DM – Introduction to graphs

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Outline

Outline

Some examples

Figure: Regional routes of VNA (as of 2017).[1]

[1] Source: https://www.viags.vn/tin-tuc/tin-chuyen-nganh/vietnam-airlines-routemap-wordwide

Some examples

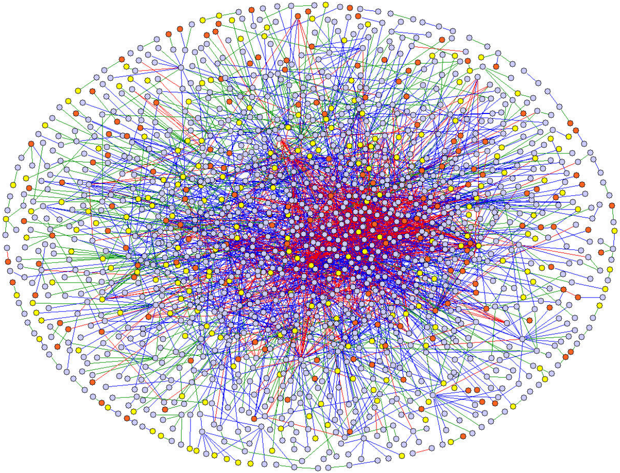
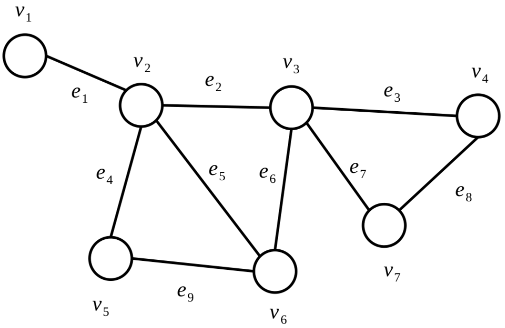


Figure: Map of human protein interaction.[2]

[2] Source: https://www.mdc-berlin.de/news/archive/2008/20080910-erwin_schr_dinger_prize_2008_goes_to_resea

Graph



- Each vertex is represented by a dot or a circle
- Each edge is represented by a line connecting its endpoints

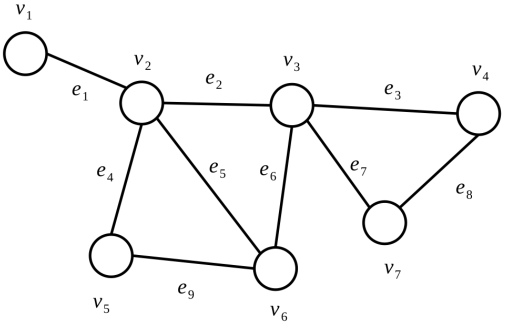
Definition

A (undirected) graph $G = (V, E)$ is defined by:

1. *A* non-empty *set V of vertices;*
2. *A set E of edges, which are unordered pairs of vertices.*

Note: In this course, we only consider *finite* graphs.

Graphs



- $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
- $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$

Multigraph and loop

Definition

- *A graph is simple if there is at most one edge between any pair of vertices.*
- *Edges connecting the same pair of vertices are called multiple edges. A graph having multiple edges is also called a multigraph.*
- *A loop is an edge connecting a vertex to itself.*

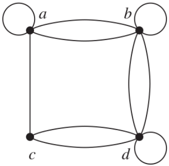


Figure: Rosen, p. 650

Note: By convention, "graphs" refer to simple graphs without loops unless otherwise stated.

Examples

Social networks:

- Each vertex represents a person
- Two friends are connected by an edge

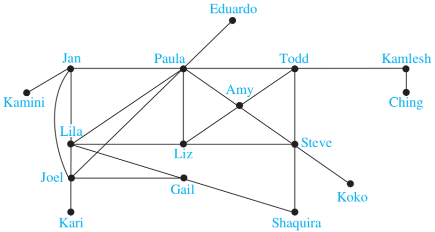


Figure: Rosen, p. 645

Examples

Niche overlap graphs (ecology):

- Each vertex represents a species
- Two competing species are connected by an edge

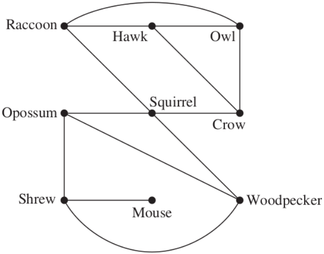


Figure: Rosen, p. 648

Examples

Protein interaction graphs (biology):

- Each vertex represents a protein
- Two proteins that interact are connected by an edge

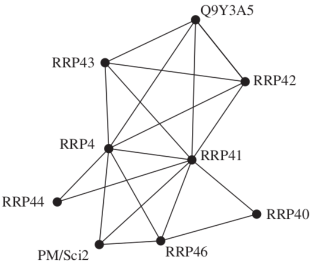


Figure: Rosen, p. 648

Examples

Computer networks:

- Each vertex represents a server
- Each edge represents a link

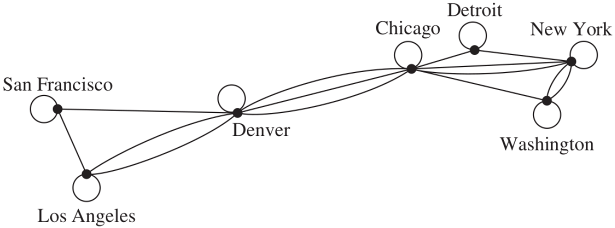


Figure: Rosen, p. 642

Outline

An example

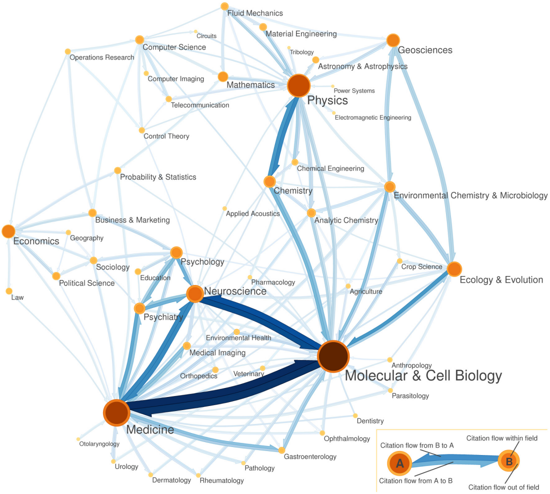


Figure: A map of science disciplines.[3]

[3] Source: http://images.math.cnrs.fr/Representer-les-mondes.html

Directed graph

Definition

A directed graph *(or* digraph*)* $G = (V, A)$ *is defined by:*

1. *A* non-empty *set* V *of* vertices *(or* nodes*);*
2. *A set* A *of* arcs*, which are* ordered *pairs of vertices.*

The notions of *simple directed graphs*, *multiple arcs*, *loops* and *directed multigraphs* are defined similarly to the undirected case.

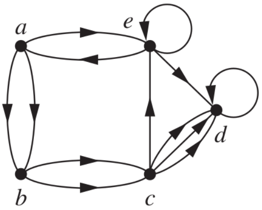


Figure: Rosen, p. 650

Examples

Round-robin tournament:

- Each node represents a team
- Each team play against every other team, an arc goes from the winning team to the losing team (assume that there are no ties)

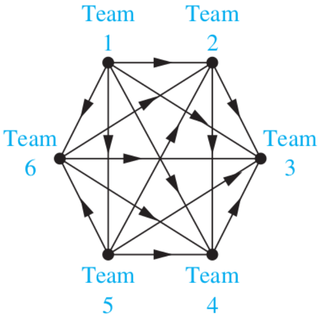


Figure: Rosen, p. 649

Examples

Dependency (software):

- Each node represents a module
- An arc (a, b) is added if module b depends on module a

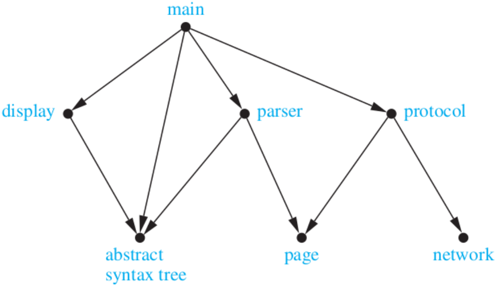


Figure: Rosen, p. 647

Examples

Precedence graph:

- Each node represents a statement
- An arc (S_i, S_j) is added if statement S_i must wait for statement S_j to be executed

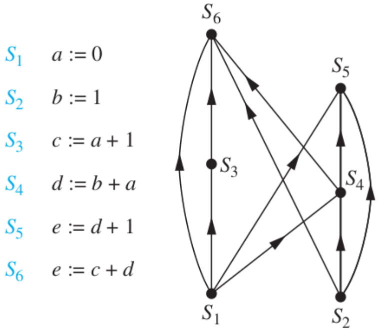


Figure: Rosen, p. 647

Examples

. . . and many other examples:

- citation graphs (academic)
- co-author graphs (academic)
- genealogical graphs
- road networks
- the Web
- etc.

Outline

Adjacency, neighborhood

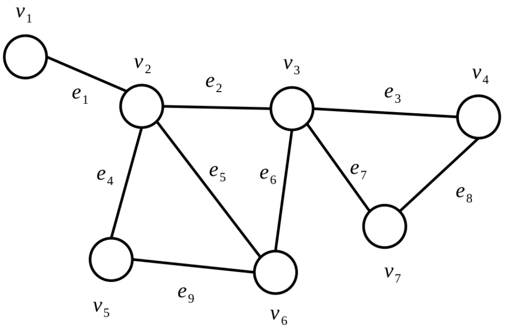
Let $G = (V, E)$ be a graph.

Definition

- *Two vertices u and v are* adjacent *if they are endpoints of the same edge e. In this case, u is a* neighbor *of v and the edge e is* incident *with the vertices u and v.*
- *The set of all neighbors of a vertex v is the* neighborhood *of v and is denoted by $N(v)$. The neighborhood of a set A of vertices is the union of the neighborhoods of the vertices in A: $N(A) = \bigcup_{v \in A} N(v)$.*

Adjacency, neighborhood

Example:

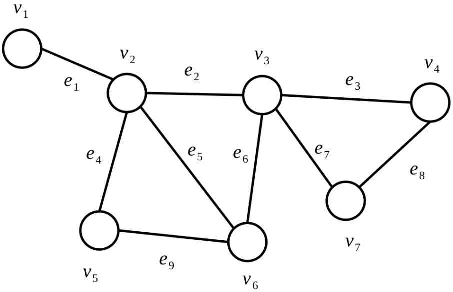


- v_1 is adjacent to v_2;
- v_4 is not adjacent to v_2;
- $N(v_2) = \{v_1, v_3, v_5, v_6\}$;
- $N(\{v_1, v_2\}) = \{v_1, v_2, v_3, v_5, v_6\}$;
- $N(\{v_2, v_3\}) = V$.

Degree

Definition

The degree *of a vertex v, denoted by* $\deg(v)$ *or simply* $d(v)$*, is the number of edges incident with it. A loop contributes 2 to the degree of its endpoint.*



$$\deg(v_1) = 1\,, \deg(v_2) = 4\,.$$

Degree

Note:

- A vertex is *isolated* if it has degree 0; it is *pendant* if it has degree 1.
- In a multigraph, the degree of a vertex may be greater than the number of its neighbors!

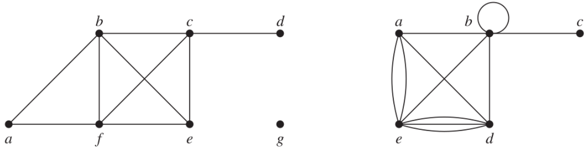


Figure: Rosen, p. 652

The handshaking theorem

Theorem

Let $G = (V, E)$ be an undirected graph. Then

$$\sum_{v \in V} \deg(v) = 2\,|E|\,.$$

Proof: Each edge contributes 2 to the sum of degrees of all the vertices.

Corollary

There are an even number of vertices of odd degrees.

Outline

Adjacency

Let $G = (V, A)$ be a digraph.

Definition

If $a = (u, v)$ is an arc, then u is adjacent to *v and v is* adjacent from *u; u is the* initial node *and v is the* end node *of a; a is an* out-going arc *of u and an* in-going arc *of v.*

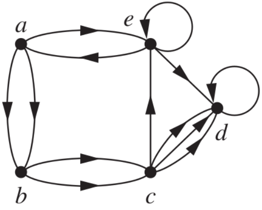


Figure: Rosen, p. 650

- a is adjacent to b, b is adjacent from a;
- b is *not* adjacent to a, a is *not* adjacent from b.
- e is adjacent to and from itself.

Degrees

Definition

The out-degree *(resp.* in-degree*) of a node v, denoted by* $\deg^+(v)$ *or* $d^+(v)$ *(resp.* $\deg^-(v)$ *or* $d^-(v)$*), is the number of out-going (resp. in-going) arcs of v.*

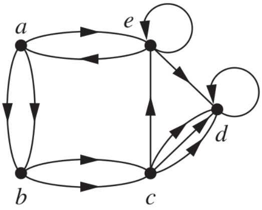


Figure: Rosen, p. 650

- $d^+(a) = 3$, $d^-(a) = 1$;
- $d^+(d) = 1$, $d^-(d) = 5$.

Theorem

Let $G = (V, A)$ be a directed graph. Then

$$\sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) = |A|.$$

Complete graphs

Definition

A complete graph *of n vertices, denoted by K_n, is a simple graph having n vertices and such that every pair of vertices is connected by an edge.*

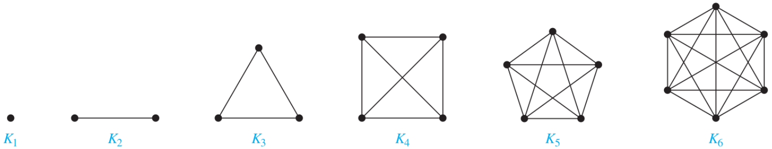


Figure: Rosen, p. 655

Paths

Definition

A path *on n vertices, denoted by P_n, is a simple graph having n vertices $\{v_1, v_2, \ldots, v_n\}$ and $n-1$ edges $\{\{v_i, v_{i+1}\} | 1 \leq i \leq n-1\}$.*

Figure: P_6

Cycles

Definition

A cycle *on $n \geq 3$ vertices, denoted by C_n, is a simple graph having n vertices $\{v_1, v_2, \ldots, v_n\}$ and n edges $\{\{v_i, v_{i+1}\} \,|1 \leq i \leq n\}$ ($v_{n+1} \equiv v_1$).*

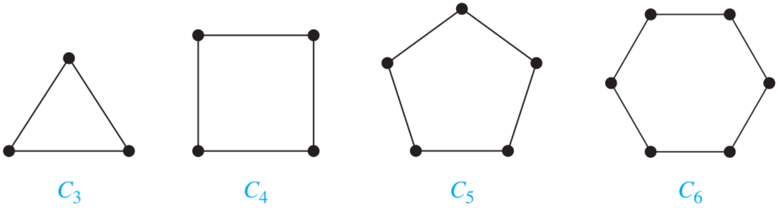


Figure: Rosen, p. 655

Wheel graphs

Definition

A wheel graph *of $n+1$ vertices ($n \geq 3$), denoted by W_n, is a simple graph obtained from C_n by adding a new vertex and connecting it to every vertex of C_n.*

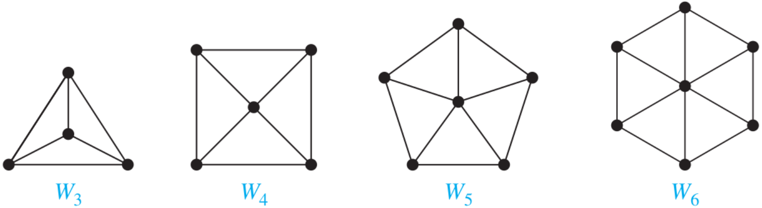


Figure: Rosen, p. 655

Hypercubes

Definition

A n-dimensional hypercube*, or an* n-cube*, denoted by Q_n, is a simple graph in which*

- *the vertices are the binary strings of length n*
- *two strings are adjacent iff. they differs at only one position.*

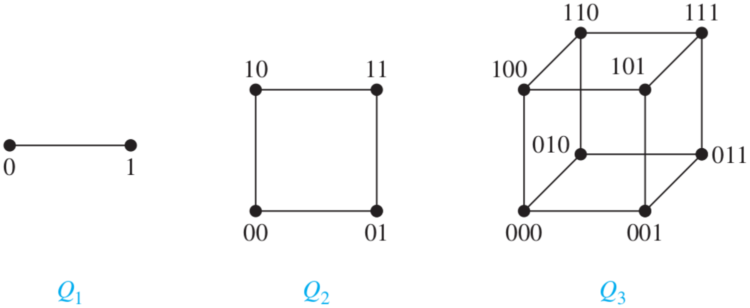


Figure: Rosen, p. 655

Hypercubes

Construction of Q_{n+1} from Q_n:

1. Consider two copies Q_n: Q_n and Q'_n
2. Change the labels of Q_n and Q'_n:
   - Q_n: $s \to 0s$
   - Q'_n: $s \to 1s$
3. Add the edges $\{0s, 1s\}$

Bipartite graphs

Definition

A graph $G = (V, E)$ is a bipartite graph *if V can be partitioned into two sets V_1 and V_2 so that every edge connects a vertex in V_1 and a vertex in V_2.*
The pair (V_1, V_2) is a bipartition *of the vertices of G.*

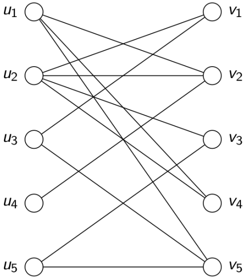


Figure: A bipartite graph.

Bipartite graphs

Examples:

- The path P_5 is a bipartite graph.
- The cycle C_6 is a bipartite graph.
- The cycle C_3 is *not* a bipartite graph.
- The complete graph K_4 is *not* a bipartite graph.
- The hypercubes Q_1, Q_2, Q_3 are bipartite graphs.

Recognizing bipartite graphs

Is this graph bipartite?

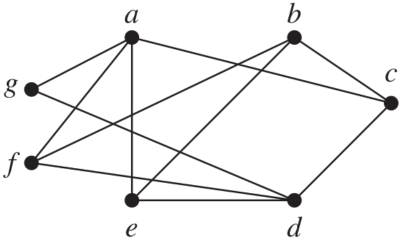


Figure: Rosen, p. 656

It is.

Recognizing bipartite graphs

Theorem

A simple graph is bipartite iff. its vertices can be colored by two different colors so that no two adjacent vertices have the same color.

Examples:

- C_n is bipartite iff. n is even.
- P_n is bipartite for all n.
- Q_n is bipartite for all n.

Complete bipartite graphs

Definition

A complete bipartite graph $K_{m,n}$ *is a bipartite graph with bipartition* $V = V_1 \cup V_2$ *such that:*

- V_1 *has* m *vertices,* V_2 *has* n *vertices;*
- *Two vertices are connected by an edge iff. one vertex is in* V_1*, the other is in* V_2.

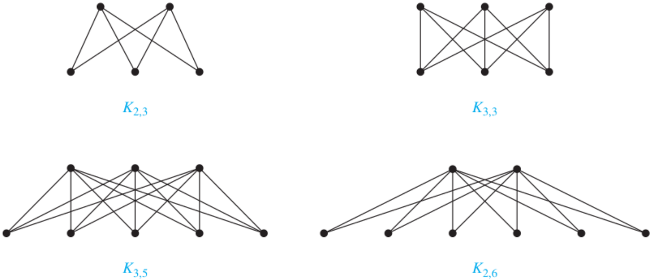


Figure: Rosen, p. 658

Subgraphs

Definition

A subgraph *of a graph* $G = (V, E)$ *is a* graph $H = (W, F)$ *where* $W \subset V, F \subset E$.

Examples:

- Every graph is a subgraph of itself.
- The path P_n is a subgraph of the cycle C_m if $n \leq m$.
- The cycle C_n is a subgraph of the wheel W_n.
- Every graph on n vertices is a subgraph of K_n.

Subgraphs

Definition

Let $G = (V, E)$ be a simple graph and let W be a subset of vertices. The subgraph induced *by W is the subgraph of G where:*

- *The set of vertices is W;*
- *The set of edges consists of all edges having both endpoints in W.*

Examples:

- C_n is the subgraph of W_n induced by the "outer" vertices.
- If $m \leq n$, K_m is the subgraph of K_n induced by a subset of m vertices.
- P_n is *not* an induced subgraph of C_n.
- Q_{n-1} is the subgraph of Q_n induced by all vertices whose labels starts with 0.

Some operations on graphs

Let $G = (V, E)$ be a simple graph.

Edge removal:

- $G - e = (V, E \setminus \{e\})$.
- Result: a subgraph of G.

Vertex removal:

- $G - v = (V \setminus \{v\}, E \setminus \{e \mid e \text{ is incidence with } v\})$.
- Result: a subgraph of G.

Edge contraction:

- Remove an edge, then merge its endpoints.
- Result: *not* necessarily a subgraph!

Outline

Adjacency list

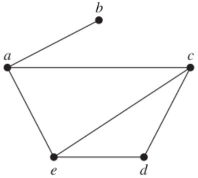


TABLE 1 **An Adjacency List for a Simple Graph.**

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Figure: Rosen, p. 668

Adjacency list

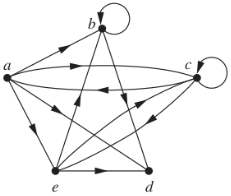


TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Figure: Rosen, p. 669

Outline

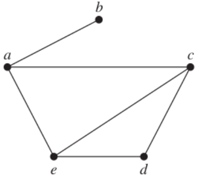
Adjacency matrix

Definition

The adjacency matrix *of a graph $G = (V, E)$ is a $V \times V$ matrix of integer entries $A = (a_{u,v})$ such that:*

$$a_{u,v} = \begin{cases} 1 \text{ if } u, v \text{ are adjacent,} \\ 0 \text{ otherwise.} \end{cases}$$

Example: (with vertex ordering a, b, c, d, e)



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Outline

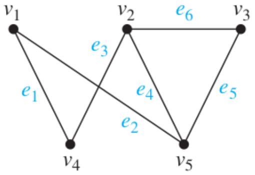
Incidence matrix

Definition

The incidence matrix *of a graph $G = (V, E)$ is a $V \times E$ matrix of integer entries $M = (m_{v,e})$ such that:*

$$m_{v,e} = \begin{cases} 1 \text{ if } e \text{ is incident with } v, \\ 0 \text{ otherwise.} \end{cases}$$

Example:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Note: The notions of adjacency matrix and incidence matrix can be extended for multigraphs, for directed graphs, etc.