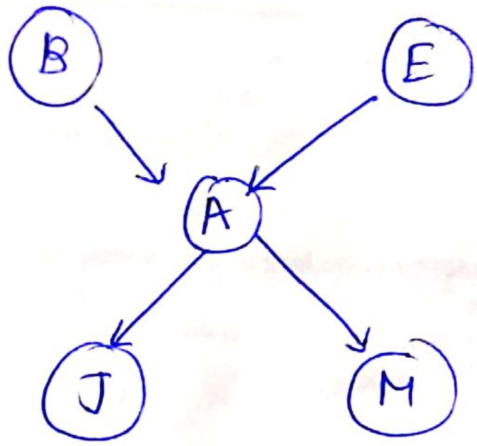
## Review.

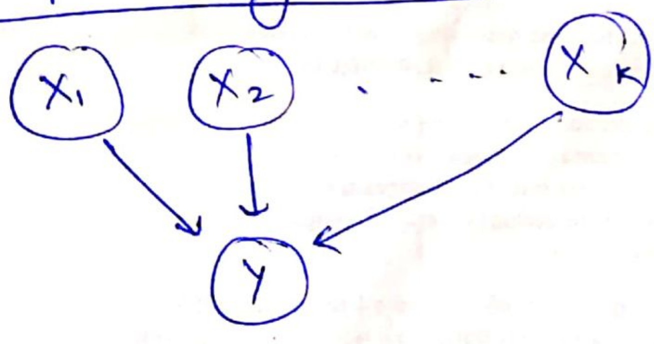



Burglary?
Earthquake?
Alarm?
John Calls?
Mary calls?

* Belief Network $BN = DAG + CPTs$
* Conditional Independence on all other ancestors given pare

$$P(x_i \mid x_1, \ldots x_{i-1}) = P(x_i \mid pa(x_i))$$

## Representing CPTs




For simplicity, assume binary variables i.e. $\{0,1\}$ are the values

$X_i \in \{0,1\}$ , $Y \in \{0,1\}$

How to represent $P(Y=1 \mid X_1, X_2, \ldots$

① Lookup table $O(2^k)$ rows store arbitrary CPT

| $X_1$ | $X_2$ | $\ldots$ | $X_k$ | $P(Y=1 \mid X_1, \ldots X_k)$ |
|---|---|---|---|---|
| 0 | 0 | | | 0.1 |
| 1 | 0 | | | 0.9 |
| $\vdots$ | $\vdots$ | | | $\vdots$ |
| $\vdots$ | $\vdots$ | | | 0.6 |

$2^k$ combos

too many combinations if k is large.

(2) Deterministic CPT

"AND" function as a CPT $P(Y=1 \mid X_1, \dots X_k) = \prod_{i=1}^{k} X_i$

"OR" function as a CPT $P(Y=0 \mid X_1, \dots X_k) = \prod_{i=1}^{k} (1 - X_i)$

(3) Noisy-OR CPT

Use $k$ numbers $p_i \in [0,1]$ to parametrize all $2^k$ entries in CPT.

$$P(Y=0 \mid X_1, X_2, \dots X_k) = \prod_{i=1}^{k} (1-p_i)^{X_i}$$

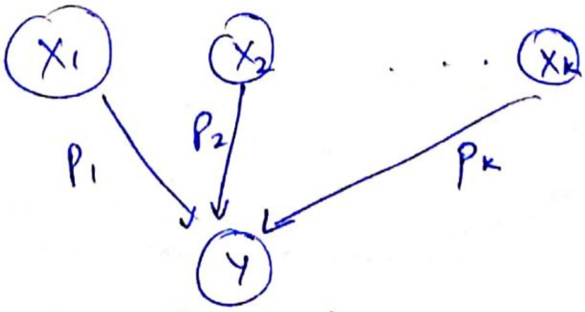
$$P(Y=1 \mid X_1, X_2, \dots X_k) = 1 - \prod_{i=1}^{k} (1-p_i)^{X_i}$$

<u>Why "Noisy-OR"?</u> Look at prob. that $Y=1$ when exactly one parent is "on" (equal to 1):

$$P(Y=1 \mid X_1=0, X_2=0, \dots X_{j-1}=0, X_j=1, X_{j+1}=0, \dots X_k=0)$$

$$= 1 - (1-p_1)^0 \cdots (1-p_j)^1 \cdots (1-p_k)^0$$

$$= 1 - (1-p_j) = p_j$$



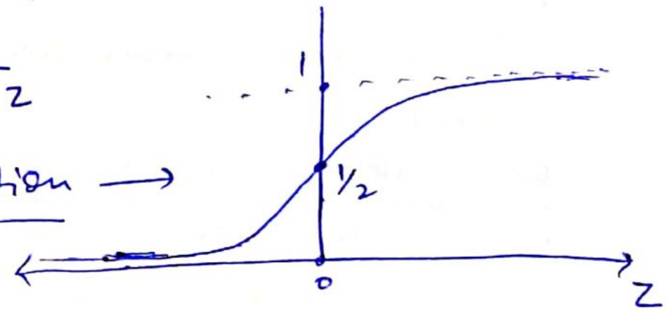

Intuitively, $p_i \in [0,1]$ is trigger prob. that $\underbrace{X_i = 1}_{\text{by itself}}$ causes $Y=1$. Setting all $p_i = 1$ for $i = 1, 2, \dots k$ recovers logical OR.

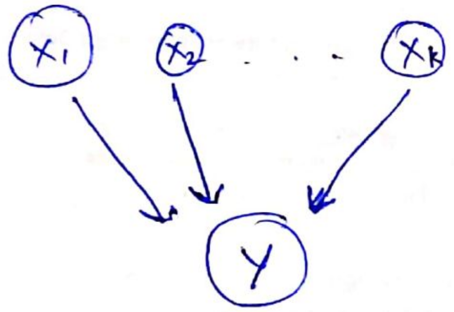
<u>Example</u> : parents are diseases and child is symptomps.
The more diseases you have, the more probability of symptomps.

(4) Sigmoid CPT

Use $k$ real numbers $\theta_i$ to parametrize $2^k$ rows of CPT. Let $\sigma(z) = \dfrac{1}{1+e^{-z}}$

Sigmoid function →




$$P(Y=1 \mid X_1, X_2, \ldots X_k) = \sigma\left(\sum_{i=1}^{k} \theta_i X_i\right)$$

Also known as logistic regression in Statistics or activation function in Neural Nets.

- If $\theta_i$ strongly negative, then $X_i=1$ tends to inhibit / supress $Y=1$
- If $\theta_i$ strongly positive, then $X_i=1$ tends to activate / excite $Y=1$
- Unlike a Noisy-OR CPT, sigmoid CPT can mix inhibition / excitation.

## Conditional Independence

- A node $X_i$ is conditionally independent of its non-parent ancestors given its parent.

$$P(X_i \mid X_1, X_2, \dots X_{i-1}) = P(X_i \mid pa(X_i))$$

- More generally:

Let X, Y, and E refer to disjoint sets of nodes in BN

When is X conditionally independent of Y given E?

When is $P(X \mid E, Y) = P(X \mid E)$ ?

When is $P(X, Y \mid E) = P(X \mid E)\, P(Y \mid E)$ ?

### d-separation

(direction-dependent (in DAGs)

Relates conditional independence to graph theoretic properties.

**Thm:** $P(X, Y \mid E) = P(X \mid E)\, P(Y \mid E)$ if and only if every (undirected) path from a node in X to a node in Y is "d-separated" by E.

**Def:** a path $\pi$ is d-separated if there exists a node $Z \in \pi$ for which one of three conditions hold:

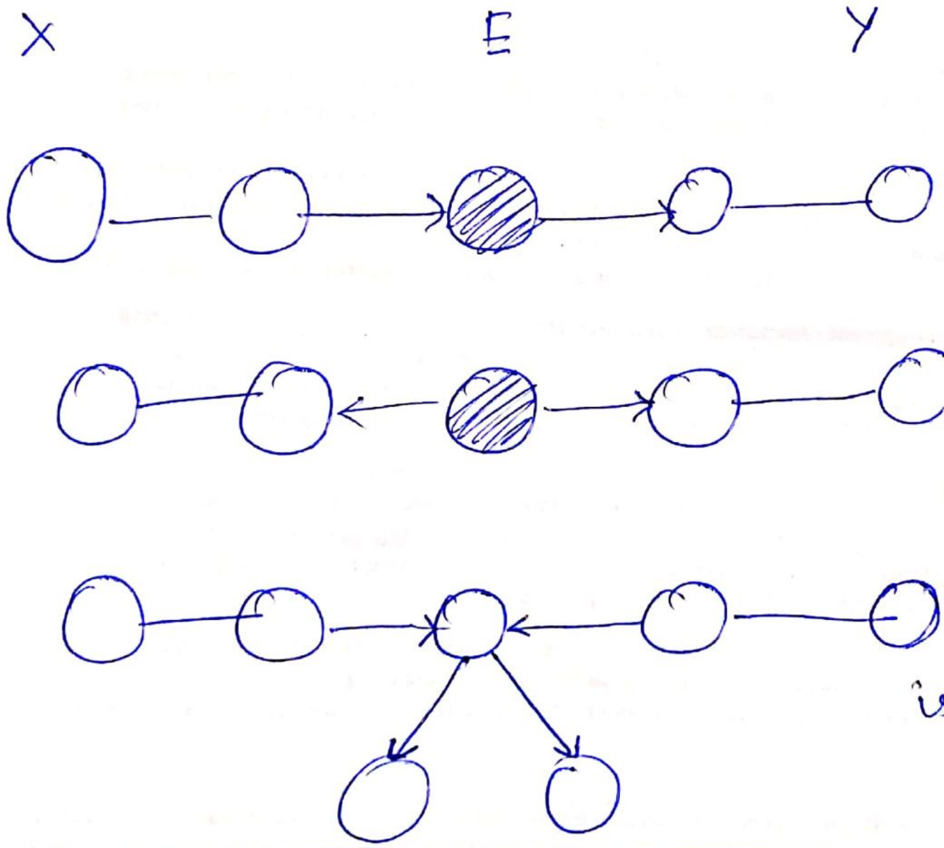
1) $Z \in E$ with $\rightarrow Z \rightarrow$ edges flow through the node Z

$\leftarrow Z \leftarrow$

2) $Z \in E$ with ← Z → edges diverge from Z

3) $Z \in E$ ~~with~~ with → Z ← edges converge on Z

descendants(Z) $\notin E$

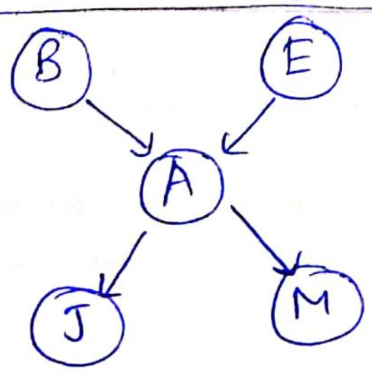



$Z \in E$ is an intervening event in a causal chain

$Z \in E$ is a common explanation or cause of X and Y.

$Z \in E$, desc(Z) $\notin E$
↓
is a common effect (unobserved).

Ex.




True or false?

1) $P(B|A,M) \overset{?}{=} P(B|A)$ True

path B → A → M

Checks out → A → (1) ✓

2) $P(J, M \mid A) \stackrel{?}{=} P(J \mid A)\, P(M \mid A)$ **TRUE**.

$J \leftarrow A \rightarrow M$

satisfies (ii)

3) $P(B) \stackrel{?}{=} P(B \mid E)$ **TRUE**

$B \rightarrow A \leftarrow E$

satisfies (iii)

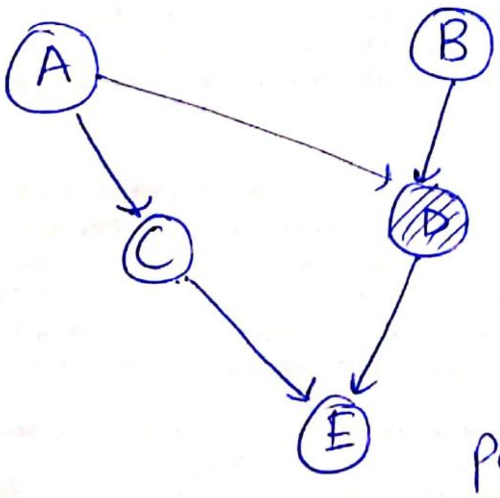
4) $P(B \mid M) \stackrel{?}{=} P(B \mid M, E)$ **FALSE**

$B \rightarrow A \leftarrow E$

A violates (i), (ii), (iii) $\rightarrow$ desc(A) is E

explaining away.

Ex.



true or false

**FALSE**.

$P(B \mid D, E) \stackrel{?}{=} P(B \mid D)$

path $B \rightarrow D \rightarrow E$

Condition (i) d-separation

path $B \rightarrow D \leftarrow A \rightarrow C \rightarrow E$

violates cond (iii).

all 3 conds. violated.

* Proof that d-sep $\iff$ conditional independence. is not trivial.

* Algorithms exist for efficient tests of d-separation.

~~*~~ ~~P~~ $\boxed{\text{Inference}}$

* Problem.

  E – set of evidence nodes

  Q – set of query nodes.

  How to compute posterior distribution $P(Q|E)$?

* Types of inference
  - diagnostic reasoning from effects to causes.

    $P(B=1 \mid M=1)$
  - causal reasoning from causes to effects

    $P(M=1 \mid B=1)$
  - explaining away (about multiple causes)

    $P(B=1 \mid A=1, E=1)$
  - mixed reasoning $P(B=1, M=1 \mid J=1, A=1)$

When can inference be done efficiently?

i.e. (polynomial time in size of DAG and CPTs)