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CSE 250A

1.1

a) left side $= P(X,Y|E) = \frac{P(X,Y,E)}{P(E)}$

right side $= P(X|Y,E)\cdot P(Y|E) = \frac{P(X,Y,E)}{P(Y,E)} \cdot \frac{P(Y,E)}{P(E)} = \frac{P(X,Y,E)}{P(E)}$

So, left side = right side, proof done

b) $P(X|Y,E) = \frac{P(Y,E|X)\cdot P(X)}{P(Y,E)}$

$= \frac{P(Y|E,X)\cdot P(E|X)\cdot P(X)}{P(Y,E)}$ using proof from a)

$= \frac{P(Y|E,X)\cdot P(X|E)\cdot P(E)}{P(Y|E)\cdot P(E)}$ bayes rule
product rule

$= \frac{P(Y|E,X)\cdot P(X|E)}{P(Y|E)}$ proof done.

c) $P(X|E) = \frac{P(X,E)}{P(E)} = \frac{\sum_y P(X,Y=y,E)}{P(E)}$ product rule

given that $\frac{P(X,Y=y,E)}{P(E)} = P(X,Y=y|E)$ product rule

so $P(X|E) = \sum_y P(X,Y=y|E)$

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1.2

a) as proved in 1.1 a) $P(X,Y|E) = P(X|Y,E) \cdot P(Y|E)$

so (1) implies that $P(X|Y,E) = P(X|E)$ which is (2)

$$P(X,Y|E) = \frac{P(X,Y,E)}{P(E)} = \frac{P(X,Y,E)}{P(X,E)} \times \frac{P(X,E)}{P(E)} = P(Y|X,E) \cdot P(X|E)$$

so (1) implies that $P(Y|X,E) = P(Y|E)$, which is (3)

b) as proved previously $P(X,Y|E) = P(X|Y,E) \cdot P(Y|E)$

so (2) implies that $P(X,Y|E) = P(X|E) \cdot P(Y|E)$ which is (1)

as proved in a), given (1), (2); (3) is also correct

So (2) can imply both (1) and (3)

c) as proved previously $P(X,Y|E) = P(Y|X,E) \cdot P(X|E)$

So (3) implies that $P(Y|X,E) = P(Y|E)$, which is (1)

given (1), as proved in a), we can imply that $P(X|Y,E) = P(X,E)$

So (3) implies (1) and (2)

Thus (1), (2), (3) are equivalent

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1.3 $X, Y, Z \in \{0, 1\}$

a) X: whether a person is infected with flu $X \in \{0,1\}$ ($X=1$ means infected)

Y: whether a person has a fever. $Y \in \{0,1\}$ ($Y=1$ means doesn't have fever)

Z: whether a person coughs $Z \in \{0,1\}$ ($Z=1$ means doesn't cough)

$P(X=1) < P(X=1 \mid Y=1) < P(X=1 \mid Y=1, Z=1)$

b) X: whether a person is dangerous $X=1$ means this person is dangerous

Y: whether a person is armed $Y=1$ means this person is armed

Z: whether a person is a police officer. $Z=1$ means yes

$P(X=1) < P(X=1 \mid Y=1) \quad P(X=1 \mid Y=1, Z=1) < P(X=1 \mid Y=1)$

c) X: whether a person stays up really late. $X=1$ means yes

Y: whether a person will be late for school, $Y=1$ means yes

Z: whether a person will lay bed in the morning $Z=1$ means yes

$P(X=1, Y=1) \neq P(X=1)\, P(Y=1)$

$P(X=1, Y=1 \mid Z=1) = P(X=1 \mid Z=1) \times P(Y=1 \mid Z=1)$

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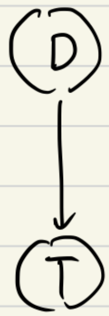
1.4 $D=1$ means cyclists use drugs $T=1$ means test result positive

a)



$P(D=1) = 0.01 \quad P(D=0) = 0.99$

$P(T\|D)$	D	T
0.95	0	0
0.1	1	0
0.05	0	1
0.90	1	1

b) $P(D=0 \mid T=0) = P(T=0 \mid D=0) \cdot P(D=0) / P(T=0) = 0.95 \times 0.99 / P(T=0)$

$P(T=0) = \sum_D P(T=0, D) = P(T=0, D=0) + P(T=0, D=1)$

$= P(D=0) \cdot P(T=0 \mid D=0) + P(D=1) \cdot P(T=0 \mid D=1) = 0.9415$ So answer $= 0.9989$

c) $P(D=1 \mid T=1) = \dfrac{P(T=1 \mid D=1) \cdot P(D=1)}{P(T=1)}$

$= \dfrac{0.9 \times 0.01}{0.01 \times 0.9 + 0.05 \times 0.99}$

$\cong 0.154$

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1.5

a) $\mathcal{H}(x) = -\sum_{i=1}^{n} p_i \log p_i$ $\qquad \nabla f = \begin{bmatrix} -(1+\log p_1) \\ -(1+\log p_2) \\ \vdots \\ -(1+\log p_n) \end{bmatrix}$

$g(x) = \sum_i p_i \qquad \nabla g = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$\nabla f = \lambda \nabla g \Rightarrow -(1+\log p_i) = \lambda$ for $i = 1, 2, \cdots, n$

so $p_1 = p_2 = p_3 = \cdots = p_n$ since $\sum_i p_i = 1$ $\quad p_i = \frac{1}{n}$

b) first we have $H(X_i) = -\sum_{x_i} P(x_i) \log P(x_i)$ & we have $\sum_i P(x_i) = 1$

$$H(X_1, \cdots, X_n) = -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n}\prod_{i=1}^{n} P(x_i) \cdot \log\left(\prod_{i=1}^{n} P(x_i)\right)$$

$$= -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_{n-1}}\prod_{i=1}^{n-1} P(x_i) \sum_{x_n} P(x_n)\left[\sum_{i=1}^{n-1}\log P(x_i) + \log P(x_n)\right]$$

$$= -\sum_{x_1} P(x_1)\sum_{x_2} P(x_2)\cdots\sum_{x_{n-1}} P(x_{n-1})\cdot\left[\sum_{x_n} P(x_n)\cdot\left(\sum_{i=1}^{n-1}\log P(x_i)\right) + \sum_{x_n} P(x_n)\cdot\log P(x_n)\right]$$

$$= -\sum_{x_1} P(x_1)\sum_{x_2} P(x_2)\cdots\sum_{x_{n-1}} P(x_{n-1})\cdot\left[\sum_{i=1}^{n-1}\log P(x_i) - H(X_n)\right]$$

$$= -\sum_{x_1} P(x_1)\times\cdots\times\sum_{x_{n-1}} P(x_{n-1})\left[-H(X_n) + \log P(x_{n-1}) + \sum_{i=1}^{n-2}\log P(x_i)\right]$$

// $\sum_{i=1}^{n-2}\log P(x_i)$

$$= -\sum_{x_1} P(x_1)\cdots\sum_{x_{n-2}} P(x_{n-2})\times\left[-H(X_n) - H(X_{n-1}) + \sum_{x_{n-1}}\left(P(x_{n-1})\cdot\sum_{i=1}^{n-2}\log P(x_i)\right)\right]$$

$$= -\sum_{x_1} P(x_1)\times\cdots\times\sum_{x_{n-2}} P(x_{n-2})\left[-H(X_n) - H(X_{n-1}) + \sum_{i=1}^{n-2}\log P(x_i)\right]$$

After that we can keep iterating

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if we keep repeating this procedure, we can find that

$$\sum_{x_j} P(x_j) \times \sum_{i=1}^{j} \log P(x_i) + C = C - H(X_j) + \sum_{i=1}^{j-1} \log P(x_i)$$ given C is constant

finally we will have $H(X_1, \cdots, X_n) = -\sum_{x_1} p(x_1) \cdot [-\sum_{i=2}^{n} H(X_i) + \log p(x_1)]$

$$= \sum_{x_1} p(x_1) \cdot \sum_{i=2}^{n} H(X_i) - \sum_{x_1} p(x_1) \log p(x_1)$$

$$= \sum_{i=2}^{n} H(X_i) + H(X_1) = \sum_{i=1}^{n} H(X_i)$$

1.6

a) $f(x) = \log(x) - (x-1)$ $\quad \frac{d}{dx} f(x) = \frac{1}{x} - 1$ given $x \in (0, +\infty)$

when $x \in (0, 1]$ $f(x)$ is monotonically increasing $x \to 0$ $f(x) \to -\infty$ $x = 1$ $f(x) = 0$

when $x \in (1, +\infty)$ $\frac{df}{dx} < 0$ $f(x)$ is monotonically decreasing $f(x) < 0$ when $x > 1$

thus $\max f(x) = f(1) = 0$

so $\log(x) \le (x-1)$, the equality holds if and onlf if $x = 1$

b) $KL(p, q) = \sum_i p_i \log(p_i / q_i) = -\sum_i p_i \log\left(\frac{q_i}{p_i}\right)$

given $\log(x) \le x - 1$ we have $-\sum_i p_i \log\left(\frac{q_i}{p_i}\right) \ge -\sum_i p_i \left(\frac{q_i}{p_i} - 1\right)$

$= -\sum_i q_i + \sum_i p_i = 0$ So $KL(p, q) \ge 0$

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as proved in a) $\log(x) \le x-1$ the equality holds if and only if $x=1$

$-\sum_i p_i \log(\frac{q_i}{p_i}) \le -\sum_i (\frac{q_i}{p_i}-1)$ the equality holds if and only if

$\frac{q_i}{p_i}=1$ for every i, so $KL(p,q) \ge 0$, the equality holds if and only if $p_i = q_i$ for every i

c) $KL(p,q) = \sum_i p_i \log(\frac{p_i}{q_i})$

$-KL(p,q) = -\sum_i p_i \log(\frac{q_i}{p_i}) = -2\sum_i p_i \log\sqrt{\frac{q_i}{p_i}} \ge -2\sum_i p_i (\sqrt{\frac{q_i}{p_i}}-1)$

$= -2\sum_i \sqrt{p_i q_i} - p_i = \sum_i 2p_i - 2\sqrt{p_i q_i}$

given $\sum_i p_i = \sum_i q_i = 1$ $\sum_i 2p_i - 2\sqrt{p_i q_i} = \sum_i p_i + q_i - 2\sqrt{p_i q_i}$

$= \sum_i (\sqrt{p_i} - \sqrt{q_i})^2$

So $KL(p,q) \ge \sum_i (\sqrt{p_i} - \sqrt{q_i})^2$

d) $X \in \{0,1\}$ $P(X=0) = P(X=1) = 0.5$

$Y \in \{0,1\}$ $P(Y=0) = 0.4$ $P(Y=1) = 0.6$

$KL(X,Y) = \sum_i x_i \log(\frac{x_i}{y_i}) = 0.5\log(\frac{0.5}{0.6}) + 0.5\log(\frac{0.5}{0.4}) \cong 0.0204$

$KL(Y,X) = \sum_i y_i \log(\frac{y_i}{x_i}) = 0.4\log(\frac{0.4}{0.5}) + 0.6\log(\frac{0.6}{0.5}) \cong 0.0201$

$KL(X,Y) \ne KL(Y,X)$

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1.7

a) $I(X,Y) = \sum_x \sum_y p(x,y) \log\left[\frac{p(x,y)}{p(x)p(y)}\right]$

$= -\sum_x \sum_y p(x,y) \log \frac{p(x)p(y)}{p(x,y)} \geq -\sum_x \sum_y p(x,y)\left(\frac{p(x)p(y)}{p(x,y)} - 1\right)$

$= -\sum_x \left(\sum_y p(x)p(y) - p(x,y)\right)$

$= -\left(\sum_x p(x) \sum_y p(y) - \sum_x \sum_y p(x,y)\right)$

$= -\left(\sum_x p(x) - \sum_x p(x)\right)$ marginalization rule

$= 0$ So $I(X,Y) \geq 0$

b) 1° we need to prove that if X, Y are independent, $I(X,Y) = 0$

X, Y independent $p(x,y) = p(x)p(y)$

$I(X,Y) = \sum_x \sum_y p(x,y) \log(1) = 0$

2° we need to prove if $I(X,Y) = 0$, X, Y are independent

$I(X,Y) = 0 \Rightarrow \frac{p(x)p(y)}{p(x,y)} = 1$

so $p(x,y) = p(x)p(y)$ which means X, Y are independent

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hw1

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1.8

a) Yes BN_1 implies Y and Z are conditional independent given X

b) No, BN_2 and BN_3 contains the same information in terms of marginal and conditional prob

c) Yes. BN_3 implies X and Z are conditional independent