Convolute and Deconvolute

convolute

1-D convolute

Equal to MATLAB function: `conv([],[],'full')`

Need `x` and `t` as column vector.

1. conv_define(x, t)

$$y[s] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[s-k]$$

2. conv_mask_slide(x, t)

$$y = x_1 \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(m+n-1,1)} + x_2 \cdot \begin{pmatrix} 0 \\ h_1 \\ h_2 \\ \vdots \\ h_{n-1} \\ h_n \\ \vdots \\ 0 \end{pmatrix}_{(m+n-1,1)} + \cdots + x_m \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ h_{n-2} \\ h_{n-1} \\ h_n \end{pmatrix}_{(m+n-1,1)}$$

3. conv_matrix_dot(x, t)

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+n-1} \end{pmatrix}_{(m+n-1,1)} = A_n \cdot x = \begin{pmatrix} h_1 & 0 & \cdots & 0 & 0 \\ h_2 & h_1 & \cdots & 0 & 0 \\ h_3 & h_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_n & h_{n-1} & \cdots & \cdots & \cdots \\ 0 & h_n & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h_n & h_{n-1} \\ 0 & 0 & \cdots & 0 & h_n \end{pmatrix}_{(m+n-1,m)} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}_{(m,1)}$$

2-D convolute

Equal to MATLAB function: `conv2([],[],'full')`

Functions `conv2_matrix_dot` has a huge memory footprint.

1. conv2_define(x, h)

$$y[s_1, s_2] = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} x[k1, k2] \cdot h[s_1 - k_1, s_2 - k_2]$$

2. conv2_mask_slide(x, h)

$$y=\sum_{i=1}^{m}\sum_{j=1}^{n}x_{i,j}\cdot\begin{pmatrix}O_{i-1,j-1} & O_{i-1,q} & O_{i-1,n-j}\\ O_{p,j-1} & h_{p,q} & O_{p,n-j}\\ O_{m-i,j-1} & O_{m-i,q} & O_{m-i,n-j}\end{pmatrix}$$

3. conv2_double_conv(x, h)

$$x=\begin{pmatrix}x_{(1,1)} & x_{(1,2)} & x_{(1,3)} & \cdots & x_{(1,n)}\\ x_{(2,1)} & x_{(2,2)} & x_{(2,3)} & \cdots & x_{(2,n)}\\ x_{(3,1)} & x_{(3,2)} & x_{(3,3)} & \cdots & x_{(3,n)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ x_{(m,1)} & x_{(m,2)} & x_{(m,3)} & \cdots & x_{(m,n)}\end{pmatrix}_{m,n}=\begin{pmatrix}X_1\\ X_2\\ X_3\\ \vdots\\ X_m\end{pmatrix}$$

$$h=\begin{pmatrix}h_{(1,1)} & h_{(1,2)} & h_{(1,3)} & \cdots & h_{(1,q)}\\ h_{(2,1)} & h_{(2,2)} & h_{(2,3)} & \cdots & h_{(2,q)}\\ h_{(3,1)} & h_{(3,2)} & h_{(3,3)} & \cdots & h_{(3,q)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ h_{(p,1)} & h_{(p,2)} & h_{(p,3)} & \cdots & h_{(p,q)}\end{pmatrix}_{p,q}=\begin{pmatrix}H_1\\ H_2\\ H_3\\ \vdots\\ H_p\end{pmatrix}$$

$$\begin{cases}X_i=\begin{bmatrix}x_{(i,1)} & x_{(i,2)} & x_{(i,3)} & \cdots & x_{(i,n)}\end{bmatrix}\\ H_i=\begin{bmatrix}h_{(i,1)} & h_{(i,2)} & h_{(i,3)} & \cdots & h_{(i,q)}\end{bmatrix}\end{cases}$$

$$y=\begin{pmatrix}Y_1\\ \vdots\\ Y_k\\ \vdots\\ Y_{m+p-1}\end{pmatrix}=\begin{pmatrix}X_1\\ X_2\\ X_3\\ \vdots\\ X_m\end{pmatrix}*\begin{pmatrix}H_1\\ H_2\\ H_3\\ \vdots\\ H_p\end{pmatrix}$$

$$Y_k=\sum_{i=-\infty}^{+\infty}(X_i*H_{k-i+1})=\sum_{i=\max(k-n+1,1)}^{\min(m,k)}(X_i*H_{k-i+1})$$

4. conv2_matrix_dot(x, h)

$$x=\begin{pmatrix}x_{(1,1)} & x_{(1,2)} & x_{(1,3)} & \cdots & x_{(1,n)}\\ x_{(2,1)} & x_{(2,2)} & x_{(2,3)} & \cdots & x_{(2,n)}\\ x_{(3,1)} & x_{(3,2)} & x_{(3,3)} & \cdots & x_{(3,n)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ x_{(m,1)} & x_{(m,2)} & x_{(m,3)} & \cdots & x_{(m,n)}\end{pmatrix}_{m,n}=\begin{pmatrix}X_1\\ X_2\\ X_3\\ \vdots\\ X_m\end{pmatrix}$$

$$h=\begin{pmatrix}h_{(1,1)} & h_{(1,2)} & h_{(1,3)} & \cdots & h_{(1,q)}\\ h_{(2,1)} & h_{(2,2)} & h_{(2,3)} & \cdots & h_{(2,q)}\\ h_{(3,1)} & h_{(3,2)} & h_{(3,3)} & \cdots & h_{(3,q)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ h_{(p,1)} & h_{(p,2)} & h_{(p,3)} & \cdots & h_{(p,q)}\end{pmatrix}_{p,q}=\begin{pmatrix}H_1\\ H_2\\ H_3\\ \vdots\\ H_p\end{pmatrix}$$

$$y=\begin{pmatrix}y_{(1,1)} & y_{(1,2)} & y_{(1,3)} & \cdots & y_{(1,n+q-1)}\\ y_{(2,1)} & y_{(2,2)} & y_{(2,3)} & \cdots & y_{(2,n+q-1)}\\ y_{(3,1)} & y_{(3,2)} & y_{(3,3)} & \cdots & y_{(3,n+q-1)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ y_{(p,1)} & y_{(p,2)} & y_{(p,3)} & \cdots & y_{(m+p-1,n+q-1)}\end{pmatrix}_{m+p-1,n+q-1}=\begin{pmatrix}Y_1\\ Y_2\\ Y_3\\ \vdots\\ Y_{m+p-1}\end{pmatrix}$$

$$\begin{cases}X_i=\begin{bmatrix}x_{(i,1)} & x_{(i,2)} & x_{(i,3)} & \cdots & x_{(i,n)}\end{bmatrix}\\ H_i=\begin{bmatrix}h_{(i,1)} & h_{(i,2)} & h_{(i,3)} & \cdots & h_{(i,q)}\end{bmatrix}\\ Y_i=\begin{bmatrix}y_{(1,1)} & y_{(1,2)} & y_{(1,3)} & \cdots & y_{(1,n+q-1)}\end{bmatrix}\end{cases}$$

$$
Ah = \begin{pmatrix}
E_1 & O & \cdots & O & O \\
E_2 & E_1 & \cdots & O & O \\
E_3 & E_2 & \cdots & O & O \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
E_p & E_{p-1} & \cdots & \cdots & \cdots \\
O & E_p & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
O & O & \cdots & E_p & E_{p-1} \\
O & O & \cdots & O & E_p
\end{pmatrix}_{[(m+n-1)\times(n+q-1),m\times n]}
$$

$$
E_k = \begin{pmatrix}
h_{(k,1)} & 0 & \cdots & 0 & 0 \\
h_{(k,2)} & h_{(k,1)} & \cdots & 0 & 0 \\
h_{(k,3)} & h_{(k,2)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h_{(k,n)} & h_{(k,n-1)} & \cdots & \cdots & \cdots \\
0 & h_n & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & h_{(k,n)} & h_{(k,n-1)} \\
0 & 0 & \cdots & 0 & h_{(k,n)}
\end{pmatrix}_{n+q-1,n}
$$

$$
\begin{aligned}
x\_\overrightarrow{vec} &= \begin{bmatrix} X_1 & X_2 & \cdots & X_m \end{bmatrix}^T \\
y\_\overrightarrow{vec} &= \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{m+p-1} \end{bmatrix}^T \\
&= Ah \cdot x\_\overrightarrow{vec}
\end{aligned}
$$