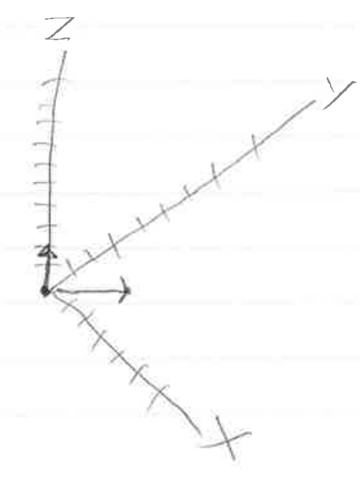
$$\nabla f = \begin{bmatrix} (2(0)^2 + (2\cdot 0)) \\ 2\cdot -1 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$f(x,y) = 2xy + x^2 + y$ at $(0, 0)$

$$\nabla f = \begin{bmatrix} 2(0)^2 + (2\cdot 0) \\ 2\cdot 0 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



E.7 (Cont'd)

iii. $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2 + x_1x_2^2$

$\nabla f(x,y) = \frac{\partial f}{\partial x_1} \quad 2x' + 2x'(x2)$

$\nabla f(x,y) \ \frac{\partial f}{\partial x_2} \quad 2x' + x'(2x')$

$\nabla f = \begin{bmatrix} 16x^2 \\ 4x^2 \end{bmatrix}$

E.8:

i. slope = 3, y-intercept (0, 0.5)

$y = mx + b$

$y = 3x + b$

$-0.5 = 3(0) + b$

$-0.5 = 0 + b$

$-0 \quad -0$

$5 = b$

$\therefore \quad y = 3x + -0.5$

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ii. $\overset{x_1}{(}4, \overset{y_1}{8}), \quad (\overset{x_2}{6}, \overset{y_2}{14})$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{(14) - (8)}{(6) - (4)}$

$m = \dfrac{6}{2} = \dfrac{3}{1}$

$y - y_1 = m(x - x_1)$

$y - (8) = \left(\frac{3}{1}\right)(x - (4))$

$y - 8 = \frac{3}{1}x - \frac{3}{1} \cdot \frac{4}{1}$

$y - 8 = \frac{3}{1}x - \frac{12}{1}$

$-\frac{12}{1} \qquad -\frac{12}{1}$

$y = \frac{3}{1}x - (-4)$

E.8 (cont'd):

iii. Passes through (3,2), perpendicular to $y = 5x + 3$

$y = \frac{5}{1}x + 3$

$m = -\frac{1}{5}$ (3, 2)

$y = mx + b$

$y = -\frac{1}{5}x + b$

$2 = -\frac{1}{5}\left(\frac{3}{1}\right) + b$

$2 = -\frac{3}{5} + b$

$-\frac{3}{5} \quad -\frac{3}{5}$

$0.6 = b$

$y = -\frac{1}{5}x + 0.6$

iv. $b = 3$, passes through $(\overset{x}{2}, \overset{y}{1})$

$y = mx + b$

$y = mx + 3$

$1 = m(2) + 3$

v. $(\overset{x}{6}, \overset{y}{4}), (\overset{x_2}{1}, \overset{y_2}{-1})$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{(-1) - 4}{1 - 6}$

$m = \frac{-5}{-5}$

$m = 1$

$y - y_1 = m(x - x_1)$

$y - (4) = (1)(x - (6))$

$y - 4 = 1x - 6$

$-4 \quad -4$

$\therefore y = 1x - 10$

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E. 9:

i.

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \quad \lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$i - \lambda I = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 5-\lambda \end{bmatrix}$$

determinate $\Rightarrow \begin{bmatrix} 2-\lambda & 0 \\ 0 & 5\lambda \end{bmatrix}$

$= (2-\lambda)(5-\lambda) - (0)(0)$

$\lambda^2 - 7\lambda + 10 = 0 \Rightarrow (\lambda - 5)(\lambda \; 2 = 0$

$\Rightarrow$ Eigenvalues

$\boxed{\lambda_1 = 5}$ $\boxed{\lambda_2 = 2}$

$(i - \lambda I)v = 0$ $\quad \lambda = 5$

$$\begin{bmatrix} 2-5 & 0 \\ 0 & 5-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \boxed{\text{Eigenvector}}$$

$$\left.\begin{matrix} -3x + 0y = 0 \\ 0x + 0y = 0 \end{matrix}\right\} \begin{matrix} -3x = 0 \Rightarrow x = 0 \\ 0 \end{matrix}$$

E9 (cont'd):

ii. $\begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix}$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$ii - \lambda I = \begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix}$$

determinate $\begin{bmatrix} 5-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix}$

$$= (5-\lambda)(5-\lambda) - (4)(1)$$

$$= 5 - 5\lambda + \lambda + \lambda^2 - 4$$

$$= \lambda^2 - 4\lambda - 1$$

$$\lambda^2 - 4\lambda - 1 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0$$

Eigenvalues

$\boxed{\lambda_1 = 6}$ $\boxed{\lambda_2 = -1}$

$(i - \lambda I)v = 0 \quad \lambda = 6$

$$\begin{bmatrix} 5-6 & 1 \\ 4 & 5-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ Eigenvectors}$$

$$\left.\begin{aligned} -1x + 1y &= 0 \\ 4x + -1y &= 0 \end{aligned}\right\} \begin{aligned} -1x &= 0 \Rightarrow x = 0 \\ 3x &= 0 \Rightarrow x = 0 \end{aligned}$$

E.9 (cont'd)

iii

$$\begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix} \quad \lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$iii - \lambda I = \begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 5 \\ 3 & 1-\lambda \end{bmatrix}$$

$$\text{determinate} \Rightarrow \begin{bmatrix} 3-\lambda & 5 \\ 3 & 1-\lambda \end{bmatrix}$$

$$= (3-\lambda)(1-\lambda) - (3)(5)$$

$$= 3 - 3\lambda + \lambda + \lambda - 15$$

$$= \lambda^2 - 2\lambda - 18$$

$$= \lambda^2 - 2\lambda - 18 = 0$$

$$= (\lambda - 4)(\lambda + 3)$$

= Eigenvalues

$$\boxed{\lambda_1 = 4} \quad \boxed{\lambda = -3}$$

$$(1 - \lambda I)v = 0 \quad \lambda = 4$$

$$\begin{bmatrix} 3-4 & 5 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 5 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ Eigen vectors}$$

$$\left.\begin{array}{l} -1x + 5y = 0 \\ 3x + -3y = 0 \end{array}\right\} \begin{array}{l} 4x = 0 \Rightarrow x = 0 \\ 1x = 0 \Rightarrow x = 0 \end{array}$$