§4 矩阵的最大秩分解

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定理 1 设 $\boldsymbol{A} \in \boldsymbol{C}_r^{m\times n}$，则存在矩阵 $\boldsymbol{B} \in \boldsymbol{C}_r^{m\times r}$，$\boldsymbol{D} \in \boldsymbol{C}_r^{r\times n}$，使得

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{D}.$$

证明： 因为 $\boldsymbol{A} \in \boldsymbol{C}_r^{m\times n}$，存在酉矩阵 $\boldsymbol{U} \in \boldsymbol{U}^{m\times m}, \boldsymbol{V} \in \boldsymbol{U}^{n\times n}$ 和 $\boldsymbol{r}$ 阶正线下三角矩阵 $\boldsymbol{L}$，使得

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{U}\begin{bmatrix}\boldsymbol{L} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{V}\\ &= \boldsymbol{U}\begin{bmatrix}\boldsymbol{L}\\ \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}_r & \boldsymbol{0}\end{bmatrix}\boldsymbol{V}\\ &= \boldsymbol{B}\boldsymbol{D},\end{aligned}$$

其中 $\boldsymbol{B} = \boldsymbol{U}\begin{bmatrix}\boldsymbol{L}\\ \boldsymbol{0}\end{bmatrix} \in \boldsymbol{C}_r^{m\times r}$，$\boldsymbol{D} = \begin{bmatrix}\boldsymbol{E}_r & \boldsymbol{0}\end{bmatrix}\boldsymbol{V} \in \boldsymbol{C}_r^{r\times n}$，得证.

矩阵最大秩分解的步骤：

(1) 进行行初等变换，化为简化行阶梯型：

$$\tilde{\boldsymbol{A}} = \begin{bmatrix}\boldsymbol{0\cdots 0} & \boldsymbol{1} & \star\cdots\star & \boldsymbol{0} & \star\cdots\star & \boldsymbol{0} & \star\cdots\star\\ \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0} & \boldsymbol{1} & \star\cdots\star & \boldsymbol{0} & \star\cdots\star\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0} & \boldsymbol{1} & \star\cdots\star\\ \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0} & \boldsymbol{0} & \boldsymbol{0\cdots 0}\end{bmatrix},$$

其中 $\star$ 表示不一定为 0 的元素，$\tilde{\boldsymbol{A}}$ 中非零首元所在的列分别为 $\boldsymbol{i_1, i_2, \cdots, i_r}$.

(2) 由 $\boldsymbol{A}$ 中第 $\boldsymbol{i_1, i_2, \cdots, i_r}$ 列构成 $\boldsymbol{B} = (\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \cdots, \boldsymbol{a}_{i_r})$.

(3) 由 $\tilde{A}$ 的非零行构成 D.

思考上述算法的证明？还有哪些计算最大秩分解算法？

例 1 求矩阵 $A=\begin{bmatrix}1&3&2&1&4\\2&6&1&0&7\\3&9&3&1&11\end{bmatrix}$ 的最大秩分解.

解： (行初等变换) 对矩阵 A 进行行初等变换化为简化行阶梯型，有

$$A\to\begin{bmatrix}1&3&0&-\frac{1}{3}&\frac{10}{3}\\0&0&1&\frac{2}{3}&\frac{1}{3}\\0&0&0&0&0\end{bmatrix}=\tilde{A},$$

则

$$B=\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix},\quad D=\begin{bmatrix}1&3&0&-\frac{1}{3}&\frac{10}{3}\\0&0&1&\frac{2}{3}&\frac{1}{3}\end{bmatrix}.$$

(列初等变换) 对矩阵 A 进行列初等变换化为简化列阶梯型，有

$$A\to\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\1&1&0&0&0\end{bmatrix}=\tilde{A},$$

则

$$B=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix},\quad D=\begin{bmatrix}1&3&2&1&4\\2&6&1&0&7\end{bmatrix}.$$

定理 2 设 $A\in C_r^{m\times n}$，且 $A=B_1D_1=B_2D_2$ 均是 A 的最大秩分解，则

(1) 存在 r 阶可逆矩阵 Q，使得

$$B_1=B_2Q,\quad D_1=Q^{-1}D_2.$$

(2) $D_1^H(D_1D_1^H)^{-1}(B_1^HB_1)^{-1}B_1^H=D_2^H(D_2D_2^H)^{-1}(B_2^HB_2)^{-1}B_2^H$.

注：

(1) $\mathrm{rank}(A)=\mathrm{rank}(A^HA)=\mathrm{rank}(AA^H)$.

(2) 若 $B\in C_r^{m\times r}$，则 $B^H\in C_r^{r\times m}$，$B^HB\in C_r^{r\times r}$，进而有

$$(B^HB)^{-1}(B^HB)=(B^HB)^{-1}B^HB=E_r.$$

(3) $\boldsymbol{D} \in \boldsymbol{C}_r^{r\times n}$，则 $\boldsymbol{D}^H \in \boldsymbol{C}_r^{n\times r}$，$\boldsymbol{D}\boldsymbol{D}^H \in \boldsymbol{C}_r^{r\times r}$，进而有

$$(\boldsymbol{D}\boldsymbol{D}^H)(\boldsymbol{D}\boldsymbol{D}^H)^{-1} = \boldsymbol{D}\boldsymbol{D}^H(\boldsymbol{D}\boldsymbol{D}^H)^{-1} = \boldsymbol{E}_r.$$

证明： (1) 因为 $\boldsymbol{B}_1\boldsymbol{D}_1 = \boldsymbol{B}_2\boldsymbol{D}_2$，则 $\boldsymbol{B}_1\boldsymbol{D}_1\boldsymbol{D}_1^H = \boldsymbol{B}_2\boldsymbol{D}_2\boldsymbol{D}_1^H$，进而得

$$\boldsymbol{B}_1 = \boldsymbol{B}_2\boldsymbol{D}_2\boldsymbol{D}_1^H\left(\boldsymbol{D}_1\boldsymbol{D}_1^H\right)^{-1} = \boldsymbol{B}_2\boldsymbol{Q}_1,$$

其中 $\boldsymbol{Q}_1 = \boldsymbol{D}_2\boldsymbol{D}_1^H\left(\boldsymbol{D}_1\boldsymbol{D}_1^H\right)^{-1}$.

同理可得 $\boldsymbol{D}_1 = (\boldsymbol{B}_1^H\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1^H\boldsymbol{B}_2\boldsymbol{D}_2 = \boldsymbol{Q}_2\boldsymbol{D}_2$，其中 $\boldsymbol{Q}_2 = (\boldsymbol{B}_1^H\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1^H\boldsymbol{B}_2$.

于是有

$$\boldsymbol{B}_1\boldsymbol{D}_1 = \boldsymbol{B}_2\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{D}_2 = \boldsymbol{B}_2\boldsymbol{D}_2,$$

即

$$\boldsymbol{B}_2^H\boldsymbol{B}_2\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{D}_2\boldsymbol{D}_2^H = \boldsymbol{B}_2^H\boldsymbol{B}_2\boldsymbol{D}_2\boldsymbol{D}_2^H,$$

因此，$\boldsymbol{Q}_1\boldsymbol{Q}_2 = \boldsymbol{E}_r$，记 $\boldsymbol{Q} = \boldsymbol{Q}_1$，则 $\boldsymbol{Q}_2 = \boldsymbol{Q}^{-1}$.

(2) 利用 (1)，有

$$\begin{aligned}
\boldsymbol{D}_1^H(\boldsymbol{D}_1\boldsymbol{D}_1^H)^{-1}(\boldsymbol{B}_1^H\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1^H &= (\boldsymbol{Q}^{-1}\boldsymbol{D}_2)^H\left[\boldsymbol{Q}^{-1}\boldsymbol{D}_2(\boldsymbol{Q}^{-1}\boldsymbol{D}_2)^H\right]^{-1}\left[(\boldsymbol{B}_2\boldsymbol{Q})^H\boldsymbol{B}_2\boldsymbol{Q}\right]^{-1}(\boldsymbol{B}_2\boldsymbol{Q})^H \\
&= \boldsymbol{D}_2^H(\boldsymbol{Q}^{-1})^H\left[\boldsymbol{Q}^{-1}\boldsymbol{D}_2\boldsymbol{D}_2^H(\boldsymbol{Q}^{-1})^H\right]^{-1}\left[\boldsymbol{Q}^H\boldsymbol{B}_2^H\boldsymbol{B}_2\boldsymbol{Q}\right]^{-1}\boldsymbol{Q}^H\boldsymbol{B}_2^H \\
&= \boldsymbol{D}_2^H(\boldsymbol{Q}^{-1})^H\boldsymbol{Q}^H(\boldsymbol{D}_2\boldsymbol{D}_2^H)^{-1}\boldsymbol{Q}\boldsymbol{Q}^{-1}(\boldsymbol{B}_2^H\boldsymbol{B}_2)^{-1}(\boldsymbol{Q}^H)^{-1}\boldsymbol{Q}^H\boldsymbol{B}_2^H \\
&= \boldsymbol{D}_2^H(\boldsymbol{D}_2\boldsymbol{D}_2^H)^{-1}(\boldsymbol{B}_2^H\boldsymbol{B}_2)^{-1}\boldsymbol{B}_2^H,
\end{aligned}$$

得证.