LBGK 方程到宏观方程的理论推导

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1. 问题描述

不含外力项的 LBGK 方程、宏观连续方程以及不含外力项的不可压缩 N-S 方程分别为式(1)、(2)、(3)，试从方程(1)推导方程(2)和(3)。

$$f_\alpha(\vec{r}+\overrightarrow{e_\alpha}\delta_t, t+\delta_t)-f_\alpha(\vec{r},t)=-\frac{1}{\tau}[f_\alpha(\vec{r},t)-f_\alpha^{eq}(\vec{r},t)] \tag{1}$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u})=0 \tag{2}$$

$$\frac{\partial\vec{u}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}=-\frac{1}{\rho}\vec{\nabla}P+v\Delta\vec{u} \tag{3}$$

2. 理论推导

该理论推导以 D2Q9 模型为例，包括两大部分推导：一是 D2Q9 模型权系数及格子声速推导，二是利用 Chapman-Enskog（CE）展开，基于 D2Q9 模型，从 LBGK 方程推导宏观方程。由于 word 软件公式编辑显示问题，$\overrightarrow{e_{\alpha\iota}}$和$\overrightarrow{e_{\alpha\jmath}}$分别表示$e_{\alpha i}$和$e_{\alpha j}$的矢量形式，$\overrightarrow{u_\iota}$和$\overrightarrow{u_\jmath}$分别表示$u_i$和$u_j$的矢量形式（即下标$i$和$j$虽然显示不同，但代表同样的含义）。

Qian 等人[1]提出的 DdQm（d 维空间 m 维粒子速度）系列模型是 LBM 的基本模型，其中 D2Q9 模型是模拟二维问题的常用模型，其采用的平衡态分布函数为：

$$f_\alpha^{eq} = \rho\omega_\alpha\left[1 + \frac{\overrightarrow{e_\alpha}\cdot\vec{u}}{c_s^2} + \frac{(\overrightarrow{e_\alpha}\cdot\vec{u})^2}{2c_s^4} - \frac{u^2}{2c_s^2}\right] \tag{4}$$

式中，c_s为格子声速；ω_α为权系数，其是温度T、粒子速度$\overrightarrow{e_\alpha}$及空间维数 D 的函数，在不考虑温度变化和 D2Q9 模型条件下，ω_α只是粒子速度$\overrightarrow{e_\alpha}$的函数[2]；对于 D2Q9 模型，$\overrightarrow{e_\alpha}$为：

$$\overrightarrow{e_\alpha} = \begin{cases} [0,0], \alpha = 0 \\ \left\{cos\left[\frac{\pi}{2}(\alpha-1)\right], sin\left[\frac{\pi}{2}(\alpha-1)\right]\right\}c, \alpha = 1,2,3,4 \\ \left\{cos\left[\frac{\pi}{4}+\frac{\pi}{2}(\alpha-5)\right], sin\left[\frac{\pi}{4}+\frac{\pi}{2}(\alpha-5)\right]\right\}\sqrt{2}c, \alpha = 5,6,7,8 \end{cases} \tag{5}$$

式中，c为声速。

根据动理论（kinetic theory），平衡态分布函数在离散的速度空间需满足以下方程（Krüger 等人的书中介绍了方程来源）[3]：

$$\sum_\alpha f_\alpha^{eq} = \rho \tag{6}$$

$$\sum_\alpha f_\alpha^{eq}\,\overrightarrow{e_\alpha} = \rho\vec{u} \tag{7}$$

$$\sum_\alpha f_\alpha^{eq}\, e_{\alpha i}e_{\alpha j} = \rho u_i u_j + p\delta_{ij} \tag{8}$$

同时，在后续推导过程中，常会遇到粒子速度的n阶张量：

$$E^{(n)} = \sum_\alpha(\overrightarrow{e_{\alpha 1}})(\overrightarrow{e_{\alpha 2}})\dots(\overrightarrow{e_{\alpha n}}) \tag{9}$$

Stephen Wolfram[4]和何雅玲[2]等人对其结果进行了相关张量分析及整理，推导所需的结果包括：

$$E^{(2n+1)} = 0, n \geq 0 \tag{10}$$

$$\sum_{\alpha=1}^{4} e_{\alpha i}e_{\alpha j} = 2c^2\delta_{ij} \tag{11}$$

$$\sum_{\alpha=5}^{8} e_{\alpha i}e_{\alpha j} = 4c^2\delta_{ij} \tag{12}$$

$$\sum_{\alpha=1}^{4} e_{\alpha i}e_{\alpha j}\, e_{\alpha k}e_{\alpha m} = 2c^4\delta_{ijkm} \tag{13}$$

$$\sum_{\alpha=5}^{8} e_{\alpha i}e_{\alpha j}\, e_{\alpha k}e_{\alpha m} = 4c^4\Delta_{ijkm} - 8c^4\delta_{ijkm} \tag{14}$$

式中，δ为 Kronecker 符号，$\Delta_{ijkm}= \delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}$。

下面将首先推导 D2Q9 模型的权系数ω_α及格子声速c_s的表达式。

将方程(4)代入方程(6)得：

$$\sum_\alpha \rho\omega_\alpha[1+\frac{\overrightarrow{e_\alpha}\cdot\vec{u}}{c_s^2}+\frac{(\overrightarrow{e_\alpha}\cdot\vec{u})^2}{2c_s^4}-\frac{u^2}{2c_s^2}]-\rho=\sum_\alpha\left[\rho\omega_\alpha\left(1-\frac{u^2}{2c_s^2}\right)+e_{\alpha i}\frac{\rho\omega_\alpha u_i}{c_s^2}+e_{\alpha i}e_{\alpha j}\frac{\rho\omega_\alpha u_i u_j}{2c_s^4}\right]-\rho=0 \tag{15}$$

根据方程(10)得：

$$\sum_\alpha e_{\alpha i}\frac{\rho\omega_\alpha u_i}{c_s^2}=0 \tag{16}$$

根据方程(5)、(11)和(12)，以及ω_α只是粒子速度$\overrightarrow{e_\alpha}$的函数，得：

$$\sum_\alpha e_{\alpha i}e_{\alpha j}\frac{\rho\omega_\alpha u_i u_j}{2c_s^4}=\frac{\rho\omega_1 u_i u_j}{2c_s^4}\sum_{\alpha=1}^{4}e_{\alpha i}e_{\alpha j}+\frac{\rho\omega_5 u_i u_j}{2c_s^4}\sum_{\alpha=5}^{8}e_{\alpha i}e_{\alpha j}=\frac{\rho\omega_1 u_i u_j}{2c_s^4}\cdot 2c^2\delta_{ij}+\frac{\rho\omega_5 u_i u_j}{2c_s^4}\cdot 4c^2\delta_{ij}=\frac{\rho\omega_1 c^2u^2}{c_s^4}+\frac{2\rho\omega_5 c^2 u^2}{c_s^4} \tag{17}$$

又：

$$\sum_\alpha \rho\omega_\alpha\left(1-\frac{u^2}{2c_s^2}\right)=\rho\left(1-\frac{u^2}{2c_s^2}\right)\left(\sum_\alpha\omega_\alpha\right)=\rho\left(1-\frac{u^2}{2c_s^2}\right)(\omega_0+4\omega_1+4\omega_5) \tag{18}$$

将方程(16)、(17)、(18)代入方程(15)得：

$$\sum_\alpha \rho\omega_\alpha[1+\frac{\overrightarrow{e_\alpha}\cdot\vec{u}}{c_s^2}+\frac{(\overrightarrow{e_\alpha}\cdot\vec{u})^2}{2c_s^4}-\frac{u^2}{2c_s^2}]-\rho=\rho\left(1-\frac{u^2}{2c_s^2}\right)(\omega_0+4\omega_1+4\omega_5)+\frac{\rho\omega_1 c^2u^2}{c_s^4}+\frac{2\rho\omega_5 c^2u^2}{c_s^4}-\rho=\rho(\omega_0+4\omega_1+4\omega_5)+\rho u^2\left(\omega_1\left(\frac{c^2}{c_s^4}-\frac{2}{c_s^2}\right)+\omega_5\left(\frac{2c^2}{c_s^4}-\frac{2}{c_s^2}\right)-\frac{\omega_0}{2c_s^2}\right)-\rho=0 \tag{19}$$

即：

$$\rho(\omega_0+4\omega_1+4\omega_5)+\rho u^2\left(\omega_1\left(\frac{c^2}{c_s^4}-\frac{2}{c_s^2}\right)+\omega_5\left(\frac{2c^2}{c_s^4}-\frac{2}{c_s^2}\right)-\frac{\omega_0}{2c_s^2}\right)=\rho \tag{20}$$

将方程(4)代入方程(7)得：

$$\sum_\alpha \rho\omega_\alpha[1+\frac{\overrightarrow{e_\alpha}\cdot\vec{u}}{c_s^2}+\frac{(\overrightarrow{e_\alpha}\cdot\vec{u})^2}{2c_s^4}-\frac{u^2}{2c_s^2}]\overrightarrow{e_\alpha}-\rho\vec{u}=\sum_\alpha\left[\rho\omega_\alpha e_{\alpha i}\left(1-\frac{u^2}{2c_s^2}\right)+\rho\omega_\alpha e_{\alpha i}e_{\alpha j}\frac{u_j}{c_s^2}+\rho\omega_\alpha e_{\alpha i}e_{\alpha j}e_{\alpha k}\frac{u_j u_k}{2c_s^4}\right]\overrightarrow{E_\iota}-\rho\vec{u}=0 \tag{21}$$

由方程(10)得：

$$\sum_\alpha \rho\omega_\alpha e_{\alpha i}\left(1-\frac{u^2}{2c_s^2}\right)\overrightarrow{E_\iota}=0 \tag{22}$$

$$\sum_\alpha \rho\omega_\alpha e_{\alpha i}e_{\alpha j}e_{\alpha k}\frac{u_j u_k}{2c_s^4}\overrightarrow{E_\iota}=0 \tag{23}$$

根据方程(5)、(11)和(12)，得：

$$\sum_\alpha \rho\omega_\alpha e_{\alpha i}e_{\alpha j}\frac{u_j}{c_s^2}\overrightarrow{E_\iota}=\left[\frac{\rho\omega_1 u_j}{c_s^2}\sum_{\alpha=1}^{4} e_{\alpha i}e_{\alpha j}+\frac{\rho\omega_5 u_j}{c_s^2}\sum_{\alpha=5}^{8} e_{\alpha i}e_{\alpha j}\right]\overrightarrow{E_\iota}=\left[\frac{\rho\omega_1 u_j}{c_s^2}2c^2\delta_{ij}+\frac{\rho\omega_5 u_j}{c_s^2}4c^2\delta_{ij}\right]\overrightarrow{E_\iota}=\rho\vec{u}(\frac{2c^2\omega_1}{c_s^2}+\frac{4c^2\omega_5}{c_s^2}) \tag{24}$$

将方程(22)、(23)、(24)代入方程(21)得：

$$\rho\vec{u}\left(\frac{2c^2\omega_1}{c_s^2}+\frac{4c^2\omega_5}{c_s^2}\right)=\rho\vec{u} \tag{25}$$

将方程(4)代入方程(8)得：

$$\sum_\alpha \rho\omega_\alpha\left[1+\frac{\overrightarrow{e_\alpha}\cdot\vec{u}}{c_s^2}+\frac{(\overrightarrow{e_\alpha}\cdot\vec{u})^2}{2c_s^4}-\frac{u^2}{2c_s^2}\right]e_{\alpha i}e_{\alpha j}-\left(\rho u_i u_j+p\delta_{ij}\right)=\sum_\alpha \rho\omega_\alpha\, e_{\alpha i}e_{\alpha j}\left(1-\frac{u^2}{2c_s^2}\right)+\sum_\alpha \rho\omega_\alpha\, e_{\alpha i}e_{\alpha j}e_{\alpha k}\frac{u_k}{c_s^2}+\sum_\alpha \rho\omega_\alpha\, e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}\frac{u_k u_m}{2c_s^4}-\left(\rho u_i u_j+p\delta_{ij}\right)=0 \tag{26}$$

由方程(10)得：

$$\sum_\alpha \rho\omega_\alpha\, e_{\alpha i}e_{\alpha j}e_{\alpha k}\frac{u_k}{c_s^2}=0 \tag{27}$$

根据方程(5)、(11)和(12)，得：

$$\sum_\alpha \rho\omega_\alpha\, e_{\alpha i}e_{\alpha j}\left(1-\frac{u^2}{2c_s^2}\right)=\rho\omega_1\left(1-\frac{u^2}{2c_s^2}\right)\sum_{\alpha=1}^{4} e_{\alpha i}e_{\alpha j}+\rho\omega_5\left(1-\frac{u^2}{2c_s^2}\right)\sum_{\alpha=5}^{8} e_{\alpha i}e_{\alpha j}=2c^2\delta_{ij}\rho\omega_1\left(1-\frac{u^2}{2c_s^2}\right)+4c^2\delta_{ij}\rho\omega_5\left(1-\frac{u^2}{2c_s^2}\right) \tag{28}$$

根据方程(5)、 (13)和(14)，得：

$$\sum_\alpha \rho\omega_\alpha\, e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}\frac{u_k u_m}{2c_s^4}=\frac{\rho\omega_1 u_k u_m}{2c_s^4}\sum_{\alpha=1}^{4} e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}+\frac{\rho\omega_5 u_k u_m}{2c_s^4}\sum_{\alpha=5}^{8} e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}=\frac{\rho\omega_1 u_k u_m}{2c_s^4}\cdot 2c^4\delta_{ijkm}+\frac{\rho\omega_5 u_k u_m}{2c_s^4}\cdot$$

$$\left(4c^4\Delta_{ijkm}-8c^4\delta_{ijkm}\right)=\frac{\rho u_k u_m c^4}{c_s^4}(\omega_1-4\omega_5)\delta_{ijkm}+\frac{2\rho\omega_5 u_k u_m c^4\Delta_{ijkm}}{c_s^4}=\frac{\rho u^2 c^4}{c_s^4}(\omega_1-4\omega_5)+\frac{2\rho\omega_5 u_k u_m c^4}{c_s^4}\cdot\left(\delta_{ij}\delta_{km}+\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}\right)=\frac{\rho u^2 c^4}{c_s^4}(\omega_1-4\omega_5)+\frac{2\rho\omega_5 c^4}{c_s^4}\left(u^2\delta_{ij}+2u_i u_j\right) \tag{29}$$

将方程(27)、(28)、(29)代入方程(26)得：

$$\sum_\alpha \rho\omega_\alpha\left[1+\frac{\overrightarrow{e_\alpha}\cdot\vec{u}}{c_s^2}+\frac{(\overrightarrow{e_\alpha}\cdot\vec{u})^2}{2c_s^4}-\frac{u^2}{2c_s^2}\right]e_{\alpha i}e_{\alpha j}-\left(\rho u_i u_j+p\delta_{ij}\right)=2c^2\delta_{ij}\rho\omega_1\left(1-\frac{u^2}{2c_s^2}\right)+4c^2\delta_{ij}\rho\omega_5\left(1-\frac{u^2}{2c_s^2}\right)+\frac{\rho u^2 c^4}{c_s^4}(\omega_1-4\omega_5)+\frac{2\rho\omega_5 c^4}{c_s^4}\left(u^2\delta_{ij}+2u_i u_j\right)-\left(\rho u_i u_j+p\delta_{ij}\right)=\frac{c^4}{c_s^4}(\omega_1-4\omega_5)\rho u^2+\frac{4\omega_5 c^4}{c_s^4}\rho u_i u_j+\left(2c^2\omega_1\left(1-\frac{u^2}{2c_s^2}\right)+4c^2\omega_5\left(1-\frac{u^2}{2c_s^2}\right)+\frac{2\omega_5 c^4 u^2}{c_s^4}\right)\rho\delta_{ij}-\left(\rho u_i u_j+p\delta_{ij}\right)=0 \tag{30}$$

移项得：

$$\frac{c^4}{c_s^4}(\omega_1-4\omega_5)\rho u^2+\frac{4\omega_5 c^4}{c_s^4}\rho u_i u_j+\left(2c^2\omega_1\left(1-\frac{u^2}{2c_s^2}\right)+4c^2\omega_5\left(1-\frac{u^2}{2c_s^2}\right)+\frac{2\omega_5 c^4 u^2}{c_s^4}\right)\rho\delta_{ij}=\left(\rho u_i u_j+p\delta_{ij}\right) \tag{31}$$

分析方程(20)，因方程右端项不含u^2，为保证等式成立，有：

$$\omega_0+4\omega_1+4\omega_5=1 \tag{32}$$

$$\omega_1\left(\frac{c^2}{c_s^4}-\frac{2}{c_s^2}\right)+\omega_5\left(\frac{2c^2}{c_s^4}-\frac{2}{c_s^2}\right)-\frac{\omega_0}{2c_s^2}=0 \tag{33}$$

分析方程(25) ，为保证等式成立，有：

$$\frac{2c^2\omega_1}{c_s^2}+\frac{4c^2\omega_5}{c_s^2}=1 \tag{34}$$

分析方程(31)，为保证等式成立，方程两端的含$u_i u_j$和δ_{ij}的项应分别相等，其余项应为 0，则：

$$\frac{c^4}{c_s^4}(\omega_1-4\omega_5)=0 \tag{35}$$

$$\frac{4\omega_5 c^4}{c_s^4}=1 \tag{36}$$

$$2c^2\omega_1\left(1-\frac{u^2}{2c_s^2}\right)+4c^2\omega_5\left(1-\frac{u^2}{2c_s^2}\right)+\frac{2\omega_5 c^4 u^2}{c_s^4}=\frac{p}{\rho} \tag{37}$$

联立方程(32)、(33)、(34)、(35)、(36)和(37)，解得：

$$\omega_0 = \frac{4}{9}, \ \omega_1 = \frac{1}{9}, \ \omega_5 = \frac{1}{36} \tag{38}$$

$$c_s^2 = \frac{c^2}{3} \tag{39}$$

$$p = \rho c_s^2 \tag{40}$$

即 D2Q9 模型的权系数ω_α及格子声速c_s的表达式推导完毕。模型宏观压力由状态方程(40)给出，与方程(6)和(7)类似，模型的宏观密度及速度定义为[2]：

$$\rho = \sum_\alpha f_\alpha \tag{41}$$

$$\vec{u} = \frac{1}{\rho} \sum_\alpha f_\alpha \overrightarrow{e_\alpha} \tag{42}$$

下面将利用 Chapman-Enskog 展开，基于 D2Q9 模型，从 LBGK 方程推导宏观方程。

CE 方法是一种多尺度技术，通常采用三种时间尺度：碰撞时间尺度K_n^0、对流时间尺度K_n^1和扩散时间尺度K_n^2，其中，K_n为 Knudsen 数，其表达式为：

$$K_n = \frac{\ell_{mfp}}{\ell} \tag{43}$$

式中，ℓ_{mfp}为分子平均自由程，ℓ为宏观特征长度。CE 方法利用 Knudsen 数将离散时间（t）和连续时间（t_1、t_2）、离散空间（$\vec{r}$）和连续空间（$\vec{r}_1$）联系起来：

$$t_1 = K_n t \tag{44}$$

$$t_2 = K_n^2 t \tag{45}$$

$$\vec{r}_1 = K_n \vec{r} \tag{46}$$

根据链式求导法则有：

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_1}\frac{dt_1}{dt}+\frac{\partial}{\partial t_2}\frac{dt_2}{dt}=K_n\frac{\partial}{\partial t_1}+K_n^2\frac{\partial}{\partial t_2} \tag{47}$$

$$\frac{\partial}{\partial \vec{r}}=\frac{\partial}{\partial \vec{r}_1}\frac{d\vec{r}_1}{d\vec{r}}\ =K_n\frac{\partial}{\partial \vec{r}_1} \tag{48}$$

CE 技术在假设分布函数$f_\alpha(\vec{r},t)$已离平衡态分布函数不远并趋于平衡态的条件下，将$f_\alpha(\vec{r},t)$展开为K_n的幂级数形式[2]：

$$f_\alpha=f_\alpha^{eq}+K_nf_\alpha^{(1)}+K_n^2f_\alpha^{(2)}+\cdots \tag{49}$$

其中，f_α^{eq}也表示为$f_\alpha^{(0)}$。对于二元函数$f(x,y)$，在(x_0,y_0)的泰勒展开为[5]:

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^nf(x_0,y_0)+O^{n+1} \tag{50}$$

式中，需注意：

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)=h\frac{\partial f(x_0,y_0)}{\partial x}+k\frac{\partial}{\partial y}f(x_0,y_0) \tag{51}$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2f(x_0,y_0)=h^2\ \frac{\partial^2 f(x_0,y_0)}{\partial x^2}+2hk\frac{\partial^2 f(x_0,y_0)}{\partial x\partial y}+k^2\ \frac{\partial^2 f(x_0,y_0)}{\partial y^2} \tag{52}$$

将方程(1)左端第一项对空间$\vec{r}$和时间t在$(\vec{r},t)$做二阶泰勒展开得：

$$f_\alpha(\vec{r},t)+\left(\overrightarrow{e_\alpha}\delta_t\cdot\frac{\partial}{\partial \vec{r}}+\delta_t\frac{\partial}{\partial t}\right)f_\alpha(\vec{r},t)+\frac{1}{2!}\left(\overrightarrow{e_\alpha}\delta_t\cdot\frac{\partial}{\partial \vec{r}}+\delta_t\frac{\partial}{\partial t}\right)^2f_\alpha(\vec{r},t)-f_\alpha(\vec{r},t)+\frac{1}{\tau}[f_\alpha(\vec{r},t)-f_\alpha^{eq}(\vec{r},t)]+O^3=0 \tag{53}$$

简写、整理并略去高阶项得：

$$\delta_t\left(\overrightarrow{e_\alpha}\cdot\frac{\partial}{\partial \vec{r}}+\frac{\partial}{\partial t}\right)f_\alpha+\frac{{\delta_t}^2}{2!}\left(\overrightarrow{e_\alpha}\cdot\frac{\partial}{\partial \vec{r}}+\frac{\partial}{\partial t}\right)^2f_\alpha+\frac{1}{\tau}[f_\alpha-f_\alpha^{eq}]=0 \tag{54}$$

即：

$$\delta_t\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}+\frac{\partial}{\partial t}\right)f_\alpha+\frac{{\delta_t}^2}{2!}\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}+\frac{\partial}{\partial t}\right)^2f_\alpha+\frac{1}{\tau}[f_\alpha-f_\alpha^{eq}]=0 \tag{55}$$

将方程(47)、(48)以及(49)代入方程(55)得：

$$\delta_t\left(K_n\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+K_n\frac{\partial}{\partial t_1}+K_n^2\frac{\partial}{\partial t_2}\right)\left(f_\alpha^{eq}+K_nf_\alpha^{(1)}+K_n^2f_\alpha^{(2)}+O^3\right)+$$
$$\frac{{\delta_t}^2}{2!}\left(K_n\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+K_n\frac{\partial}{\partial t_1}+K_n^2\frac{\partial}{\partial t_2}\right)^2\left(f_\alpha^{eq}+K_nf_\alpha^{(1)}+K_n^2f_\alpha^{(2)}+O^3\right)+$$
$$\frac{1}{\tau}\left[\left(f_\alpha^{eq}+K_nf_\alpha^{(1)}+K_n^2f_\alpha^{(2)}+O^3\right)-f_\alpha^{eq}\right]=0 \qquad (56)$$

忽略K_n数的三次方及以上次方项，化简得：

$$K_n\left[\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{eq}+\frac{1}{\tau\delta_t}f_\alpha^{(1)}\right]+K_n^2\left[\frac{\partial f_\alpha^{eq}}{\partial t_2}+\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{(1)}+\right.$$
$$\left.\frac{\delta_t}{2!}\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)^2f_\alpha^{eq}+\frac{1}{\tau\delta_t}f_\alpha^{(2)}\right]+O^3=0 \qquad (57)$$

根据K_n数定义（方程(43)）知$K_n>0$，忽略高阶项得：

$$\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{eq}+\frac{1}{\tau\delta_t}f_\alpha^{(1)}=0 \qquad (58)$$

$$\frac{\partial f_\alpha^{eq}}{\partial t_2}+\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{(1)}+\frac{\delta_t}{2!}\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)^2f_\alpha^{eq}+\frac{1}{\tau\delta_t}f_\alpha^{(2)}=0 \qquad (59)$$

将方程(58)变形得：

$$\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{eq}=-\frac{1}{\tau\delta_t}f_\alpha^{(1)} \qquad (60)$$

代入方程(59)得：

$$\frac{\partial f_\alpha^{eq}}{\partial t_2}+\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{(1)}+\frac{\delta_t}{2!}\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)\left(-\frac{1}{\tau\delta_t}f_\alpha^{(1)}\right)+\frac{1}{\tau\delta_t}f_\alpha^{(2)}=0$$
$$(61)$$

整理得：

$$\frac{\partial f_\alpha^{eq}}{\partial t_2}+\left(1-\frac{1}{2\tau}\right)\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{(1)}+\frac{1}{\tau\delta_t}f_\alpha^{(2)}=0 \qquad (62)$$

根据方程(6)、(7)、(41)、(42)和(49)得：

$$\sum_\alpha f_\alpha^{(n)}=0,\qquad \sum_\alpha f_\alpha^{(n)}\overrightarrow{e_\alpha}=0\qquad n=1,2,3,\ldots \qquad (63)$$

对方程(58)求速度的零阶矩得：

$$\sum_\alpha\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{eq}+\sum_\alpha\frac{1}{\tau\delta_t}f_\alpha^{(1)}=0 \qquad (64)$$

根据方程(6)和(63)得：

$$\frac{\partial}{\partial t_1}\sum_\alpha f_\alpha^{eq}+\sum_\alpha\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1f_\alpha^{eq}=\frac{\partial\rho}{\partial t_1}+\sum_\alpha\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1f_\alpha^{eq}=0 \qquad (65)$$

由于$\overrightarrow{e_\alpha}$为常矢量，则[6]：

$$\sum_\alpha \vec{\nabla}_1 \cdot \left(f_\alpha^{eq}\overrightarrow{e_\alpha}\right) = \sum_\alpha \vec{\nabla}_1 f_\alpha^{eq} \cdot \overrightarrow{e_\alpha} \tag{66}$$

考虑方程右端项：

$$\vec{\nabla}_1 f_\alpha^{eq} \cdot \overrightarrow{e_\alpha} = \overrightarrow{E_\iota}\frac{\partial}{\partial r_{1i}} f_\alpha^{eq} e_{\alpha j} \cdot \overrightarrow{E_J} = \frac{\partial}{\partial r_{1i}} f_\alpha^{eq} e_{\alpha j}\delta_{ij} = \frac{\partial f_\alpha^{eq} e_{\alpha i}}{\partial r_{1i}} = e_{\alpha i}\frac{\partial f_\alpha^{eq}}{\partial r_{1i}} + f_\alpha^{eq}\frac{\partial e_{\alpha i}}{\partial r_{1i}} \tag{67}$$

式中$\overrightarrow{E_\iota}$和$\overrightarrow{E_J}$为单位正交基矢量，对于任意给定的α（α取值范围内），有：

$$\frac{\partial e_{\alpha i}}{\partial r_{1i}} = 0 \tag{68}$$

代入方程(67)得

$$\vec{\nabla}_1 f_\alpha^{eq} \cdot \overrightarrow{e_\alpha} = e_{\alpha i}\frac{\partial f_\alpha^{eq}}{\partial r_{1i}} \tag{69}$$

又：

$$\overrightarrow{e_\alpha} \cdot \vec{\nabla}_1 f_\alpha^{eq} = e_{\alpha i}\overrightarrow{E_\iota} \cdot \overrightarrow{E_J}\frac{\partial}{\partial r_{1j}} f_\alpha^{eq} = e_{\alpha i}\frac{\partial}{\partial r_{1j}} f_\alpha^{eq}\delta_{ij} = e_{\alpha i}\frac{\partial f_\alpha^{eq}}{\partial r_{1i}} \tag{70}$$

则联立公式 (66)、(69)、(70)得：

$$\sum_\alpha \overrightarrow{e_\alpha} \cdot \vec{\nabla}_1 f_\alpha^{eq} = \sum_\alpha \vec{\nabla}_1 \cdot \left(f_\alpha^{eq}\overrightarrow{e_\alpha}\right) \tag{71}$$

代入方程(65)并联立方程(7)得：

$$\frac{\partial\rho}{\partial t_1} + \sum_\alpha \vec{\nabla}_1 \cdot \left(f_\alpha^{eq}\overrightarrow{e_\alpha}\right) = \frac{\partial\rho}{\partial t_1} + \vec{\nabla}_1 \cdot (\rho\vec{u}) = 0 \tag{72}$$

对方程(58)求速度的一阶矩得：

$$\sum_\alpha \left(\overrightarrow{e_\alpha} \cdot \vec{\nabla}_1 + \frac{\partial}{\partial t_1}\right)\left(f_\alpha^{eq}\overrightarrow{e_\alpha}\right) + \sum_\alpha \frac{1}{\tau\delta_t} f_\alpha^{(1)}\overrightarrow{e_\alpha} = 0 \tag{73}$$

根据方程(63)得：

$$\sum_\alpha\left(\overrightarrow{e_{\alpha\iota}} \cdot \vec{\nabla}_1\right)\left(f_\alpha^{eq}\overrightarrow{e_{\alpha J}}\right) + \sum_\alpha \frac{\partial}{\partial t_1}\left(f_\alpha^{eq}\overrightarrow{e_{\alpha J}}\right) = 0 \tag{74}$$

即：

$$\frac{\partial\rho\overrightarrow{u_J}}{\partial t_1} + \sum_\alpha\left(\overrightarrow{e_{\alpha\iota}} \cdot \vec{\nabla}_1\right)\left(f_\alpha^{eq}\overrightarrow{e_{\alpha J}}\right) = 0 \tag{75}$$

由：

$$\vec{\nabla}_1 \cdot \left(f_\alpha^{eq}\overrightarrow{e_{\alpha i} e_{\alpha j}}\right) = \overrightarrow{E_k}\frac{\partial}{\partial r_{1k}} \cdot \left(f_\alpha^{eq} e_{\alpha i} e_{\alpha j}\overrightarrow{E_i}\overrightarrow{E_j}\right) = \frac{\partial f_\alpha^{eq} e_{\alpha i} e_{\alpha j}}{\partial r_{1k}}\overrightarrow{E_k} \cdot \overrightarrow{E_i}\overrightarrow{E_j} =$$

$$\frac{\partial f_\alpha^{eq} e_{\alpha i} e_{\alpha j}}{\partial r_{1k}}\delta_{ki}\overrightarrow{E_j} = \frac{\partial f_\alpha^{eq} e_{\alpha i} e_{\alpha j}}{\partial r_{1i}}\overrightarrow{E_j} = \left(e_{\alpha i}\frac{\partial f_\alpha^{eq} e_{\alpha j}}{\partial r_{1i}} + f_\alpha^{eq} e_{\alpha j}\frac{\partial e_{\alpha i}}{\partial r_{1i}}\right)\overrightarrow{E_j} \tag{76}$$

式中$\overrightarrow{E_k}$为单位基矢量，其与$\overrightarrow{E_i}$和$\overrightarrow{E_j}$三者相互正交，根据方程(68)得：

$$\vec{\nabla}_1 \cdot \left(f_\alpha^{eq}\overrightarrow{e_{\alpha i} e_{\alpha j}}\right) = e_{\alpha i}\frac{\partial f_\alpha^{eq} e_{\alpha j}}{\partial r_{1i}}\overrightarrow{E_j} = e_{\alpha i}\frac{\partial f_\alpha^{eq}\overrightarrow{e_{\alpha j}}}{\partial r_{1i}} = \left(\overrightarrow{e_{\alpha i}} \cdot \vec{\nabla}_1\right)\left(f_\alpha^{eq}\overrightarrow{e_{\alpha j}}\right) \tag{77}$$

代入方程(74)得：

$$\frac{\partial \rho \overrightarrow{u_j}}{\partial t_1} + \sum_\alpha \vec{\nabla}_1 \cdot \left(f_\alpha^{eq}\overrightarrow{e_{\alpha i} e_{\alpha j}}\right) = 0 \tag{78}$$

根据方程(8)得：

$$\frac{\partial \rho \overrightarrow{u_j}}{\partial t_1} + \vec{\nabla}_1 \cdot \left(\rho \overrightarrow{u_i}\overrightarrow{u_j} + p\delta_{ij}\overrightarrow{E_i}\overrightarrow{E_j}\right) = 0 \tag{79}$$

考虑方程左端第二项及$\overrightarrow{E_i}$、$\overrightarrow{E_j}$和$\overrightarrow{E_k}$的相互正交性：

$$\vec{\nabla}_1 \cdot \left(\rho \overrightarrow{u_i}\overrightarrow{u_j} + p\delta_{ij}\overrightarrow{E_i}\overrightarrow{E_j}\right) = \overrightarrow{E_k}\frac{\partial \rho u_i u_j}{\partial r_{1k}} \cdot \overrightarrow{E_i}\overrightarrow{E_j} + \overrightarrow{E_k}\frac{\partial p}{\partial r_{1k}}\delta_{ij} \cdot \overrightarrow{E_i}\overrightarrow{E_j} =$$

$$\frac{\partial \rho u_i u_j}{\partial r_{1k}}\delta_{ki}\overrightarrow{E_j} + \frac{\partial p}{\partial r_{1k}}\delta_{ij}\delta_{ki}\overrightarrow{E_j} = \frac{\partial \rho u_i u_j}{\partial r_{1i}}\overrightarrow{E_j} + \frac{\partial p}{\partial r_{1i}}\delta_{ij}\overrightarrow{E_j} \tag{80}$$

代入方程(79)得：

$$\frac{\partial \rho u_j}{\partial t_1} + \frac{\partial \rho u_i u_j}{\partial r_{1i}} = -\frac{\partial p}{\partial r_{1i}} \text{ 或 } \frac{\partial \rho u_j}{\partial t_1} + \frac{\partial \rho u_i u_j}{\partial r_{1i}} = -\frac{\partial p}{\partial r_{1j}} \tag{81}$$

方程(72)和(81)即为t_1尺度上的宏观方程。

对方程(62)求速度的零阶矩得：

$$\sum_\alpha \frac{\partial f_\alpha^{eq}}{\partial t_2} + \sum_\alpha \left(1 - \frac{1}{2\tau}\right)\left(\overrightarrow{e_\alpha} \cdot \vec{\nabla}_1 + \frac{\partial}{\partial t_1}\right) f_\alpha^{(1)} + \sum_\alpha \frac{1}{\tau\delta_t} f_\alpha^{(2)} = 0 \tag{82}$$

根据方程(6)和(63)得：

$$\frac{\partial \rho}{\partial t_2} = 0 \tag{83}$$

对方程(62)求速度的一阶矩得：

$$\sum_\alpha \frac{\partial f_\alpha^{eq}}{\partial t_2}\overrightarrow{e_{\alpha j}} + \sum_\alpha \left(1 - \frac{1}{2\tau}\right)\left(\overrightarrow{e_{\alpha i}} \cdot \vec{\nabla}_1 + \frac{\partial}{\partial t_1}\right)\left(f_\alpha^{(1)}\overrightarrow{e_{\alpha j}}\right) + \sum_\alpha \frac{1}{\tau\delta_t} f_\alpha^{(2)}\overrightarrow{e_{\alpha j}} = 0$$

(84)

根据方程(7)和(63)得：

$$\frac{\partial \overrightarrow{u_j}}{\partial t_2}+(1-\frac{1}{2\tau})\sum_\alpha\left(\overrightarrow{e_{\alpha i}}\cdot\vec{\nabla}_1\right)(f_\alpha^{(1)}\overrightarrow{e_{\alpha j}})=0 \tag{85}$$

考虑方程左端第二项，根据方程(77)同理可得：

$$\left(\overrightarrow{e_{\alpha i}}\cdot\vec{\nabla}_1\right)\left(f_\alpha^{(1)}\overrightarrow{e_{\alpha j}}\right)=\vec{\nabla}_1\cdot\left(f_\alpha^{(1)}\overrightarrow{e_{\alpha i}e_{\alpha j}}\right) \tag{86}$$

代入方程(85)得：

$$\frac{\partial \overrightarrow{u_j}}{\partial t_2}+(1-\frac{1}{2\tau})\vec{\nabla}_1\cdot\left(\sum_\alpha f_\alpha^{(1)}\overrightarrow{e_{\alpha i}e_{\alpha j}}\right)=0 \tag{87}$$

整理得：

$$\frac{\partial u_j}{\partial t_2}\overrightarrow{E_j}+\left(1-\frac{1}{2\tau}\right)\overrightarrow{E_k}\frac{\partial\left(\sum_\alpha f_\alpha^{(1)}e_{\alpha i}e_{\alpha j}\right)}{\partial r_{1k}}\cdot\overrightarrow{E_i}\overrightarrow{E_j}=\frac{\partial u_j}{\partial t_2}\overrightarrow{E_j}+\left(1-\frac{1}{2\tau}\right)\frac{\partial\left(\sum_\alpha f_\alpha^{(1)}e_{\alpha i}e_{\alpha j}\right)}{\partial r_{1k}}\delta_{ki}\overrightarrow{E_j}=0 \tag{88}$$

即：

$$\frac{\partial u_j}{\partial t_2}+\left(1-\frac{1}{2\tau}\right)\frac{\partial}{\partial r_{1i}}\left(\sum_\alpha f_\alpha^{(1)}e_{\alpha i}e_{\alpha j}\right)=0 \tag{89}$$

由方程(58)可得：

$$f_\alpha^{(1)}=-\tau\delta_t\left(\overrightarrow{e_\alpha}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{eq} \tag{90}$$

则方程(89)左端第二项求和号部分可变为：

$$\sum_\alpha f_\alpha^{(1)}e_{\alpha i}e_{\alpha j}=-\tau\delta_t\sum_\alpha e_{\alpha i}e_{\alpha j}\left(\overrightarrow{e_{\alpha k}}\cdot\vec{\nabla}_1+\frac{\partial}{\partial t_1}\right)f_\alpha^{eq}=-\tau\delta_t[\sum_\alpha e_{\alpha i}e_{\alpha j}\frac{\partial f_\alpha^{eq}}{\partial t_1}+\sum_\alpha e_{\alpha i}e_{\alpha j}(\overrightarrow{e_{\alpha k}}\cdot\vec{\nabla}_1)f_\alpha^{eq}] \tag{91}$$

由方程(68)同理可得：

$$e_{\alpha i}e_{\alpha j}\frac{\partial f_\alpha^{eq}}{\partial t_1}=\frac{\partial f_\alpha^{eq}e_{\alpha i}e_{\alpha j}}{\partial t_1} \tag{92}$$

$$e_{\alpha i}e_{\alpha j}e_{\alpha k}\frac{\partial f_\alpha^{eq}}{\partial r_1}=\frac{\partial f_\alpha^{eq}e_{\alpha i}e_{\alpha j}e_{\alpha k}}{\partial r_1} \tag{93}$$

则方程(91)可变为：

$$\sum_\alpha f_\alpha^{(1)}e_{\alpha i}e_{\alpha j}=-\tau\delta_t\left[\sum_\alpha\frac{\partial f_\alpha^{eq}e_{\alpha i}e_{\alpha j}}{\partial t_1}+\sum_\alpha e_{\alpha i}e_{\alpha j}e_{\alpha k}\overrightarrow{E_k}\cdot\overrightarrow{E_m}\frac{\partial f_\alpha^{eq}}{\partial r_{1m}}\right]=$$

$$-\tau\delta_t\left[\sum_\alpha \frac{\partial f_\alpha^{eq} e_{\alpha i} e_{\alpha j}}{\partial t_1} + \sum_\alpha \frac{\partial f_\alpha^{eq} e_{\alpha i} e_{\alpha j} e_{\alpha k}}{\partial r_{1m}} \delta_{km}\right] =$$

$$-\tau\delta_t\left[\frac{\partial}{\partial t_1}\left(\sum_\alpha f_\alpha^{eq} e_{\alpha i} e_{\alpha j}\right) + \frac{\partial}{\partial r_{1k}}\left(\sum_\alpha f_\alpha^{eq} e_{\alpha i} e_{\alpha j} e_{\alpha k}\right)\right] \tag{94}$$

根据方程(8)得：

$$\sum_\alpha f_\alpha^{(1)} e_{\alpha i} e_{\alpha j} = -\tau\delta_t\left[\frac{\partial}{\partial t_1}\left(\rho u_i u_j + \rho c_s^2 \delta_{ij}\right) + \frac{\partial}{\partial r_{1k}}\left(\sum_\alpha f_\alpha^{eq} e_{\alpha i} e_{\alpha j} e_{\alpha k}\right)\right] \tag{95}$$

其中：

$$\frac{\partial}{\partial t_1}\left(\rho u_i u_j + \rho c_s^2 \delta_{ij}\right) = \frac{\partial}{\partial t_1}\left(\rho u_i u_j\right) + \frac{\partial}{\partial t_1}\left(\rho c_s^2 \delta_{ij}\right) = \frac{\partial}{\partial t_1}\left(\rho u_i u_j\right) + c_s^2 \delta_{ij} \frac{\partial \rho}{\partial t_1} \tag{96}$$

由方程(72)得：

$$\frac{\partial \rho}{\partial t_1} = -\vec{\nabla}_1 \cdot (\rho \vec{u}) \tag{97}$$

代入方程(96)得：

$$\frac{\partial}{\partial t_1}\left(\rho u_i u_j + \rho c_s^2 \delta_{ij}\right) = \frac{\partial}{\partial t_1}\left(\rho u_i u_j\right) - c_s^2 \delta_{ij} \vec{\nabla}_1 \cdot (\rho \vec{u}) \tag{98}$$

由于：

$$\frac{\partial}{\partial t_1}\left(\rho u_i u_j\right) = u_i \frac{\partial}{\partial t_1}\left(\rho u_j\right) + \rho u_j \frac{\partial u_i}{\partial t_1} \tag{99}$$

$$\frac{\partial}{\partial t_1}\left(\rho u_i u_j\right) = u_j \frac{\partial}{\partial t_1}\left(\rho u_i\right) + \rho u_i \frac{\partial u_j}{\partial t_1} \tag{100}$$

$$\frac{\partial}{\partial t_1}\left(\rho u_i u_j\right) = u_i u_j \frac{\partial \rho}{\partial t_1} + \rho \frac{\partial u_i u_j}{\partial t_1} \tag{101}$$

又：

$$\rho \frac{\partial u_i u_j}{\partial t_1} = \rho u_j \frac{\partial u_i}{\partial t_1} + \rho u_i \frac{\partial u_j}{\partial t_1} \tag{102}$$

则由方程(99)加方程(100)减去方程(101)，并联立方程(102)可得：

$$\frac{\partial}{\partial t_1}\left(\rho u_i u_j\right) = u_i \frac{\partial}{\partial t_1}\left(\rho u_j\right) + u_j \frac{\partial}{\partial t_1}\left(\rho u_i\right) - u_i u_j \frac{\partial \rho}{\partial t_1} \tag{103}$$

由方程(40) 、(80)和(81)可得：

$$\frac{\partial \rho u_j}{\partial t_1} = -\frac{\partial}{\partial r_{1k}}\left(\rho u_j u_k + \rho c_s^2 \delta_{jk}\right) \tag{104}$$

同理可将上式j变为i得：

$$\frac{\partial \rho u_i}{\partial t_1} = -\frac{\partial}{\partial r_{1k}}(\rho u_i u_k + \rho c_s^2 \delta_{ik}) \tag{105}$$

则将方程 (97)、(104)、(105)代入方程(103)得：

$$\frac{\partial}{\partial t_1}(\rho u_i u_j) = -u_i \frac{\partial}{\partial r_{1k}}(\rho u_j u_k + \rho c_s^2 \delta_{jk}) - u_j \frac{\partial}{\partial r_{1k}}(\rho u_i u_k + \rho c_s^2 \delta_{ik}) + u_i u_j \vec{\nabla}_1 \cdot (\rho \vec{u}) = -u_i \frac{\partial}{\partial r_{1k}}(\rho u_j u_k) - u_i c_s^2 \frac{\partial \rho}{\partial r_{1j}} - u_j \frac{\partial}{\partial r_{1k}}(\rho u_i u_k) - u_j c_s^2 \frac{\partial \rho}{\partial r_{1i}} + u_i u_j \overrightarrow{E_k} \frac{\partial \rho u_m}{\partial r_{1k}} \cdot \overrightarrow{E_m} = -u_i c_s^2 \frac{\partial \rho}{\partial r_{1j}} - u_j c_s^2 \frac{\partial \rho}{\partial r_{1i}} - u_i \frac{\partial}{\partial r_{1k}}(\rho u_j u_k) - u_j \frac{\partial}{\partial r_{1k}}(\rho u_i u_k) + u_i u_j \frac{\partial \rho u_k}{\partial r_{1k}} \tag{106}$$

根据方程(103)的推导，同理可得：

$$-\frac{\partial}{\partial r_{1k}}(\rho u_i u_j u_k) = -u_i \frac{\partial}{\partial r_{1k}}(\rho u_j u_k) - u_j \frac{\partial}{\partial r_{1k}}(\rho u_i u_k) + u_i u_j \frac{\partial \rho u_k}{\partial r_{1k}} \tag{107}$$

代入方程(106)得：

$$\frac{\partial}{\partial t_1}(\rho u_i u_j) = -u_i c_s^2 \frac{\partial \rho}{\partial r_{1j}} - u_j c_s^2 \frac{\partial \rho}{\partial r_{1i}} - \frac{\partial}{\partial r_{1k}}(\rho u_i u_j u_k) \tag{108}$$

根据方程(4)，可将方程(95)式右端第二项变形为：

$$\frac{\partial}{\partial r_{1k}}\left(\sum_\alpha f_\alpha^{eq} e_{\alpha i} e_{\alpha j} e_{\alpha k}\right) = \frac{\partial}{\partial r_{1k}}\left(\sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k}\, \rho \omega_\alpha [1 + \frac{\overrightarrow{e_\alpha} \cdot \vec{u}}{c_s^2} + \frac{(\overrightarrow{e_\alpha} \cdot \vec{u})^2}{2c_s^4} - \frac{u^2}{2c_s^2}]\right) = \frac{\partial}{\partial r_{1k}}(\sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k}\, \rho \omega_\alpha \left(1 - \frac{u^2}{2c_s^2}\right) + \sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k} e_{\alpha m}\, \rho \omega_\alpha \frac{u_m}{c_s^2} + \sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k} e_{\alpha m} e_{\alpha n}\, \rho \omega_\alpha \frac{u_m u_n}{2c_s^4}) \tag{109}$$

根据方程(10)可得

$$\sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k}\, \rho \omega_\alpha \left(1 - \frac{u^2}{2c_s^2}\right) = 0 \tag{110}$$

$$\sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k} e_{\alpha m} e_{\alpha n}\, \rho \omega_\alpha \frac{u_m u_n}{2c_s^4} = 0 \tag{111}$$

代入方程(109)得：

$$\frac{\partial}{\partial r_{1k}}\left(\sum_\alpha f_\alpha^{eq} e_{\alpha i} e_{\alpha j} e_{\alpha k}\right) = \frac{\partial}{\partial r_{1k}}\left(\sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k} e_{\alpha m}\, \rho \omega_\alpha \frac{u_m}{c_s^2}\right) \tag{112}$$

又：

$$\sum_\alpha e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}\,\rho\omega_\alpha\frac{u_m}{c_s^2}=\omega_1\frac{\rho u_m}{c_s^2}\sum_{\alpha=1}^4 e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}+\omega_5\frac{\rho u_m}{c_s^2}\sum_{\alpha=5}^8 e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}\tag{113}$$

根据方程(13)和(14)得：

$$\sum_\alpha e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}\,\rho\omega_\alpha\frac{u_m}{c_s^2}=\frac{\rho u_m}{9c_s^2}\cdot 2c^4\delta_{ijkm}+\frac{\rho u_m}{36c_s^2}\cdot\left(4c^4\Delta_{ijkm}-8c^4\delta_{ijkm}\right)=\frac{\rho u_m}{9c_s^2}c^4\Delta_{ijkm}\tag{114}$$

根据方程(39)得：

$$\sum_\alpha e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha m}\,\rho\omega_\alpha\frac{u_m}{c_s^2}=\frac{\rho u_m}{9c_s^2}\cdot 9c_s^4\Delta_{ijkm}=\rho u_m c_s^2\Delta_{ijkm}=\rho u_m c_s^2\left(\delta_{ij}\delta_{km}+\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}\right)=\rho c_s^2\left(\delta_{ij}u_k+\delta_{ik}u_j+\delta_{jk}u_i\right)\tag{115}$$

代入方程(112)得：

$$\frac{\partial}{\partial r_{1k}}\left(\sum_\alpha f_\alpha^{eq}e_{\alpha i}e_{\alpha j}e_{\alpha k}\right)=\frac{\partial}{\partial r_{1k}}\left(\rho c_s^2\left(\delta_{ij}u_k+\delta_{ik}u_j+\delta_{jk}u_i\right)\right)=\left(\frac{\partial}{\partial r_{1k}}(\rho u_k c_s^2\delta_{ij})+\frac{\partial}{\partial r_{1k}}(\rho c_s^2\delta_{ik}u_j)+\frac{\partial}{\partial r_{1k}}(\rho c_s^2\delta_{jk}u_i)\right)=c_s^2\vec{\nabla}_1\cdot(\rho\vec{u})\delta_{ij}+\rho c_s^2\frac{\partial u_j}{\partial r_{1i}}+\rho c_s^2\frac{\partial u_i}{\partial r_{1j}}+c_s^2u_j\frac{\partial\rho}{\partial r_{1i}}+c_s^2u_i\frac{\partial\rho}{\partial r_{1j}}\tag{116}$$

将方程(98)、(108)和(116)代入方程(95)得：

$$\sum_\alpha f_\alpha^{(1)}e_{\alpha i}e_{\alpha j}=-\tau\delta_t\left[-u_ic_s^2\frac{\partial\rho}{\partial r_{1j}}-u_jc_s^2\frac{\partial\rho}{\partial r_{1i}}-\frac{\partial}{\partial r_{1k}}(\rho u_iu_ju_k)-c_s^2\delta_{ij}\vec{\nabla}_1\cdot(\rho\vec{u})+c_s^2\vec{\nabla}_1\cdot(\rho\vec{u})\delta_{ij}+\rho c_s^2\frac{\partial u_j}{\partial r_{1i}}+\rho c_s^2\frac{\partial u_i}{\partial r_{1j}}+c_s^2u_j\frac{\partial\rho}{\partial r_{1i}}+c_s^2u_i\frac{\partial\rho}{\partial r_{1j}}\right]=-\tau\delta_t\left[-\frac{\partial}{\partial r_{1k}}\left(\rho u_iu_ju_k\right)+\rho c_s^2\left(\frac{\partial u_j}{\partial r_{1i}}+\frac{\partial u_i}{\partial r_{1j}}\right)\right]\tag{117}$$

代入方程(89)得：

$$\frac{\partial u_j}{\partial t_2}+\left(1-\frac{1}{2\tau}\right)\frac{\partial}{\partial r_{1i}}\left\{-\tau\delta_t\left[-\frac{\partial}{\partial r_{1k}}\left(\rho u_iu_ju_k\right)+\rho c_s^2\left(\frac{\partial u_j}{\partial r_{1i}}+\frac{\partial u_i}{\partial r_{1j}}\right)\right]\right\}=0\tag{118}$$

由方程(72)乘以K_n加上方程(83)乘以K_n^2得：

$$K_n\frac{\partial\rho}{\partial t_1}+K_n\vec{\nabla}_1\cdot(\rho\vec{u})+K_n^2\frac{\partial\rho}{\partial t_2}=0\tag{119}$$

联立方程(47)和(48)得：

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{120}$$

方程(120)即为宏观连续方程(2)。由方程(81)乘以K_n加上方程(118)乘以K_n^2得：

$$K_n \frac{\partial \rho u_j}{\partial t_1} + K_n \frac{\partial \rho u_i u_j}{\partial r_{1i}} + K_n^2 \frac{\partial u_j}{\partial t_2} + K_n^2 \left(1 - \frac{1}{2\tau}\right) \frac{\partial}{\partial r_{1i}} \left\{-\tau \delta_t \left[-\frac{\partial}{\partial r_{1k}} \left(\rho u_i u_j u_k\right) + \rho c_s^2 \left(\frac{\partial u_j}{\partial r_{1i}} + \frac{\partial u_i}{\partial r_{1j}}\right)\right]\right\} = -K_n \frac{\partial p}{\partial r_{1j}} \tag{121}$$

联立方程(47)和(48)得：

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_i u_j}{\partial r_i} = -\frac{\partial p}{\partial r_j} + K_n^2 \left(1 - \frac{1}{2\tau}\right) \frac{\partial}{\partial r_{1i}} \left\{\tau \delta_t \left[-\frac{\partial}{\partial r_{1k}} \left(\rho u_i u_j u_k\right) + \rho c_s^2 \left(\frac{\partial u_j}{\partial r_{1i}} + \frac{\partial u_i}{\partial r_{1j}}\right)\right]\right\} \tag{122}$$

考虑方程右端第二项：

$$K_n^2 \left(1 - \frac{1}{2\tau}\right) \frac{\partial}{\partial r_{1i}} \left\{\tau \delta_t \left[-\frac{\partial}{\partial r_{1k}} \left(\rho u_i u_j u_k\right) + \rho c_s^2 \left(\frac{\partial u_j}{\partial r_{1i}} + \frac{\partial u_i}{\partial r_{1j}}\right)\right]\right\} = K_n \frac{\partial}{\partial r_{1i}} \left\{\delta_t (\tau - \frac{1}{2}) \left[-K_n \frac{\partial}{\partial r_{1k}} \left(\rho u_i u_j u_k\right) + K_n \rho c_s^2 \left(\frac{\partial u_j}{\partial r_{1i}} + \frac{\partial u_i}{\partial r_{1j}}\right)\right]\right\} = \frac{\partial}{\partial r_i} \left\{\delta_t (\tau - \frac{1}{2}) \left[-\frac{\partial}{\partial r_k} \left(\rho u_i u_j u_k\right) + \rho c_s^2 \left(\frac{\partial u_j}{\partial r_i} + \frac{\partial u_i}{\partial r_j}\right)\right]\right\} = \frac{\partial}{\partial r_i} \left\{\rho c_s^2 \delta_t \left(\tau - \frac{1}{2}\right) \left(\frac{\partial u_j}{\partial r_i} + \frac{\partial u_i}{\partial r_j}\right) - c_s^2 \delta_t (\tau - \frac{1}{2}) \cdot \frac{1}{c_s^2} \frac{\partial}{\partial r_k} \left(\rho u_i u_j u_k\right)\right\} \tag{123}$$

令：

$$v = c_s^2 \delta_t (\tau - \frac{1}{2}) \tag{124}$$

代入公式(123)得：

$$K_n^2 \left(1 - \frac{1}{2\tau}\right) \frac{\partial}{\partial r_{1i}} \left\{\tau \delta_t \left[-\frac{\partial}{\partial r_{1k}} \left(\rho u_i u_j u_k\right) + \rho c_s^2 \left(\frac{\partial u_j}{\partial r_{1i}} + \frac{\partial u_i}{\partial r_{1j}}\right)\right]\right\} = \frac{\partial}{\partial r_i} \left\{\rho v \left(\frac{\partial u_j}{\partial r_i} + \frac{\partial u_i}{\partial r_j}\right) - \frac{v}{c_s^2} \frac{\partial}{\partial r_k} \left(\rho u_i u_j u_k\right)\right\} \tag{125}$$

代入公式(122)得：

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_i u_j}{\partial r_i} = -\frac{\partial p}{\partial r_j} + \frac{\partial}{\partial r_i} \left\{\rho v \left(\frac{\partial u_j}{\partial r_i} + \frac{\partial u_i}{\partial r_j}\right) - \frac{v}{c_s^2} \frac{\partial}{\partial r_k} \left(\rho u_i u_j u_k\right)\right\} \tag{126}$$

即：

$$\frac{\partial}{\partial t} (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) = -\vec{\nabla} P + \vec{\nabla} \cdot \left[\rho v \left(\vec{\nabla} \vec{u} + (\vec{\nabla} \vec{u})^T\right) - \frac{v}{c_s^2} \vec{\nabla} \cdot (\rho \vec{u} \vec{u} \vec{u})\right]$$

(127)

式中，T表示转置。当密度ρ为常数，且流动为低马赫数流动，$\vec{\nabla}\cdot(\rho\vec{u}\vec{u}\vec{u})$项可以忽略[2]，又：

$$\vec{\nabla}\cdot\left(\vec{\nabla}\vec{u}\right)^T=\overrightarrow{E_k}\frac{\partial}{\partial r_k}\left(\frac{\partial u_i}{\partial r_j}\right)\cdot\overrightarrow{E_\iota}\overrightarrow{E_J}=\frac{\partial}{\partial r_i}\left(\frac{\partial u_i}{\partial r_j}\right)\overrightarrow{E_J} \tag{128}$$

交换右端偏微分顺序得[2]：

$$\vec{\nabla}\cdot\left(\vec{\nabla}\vec{u}\right)^T=\frac{\partial}{\partial r_j}\left(\frac{\partial u_i}{\partial r_i}\right)\overrightarrow{E_J}=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right)=0 \tag{129}$$

与方程(76)类似，可推得：

$$\vec{\nabla}\cdot(\rho\vec{u}\vec{u})=\left(\vec{u}\cdot\vec{\nabla}\right)(\rho\vec{u})+\rho\vec{u}\vec{\nabla}\cdot\vec{u} \tag{130}$$

又当密度ρ为常数时，宏观连续方程(120)变为：

$$\vec{\nabla}\cdot\vec{u}=0 \tag{131}$$

代入方程(130)得：

$$\vec{\nabla}\cdot(\rho\vec{u}\vec{u})=\left(\vec{u}\cdot\vec{\nabla}\right)(\rho\vec{u}) \tag{132}$$

联立方程(127)、(129)、(132)得：

$$\frac{\partial\vec{u}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}=-\frac{1}{\rho}\vec{\nabla}P+v\Delta\vec{u} \tag{130}$$

方程(136)即为宏观不含外力项的不可压缩 N-S 方程(3)，推导完成。

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