CS1231 Chapter 4

Sets

4.1 Basics

Definition 4.1.1. (1) A *set* is an unordered collection of objects.

(2) These objects are called the *members* or *elements* of the set.

(3) Write

$x \in A$	for	x is an element of A;
$x \notin A$	for	x is not an element of A;
$x, y \in A$	for	x, y are elements of A;
$x, y \notin A$	for	x, y are not elements of A; etc.

(4) We may read $x \in A$ also as "x is in A" or "A *contains* x (as an element)".

Warning 4.1.2. Some use "contains" for the subset relation, but in this module we do *not*. We will also use "contains" for the subgraph relation in Chapters 11 and 12. The context should make it clear which meaning is intended.

Definition 4.1.3 (roster notation)**.** (1) The set whose only elements are $x_1, x_2, \ldots, x_n$ is denoted $\{x_1, x_2, \ldots, x_n\}$.

(2) The set whose only elements are $x_1, x_2, x_3, \ldots$ is denoted $\{x_1, x_2, x_3, \ldots\}$.

Note 4.1.4. For all objects $x_1, x_2, \ldots, x_n, z$,

$$z \in \{x_1, x_2, \ldots, x_n\} \quad \Leftrightarrow \quad z \text{ appears in the list } x_1, x_2, \ldots, x_n.$$

Example 4.1.5. (1) The only elements of $A = \{1, 5, 6, 3, 3, 3\}$ are 1, 5, 6 and 3. So $6 \in A$ but $7 \notin A$.

(2) The only elements of $B = \{0, 2, 4, 6, 8, \ldots\}$ are the non-negative even integers. So $4 \in B$ but $5 \notin B$.

Definition 4.1.6 (set-builder notation)**.** Let U be a set and $P(x)$ be a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted

$$\{x \in U : P(x)\} \quad \text{or} \quad \{x \in U \mid P(x)\}.$$

This is read as "the set of all x in U such that $P(x)$".

Note 4.1.7. Let U be a set and $P(x)$ be a predicate over U. For all objects z,

$$z \in \{x \in U : P(x)\} \quad \Leftrightarrow \quad z \in U \text{ and } P(z) \text{ is true.}$$

Example 4.1.8. (1) The elements of $C = \{x \in \mathbb{Z}_{\geqslant 0} : x \text{ is even}\}$ are precisely the elements of $\mathbb{Z}_{\geqslant 0}$ that are even, i.e., the non-negative even integers. So $6 \in C$ but $7 \notin C$.

(2) The elements of $D = \{x \in \mathbb{Z} : x \text{ is a prime number}\}$ are precisely the elements of $\mathbb{Z}$ that are prime numbers, i.e., the prime integers. So $7 \in D$ but $9 \notin D$.

Definition 4.1.9 (replacement notation)**.** Let A be a set and $t(x)$ be (the name of) an object for each element x of A. Then the set of all objects of the form $t(x)$ where x ranges over the elements of A is denoted

$$\{t(x) : x \in A\} \quad \text{or} \quad \{t(x) \mid x \in A\}.$$

This is read as "the set of all $t(x)$ where $x \in A$".

Note 4.1.10. Let A be a set and $t(x)$ be an object for each element x of A. For all objects z,

$$z \in \{t(x) : x \in A\} \quad \Leftrightarrow \quad \exists x \in A \quad z = t(x).$$

Example 4.1.11. (1) The elements of $E = \{x + 1 : x \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x + 1$ where $x \in \mathbb{Z}_{\geqslant 0}$, i.e., the positive integers. So $1 = 0 + 1 \in E$ but $0 \notin E$.

(2) The elements of $F = \{x - y : x, y \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x - y$ where $x, y \in \mathbb{Z}_{\geqslant 0}$, i.e., the integers. So $-1 = 1 - 2 \in F$ but $\sqrt{2} \notin F$.

Definition 4.1.12. Two sets are *equal* if they have the same elements, i.e., for all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z \ (z \in A \Leftrightarrow z \in B).$$

Example 4.1.13. $\{1, 5, 6, 3, 3, 3\} = \{1, 5, 6, 3\} = \{1, 3, 5, 6\}$.

Slogan 4.1.14. Order and repetition do not matter.

Explanation. When specifying a set using roster notation, although one may choose to list the elements in different orders or with repetition in the specification, the set being specified stays the same, according to the definition of set equality; cf. Example 4.1.13.

In general, the elements of a set come in no inherent order. If there were such an order, then re-ordering the elements of a set A should give a set B that does not equal A because objects with different properties must (by convention) be different, but B equals A since they have the same elements; this is a contradiction. There is hence *no* concept of the first element, the second element, etc. of a set. Similarly, there is *no* concept of the number of times an element appears in a set. $\square$

Example 4.1.15.
$$\begin{aligned}\{y^2 : y \text{ is an odd integer}\} &= \{x \in \mathbb{Z} : x = y^2 \text{ for some odd integer } y\} \\ &= \{1^2, 3^2, 5^2, \dots\}.\end{aligned}$$

Example 4.1.16. $\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Proof. ($\Rightarrow$) Take any $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. Then $z \in \mathbb{Z}$ and $z^2 = 1$. So

$$\begin{aligned} & z^2 - 1 = (z - 1)(z + 1) = 0. \\ \therefore \quad & z - 1 = 0 \quad \text{or} \quad z + 1 = 0. \\ \therefore \quad & z = 1 \quad \text{or} \quad z = -1. \end{aligned}$$

This means $z \in \{1, -1\}$.

($\Leftarrow$) Take any $z \in \{1, -1\}$. Then $z = 1$ or $z = -1$. In either case, we have $z \in \mathbb{Z}$ and $z^2 = 1$. So $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. $\square$

Exercise 4.1.17. Write down proofs of the claims made in Example 4.1.11. In other words, prove that $E = \mathbb{Z}^+$ and $F = \mathbb{Z}$, where E and F are as defined in Example 4.1.11. ✎ 4a

= means prove both -> and <-

Theorem 4.1.18. There exists a unique set with no element, i.e.,

- there is a set with no element; and (existence part)
- for all sets A, B, if both A and B have no element, then $A = B$. (uniqueness part)

Proof.
- (existence part) The set $\{\}$ has no element.
- (uniqueness part) Let A, B be sets with no element. Then vacuously,

$$\forall z\ (z \in A \Rightarrow z \in B) \quad \text{and} \quad \forall z\ (z \in B \Rightarrow z \in A)$$

because the hypotheses in the implications are never true. So $A = B$. □

Definition 4.1.19. The set with no element is called the *empty set*. It is denoted by $\varnothing$.

4.2 Subsets

Definition 4.2.1. Let A, B be sets. Call A a *subset* of B, and write $A \subseteq B$, if

$$\forall z\ (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that B *includes* A, and write $B \supseteq A$ in this case.

Note 4.2.2. We avoid using the symbol $\subset$ because it may have different meanings to different people.

Example 4.2.3. (1) $\{1, 5, 2\} \subseteq \{5, 2, 1, 4\}$ but $\{1, 5, 2\} \not\subseteq \{2, 1, 4\}$.

(2) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Remark 4.2.4. Let A, B be sets.

(1) $$A \not\subseteq B \quad \Leftrightarrow \quad \exists z\ (z \in A \text{ and } z \notin B).$$

(2) $$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

(3) $$A \subseteq A.$$

Definition 4.2.5. Let A, B be sets. Call A a *proper subset* of B, and write $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is *proper* or *strict*.

Example 4.2.6. All the inclusions in Example 4.2.3 are strict.

Proposition 4.2.7. The empty set is a subset of any set, i.e., for any set A,

$$\varnothing \subseteq A.$$

Proof. Vacuously,

$$\forall z\ (z \in \varnothing \Rightarrow z \in A)$$

because the hypothesis in the implication is never true. So $\varnothing \subseteq A$ by the definition of $\subseteq$. □

Note 4.2.8. Sets can be elements of sets.

Example 4.2.9. (1) The set $A = \{\varnothing\}$ has exactly 1 element, namely the empty set. So A is not empty.

(2) The set $B = \{\{1\}, \{2, 3\}\}$ has exactly 2 elements, namely $\{1\}, \{2, 3\}$. So $\{1\} \in B$, but $1 \notin B$.

Note 4.2.10. Membership and inclusion can be different.

check to see if {n} appears in the list i.e : it is an element in C

In $\in$, it is an element! not a set. Therefore, directly check the element

Question 4.2.11. Let $C = \{\{1\}, 2, \{3\}, 3, \{\{4\}\}\}$. Which of the following are true? ✎ 4b

check to see if n appears in the list

In subsets($\subseteq$), both are sets, compare the elements in both sets

- $\{1\} \in C$. True
- $\{2\} \in C$.
- $\{3\} \in C$. True
- $\{4\} \in C$.
- $\{1\} \subseteq C$.
- $\{2\} \subseteq C$. True
- $\{3\} \subseteq C$. True
- $\{4\} \subseteq C$.

Definition 4.2.12. Let A be a set. The set of all subsets of A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example 4.2.13. (1) $\mathcal{P}(\varnothing) = \{\varnothing\}$.

(2) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.

(3) $\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

(4) The following are subsets of $\mathbb{N}$ and thus are elements of $\mathcal{P}(\mathbb{N})$.

$$\begin{aligned}
&\varnothing, \{0\}, \{1\}, \{2\}, \ldots \{0, 1\}, \{0, 2\}, \{0, 3\} \ldots \{1, 2\}, \{1, 3\}, \{1, 4\} \ldots \\
&\{2, 3\}, \{2, 4\}, \{2, 5\} \ldots \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \ldots \\
&\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \ldots \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \ldots \\
&\mathbb{N}, \mathbb{N}_{\geqslant 1}, \mathbb{N}_{\geqslant 2}, \ldots \{0, 2, 4, \ldots\}, \{1, 3, 5, \ldots\}, \{2, 4, 6, \ldots\}, \{3, 5, 7, \ldots\}, \ldots \\
&\{x \in \mathbb{N} : (x-1)(x-2) < 0\}, \{x \in \mathbb{N} : (x-2)(x-3) < 0\}, \ldots \\
&\{3x + 2 : x \in \mathbb{N}\}, \{4x + 3 : x \in \mathbb{N}\}, \{5x + 4 : x \in \mathbb{N}\}, \ldots
\end{aligned}$$

4.3 Boolean operations

Definition 4.3.1. Let A, B be sets.

(1) The *union* of A and B, denoted $A \cup B$, is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Read $A \cup B$ as "A union B".

(2) The *intersection* of A and B, denoted $A \cap B$, is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Read $A \cap B$ as "A intersect B".

(3) The *complement* of B in A, denoted $A - B$ or $A \setminus B$, is defined by

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Read $A \setminus B$ as "A minus B".

Everything in A but NOT in B

Everything in A but NOT in B

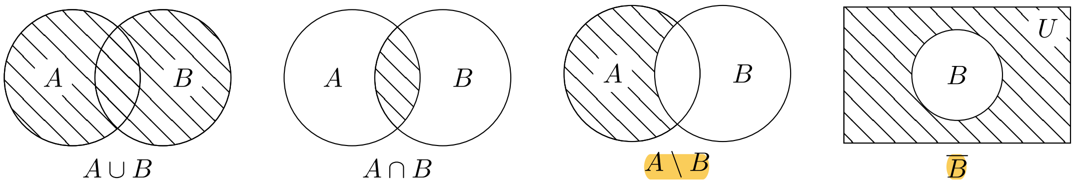


Figure 4.1: Boolean operations on sets

Convention and terminology 4.3.2. When working in a particular context, one usually works within a fixed set U which includes all the sets one may talk about, so that one only needs to consider the elements of U when proving set equality and inclusion (because no other object can be the element of a set). This U is called a *universal set*.

Definition 4.3.3. Let B be a set. In a context where U is the universal set (so that implicitly $U \supseteq B$), the *complement* of B, denoted $\overline{B}$ or B^{c}, is defined by

$$\overline{B} = U \setminus B.$$

Example 4.3.4. Let $A = \{x \in \mathbb{Z} : x \leqslant 10\}$ and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 15\}$. Then

$$\begin{aligned} A \cup B &= \{x \in \mathbb{Z} : (x \leqslant 10) \vee (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x \leqslant 15\}; \\ A \cap B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 10\}; \\ A \setminus B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge \neg(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x < 5\}; \\ \overline{B} &= \{x \in \mathbb{Z} : \neg(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : (x < 5) \vee (x > 15)\}, \end{aligned}$$

in a context where $\mathbb{Z}$ is the universal set. To show the first equation, one shows

$$\forall x \in \mathbb{Z} \;\; \big((x \leqslant 10) \vee (5 \leqslant x \leqslant 15) \Leftrightarrow (x \leqslant 15)\big), \qquad \text{etc.}$$

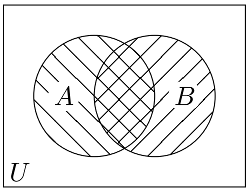
Theorem 4.3.5 (set identities)**.** For all set A, B, C in a context where U is the universal set, the following hold.

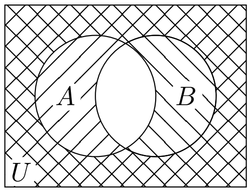
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotence	$A \cup A = A$	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Identities	$A \cup \varnothing = A$	$A \cap U = A$
Annihilators	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Complement	$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$
Double Complement Law	$\overline{(\overline{A})} = A$	
Top and bottom	$\overline{\varnothing} = U$	$\overline{U} = \varnothing$
Set difference	$A \setminus B = A \cap \overline{B}$	

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Venn Diagrams. In the left diagram below, hatch the regions representing A and B with and respectively. In the right diagram below, hatch the regions representing $\overline{A}$ and $\overline{B}$ with and respectively.





Then the region represents $\overline{A \cup B}$ in the left diagram, and the region represents $\overline{A} \cap \overline{B}$ in the right diagram. Since these regions occupy the same region in the respective diagrams, we infer that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Note 4.3.6. This argument depends on the fact that each possibility for membership in A and B is represented by a region in the diagram.

Proof using a truth table. The rows in the following table list all the possibilities for an element $x \in U$:

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in \overline{A \cup B}$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in \overline{A} \cap \overline{B}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns under "$x \in \overline{A \cup B}$" and "$x \in \overline{A} \cap \overline{B}$" are the same, for any $x \in U$,

$$x \in \overline{A \cup B} \quad \Leftrightarrow \quad x \in \overline{A} \cap \overline{B}$$

no matter in which case we are. So $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$

Direct proof. Let $z \in U$. Then

$$\begin{array}{clll}
 & z \in \overline{A \cup B} & & \\
\Leftrightarrow & z \notin A \cup B & & \text{by the definition of } \overline{\cdot}; \\
\Leftrightarrow & \neg\big((z \in A) \vee (z \in B)\big) & & \text{by the definition of } \cup; \\
\Leftrightarrow & (z \notin A) \wedge (z \notin B) & & \text{by De Morgan's Laws for propositions;} \\
\Leftrightarrow & (z \in \overline{A}) \wedge (z \in \overline{B}) & & \text{by the definition of } \overline{\cdot}; \\
\Leftrightarrow & z \in \overline{A} \cap \overline{B} & & \text{by the definition of } \cap.
\end{array}$$
$\square$

Example 4.3.7. Under the universal set U, show that $(A \cap B) \cup (A \setminus B) = A$ for all sets A, B.

Proof.
$$\begin{aligned}
(A \cap B) \cup (A \setminus B) &= (A \cap B) \cup (A \cap \overline{B}) && \text{by the set identity on set difference;} \\
&= A \cap (B \cup \overline{B}) && \text{by distributivity;} \\
&= A \cap U && \text{by the set identity on complement;} \\
&= A && \text{as } U \text{ is an identity for } \cap.
\end{aligned}$$
$\square$

Definition 4.3.8. Two set A and B are *disjoint* if $A \cap B = \varnothing$.

Example 4.3.9. Let A, B be sets. Show the following.

(1) $A \cap B \subseteq A$.

(2) $A \subseteq A \cup B$.

By the definition of ⊆, we need to show that
∀z (z ∈ A ⇒ z ∈ A∪B)
Let z ∈ A. Then z ∈ A∪B by the definition of ∪.
In particular, we know z∈ A∪B, as required.

4.3.10:
Assume A ⊆ B and A ⊆ C. Take any z ∈ A. Then z ∈ B and z ∈ C as A ⊆ B and A ⊆ C by assumption. So z ∈ B ∩ C by the definition of ∩. As the choice of z in A was arbitrary, this shows A ⊆ B ∩ C.

Proof. (1) By the definition of $\subseteq$, we need to show that

$$\forall z\ (z \in A \cap B \Rightarrow z \in A).$$

Let $z \in A \cap B$. Then $z \in A$ and $z \in B$ by the definition of $\cap$. In particular, we know $z \in A$, as required.

Proved on previous page

(2) This is left as an exercise. □ ✎ 4c

Exercise 4.3.10. Show that for all sets A, B, C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$. ✎ 4d

Question 4.3.11. Is the following true? ✎ 4e

$$(A \setminus B) \cup (B \setminus C) = A \setminus C \quad \text{for all sets } A, B, C.$$

False

4.4 Russell's Paradox (extra material)

Example 4.4.1. (1) $\varnothing \notin \varnothing$.

(2) $\mathbb{Z} \notin \mathbb{Z}$.

(3) $\{\varnothing\} \notin \{\varnothing\}$.

Question 4.4.2. Is there a set x such that $x \in x$? ✎ 4f

Theorem 4.4.3 (Russell 1901)**.** There is no set R such that

$$\forall x\ (x \in R \quad \Leftrightarrow \quad x \notin x). \tag{$*$}$$

In words, there is no set R whose elements are precisely the sets x that are not elements of themselves.

Proof. We prove this by contradiction. Suppose R is a set satisfying $(*)$. Applying $(*)$ to $x = R$ gives

$$R \in R \quad \Leftrightarrow \quad R \notin R. \tag{$\dagger$}$$

Split into two cases.

- Case 1: assume $R \in R$. Then $R \notin R$ by the $\Rightarrow$ part of $(\dagger)$. This contradicts our assumption that $R \in R$.
- Case 2: assume $R \notin R$. Then $R \in R$ by the $\Leftarrow$ part of $(\dagger)$. This contradicts our assumption that $R \notin R$.

In either case, we get a contradiction. So the proof is finished. □

Question 4.4.4 (tongue-in-cheek)**.** Can you write a proof of Theorem 4.4.3 that does not mention contradiction? ✎ 4g

Tutorial exercises

4.1. Which of the following are true? Which of them are false?

(a) $\varnothing \in \varnothing$.

(b) $\varnothing \subseteq \varnothing$.

(c) $\varnothing \in \{\varnothing\}$.

(d) $\varnothing \subseteq \{\varnothing\}$.

(e) $\{\varnothing, 1\} = \{1\}$.

(f) $1 \in \{\{1, 2\}, \{2, 3\}, 4\}$.

(g) $\{1,2\} \subseteq \{3,2,1\}$.

(h) $\{3,3,2\} \subsetneq \{3,2,1\}$.

4.2. Let $A = \{2n+1 : n \in \mathbb{Z}\}$ and $B = \{2n-1 : n \in \mathbb{Z}\}$. Is $A \overset{\text{yes}}{=} B$? Prove that your answer is correct.

4.3. Write down $\mathcal{P}(\mathcal{P}(\varnothing))$ in roster notation.

4.4. Let $U = \{5, 6, 7, \ldots, 12\}$. Write down the following sets in roster notation.

(a) $\{n \in U : n \text{ is even}\}$.

(b) $\{m - n : m, n \in U\}$.

(c) $\{-5, -4, -3, \ldots, 5\} \setminus \{1, 2, 3, \ldots, 10\}$.

(d) $\overline{\{5,7,9\} \cup \{9,11\}}$, in a context where U is the universal set.

4.5. In this exercise, we study the notion of *symmetric difference* of sets, which is the set-theoretic counterpart of the exclusive-or operation in propositional logic.

For sets A and B, define $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

(a) Let $A = \{1, 4, 9, 16\}$ and $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$. Write down $A \triangle B$ in roster notation.

(b) Show that, for all sets A, B,

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

4.6. In this exercise, we investigate one way of expressing the subset relation in terms of Boolean operations.

Let A, B be sets. Show that $A \subseteq B$ if and only if $A \cup B = B$.

4.7. Let $A = \{5, 6, 7, 8, 9\}$. Consider

$$\mathscr{C}_1 = \{\{5,6\}, \{7,8\}\} \quad \text{and} \quad \mathscr{C}_2 = \{\{5\}, \{6,7,8\}, \{8,9\}\}.$$

Define $P_1(A, \mathscr{C})$ and $P_2(A, \mathscr{C})$ to be respectively the sentences

$$\forall x \in A \ \ \exists S \in \mathscr{C} \ \ (x \in S) \quad \text{and} \quad \forall S_1, S_2 \in \mathscr{C} \ \ (S_1 \cap S_2 \neq \varnothing \Rightarrow S_1 = S_2).$$

Which of $P_1(A, \mathscr{C}_1), P_2(A, \mathscr{C}_1), P_1(A, \mathscr{C}_2), P_2(A, \mathscr{C}_2)$ are true? Which of them are false? Briefly explain your answers.

We will encounter P_1 and P_2 again in Chapter 6.

4.8. De Morgan's Laws, as we stated them, is about two sets only. Nevertheless, it is readily generalizable to any finite number of sets, as we will show in this exercise.

Work in a context with a universal set. Prove by induction that for all $n \in \mathbb{Z}^+$ and all sets $A_0, A_1, A_2, \ldots,$

$$\overline{A_0 \cup A_1 \cup \ldots A_n} = \overline{A}_0 \cap \overline{A}_1 \cap \cdots \cap \overline{A}_n.$$

Extra exercises

4.9. Let A and B be sets. Define $A_0 = A \setminus B$ and $B_0 = B$. Prove that $A_0 \cap B_0 = \varnothing$ and $A_0 \cup B_0 = A \cup B$.

4.10. In this exercise, we investigate another way of expressing the subset relation in terms of Boolean operations; cf. Tutorial Exercise 4.6.

Let A, B be sets. Show that $A \subseteq B$ if and only if $A \cap B = A$.

4.11. Similar to $\sum$ for addition and $\prod$ for multiplication, we have $\bigcap$ for intersection and $\bigcup$ for union. This notation will be useful in the tutorial exercises for Chapter 8.

Let $A_0, A_1, A_2, \dots$ be sets. For each $n \in \mathbb{N}$, define

$$\bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap \cdots \cap A_n \quad \text{and} \quad \bigcap_{i=0}^{\infty} A_i = \{x : x \in A_i \text{ for all } i \in \mathbb{N}\},$$
$$\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \cdots \cup A_n \quad \text{and} \quad \bigcup_{i=0}^{\infty} A_i = \{x : x \in A_i \text{ for some } i \in \mathbb{N}\}.$$

Fix $n \in \mathbb{N}$. Consider the case when each $A_i = \{x \in \mathbb{Q} : \frac{1}{i+1} \leqslant x \leqslant i+1\}$. What are $\bigcap_{i=0}^{n} A_i$, $\bigcap_{i=0}^{\infty} A_i$, $\bigcup_{i=0}^{n} A_i$, and $\bigcup_{i=0}^{\infty} A_i$?