Problem 2:

V-Rectangular box: $f(x,y,z) = xyz8$

sphere: $g(x,y,z) = x^2+y^2+z^2-1=0$

assume:
$x,y,z>0$

① $z=\sqrt{1-x^2-y^2}$ or $-\sqrt{1-x^2-y^2}$

$$f(x,y,\sqrt{1-x^2-y^2}) = 8xy\sqrt{1-x^2-y^2}$$

$$\nabla f(x,y,\sqrt{1-x^2-y^2}) = 8y\sqrt{1-x^2-y^2} + \frac{8xy(-2x)}{2\sqrt{1-x^2-y^2}}$$

$$8y\sqrt{1-x^2-y^2} - \frac{8x^2y}{\sqrt{1-x^2-y^2}}$$

$$\frac{8y(1-x^2-y^2)-8x^2y}{\sqrt{1-x^2-y^2}}$$

$$= \left\langle \frac{8y(1-2x^2-y^2)}{\sqrt{1-x^2-y^2}}, \frac{8x(1-2y^2-x^2)}{\sqrt{1-x^2-y^2}} \right\rangle$$

$8y(1-y^2-2x^2)=0$ $\qquad$ $8x(1-x^2-2y^2)=0$

$1-y^2-2x^2=0$ $\qquad$ $1-x^2-2y^2=0$

$-y^2-2x^2=-1$ $\qquad$ $-x^2-2y^2=-1$

$$-2x^2+x^2-y^2+2y^2=0$$

$$-x^2+y^2=0$$

$$-x^2=-y^2$$

$$x^2=y^2$$

$$y=\pm x$$

$$y=x$$

solve for $x$ and $y$

$1-y^2-2y^2=0$ $\qquad$ $1-x^2-2x^2=0$

$1-3y^2=0$ $\qquad$ $1-3x^2=0$

$-3y^2=-1$ $\qquad$ $x=\sqrt{\frac{1}{3}}$

$y=\sqrt{\frac{1}{3}}$ , $x=\sqrt{\frac{1}{3}}$

Plug into $f(x,y,z)=8\sqrt{\frac{1}{3}}\sqrt{\frac{1}{3}}\sqrt{1-\frac{1}{3}-\frac{1}{3}}$

max volume $=\frac{8}{3}\sqrt{\frac{1}{3}}$

②

$$\nabla f(x,y,z)=\lambda\,\Delta g(x,y,z)$$

$$\langle yz, xz, xy\rangle=\lambda\langle 2x, 2y, 2z\rangle$$

① $8yz=\lambda 2x$

② $8xz=\lambda 2y$

③ $8xy=\lambda 2z$

④ $x^2+y^2+z^2=1 \longrightarrow 3x^2=1$

① $z = \dfrac{\lambda 2x}{8y}$

② $z = \dfrac{\lambda 2y}{8x}$

$\Rightarrow \dfrac{\lambda 2x}{8y} = \dfrac{\lambda 2y}{8x}$

$\cancel{8\lambda}2x^2 = \cancel{8\lambda}2y^2$

$x^2 = y^2$

$y = \pm x$

① $y = \dfrac{\lambda 2x}{8z}$

③ $y = \dfrac{\lambda 2z}{8x}$

$\Rightarrow \dfrac{\lambda 2x}{8z} = \dfrac{\lambda 2z}{8x}$

$\cancel{8\lambda}2x^2 = \cancel{8\lambda}2z^2$

$x^2 = z^2$

$z = \pm x$

$x^2 = z^2 = y^2$

$x^2 = \frac{1}{3}$

$x = \sqrt{\frac{1}{3}}$ or $x = -\sqrt{\frac{1}{3}}$

so, points are...

$\left(\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}\right)$ $\left(-\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}\right)$

$\left(\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}\right)$ $\left(-\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}\right)$

$\left(\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}\right)$ $\left(-\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}\right)$

$\left(\sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}\right)$ $\left(-\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}, -\sqrt{\tfrac{1}{3}}\right)$

$f(x,y,z) = xyz8$

max volume when there are no or an even number of negatives

which $= \frac{8}{3}\sqrt{\frac{1}{3}}$