Finite Element Solver for the 2D Wave Equation

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This report presents the implementation of a Finite Element Method (FEM) solver for the 2D wave equation using C++. The solver utilizes the Newmark method for time-stepping and solves the wave equation on a rectangular domain. This report discusses the implementation details, the choice of discretization methods, properties of the method, as well as computational and algorithmic aspects. The simulation results are analyzed and error convergence is tested. The challenges in numerical dissipation and dispersion, and the trade-offs between computational complexity and accuracy are also discussed.

Contents

1 Introduction

The wave equation is a second-order partial differential equation that describes the propagation of waves, such as sound waves, light waves, and seismic waves, through a medium. In this report, we address the 2D wave equation with specified initial conditions and boundary conditions using the Finite Element Method (FEM) and time-stepping schemes. This report will cover the theoretical background of the wave equation, discretization strategies, and numerical methods used, as well as present implementation details in C++.

2 Theoretical Background

2.1 The Wave Equation

The general form of the wave equation in two spatial dimensions is:

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(x,t),$$

where $u(x,t)$ represents the displacement at position $x = (x_1, x_2)$ and time t, and ∇^2 is the Laplacian operator. The term $f(x,t)$ represents the external forcing term.

2.2 Boundary and Initial Conditions

The wave equation is solved subject to boundary conditions on the domain Ω and initial conditions for the displacement and velocity:

$$u(t=0) = u_0(x), \quad \frac{\partial u}{\partial t}(t=0) = u_1(x), \quad \text{on}\, \partial\Omega.$$

In this study, we use Dirichlet boundary conditions $u = g(x,t)$, which specify the value of the solution on the boundary of the domain, on $\partial\Omega$, where $\partial\Omega$ is the boundary of the domain. This condition implies that the displacement $u(x,t)$ at the boundary of the domain is fixed, meaning that the value of the wave is specified at all times during the simulation.

2.3 Finite Element Method (FEM)

The Finite Element Method is a numerical technique for solving partial differential equations by discretizing the spatial domain Ω into a finite number of elements. The weak form of the wave equation is derived by multiplying the equation by test functions and integrating by parts.

For the 2D wave equation, we use the following weak formulation:

$$\int_\Omega \frac{\partial^2 u}{\partial t^2} v \, d\Omega + \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega,$$

where v is the test function. The Galerkin method is a popular variational method for solving partial differential equations. In this approach, the solution is approximated by a linear combination of basis (or trial) functions, and the residual error is projected onto the subspace spanned by the test functions.

Consider a boundary value problem:

$$\mathcal{L}(u) = f \quad \text{in} \quad \Omega,$$

with boundary conditions:

$$u = g \quad \text{on} \quad \partial\Omega.$$

The Galerkin method approximates the solution $u(x)$ by a linear combination of trial functions:

$$u_h = \sum_{i=1}^{N} u_i \phi_i(x),$$

where $\phi_i(x)$ are the trial (basis) functions, and u_i are the unknown coefficients.

We then multiply the differential equation by a test function v_h and integrate over the domain Ω:

$$\int_\Omega v_h \mathcal{L}(u_h) \, d\Omega = \int_\Omega v_h f \, d\Omega.$$

For the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{in} \quad \Omega,$$

with boundary conditions $u = g$ on $\partial\Omega$, we multiply the equation by a test function $v(x,t)$ and integrate:

$$\int_\Omega v \frac{\partial^2 u}{\partial t^2} \, d\Omega - \int_\Omega \nabla v \cdot \nabla u \, d\Omega = \int_\Omega v f \, d\Omega.$$

Next, we approximate the solution $u(x,t)$ by a linear combination of basis functions:

$$u_h(x,t) = \sum_{i=1}^{N} u_i(t) \phi_i(x),$$

where $\phi_i(x)$ are the trial functions, and $u_i(t)$ are the unknown time-dependent coefficients.

Finally, applying the Galerkin method leads to the system of equations:

$$\sum_{i=1}^{N} M_{ij} \frac{d^2 u_i}{dt^2} + K_{ij} u_j = F_i(t),$$

where:

- $M_{ij} = \int_\Omega \phi_i \phi_j \, d\Omega$ is the mass matrix,
- $K_{ij} = \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \, d\Omega$ is the stiffness matrix,
- $F_i(t) = \int_\Omega \phi_i f \, d\Omega$ is the force term.

The Galerkin approach is employed to approximate the solution of the wave function, using basis functions defined over the elements.

2.4 Time Stepping Methods

Time-stepping methods are essential techniques used to solve time-dependent partial differential equations (PDEs), such as the wave equation, by discretizing time into small intervals. The goal is to approximate the solution at each time step, evolving the solution forward in time based on initial and boundary conditions. For the wave equation, the solution evolves over time, and the behavior of the solution at each time step depends on both the previous and current states. Time-stepping methods allow us to iteratively compute the solution at each time step.

2.5 Time Discretization: Newmark Method

For time-stepping, we employ the Newmark method, which is widely used for dynamic systems. The Newmark method provides a numerical solution by discretizing the time domain. In this case, we update the solution at each time step based on the previous and current values:

$$u^{k+1} = u^k + \Delta t \cdot \dot{u}^k + \frac{\Delta t^2}{2} \ddot{u}^k,$$

where u^k is the displacement at time step k, $\dot{u}^k$ is the velocity, and $\ddot{u}^k$ is the acceleration. The method ensures stability for larger time steps and is suitable for wave propagation problems.
The Newmark method is particularly effective for problems involving structural dynamics and mechanical systems, where the system exhibits oscillatory behavior

or wave propagation, such as in seismic waves or vibrations of structures. By using a combination of displacement, velocity, and acceleration, it captures the dynamics of the system over time. Moreover, it is implicit in nature, allowing for unconditionally stable solutions that are beneficial for stiff problems or when large time steps are required.

2.6 Time Discretization: Crank-Nicolson Method

Another commonly used time-stepping scheme is the Crank-Nicolson method. The Crank-Nicolson method is an implicit scheme that is typically applied to solve partial differential equations, including the wave equation. In the Crank-Nicolson scheme, the update from one time step to the next is given by:

$$u^{k+1} = u^k + \frac{\Delta t}{2}\left(\dot{u}^k + \dot{u}^{k+1}\right),$$

where u^k represents the displacement at time step k, and $\dot{u}^k$ and $\dot{u}^{k+1}$ are the velocities at time steps k and $k+1$, respectively. The method averages the velocity between the current and the next time step.
The Crank-Nicolson method is popular in solving diffusion and wave equations because it offers second-order accuracy in both time and space. Unlike the Newmark method, which explicitly includes acceleration terms, the Crank-Nicolson method focuses on averaging the velocity and provides a better balance between stability and accuracy.

2.7 Comparison of Newmark and Crank-Nicolson Methods

Both the Newmark and Crank-Nicolson methods are widely used for solving time-dependent problems, but they differ in their approaches and applications.

- **Stability:** Both methods are stable for a wide range of time steps, but the Newmark method is more flexible. The Crank-Nicolson method is implicit and unconditionally stable, making it well-suited for problems involving stiff systems. In contrast, the Newmark method may require smaller time steps to maintain stability for certain types of problems.

- **Accuracy:** The Crank-Nicolson method provides second-order accuracy in both space and time, which makes it ideal for diffusion problems. The Newmark method also offers second-order accuracy but typically requires a finer discretization to achieve the same level of accuracy as Crank-Nicolson for wave propagation problems.

- **Applicability:** The Newmark method is commonly used for structural dynamics and wave propagation problems in engineering applications, where the system's behavior can involve both oscillations and large displacements. The Crank-Nicolson method, on the other hand, is more commonly used for heat conduction, diffusion problems, and solving parabolic partial differential equations.

- **Computational Efficiency:** The Newmark method is often more computationally efficient for wave propagation problems, as it only requires the solution of simpler linear equations compared to the Crank-Nicolson method. The Crank-Nicolson method involves solving a system of equations at each time step, making it computationally more expensive, but it offers a higher level of accuracy for certain types of problems.

In summary, while both methods are used to solve time-dependent problems, the choice of method depends on the problem's specific characteristics, such as stability, accuracy, and computational cost. The Newmark method is often favored for large-scale wave propagation problems due to its simplicity and stability for large time steps, whereas the Crank-Nicolson method is preferred for more accurate solutions in diffusion-type problems, especially when high accuracy is required over time and space.

3 Numerical Method

3.1 Spatial Discretization

The spatial domain is discretized into triangular or quadrilateral elements, depending on the geometry of the problem. The local element matrices for stiffness and mass are assembled using shape functions and their derivatives.

The global matrices are constructed by assembling the local element matrices and then solving the global system of equations at each time step. For a simple mesh with n elements, the system of equations is:

$$M\ddot{u} + Ku = F,$$

where M is the mass matrix, K is the stiffness matrix, F is the external force vector, and u is the displacement vector.

3.2 Time-Stepping Scheme

The Newmark method is implemented following these key steps:

- Initialization: begin by setting the initial conditions for displacement and velocity at time $t = 0$. These initial values are denoted as u^0 (initial displacement) and $\dot{u}^0$ (initial velocity);

- Time integration: for each subsequent time step, the displacement and velocity are updated using the Newmark method. The displacement at the next time step is computed based on the previous displacement, velocity, and acceleration using the following equation:

$$u^{k+1} = u^k + \Delta t \cdot \dot{u}^k + \frac{\Delta t^2}{2}\ddot{u}^k,$$

  where u^{k+1} is the updated displacement, $\dot{u}^k$ is the velocity, and $\ddot{u}^k$ is the acceleration at time step k. The velocity is updated similarly, using the displacement and acceleration information from the previous time step.

- Post-processing: after each time step, the displacement and velocity are stored for analysis and visualization. These results can be visualized to track the evolution of the system over time and further analyzed to assess the dynamics of the system.

The solution of the system of equations at each time step involves solving a set of linear equations, which can be efficiently handled using sparse direct solvers or iterative solvers, particularly for large-scale systems. Sparse solvers are used to exploit the sparse structure of the system matrices, ensuring that computational resources are utilized efficiently, especially when dealing with problems involving a large number of degrees of freedom. In addition to the Newmark method, the Crank-Nicolson method can also be used for time-stepping in wave propagation problems. This method is based on a central difference approach, where the solution is implicitly updated by averaging the values from the current and next time steps. The Crank-Nicolson method is commonly used for parabolic and wave problems due to its stability and accuracy.

- Initialization: similar to the Newmark method, the initial conditions for displacement and velocity are specified at $t = 0$ as u^0 (initial displacement) and $\dot{u}^0$ (initial velocity).

- Time integration: for each time step, the displacement and velocity are updated using the Crank-Nicolson method. This method combines the current and next time step values, resulting in a time discretization that is second-order accurate and unconditionally stable. The displacement update equation for Crank-Nicolson is given by:

$$u^{k+1} = u^k + \frac{\Delta t}{2}\left(\dot{u}^k + \dot{u}^{k+1}\right),$$

where u^{k+1} is the updated displacement at time t^{k+1}, and $\dot{u}^k$ and $\dot{u}^{k+1}$ are the velocities at time steps k and $k+1$, respectively.

The velocity update equation is similarly derived and implicitly coupled with the displacement. The main advantage of Crank-Nicolson is that it involves averaging, which reduces numerical dissipation and provides better stability for larger time steps.

- Post-processing: as with the Newmark method, after each time step, the results (displacement and velocity) are stored for visualization and analysis. The results can be used to examine the dynamics of the system and track the evolution of the wave propagation over time.

The Crank-Nicolson method generally requires the solution of a system of linear equations for each time step. This can be efficiently handled using direct or iterative solvers, similar to the Newmark method. The implicit nature of the Crank-Nicolson scheme means that the system of equations must be solved at each time step, but the method is more stable for larger time steps than explicit schemes.