Lecture 5 Oct. 10^{th}

$\boxed{\text{Review}}$

* $P(X, Y \mid E) = P(X \mid E)P(Y \mid E)$ when all path from X to Y are d-separated by E.

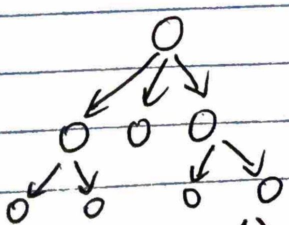
* A path π is d-separated if there exists a node ~~$z \in E$~~ $z \in \pi$ for which
  1) $z \in E$: $\rightarrow z \rightarrow$ (causuel chain)
  2) $z \in E$: $\leftarrow z \rightarrow$ (common cause)
  3) $z \notin E$, $\underbrace{\text{descendent}}$ $(z) \notin E$.
     child / grand children / ---- (not necessarily immediate)
     $\rightarrow z \leftarrow$, $\swarrow \searrow$ (unobserved common effect).

$\boxed{\text{Inference}}$

Query nodes Q
Evidence nodes E
How to compute $P(Q \mid E)$?
More precisely, how to express $P(Q \mid E)$, in terms of CPTs $\underbrace{P(X_i \mid pa(X_i))}_{\text{Assumed to be given.}}$?

Question : * When can inference be done efficiently?

Answer : In polytree (- a graph w/o loops
- singly connect network
- between $\forall$ 2 nodes there's at most one path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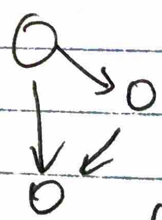
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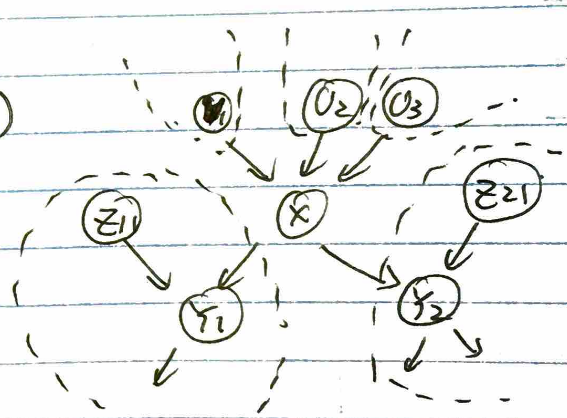
* Strategies :
  1) Bayes Rule
  2) Product Rule
  3) Marginal / conditional indep
  4) Combination of 1) ~ 3)

1) express $P(Q|E)$ in terms of conditional probs that repeat order of DAG
2) express joint predictions in terms of individual predictions.
3) introduce nodes to predict children from parents (using CPTs)
4) remove non-parent ancestors from RHS of conditional bar.

<u>Goal</u> : Compute $P(X=x|E)$



Node X
Evidence E
Assume $X \notin E$.
Parents : $U_1, U_2, U_3, \cdots$
Children : $Y_1, Y_2, \cdots$
Spouses : Z_{jk} (parents of jth child excluding X).

Draw one box for each parent & child.
claim : <u>boxes don't overlap for polytree</u>.

Types of Evidence : E_X^+ — evidence connected to X thru parents.
E_X^- — ---- children
$E = E_X^+ \cup E_X^-$

General Strategy — recursion

$$P(X=x|E) = P(X=x|E_x^-, E_x^+)$$

$$= \frac{P(E_x^-|X=x, E_x^+)\, P(X=x|E_x^+)}{P(E_x^-|E_x^+)}$$

$\boxed{\text{Bayes Rule}}$

$$= P(E_x^-|X=x)\, P(X=x|E_x^+) / P(E_x^-|E_x^+)$$

$\boxed{\text{CI + d-sep rule 1)}}$

- Plan of attack: - compute terms in numerator
  - compute denominator via

Since: $\sum_x P(X=x|E) = 1 = \frac{\sum_x P(E_x^-|X=x)\, P(X=x|E_x^+)}{P(E_x^-|E_x^+)}$

$$\Rightarrow \quad P(E_x^-|E_x^+) = \sum_x P(E_x^-|X=x)\, P(X=x|E_x^+)$$

* Recursion through parents:

$$P(X=x|E_x^+) = \cancel{\sum_{\vec{u}} P(E_x|X=x}$$

$$= \sum_{\vec{u}} P(X=x, \vec{U}=\vec{u}\,|\,E_x^+)$$

where $\vec{U} = (u_1, u_2, \ldots)$

$$= \sum_{\vec{u}} P(\vec{U}=\vec{u}|E_x^+)\, P(X=x|\vec{U}=\vec{u}, E_x^+)$$

$$= \sum_{\vec{u}} P(\vec{U}=\vec{u}|E_x^+)\, P(X=x|\vec{U}=\vec{u})$$

$\boxed{\text{(CI)}}$

$$= \sum_{\vec{u}} P(X=x \mid U=\vec{u}) \left\{ \prod_{i=1}^{n} P(U_i = u_i \mid E_X^+) \right\}$$

$\boxed{(\text{CI thru d-sep } ③)}$

$\boxed{\begin{array}{l} \text{Let } E_{U_i \setminus X} = \text{evidence connected to } U_i \text{ not thru } X. \\ E_X^+ = E_{U_1 \setminus X} \cup E_{U_2 \setminus X} \cup \cdots\cdots \end{array}}$

$$= \underbrace{\sum_{\vec{u}} P(X=x \mid \vec{U}=\vec{u})}_{\text{CPT given}} \left\{ \underbrace{\prod_{i=1}^{n} P(U_i = u_i \mid E_{U_i \setminus X})}_{\text{recursive instance.}} \right\}$$

$\boxed{CI}$ d-sep ③

* Recursion through children.

$$P(E_X^- \mid X=x) = \prod_{j=1}^{n} P(E_{Y_j \setminus X} \mid X=x)$$

can be computed recursively

$\boxed{CI}$ d-sep ③

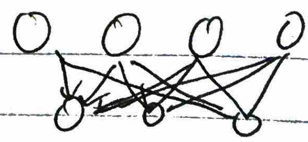
Next steps: introduce children Y_j's., spouses, Z_{jk}'s.

* Termination Conditions:
  1) Root node (no parent)
  2) leaf node (no children)
  3) Evidence node (trivial)

* Runtime Analysis:

linear in # nodes and size of CPTs.

Q: What about BNs?



(bipartite)