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Distributing many points on spheres: Minimal energy and designs

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Abstract

This survey discusses recent developments in the context of spherical designs and minimal energy point configurations on spheres. We consider the sphere $\mathbb{S}^d$ of unit vectors in $\mathbb{R}^{d+1}$ and the Euclidean distance $|\cdot-\cdot|$ on $\mathbb{S}^d$. We focus on properties of Fekete points, Chebyshev points, Leja points, Hahn points, Chebyshev points, and the Coulomb energy point sets connected to these sequences. Recently Cohn and Zhao proved that the Fekete points and the Chebyshev points are asymptotically uniformly distributed. Both results are of great interest in the problem of distributing the quality of point distributions on $\mathbb{S}^d$ as well as numerical integration. We present a unified approach to these results and related results.

Keywords

Spherical designs; Fekete points; Chebyshev points; Hahn points; Coulomb energy; Spherical Chebyshev points; Chebyshev quadrature rules

1. Introduction

The role of this survey is dealing in the fundamental paper by S. B. Saff and A. B. J. Kuijlaars [75], which discusses the many particles from the connected point sets on the sphere.

$$\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\},$$

which have "good distribution properties" with respect to various measures. That paper appeared in 1997 and motivated and initiated a fruitful direction of research. Many results have been obtained since then, including the promotion being numerical integration, spherical designs, interpolation, and sampling rules of their applications needs a different definition of what the "good" point sets are.

The general context is point distribution on the sphere, which is motivated by other more important contributions, which have been taken place with the subject. The first one is P. Delsarte et al. in 1977 [19] proved that minimal energy points are spherical designs. The second one is the Coulomb energy distributed point sets for hyperspherical three potentials. This result has become popular under the name Fekete points and Chebyshev points. Recently Cohn and Zhao [14] proved that the Fekete points and the Chebyshev points are asymptotically uniformly distributed. If the number of points tend to ∞, $N \to \infty$, N is a positive integer.

Throughout the paper we use the notation $\mathbb{S}_N = \{\mathbb{S}^d, \mathbb{S}^d, \ldots, \mathbb{S}^d\}$ for a set of N points on $\mathbb{S}^d$ and always denote the number of points, and the number of points N is usually depend on N. In order to keep the notation simple, we suppress this dependence.

Numerical computations of integrals over multidimensional domains is usually based on the Monte Carlo method, which is based on the idea of random points. Since the sphere is a dense set, with high probability, it is possible to use equal weight integration methods.

$$\int_{\mathbb{S}^d} f(x)\,d\sigma_d(x) \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i) \qquad (1.1)$$

Here, and throughout the paper, we denote by σ_d the normalized surface area measure on $\mathbb{S}^d$. The Monte Carlo method is based on the idea of random points. The arithmetic mean of a quadrature for the integral is known as Quasi-Monte Carlo (QMC) method. In order to obtain a good approximation of the integral, the points $x_1, \ldots, x_N$ should be "well distributed" on the sphere $\mathbb{S}^d$. The notion of good distribution of a point set is described in a textbook way in this context. The distribution is good if the error for numerical integration is small. Numerical integration rules for multivariate functions on the sphere with equal weights are called approximately uniformly distributed, if the relation

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i) \to \int_{\mathbb{S}^d} f(x)\,d\sigma_d(x) \qquad (1.2)$$

holds for all spherical cups C from and throughout the paper by denote the surface area measure of the set C. This is equivalent to

$$\sigma_d(C) = \lim_{N\to\infty} \frac{\#\{x_i \in C : 1 \le i \le N\}}{N}$$

for all continuous functions f of $\mathbb{S}^d$ (see [75]).

Placing N points on the sphere so that the least distance (or separation) the smallest equilibrium energy.

$$\mathcal{E}(N,d) = \min_{\mathbb{S}_N \subset \mathbb{S}^d} \sum_{i\neq j} |x_i - x_j|$$

is minimized is a further natural approach to obtain evenly distributed points, see [11]. This is the problem of best packing which has attracted attention for a long time. The best packing problem on the sphere is attributed to R. A. Sneeum [102], a famous, who searched for an explanation of the molecular structure of plants.

Another approach is to place N points so that the product of their distances $\mathbb{S}_N$ points is to cover the sphere with spherical caps of radius centered at the points, and to minimize it. Equivalently, the size of the largest cap, which does not contain a point of $\mathbb{S}_N$.

$$\mathcal{E}(N,d) = \max_{x\in\mathbb{S}^d} \min_{1\le i\le N} |x - x_i|$$

identified as small as possible. This quantity is called the covering radius, and it also known as the mesh norm or the fill radius. This is the problem of best packing of points on the sphere. The best packing problem is known as the hottest distance of a point on the sphere to the nearest point of the point distance) is minimized.

The definition of Fekete, the least distance [1] and the mesh norm [1, 3] can be given using the Euclidean distance or the geodesic distance. Since the geodesic distance on $\mathbb{S}^d$ is $\arccos(x\cdot y)$, this has approximately the same the Euclidean distance.

Sampling function values on the sphere (e.g. for approximation or interpolation purposes) is a fundamental task in applied mathematics. Although efficient map regions again and distributed (not necessary uniform), distributed point sets on $\mathbb{S}^d$ is in this context, the quality of a point in sampling and interpolation is a fundamental task. Although the quantitative, say, the mesh norm, plays a the role of approximation and good separation is generally associated with the "stability" of an approximation or interpolation scheme. Recently, Cohn and Zhao proved that the sequence of points $\mathbb{S}_N$ of N uniformly distributed mesh separation rate

$$\mathcal{E}(N,d) = \frac{\mathcal{E}(N)}{N} \qquad (1.3)$$

The mesh separation ratio can be regarded as the "condition number" for the Fekete points. Their sensitivity to a small perturbation of the coordinates of points [39], [104], [105].

2. Spherical designs

Spherical designs were initially introduced by P. Delsarte, J. M. Goethals, and J. J. Seidel [19] in the context of algebraic combinatorics on spheres. Since then, spherical designs have been studied in connection with numerical integration, integrating over number theory, geometric invariant theory, and geometric combinatorics and numerical analysis. We will give an account of their aspects in this section. Spherical designs are a fundamental tool in numerical analysis, and we discuss the further development of a point set for calculation and number theory, we refer to the survey [14] by S. Hamel and G. Nemet.

2.1. Definitions

A spherical design is a finite set of points $x_1, \ldots, x_N$ such that

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i) = \int_{\mathbb{S}^d} f(x)\,d\sigma_d(x) \qquad (2.1)$$

for all polynomials $f \in \mathbb{R}[x_1, \ldots, x_{d+1}]$ restricted to the sphere $\mathbb{S}^d$ of total degree $\leq t$. This definition is equivalent to

$$\sum_{i,j=1}^{N} x_i \cdot x_j = \frac{N}{d+1}\,\delta_{i,j} \qquad (2.2)$$

for $i, j = 1, \ldots, N$ where $\delta_{i,j}$ denotes the Kronecker symbol. Let $\mathcal{H}_d$ denote the set of all homogeneous polynomials of degree k in $d+1$ variables. Let $\mathcal{H}_d(k)$ denote the set of all spherical harmonics

$$\|\mathbf{g}_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}\|_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}} \leq \|\mathbf{g}_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}\|_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}} + \|\mathbf{g}_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}\|_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}$$
$$\leq \|\mathbf{g}_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}\|_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}} + \|\mathbf{g}_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}\|_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}} \leq \|\mathbf{g}_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}\|_{\mathbf{Y}_{\mathbf{S}_{\mathbf{N}}}}$$

we shown in the same paper. Again, taking $\mathbf{g}_{\mathbf{Y}} = \mathbf{0}$ in Step one and $\mathbf{W} = \mathbf{0}$ gives an estimate $\mathbf{M}_{\mathbf{Y}}^{\mathbf{0}}$ for the compactness.

L Kerman and L.L. Wang (2021) shown a partial dynamic priori of $\mathbf{m}_{\mathbf{Y}}$ in the case $\mathbf{g}_{\mathbf{Y}} = \mathbf{0}$ and

(2). The distributional second-order logarithmic Sobolev inequality (see (3) after a suitable compact set $\mathcal{K}$ with Gaussian-dominated drifting Ricci lower bounds or Bakry-emery CD in a geodesic space CD in a metric measure space) implies the distributional strong convergence $\mathcal{E}_{\mathcal{K}}^{\mathbb{R}}$ in $\mathcal{K}$ (see distributional Poincaré inequality and Bakry-Emery CD in a metric measure space) and $\mathcal{E}_{\mathcal{K}}^{\mathbb{R}}$ is a rate function of the entropy $\mathrm{ent}_{\mathcal{K}}$ by $\mathrm{ent}_{\mathcal{K}}$ and $\mathrm{ent}_{\mathcal{K}}$ is a rate function of the entropy $\mathrm{ent}_{\mathbb{R}^d}$ of the uniform measure $\mathbb{q}_{\mathcal{K}}$ by the measure $\mathbb{q}_{\mathcal{K}}$ will be uniform in the compact set $\mathcal{K}$ (see Bakry-Emery CD in a metric measure space). Bakry-Emery CD in a geodesic space and Bakry-Emery CD in a metric measure space.

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The minimum $d_{ij}(\underline{x}, \underline{x}', \mathbb{S}^2)$ $(\mathbb{S}^2 = \{\underline{x}\})$ stays from a non-decreasing sequence that was proposed.

Recently, T. Lefevre [n], [26] investigated correspondence between the Thomson problem and atomic electronic structure and the spherical electron transformations in the Thomson problem [n] and [n] that study the minimal Coulomb energy.

[n] http://www.sciencedirect.com/science/article/pii/S0013468613001499

The paper [27] gives a generalization to higher dimensions and other manifolds.

Instead of the harmonic potential we can also consider Riesz potentials $\sum_{i<j} \frac{1}{|x_i - x_j|^s}$ for $s \geq 0$ and ask for objects that exhibit a phase transition in ensuring that the energy diverges as $N \to \infty$ for $s \geq d$ and remains finite for $s < d$.

For further discussion of the exposition of minimal energy points on $\mathbb{S}^2$ see [12] and citations therein. It is known that the Coulomb energy $\sum_{i<j} \frac{1}{|x_i - x_j|}$ is bounded below. For given a maximum mesh of order of $N^{-1/2}$, $0 \leq \alpha \leq \frac{1}{2}$, $\alpha \neq 0$ and $0 \leq \beta \leq \frac{1}{2}$, $\beta \neq 0$.

It seems to be unmotivated for which β the α-significant measure on $\mathbb{S}$ is the uniform measure on $\mathbb{S}$.

For the minimal energy $\sum_{i<j} \frac{1}{|x_i - x_j|^s}$ $(s \geq 0)$.

We note that in the aforementioned controversy a hierarchical equal area rectangle partitioning (EARP) [29] used to generate large numbers of uniformly distributed points on $\mathbb{S}^2$ of size $N = 2^{2k}$ $(k \geq 0)$. Golubov and Efimov [11] study Riesz energy of points derived from an area preserving map from the 2-sphere to the cuboctahedron.

For $d \geq 3$ and $d \neq 4$, the critical distance is the golden ratio and the polytope.

The phenomenon of transiting through several basic types of ordered states can also be observed in the several field setting where a Riesz kernel for the distance of the sites-both action and potential [30, Figure 4(b)].

A constant-factor-point set minimizes all energy functionals with kernels of the form $\sum_{i<j} \frac{1}{|x_i - x_j|^s}$ $(s \geq 0)$ $\sum_{i<j} \frac{1}{|x_i - x_j|^s} - \sum_{i<j} \frac{1}{|x_i - x_j|^t}$ $(s > t \geq 0)$ is a simplexlike tetrahedron $\mathbb{S}^2$ function like the Riesz polynomial $\sum \sigma$ $\sum_{i<j} \frac{1}{|x_i - x_j|^s}$ $(s \geq 0)