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System Identification Via Linear Least Squares

A Report On The Characterisation Of
A Simple Cart System
Using Matlab

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Introduction

The aim of this report is to detail the derivation and testing of a model of a simple cart system via a linear least squares based system identification. The cart system to be identified is a simple system involving a cart driven by a voltage V.

To simplify the system the overall force on the cart is modeled as a proportional constant $\left(\bar{\beta}\right)$ relating the voltage to the force provided by the DC motor and gears along with a simple second order damping term $\left(\bar{C}\right)$; along with the normal Newton's second law of motion for the acceleration. This results in a pair of equations describing the system:

$$F\left(t\right) = \bar{\beta}V - \bar{C}\dot{y}\,, \qquad M_C\,\ddot{y} = F\left(t\right) \tag{1}$$

By rearranging and simplifying these equations we can get a single second order equation describing the system:

$$\ddot{y} = \beta V - C\dot{y}\,, \qquad \beta = \frac{\bar{\beta}}{M_C}\,, \qquad C = \frac{\bar{C}}{M_C} \tag{2}$$

This can also be Laplace transformed into a transfer function:

$$G\left(s\right) = \frac{Y\left(s\right)}{R\left(s\right)} = \frac{\beta K_p}{s^2 + Cs + \beta K_p} \tag{3}$$

The controller used on the system was a proportional controller with an input signal of:

$$r\left(t\right) = 0.06\sin\left(\omega t\right) \tag{4}$$

This gives an overall system of:

$$\ddot{y} + C\dot{y} + \beta K_p y = \beta K_p r\left(t\right)\,, \qquad K_p = 60 \tag{5}$$

A frequency response experiment for this system was run for a range of 12 frequencies, 0.6–2.8 Hz with a step of 0.2.

By re-writing the system with $K = \beta K_p$ and $\bar{K} = 0.06\beta K_p$ the analytical solution was found to be:

$$y_{\text{model}} = A_1\cos\left(\omega t\right) + A_2\sin\left(\omega t\right) \tag{6}$$

where,

$$A_1 = -\frac{\bar{K}C\omega}{C^2\omega^2 + \omega^4 - 2\omega^2 K + K^2} \tag{7}$$

and,

$$A_2 = \frac{\bar{K}\left(\omega^2 - K\right)}{C^2\omega^2 + \omega^4 - 2\omega^2 K + K^2} \tag{8}$$

The system identification method used was linear least squares. This is simply minimizing the function:

$$\varepsilon_{\text{error}} = \sum_{i=1}^{n}\left(y_{\text{model}}\left(t_i\right) - y_i\right)^2 = \sum_{i=1}^{n}\left(A_1\cos\left(\omega t\right) + A_2\sin\left(\omega t\right) - y_i\right)^2 \tag{9}$$

This is at a minimum when $\frac{\delta\varepsilon_{\text{error}}}{\delta A_1} = 0$ and $\frac{\delta\varepsilon_{\text{error}}}{\delta A_2} = 0$. Solving this gives:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \left(M^T M\right)^{-1} M^T b \tag{10}$$

where:

$$M = \begin{pmatrix} \cos(\omega t_1) & \sin(\omega t_1) \\ \vdots & \vdots \\ \cos(\omega t_n) & \sin(\omega t_n) \end{pmatrix}, \qquad b = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \tag{11}$$

This allows the A_1 and A_2 values to be calculated, there still needs to be a way to convert these into the C and β values. By re-arranging Equations (7) and (8) we get:

$$\beta = \frac{\omega^2 \left(A_1^2 + A_2^2\right)}{K_p \left(A_1^2 + A_2^2\right) - 0.06 K_p A_2}, \qquad C = -\frac{00.06\beta K_p A_1}{\omega \left(A_1^2 + A_2^2\right)} \tag{12}$$

Section One and Two

Wherein a Function To Fake the Model Data is Written and Tested

Before developing a function to identify the system a test function was developed to later use on verifying the identification function. The first step in the development of this test function was creating a Matlab function that takes in the C and β parameters that define the system along with a frequency to evaluate it at (ω) and a time to evaluate it for (t_{end}). The function was simply Equation (6) evaulated at the specified C, β and ω with t ranging from 0–t_{end} at a frequency of 1 kHz.

As a verification of this function an additional function utilising Matlab's in-built `lsim` function was produced. This function was based around the transfer function from Equation (3) along with the input from Equation (4) to calculate the output.

Both models created were then plotted together with input values $C = 5$, $\beta = 1$, $K_p = 60$, $f = 1.8$ and $t_{\text{end}} = 3$. This can be seen in Fig. 1. Looking at this it can be seen that the steady state response of the two systems is tending to be the same. The major difference is in the initial response, this is because the model used in the first equation is purely a steady state model whereas the transfer function based model does base its output off the initial conditions which are assumed to be zero.

Section Three (a)

Wherein A Function To Identify the System is Written

After writing this test function the actual identification function could be developed. This involved simply creating a function that matched Equation (10), the function takes in the y vector, t vector and ω frequency and produces the corresponding A_1 and A_2. From these two values the b and C values were able to be derived as Equation (12) showed.

Section Three (b)

Wherein The Identification Function is Tested

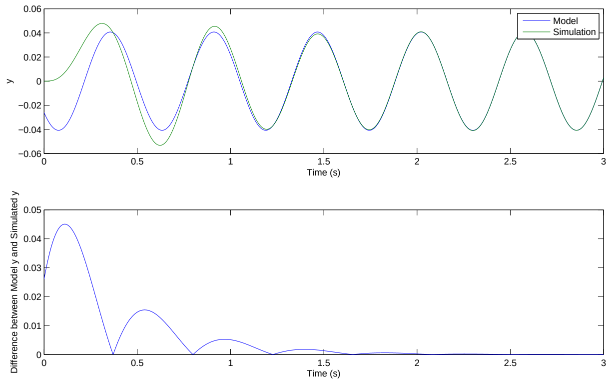

Figure 1: Model vs Matlabs simulated values

Noise Level	Median (C)	90% CI (C)	Median (b)	90% CI (b)
0.00	5.00000	0.00000	1.00000	0.00000
0.05	4.99905	0.00164	0.99985	0.00018
0.10	5.00216	0.00388	0.99993	0.00030
0.15	4.99518	0.00512	0.99969	0.00062
0.20	5.00194	0.00802	0.99885	0.00075
0.25	4.99897	0.00869	0.99937	0.00090
0.30	4.99082	0.01174	1.00069	0.00115
0.35	4.99939	0.01295	0.99886	0.00125
0.40	5.00423	0.01503	0.99959	0.00162

Table 1: Testing the Identification Function

To check the identification function a series of “noisy” sample data streams were created and run through the function. These were based off the test function developed earlier with each y datum multiplied by a normally distributed value. The value used had a mean of 1 and a standard deviation varied between 0 and 0.4 in steps of 0.05. Each of these standard deviations was simulated 100 times and the median and confidence intervals of the series was calculated.

Table 1 and Figure 2 show the median and confidence interval calculated using the test function with $C = 5$, $\beta = 1$, $K_p = 60$, $f = 1.8$, $t_{\text{end}} = 1$ and a varying standard deviation. From these it was seen that the identification of the C and β values are quite accurate, the 90% confidence interval is less than 0.3% of the found value and the difference from the correct value is less than 0.2% in the worst case.

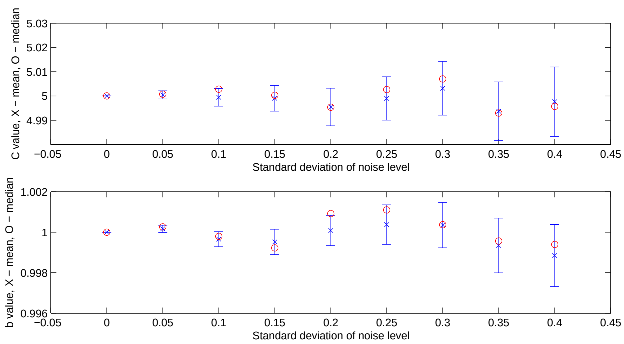

Figure 2: Testing the Identification Function

Frequency	Relative Median error	90% CI
0.6	0.26936	0.26175
0.8	0.30116	0.25065
1.0	0.32473	0.24677
1.2	0.31176	0.23621
1.4	0.24266	0.20618
1.6	0.11872	0.12534
1.8	0.02376	0.03305
2.0	0.11852	0.13654
2.2	0.18552	0.21159
2.4	0.22813	0.21008
2.6	0.28859	0.32453
2.8	0.28994	0.28366

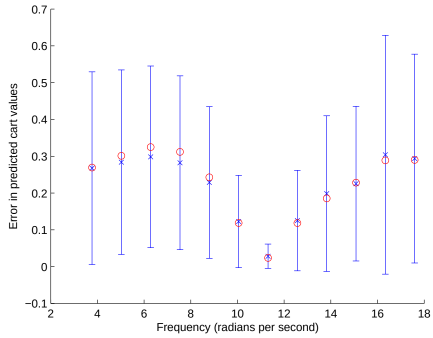

Figure 3: Error in the derived model

Section Four (a)

Wherein the Cart System is Identified

Now that the function to identify the system had been developed, tested and found to be accurate the actual characterisation of the cart could begin. The characterisation simply involved taking in the provided datasets and running them through the identification function. This resulted in a C and b value for each frequency at which the system was tested. As instructed the values derived from the seventh data set with a frequency of 1.8 Hz was used as the general model. The results of this identification were a C value of 5.83523 and a β vale of 1.93596.

Section Four (b)

Wherein the Identification is Ridiculed

To check the results of the identification the model was used to generate data at all 12 frequencies tested and this generated data was compared to the measurements. The comparison was performed by finding the absolute difference between the generated data and the measured data for each frequency, scaling this difference by the mean of the absolute measured data for that frequency and finally finding the median and 90% confidence interval of the relative error.

Figure and Table 3 shows these medians and the 90% confidence interval found. Obviously the model is most accurate at the frequency it was derived from, the relative error is down to 0–6% showing the very good accuracy. The worst case is at $f = 2.6$, $\omega = 16.3$ with a 90% CI of the error in the range 0–63%, this is obviously an unacceptable level of error so the model is really only valid for the one frequency it was derived for, and maybe the ones either side depending on the use case.

The relative errors from each frequency were then concatenated and the median and 90% CI of the overall error was calculated. This came out to a median error of 0.21840 and a 90% CI of 0–49%. Again a most likely case of 22% error and a worst case of around 49% error is likely unacceptable for the majority of use cases of this model.

Figure 4 shows a histogram of the error values. Looking at this the bell shape of a Gaussian distribution can be easily seen.

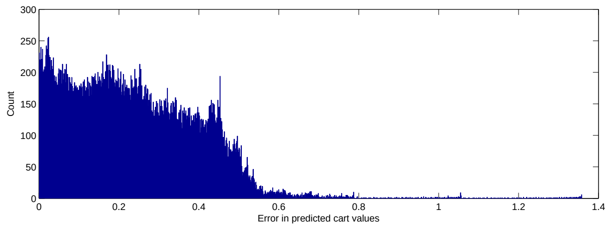

Figure 4: Histogram of the error values

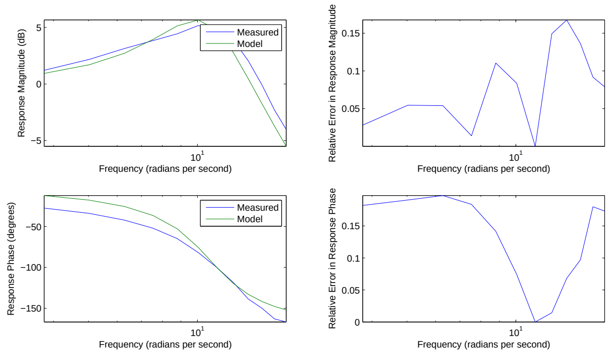

Figure 5: Bode plot and error plot of the model vs measured

Section Five (a)

Wherein the Model and Measured Are Compared

Another test of the model was to compare it to the measured Bode plot. Getting the model values was easily accomplished by taking the transfer function from Equation (3) and substituting the known C and β values into it. Getting the measured values was a little more difficult, in the end the method chosen was the same as that used to getting the model values; at each frequency the identification function was used to get the average C and β values from that data set, then the transfer function derived from the value was evaluated at that frequency.

Figure 5 shows the magnitude and phase plots found, along with the absolute relative error. As can be seen these plots are quite accurate, there is less than 17% error in both the magnitude and phase.

Section Five (b)

Wherein All the Models Are Compared

To ensure that the model chosen was the best of the available models the rest of the data sets were also used to derive a model, then the Bode plots of these were all compared to the measured data. The results of the comparison can be seen in Figure 6. This shows that all the models have very good magnitude responses at low frequencies, but as they approach the bend they diverge quite a bit from the measured values before settling into a path pretty much parallel to the correct values. They are all quite a bit better in the phase response, but none of them really manage to distinguish themselves by being amazingly better.

To help parse the info contained in the comparison an additional plot show in Figure

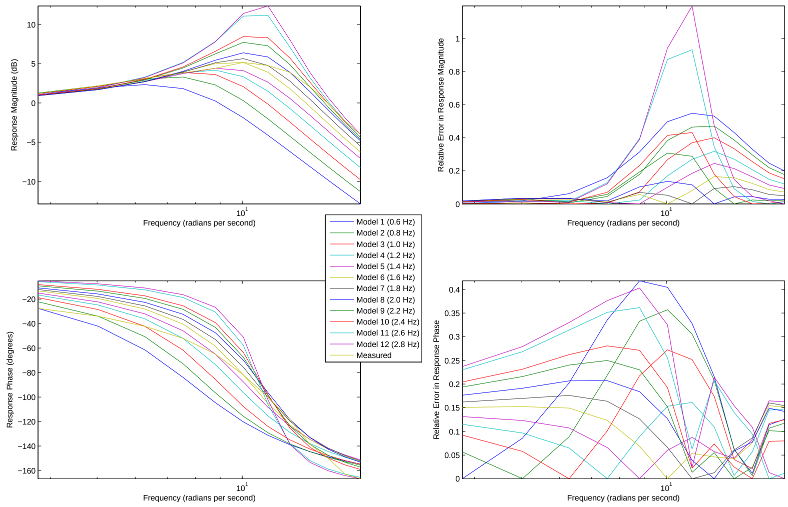

Figure 6: Bode plot and error plot of all models

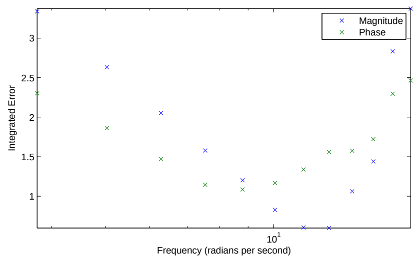

Figure 7: Integrated errors of each model

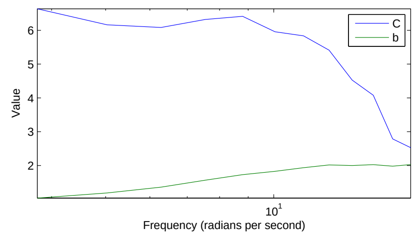

Figure 8: C and β values of each model

7 was created. This shows the sum of the errors from the last figure at each frequency for both magnitude and phase. From this it can be seen that there are two models with much better magnitude response than the rest, these are models 7 and 8 at frequencies of 1.8 and 2.0 Hz. Out of these 7 has a better phase response, so we can now see why it was chosen as the model to use from the start.

Figure 8 also shows the C and β values found for each model.