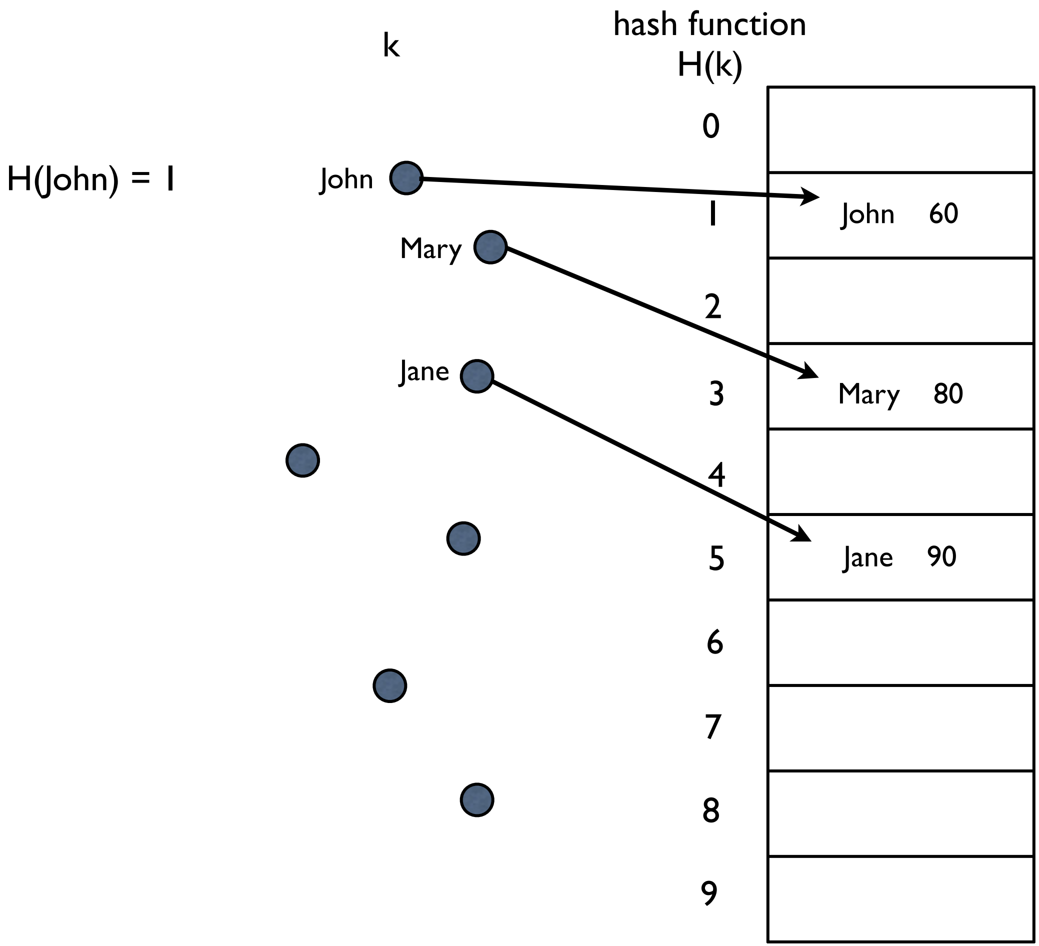
Data Structure:
Hashing

hashing

- hashing is a technique used for performing insertion, deletion, and finding in constant time
- tree operations such as FindMin, FindMax, and the printing all elements in sorted order are not supported

- hash table is an array of fixed size, containing the keys
- hash function maps each key to some cell in the hash table
  - should be easy to compute
  - should minimize the number of collision
  - uniform hash function, the probability of $h(k) = i$ is $1/b$ for all i (b is bucket size)
- collision occurs when different keys are mapped to the same cell

Hashing



k
hash function
H(k)
H(John) = 1
John
Mary
Jane
0
1
2
3
4
5
6
7
8
9
John 60
Mary 80
Jane 90

Hash functions

- adding all characters (alphabets) in the key
  - for example, h(abc) = h(bca) = 1+2+3 = 6   (a=1, b=2, c=3)
  - all ordering information is lost
  - the number of hash function value is too small, considering the number of possible keys
    - for example, length(key) = 8
    - the number of hash function value H(key) = 26 * 8 = 208
    - the number of possible keys = 26^8

- polynomial function (using horner's rule)
  - $h(k) = k_1 + 27k_2 + 27^2k_3 = ((k_3) * 27 + k_2) * 27 + k_1$
  - number gets easily too big

- division
  - h(k) = k mod m, where m is the size of hash table
  - good choice for m is a prime number

resolving collision

separate chaining:
put keys that collide in a list associated with index

open addressing:
when a new key collides, find next empty slot and put it there

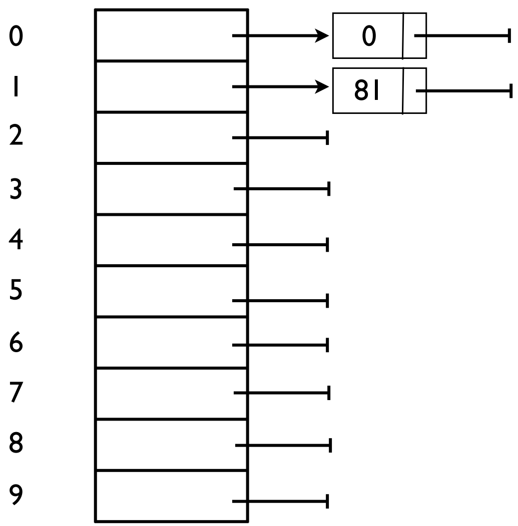
resolving collision: separate chaining (open hashing)

- keep a list of all elements that hash to the same value
- operations
  - Find: use hash function to determine which list to traverse
  - Insert: traverse down the list to check whether the element is in the list
    if not, it is inserted at the front (or at the end)

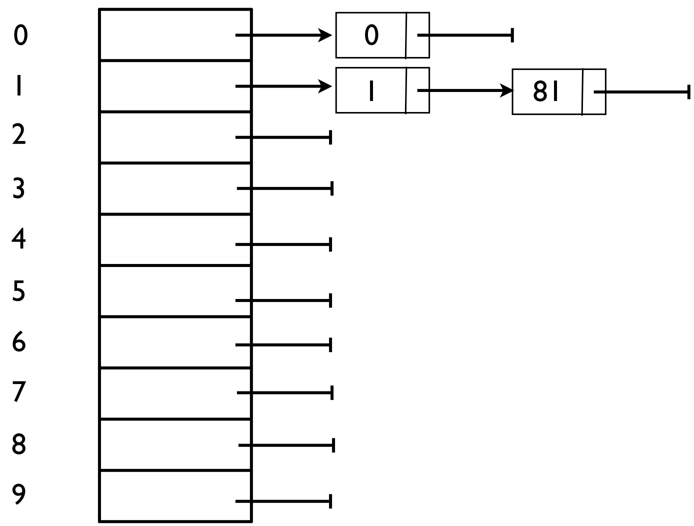
resolving collision: separate chaining (open hashing)

H(x) = x %10

insert (81)
insert(0)

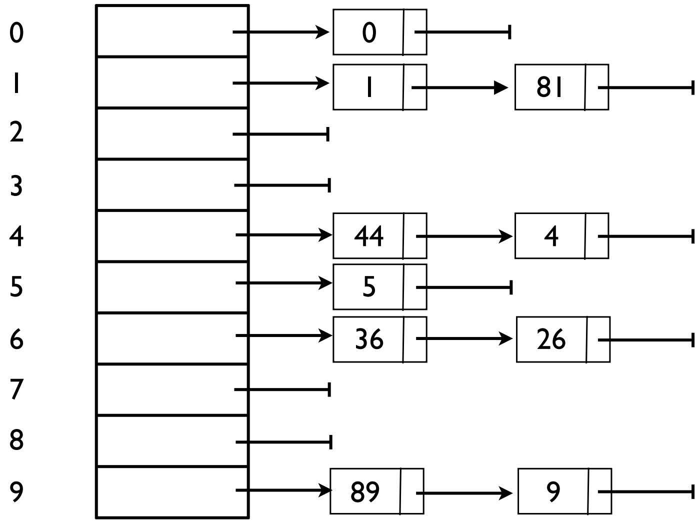


insert(1)



resolving collision: separate chaining (open hashing)

insert (81)
insert(0)
insert(1)
insert(4)
insert(26)
insert(5)
insert(9)
insert(44)
insert(36)
insert(89)



resolving collision: separate chaining

```
typedef struct  ListNode* Position;
typedef Position List;

struct ListNode {
        ElementType    Element;
        Position    Next;
}

struct HashTbl{
        int TableSize;
        List*  TheLists;
}
```

resolving collision: separate chaining

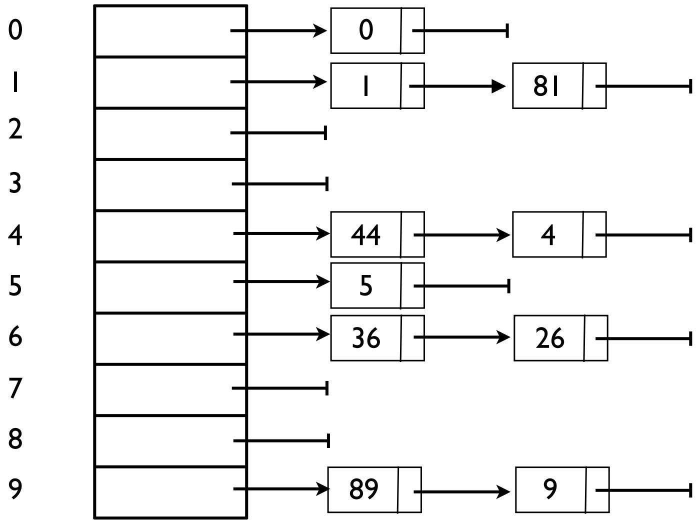
```
Position Find (ElementType Key, HashTable H){

    Position P;
    List L;

    L = H -> TheLists [ Hash(key,  H->TableSize)];
    P = L -> Next;

    while (P != NULL && P->Element != Key)
        P = P->Next;

    return P;
}
```



resolving collision: separate chaining

```
void Insert (ElementType Key,   HashTable H){

        Position Pos, newCell;
        List L;

        Pos = Find(Key,   H);

        if (Pos == NULL){

                NewCell = malloc(sizeof (struct ListNode));
                NewCell ->Element = Key;

                L = H->TheLists[Hash(Key, H->TableSize)];
                NewCell ->Next = L->Next;
                L->Next = NewCell;
        }
}
```

resolving collision: separate chaining

- load factor: the ratio of the number of elements in the hash table to the table size

  $\lambda = n / m$

  n is the number of keys in the table, m is the size of the table

- successful search (i.e. no clustering): 1 (hash function) + ($\lambda/2$) =O(1)
- unsuccessful search: 1 + λ =O(1)
- needs extra space and operation for pointers and new nodes

resolving collision: open addressing (closed hashing)

- all the keys are stored in the table without pointers
- use special value Del to determine which entries have keys & which don't.
- if a collision occurs, alternative cells are tried until an empty cell is found
- try $h_0(key)$, $h_1(key)$, $h_2(key)$, . . .
  - where $h_i(key) = (Hash(key) + F(i)) \bmod m$
  - F(i) is the collision resolution strategy
  - linear probing: F(i) is a linear function, $F(i) = i$

    for example, $h_1(key) = (Hash(key) + 1)$, $h_2(key) = (Hash(key) + 2)$, . . .
  - quadratic probing: F(i) is a quadratic function, $F(i) = i^2$

    for example, $h_1(key) = (Hash(key) + 1)$, $h_2(key) = (Hash(key) + 4)$, . . .

resolving collision: linear probing

- F(i) is a linear function. for example, F(i) = i

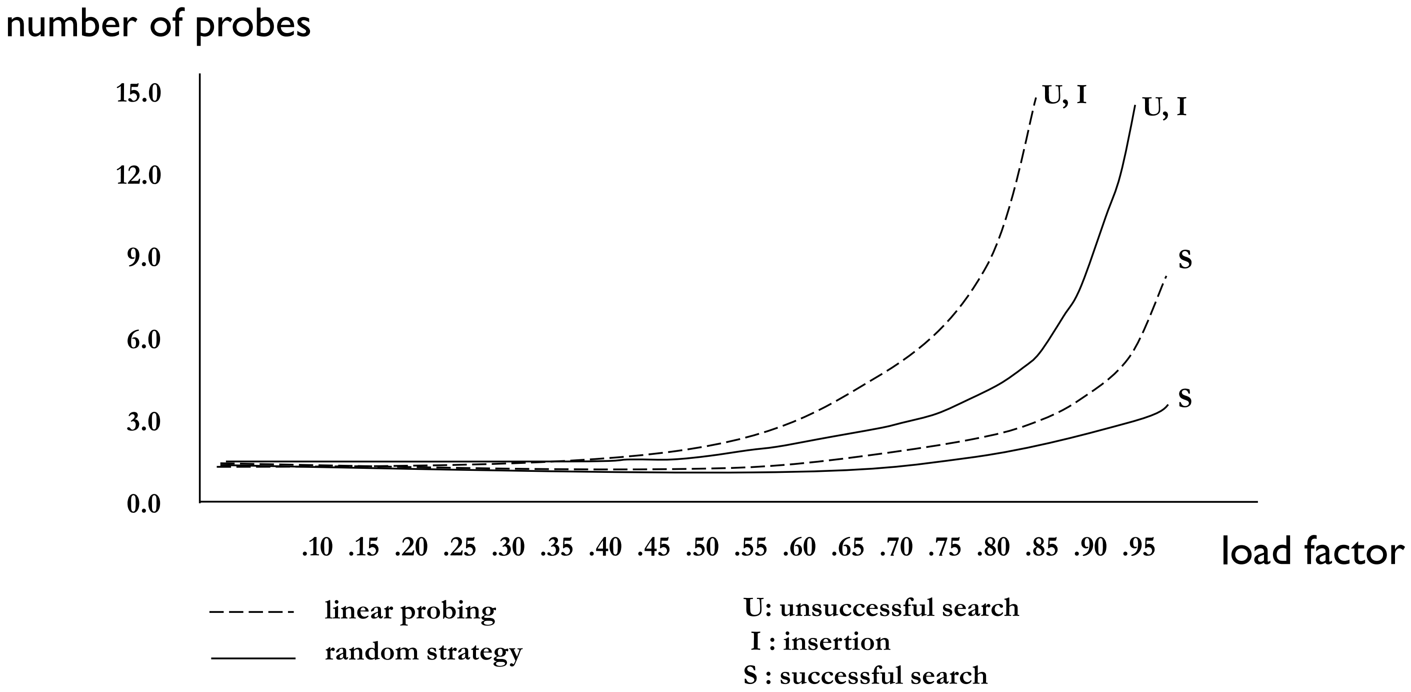
  inserting keys: 89, 18, 49, 58, 69

0		49	49	49
1			58	58
2				69
3				
4				
5				
6				
7				
8	18	18	18	18
9	89	89	89	89

resolving collision: linear probing

- primary clustering: any key that hashes into the cluster will require several attempts to resolve the collision and then it will add to the cluster
- secondary clustering: keys with different hash values have nearly the same probe sequence.
- expected number of probes
  - successful search $S = \frac{1}{2}(1 + \frac{1}{1-\lambda})$
  - unsuccessful search $U = \frac{1}{2}(1 + \left(\frac{1}{1-\lambda}\right)^2)$
  - as λ approaches to 1, the search time grows to infinity
  - linear probing does well if the table is less than 75% full

resolving collision: linear probing



Deletions in closed hashing

- Use special value Del to distinguish deleted and empty locations

delete(42), find(31)

0	10		0	10
1	50		1	50
2	42		2	Del
3	92		3	92
4	31		4	31
5			5	
6			6	
7			7	
8			8	18
9			9	89

If we see Del during probing

- find( ): keep searching until empty
- Insert(): reuse the Del location for placing a new key

resolving collision: quadratic probing

- a collision resolution method that eliminates the primary clustering problem of linear probing
- collision function $F(i) = i^2$, $h_i(key) = (Hash(key) + F(i)) \bmod m$

inserting keys: 89, 18, 49, 58, 69

0		49	49	49
1				
2			58	58
3				69
4				
5				
6				
7				
8	18	18	18	18
9	89	89	89	89

resolving collision: quadratic probing

One tricky question

- In linear probing, it is guaranteed that as long as there is one free location in the table, we will eventually find it without repeating any probe locations
- Is this also true for quadratic probing?

Fortunately, quadratic probing does a good job of visiting different locations => It can be formally proved that if m is prime, the first m/2 locations that quadratic probing visits will be distinct.

resolving collision: quadratic probing

Theorem. If quadratic probing is used and the table size m is prime, then an element can always be inserted if the table is at least half empty.

Proof:

Prove the first $m/2$ locations that quadratic probing visits will be distinct.
Let us use contradiction.

For $0 \le i < j \le \left\lfloor \frac{m}{2} \right\rfloor$

$$\begin{aligned} h(x) + i^2 &\equiv h(x) + j^2 \pmod{m} \\ i^2 &\equiv j^2 \pmod{m} \\ i^2 - j^2 &\equiv 0 \pmod{m} \\ (i-j)(i+j) &\equiv 0 \pmod{m} \end{aligned}$$

This means that $(i-j)(i+j)$ is a multiple of m. Since m is a prime, either $(i-j)$ or $(i+j)$ must be a multiple of m. Since $i \neq j$ and $i, j \le \left\lfloor \frac{m}{2} \right\rfloor$, neither $(i-j)$ nor $(i+j)$ can be a multiple of m.

resolving collision: double hashing

- use other hash function for random probing
- for example,($h_i(key)$ = (Hash(key) + F(i)) mod m)

  Hash(key) = key mod m

  F(i)= i*$Hash_2$(key), $Hash_2$(key) = R - (key mod R)

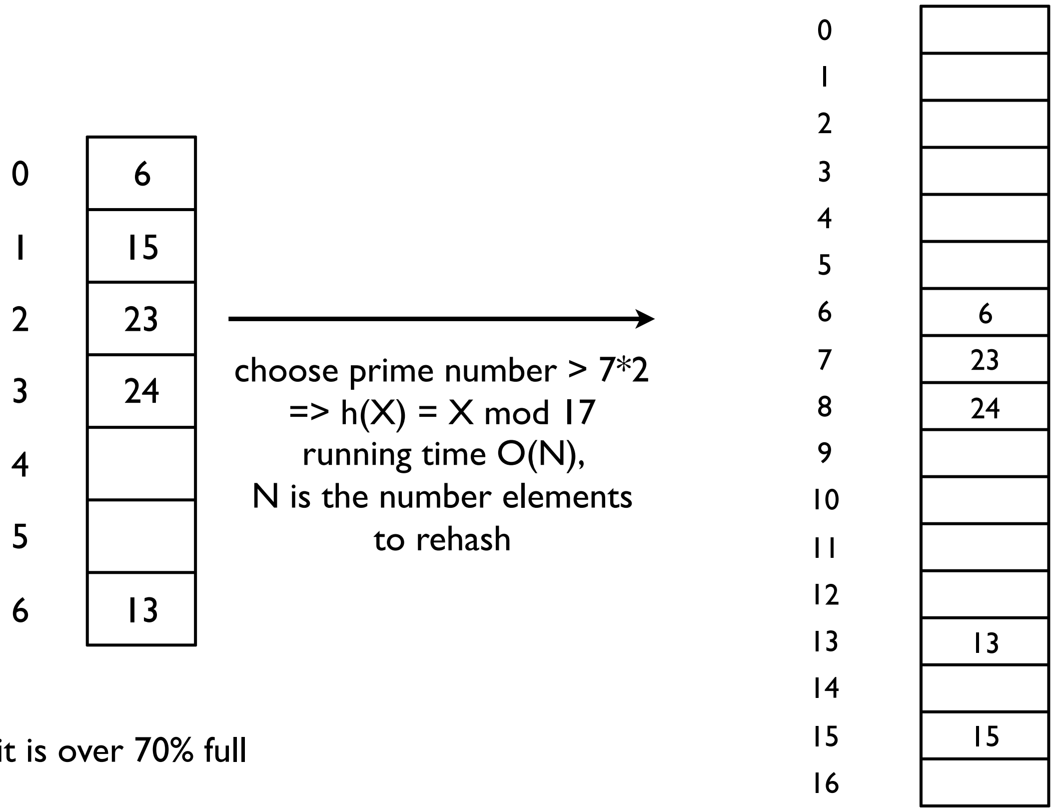
R=7
inserting keys: 89, 18, 49, 58, 69

0				69
1				
2				
3			58	58
4				
5				
6		49	49	49
7				
8	18	18	18	18
9	89	89	89	89

49: $Hash_2(49) = 7 - 0 = 7$
58: $Hash_2(58) = 7 - 2 = 5$
69: $Hash_2(69) = 7 - 6 = 1$

rehashing

- if the table gets too full, the running time for the operations start taking too long
- build another table that is about twice as big



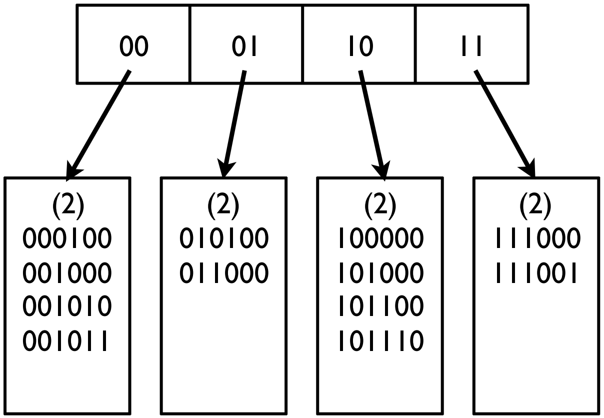
it is over 70% full

extendible hashing

- what if the hash table is too large to fit in main memory?
  - locality is important for large data structure since disk access is costly but memory access is cheap
  - efficient probing is the lack of locality
  - need a method to reduce the number of disk access

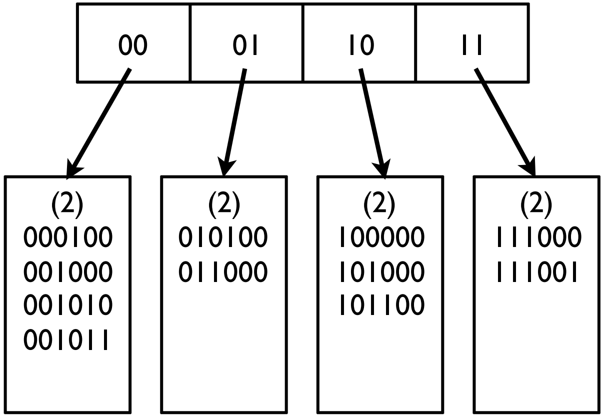
extendible hashing

- The hash table is broken into a number of smaller hash tables, each is called a *bucket*.
- The maximum size of each bucket is the size of a disk page.
- To find which bucket to search for, we store a data structure called *directory* in main memory, and each entry in the directory holds a disk address of the corresponding bucket.
- Each bucket can hold as many records that can be fit in one page, and we will try to keep each bucket at least half full.

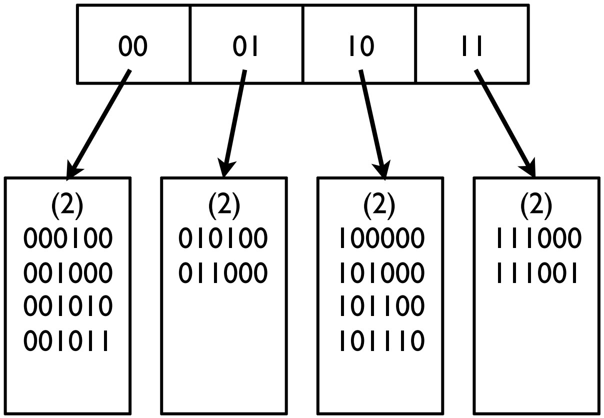


D: the number of bits used by the root
$D = 2$
d_L: the number of leading bits that all elements of some leaf L have in common
$d_L = 2$

extendible hashing



insert 101110



extendible hashing