LET'S PLAY SOME WATER (NOT SO) FUN GAME!

Problem:

We study a simple game consisting in a grid of tiles, separated or not by walls, forming different groups of tiles. Numbers figure on the end of some lines/columns. They correspond to a number of flood tiles in this column/line. The goal of the game is to find a solution, consisting of a combination of tiles flooded satisfying the number on each end of line/column and natural rules of physic.

R_side_numbers:

-Each line/column must contain the exact number of flooded tiles indicated on the end of the said line/column.

-Water must follow the following rules:

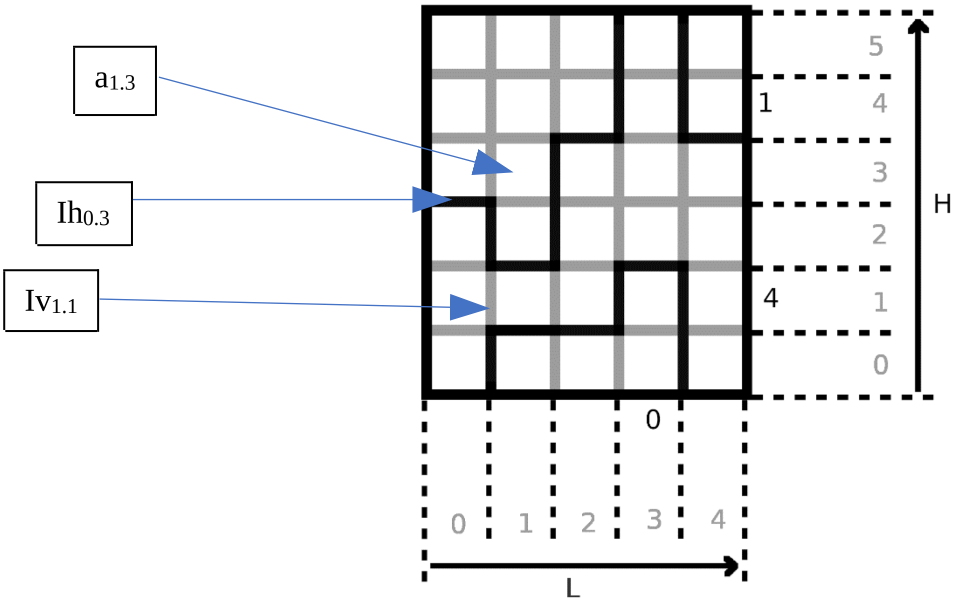
R_water_down:

A tile without water cannot be under a tile of water if there is no wall to stop said water.

R_water_level:

In the same group of tiles, if the group contains water, the level must be the same in all the group even if it is U shaped (or n shaped, for "simplicity").

Modelling:

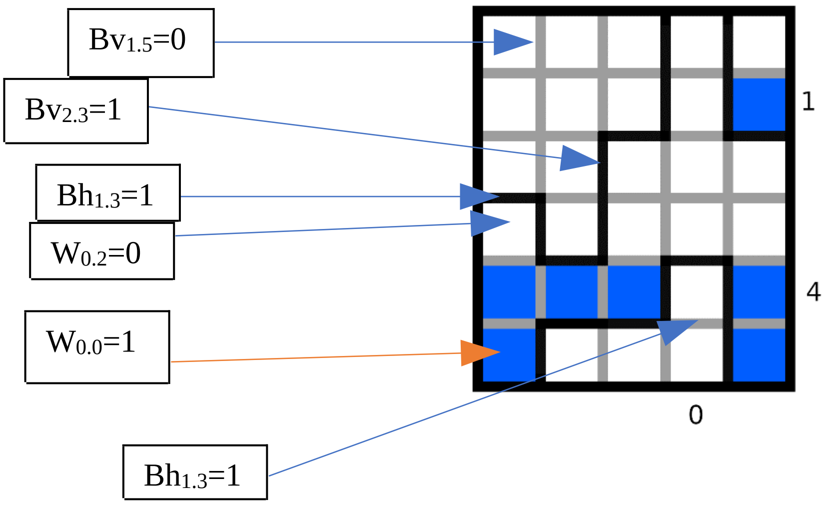


Definitions:

Let L being the horizontal size of the grid.
Let H being the vertical size of the grid.

$\forall 0 \le i < L, 0 \le j < H$ we name $a_{i,j}$ the tile of coordinates i,j
$\forall 0 < i < L, 0 \le j < H$ we name $Iv_{i,j}$ the (vertical) intersection of the tiles $a_{i,j}$ and $a_{i-1,j}$
$\forall 0 \le i < L, 0 < j < H$ we name $Ih_{i,j}$ the (horizontal) intersection of the tiles $a_{i,j}$ and $a_{i,j-1}$
$\forall 0 \le i < L$ we name Zc_i the number aside to the column i.
$\forall 0 < j < H$ we name Zl_j the number aside to the line j.

$\forall i,j$, $W_{i,j} = 1 \Leftrightarrow$ water in $a_{i,j}$.
$\forall i,j$, $Bh_{i,j} = 1 \Leftrightarrow$ horizontal barrier at $Ih_{i,j}$
$\forall i,j$, $Bv_{i,j} = 1 \Leftrightarrow$ vertical barrier at $Iv_{i,j}$

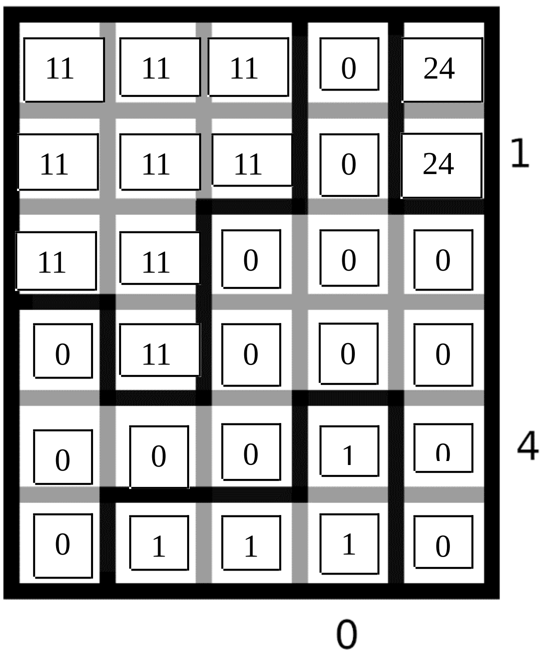


Modelling the "water's physics":

[schema]

We attribute a number to each tile such that each tile inside a same group of tiles have the same value, in order to be able to say if two tiles are in the same group (i.e. if there's a path between these tiles). For each tile $a_{i,j}$, we can access this number at $a_{i,j}$.val.

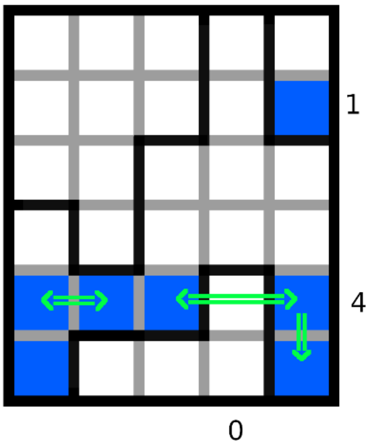
To attribute such values, a way to proceed is to initialize the values as the following:
$\forall$i,j $a_{i,j}$.val=j*h+i
And then use the following rule to get the same values in each group:
$\forall$i,j $\neg Bh_{i,j}$ => $a_{i,j}$.val = min($a_{i,j}$.val ; $a_{i,j-1}$.val)
$\forall$i,j $\neg Bv_{i,j}$ => $a_{i,j}$.val = min($a_{i,j}$.val ; $a_{i-1,j}$.val)



Once $a_{i,j}$.val is defined, the following rules model the fact that water must follow the rules of physics:

R_water_down:
$\forall$i in {0,...,L-1},j in {0,...,H-1}:
$(ai,j.val=ai,j-1.val)\Rightarrow(Wi,j\Rightarrow Wi,j-1)$

R_water_level
$\forall$i1,i2 in {0,...,L-1},j in {0,...,H-1}:
$(ai1,j.val=ai2,j.val)\Rightarrow(Wi1,j\Leftrightarrow Wi2,j)$



Modelling the fact that Zc_i and Zl_j represents the number of tiles of water on the line or column (application of R_side_numbers) :

Since there happens the same thing on each line with Zl_i and each column with Zc_j, let's say that S is either L or H, and N is the other.

We will then define Z and Wz_i as follows:
<u>if S = H:</u>
for any 0≤i<S, Z is Zc_i
for any 0≤j<N, Wz_j is $W_{i,j}$ (representing the fact of water being on the tiles of the same column as Z)
<u>if S = L:</u>
for any 0≤i<S, Z is Zl_i
for any 0≤j<N, Wz_j is $W_{j,i}$ (representing the fact of water being on the tiles of the same line as Z)

Since we want exactly Z tiles to be filled of water, a way to model it is by making a disjunction of each model, but transforming it to a conjunction normal form by developing is long and grindy. Instead, we can model the fact that any assignment with at least Z+1 tiles filled of water should be unsatisfiable, and that any assignment with at most Z-1 tiles filled of water should be unsatisfiable.

To model the fact that any assignment with at most Z-1 tiles filled of water should be unsatisfiable, we do a conjunction of all possible clauses containing exactly N-(Z-1) elements of the form Wz_i. (like this, any assignment of at most Z-1 elements that are true have a clause that they can't validate).
If Z = 0, no need to add a formula for this rule. Otherwise, the formula is the following:

$$\prod_{x_0=0}^{Z-1}\left(\prod_{x_1=x_0}^{Z}\prod_{x_2=x_1}^{Z+1}\cdots\prod_{x_{N-Z}=x_{N-Z-1}}^{N-1}\left(\sum_{i\in\{x_0,\ \ldots,x_{N-Z}\}} Wz_i\right)\right)$$

(The product symbolizes a succession of conjunctions and the sum, a succession of disjunctions.)

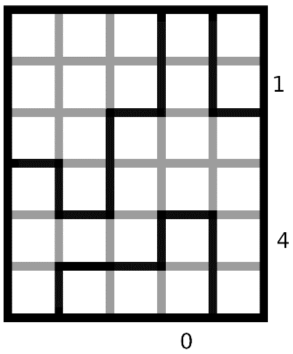
To model the fact that any assignment with at least Z+1 tiles filled of water should be unsatisfiable, it is the same reasoning, but with a conjunction of all possible clauses containing exactly Z+1 elements of the form $\neg Wz_i$ (note the negation, and so it is the same as verifying that at most N-Z elements of the form Wz_i are false, which is directly equivalent to saying that there is at most Z tiles filled)
If Z = N, no need to add a formula for this rule. Otherwise, the formula is the following:

$$\prod_{x_0=0}^{N-Z-1}\left(\prod_{x_1=x_0}^{N-Z}\prod_{x_2=x_1}^{N-Z+1}\cdots\prod_{x_Z=x_{Z-1}}^{N-1}\left(\sum_{i\in\{x_0,\ldots,x_Z\}}\neg Wz_i\right)\right)$$

And so, the conjunction of these two formulas models the fact that exactly Z of the Wz_i are true, and the other false.

Example:

Here we take the following grid:



In this example we need a model satisfying all the above-mentioned rules:

R_water_down:
$(\neg W_{0.5} \lor W_{0.4}) \land (\neg W_{0.4} \lor W_{0.3}) \land (\neg W_{0.2} \lor W_{0.1}) \land (\neg W_{0.1} \lor W_{0.0})$
$\land (\neg W_{1.5} \lor W_{1.4}) \land (\neg W_{1.4} \lor W_{1.3}) \land (\neg W_{1.3} \lor W_{1.2})$
$\land (\neg W_{2.5} \lor W_{2.4}) \land (\neg W_{2.3} \lor W_{2.2}) \land (\neg W_{2.2} \lor W_{2.1})$
$\land (\neg W_{3.5} \lor W_{3.4}) \land (\neg W_{3.4} \lor W_{3.3}) \land (\neg W_{3.3} \lor W_{3.2}) \land (\neg W_{3.1} \lor W_{3.0})$
$\land (\neg W_{4.5} \lor W4_{.4}) \land (\neg W_{4.3} \lor W_{4.2}) \land (\neg W_{4.2} \lor W_{4.1}) \land (\neg W_{4.1} \lor W_{4.0})$

R_water_level:
$(\neg W_{0.5} \lor W_{1.5}) \land (W_{0.5} \lor \neg W_{1.5}) \land (\neg W_{1.5} \lor W_{2.5}) \land (W_{1.5} \lor \neg W_{2.5})$
$\land (\neg W_{0.4} \lor W_{1.4}) \land (W_{0.4} \lor \neg W_{1.4}) \land (\neg W_{1.4} \lor W_{2.4}) \land (W_{1.4} \lor \neg W_{2.4})$
$\land (\neg W_{0.3} \lor W_{1.3}) \land (W_{0.3} \lor \neg W_{1.3}) \land (\neg W_{2.3} \lor W_{3.3}) \land (W_{2.3} \lor \neg W_{3.3}) \land (\neg W_{3.3} \lor W_{4.3}) \land (W_{3.3} \lor \neg W_{4.3})$
$\land (\neg W_{2.2} \lor W_{3.2}) \land (W_{2.2} \lor \neg W_{3.2}) \land (\neg W_{3.2} \lor W_{4.2}) \land (W_{3.2} \lor \neg W_{4.2})$
$\land (\neg W_{0.1} \lor W_{1.1}) \land (W_{0.1} \lor \neg W_{1.1}) \land (\neg W_{1.1} \lor W_{2.1}) \land (W_{1.1} \lor \neg W_{2.1})$
$\land (\neg W_{1.0} \lor W_{2.0}) \land (W_{1.0} \lor \neg W_{2.0}) \land (\neg W_{2.0} \lor W_{3.0}) \land (W_{2.0} \lor \neg W_{3.0})$

R_side_numbers:
For the line 1 ($Zl_1 = 4$) :
$S = 6, Z = Zl_1 = 4, N = 5$
$N\text{-}Z = 1$
$(W_{0,1} \lor W_{1,1}) \land (W_{0,1} \lor W_{2,1}) \land (W_{0,1} \lor W_{3,1}) \land (W_{0,1} \lor W_{4,1})$
$\land (W_{1,1} \lor W_{2,1}) \land (W_{1,1} \lor W_{3,1}) \land (W_{1,1} \lor W_{4,1})$
$\land (W_{2,1} \lor W_{3,1}) \land (W_{2,1} \lor W_{4,1})$
$\land (W_{3,1} \lor W_{4,1})$

$\land (\neg W_{0,1} \lor \neg W_{1,1} \lor \neg W_{2,1} \lor \neg W_{3,1} \lor \neg W_{4,1})$

For the line 4 ($Zl_4 = 1$):
$S = 6, Z = Zl_4 = 1, N = 5$
$N\text{-}Z = 4$
$(W_{0,4} \lor W_{1,4} \lor W_{2,4}\ W_{3,4} \lor W_{4,4})$

$\land (\neg W_{0,4} \lor \neg W_{1,4}) \land (\neg W_{0,1} \lor \neg W_{2,4}) \land (\neg W_{0,1} \lor \neg W_{3,4}) \land (\neg W_{0,1} \lor \neg W_{4,4})$
$\land (\neg W_{1,4} \lor \neg W_{2,4}) \land (\neg W_{1,1} \lor \neg W_{3,4}) \land (\neg W_{1,1} \lor \neg W_{4,4})$
$\land (\neg W_{2,4} \lor \neg W_{3,4}) \land (\neg W_{2,1} \lor \neg W_{4,4})$
$\land (\neg W_{3,4} \lor \neg W_{4,4})$

For the column 3 ($Zc_3 = 0$):
$S = 5, Z = Zc_3 = 0, N = 6$
$N\text{-}Z = 6$
$\neg W_{3,0} \land \neg W_{3,1} \land \neg W_{3,2} \land \neg W_{3,3} \land \neg W_{3,4} \land \neg W_{3,5}$