Tutorial solutions for Chapter 7

Sometimes there are other correct answers.

7.1. (a) **No.** One can write this condition symbolically as $\forall x \in A\ \ \exists y \in B\ \ y = f(x)$. The function $f\colon \{\mathrm{a}, \mathrm{b}\} \to \{0, 1\}$ satisfying $f(\mathrm{a}) = 0 = f(\mathrm{b})$ is not surjective but it satisfies the condition there.

(b) **No.** One can write this condition symbolically as $\exists y \in B\ \ \exists x \in A\ \ y = f(x)$. The function $f\colon \{\mathrm{a}, \mathrm{b}\} \to \{0, 1\}$ satisfying $f(\mathrm{a}) = 0 = f(\mathrm{b})$ is not surjective but it satisfies the condition there.

(c) **No.** One can write this condition symbolically as $\exists y \in B\ \ \forall x \in A\ \ y = f(x)$. The function $f\colon \{\mathrm{a}, \mathrm{b}\} \to \{0, 1\}$ satisfying $f(\mathrm{a}) = 0 = f(\mathrm{b})$ is not surjective but it satisfies the condition there.

(d) **Yes.** One can write this condition symbolically as $\neg\exists y \in B\ \ \forall x \in A\ \ y \neq f(x)$. This is equivalent to the surjectivity of f by Theorem 2.4.9.

Moral. Sometimes it is easier to see the logical structure of a proposition when it is written symbolically.

7.2. (a) **No.** One can write this symbolically as $\forall x_1, x_2 \in A\ \ \big(x_1 = x_2 \to f(x_1) = f(x_2)\big)$. The function $f\colon \{\mathrm{a}, \mathrm{b}\} \to \{0, 1\}$ satisfying $f(\mathrm{a}) = 0 = f(\mathrm{b})$ is not injective but it satisfies the condition there.

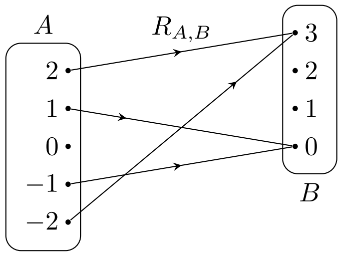
(b) **Yes.** One can write this symbolically as $\forall x_1, x_2 \in A\ \ \big(x_1 \neq x_2 \to f(x_1) \neq f(x_2)\big)$. This is equivalent to the injectivity of f because any conditional proposition is equivalent to its contrapositive.

(c) **Yes.** One can write this symbolically as $\neg\exists x_1, x_2 \in A\ \ \big(x_1 \neq x_2 \wedge f(x_1) = f(x_2)\big)$. This is equivalent to the injectivity of f by Theorem 2.4.9 and Example 1.4.23.

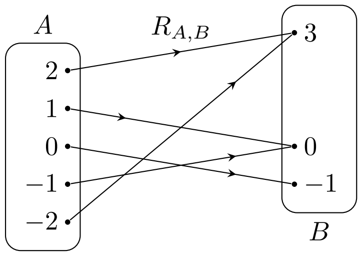
(d) **No.** One can write this symbolically as $\exists x_1, x_2 \in A\ \ \big(x_1 \neq x_2 \wedge f(x_1) = f(x_2)\big)$. The function $f\colon \{\mathrm{a}, \mathrm{b}\} \to \{0, 1\}$ satisfying $f(\mathrm{a}) = 0 = f(\mathrm{b})$ is not injective but it satisfies the condition there.

Moral. Sometimes it is easier to see the logical structure of a proposition when it is written symbolically.

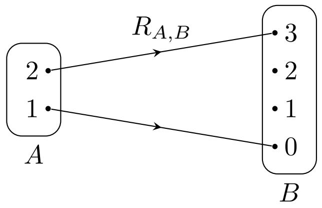
7.3. (a) **No**, because 0 is not related to any element of B, violating (F1).



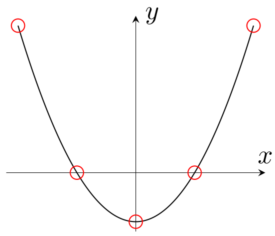
(b) Let $B = \{-1, 0, 3\}$. There is no other choice.



(c) Say $A = \{1, 2\}$. There are exactly 8 other examples.



Extra information.



Moral. The same equation can define different relations: some are functions, some are not; some are surjective, some are not; some are injective, some are not.

7.4. (a) (F1) Let $n \in \mathbb{Z}$. Then n is either even or odd by Proposition 3.2.21. If n is even, then the definition of even integers gives $x \in \mathbb{Z}$ such that $n = 2x$. If n is odd, then the definition of odd integers gives $x \in \mathbb{Z}$ such that $n = 2x + 1$. In all cases, we get $x \in \mathbb{Z}$ such that $n = 2x$ or $n = 2x + 1$, i.e., such that $(n, x) \in f$.

(F2) Let $n, x_1, x_2 \in \mathbb{Z}$ such that $(n, x_1), (n, x_2) \in f$. Recall that n is either even or odd by Proposition 3.2.21.

Consider the case when n is even. As $(n, x_1) \in f$, the definition of f tells us $n = 2x_1$ or $n = 2x_1 + 1$. The latter cannot be true because it implies n is odd, and no integer is both even and odd by Proposition 3.2.17. So it must be the case that $n = 2x_1$. Similarly, as $(n, x_2) \in f$, we derive that $n = 2x_2$. Combining the two, we have $2x_1 = n = 2x_2$. Thus $x_1 = x_2$.

Consider the case when n is odd. As $(n, x_1) \in f$, the definition of f tells us $n = 2x_1$ or $n = 2x_1 + 1$. The former cannot be true because it implies n is even, and no integer is both even and odd by Proposition 3.2.17. So it must be the case that $n = 2x_1 + 1$. Similarly, as $(n, x_2) \in f$, we derive that $n = 2x_2 + 1$. Combining the two, we have $2x_1 + 1 = n = 2x_2 + 1$. It follows that $x_1 = x_2$. $\square$

(b) This f is **surjective**, as shown below.

Let $x \in \mathbb{Z}$. Define $n = 2x$. Then $(n, x) \in f$ by the definition of f. $\square$

(c) This f is **not injective**, as shown below.

Note that $0 = 2 \times 0$ and $1 = 2 \times 0 + 1$. So $(0, 0), (1, 0) \in f$ by the definition of f, but $(0, 0) \neq (1, 0)$. $\square$

(d) The range of f is $\mathbb{Z}$.

Additional explanation. Note that the codomain of f is $\mathbb{Z}$. So we know $\operatorname{range}(f) \subseteq \mathbb{Z}$ by Remark 7.2.3(2). Conversely, by (b), for every $x \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $f(n) = x$. This shows $\mathbb{Z} \subseteq \operatorname{range}(f)$. Hence $\operatorname{range}(f) = \mathbb{Z}$. $\square$

Additional comment. Here $f(n)$ is the quotient when n is divided by 2, or equivalently $f(n) = \lfloor n/2 \rfloor$, for each $n \in \mathbb{Z}$.

7.5. (a) This g is **surjective**, as shown below.
Note that $g(\mathrm{T},\mathrm{T},\mathrm{F}) = \mathrm{T}$ and $g(\mathrm{T},\mathrm{T},\mathrm{T}) = \mathrm{F}$. So for every $y \in \{\mathrm{T},\mathrm{F}\}$, there exists $(p_0, q_0, r_0) \in \{\mathrm{T},\mathrm{F}\}^3$ such that $g(p_0, q_0, r_0) = y$. $\square$

(b) This g is **not injective**, as shown below.
Note that $g(\mathrm{T},\mathrm{T},\mathrm{T}) = \mathrm{F} = g(\mathrm{T},\mathrm{F},\mathrm{T})$, where $(\mathrm{T},\mathrm{T},\mathrm{T}) \neq (\mathrm{T},\mathrm{F},\mathrm{T})$. $\square$

(c) The function $f\colon \{\mathrm{T},\mathrm{F}\} \to \{\mathrm{T},\mathrm{F}\}^3$ that satisfies $f(\mathrm{T}) = (\mathrm{T},\mathrm{T},\mathrm{F})$ and $f(\mathrm{F}) = (\mathrm{T},\mathrm{T},\mathrm{T})$ has the required property.

Additional information.

p	q	r	$p \vee q$	$\neg r$	$p \vee q \to \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

7.6. Assume f is surjective and $g \circ f = \mathrm{id}_A$. Let $y_1, y_2 \in B$ such that $g(y_1) = g(y_2)$. Apply the surjectivity of f to find $x_1, x_2 \in A$ satisfying $f(x_1) = y_1$ and $f(x_2) = y_2$. Substituting these into $g(y_1) = g(y_2)$ gives

$$
\begin{array}{lll}
 & g(f(x_1)) = g(f(x_2)) & \\
\therefore & (g \circ f)(x_1) = (g \circ f)(x_2) & \text{by Proposition 7.3.1.} \\
\therefore & \mathrm{id}_A(x_1) = \mathrm{id}_A(x_2) & \text{as } g \circ f = \mathrm{id}_A \text{ by assumption.} \\
\therefore & x_1 = x_2 & \text{by the definition of } \mathrm{id}_A. \\
\therefore & y_1 = f(x_1) = f(x_2) = y_2 & \text{by the choice of } x_1, x_2.
\end{array}
$$
$\square$

7.7. The induction step breaks down when $k = 1$, as explained below.

When $k = 1$, the whole campus has only a and b in it, because it has exactly $k + 1 = 2$ people in it. So no person stayed on the campus throughout. Therefore, while it is true that "the people who stayed on the campus throughout have the same birthday as both a and b", it is only true vacuously. In this case, one cannot deduce that a and b have the same birthday.

Moral. In the induction step, one needs to carefully check that the proof of $P(k+1)$ from $P(k)$ works for *all* relevant k's. Even the failure of one of them can make the whole proof fail very badly.

Extra exercises

7.8. f is not a function $\mathbb{Q} \to \mathbb{Q}$, while g is.

Additional explanations.

- Recall that $y = \pm x$ stands for "$y = x$ or $y = -x$". So $1 = \pm 1$ and $-1 = \pm 1$. Thus $(1, 1), (1, -1) \in f$, where $1 \neq -1$. This shows that f does not satisfy (F2).

- g is the function $\mathbb{Q} \to \mathbb{Q}$ that satisfies

$$g(x) = \frac{1}{x^2 + 1}$$

for all $x \in \mathbb{Q}$.

7.9. Assume f^{-1} is a function $B \to A$.

(Surjectivity) Let $y \in B$. Use (F1) for f^{-1} to find $x \in A$ such that $(y, x) \in f^{-1}$. Then $(x, y) \in f$ by the definition of f^{-1}. So $f(x) = y$.

(Injectivity) Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. Define $y = f(x_1)$. Then $y = f(x_2)$ too because $f(x_1) = f(x_2)$. So $(x_1, y) \in f$ and $(x_2, y) \in f$. These imply $(y, x_1), (y, x_2) \in f^{-1}$ via the definition of f^{-1}. Hence $x_1 = x_2$ as f^{-1} satisfies (F2). □

7.10. (a) (F1) Let $x \in \mathbb{Z}$. Then $2x \in \mathbb{Z}$. So $(x, 2x) \in g$.

(F2) For g, let $x, n_1, n_2 \in \mathbb{Z}$ such that $(x, n_1), (x, n_2) \in g$. Then $n_1 = 2x$ and $n_2 = 2x$ by the definition of g. So $n_1 = 2x = n_2$. □

(b) g is **not surjective**, as shown below.

Consider $-1231 \in \mathbb{Z}$. It is odd by Example 3.1.3(2). So -1231 is not even by Proposition 3.2.17. This means $-1231 \neq 2x$ for any $x \in \mathbb{Z}$. Thus $g(x) \neq -1231$ for any $x \in \mathbb{Z}$ according to the definition of g.

(c) g is **injective**, as shown below.

Let $x_1, x_2 \in \mathbb{Z}$ such that $g(x_1) = g(x_2)$. Then $2x_1 = 2x_2$ by the definition of g. So $x_1 = x_2$.

(d) ($\subseteq$) Let $(x, y) \in f \circ g$. Use the definition of $f \circ g$ to find $n \in \mathbb{Z}$ such that $(x, n) \in g$ and $(n, y) \in f$. In view of the definitions of g and f, the former means $n = 2x$, while the latter means $n = 2y$ or $n = 2y + 1$. As $n = 2x$, we know n is even. So n is not odd by Proposition 3.2.17. Thus $n \neq 2y + 1$. It follows that $n = 2y$, from which we deduce $2y = n = 2x$. Hence $x = y$. This implies $(x, y) = (x, x) \in \mathrm{id}_{\mathbb{Z}}$.

($\supseteq$) Let $(x, y) \in \mathrm{id}_{\mathbb{Z}}$. Then $x = x$ by the definition of id. In view of the definitions of g and f, we know $(x, 2x) \in g$ and $(2x, x) \in f$. So $(x, y) = (x, x) \in f \circ g$.

It follows that $f \circ g = \mathrm{id}_{\mathbb{Z}}$.

Consider $1 \in \mathbb{Z}$. Since $1 = 2 \times 0 + 1$, we know $f(1) = 0$ from the definition of f. So $(g \circ f)(1) = g(f(1)) = g(0) = 2 \times 0 = 0 \neq 1 = \mathrm{id}_{\mathbb{Z}}(1)$. This shows $g \circ f \neq \mathrm{id}_{\mathbb{Z}}$. □

(e) **No**. Since $1 = 2 \times 0 + 1$, we know $(1, 0) \in f$ from the definition of f. So $(0, 1) \in f^{-1}$ by the definition of f^{-1}. However, we know $(0, 1) \notin g$ because $1 \neq 2 \times 0$. So $g \neq f^{-1}$. □

Alternative proof. We know from Tutorial Exercise 7.4 that f is not injective, and thus not bijective. So Extra Exercise 7.9 tells us f^{-1} is not a function. Since g is a function by (a), it must be the case that $g \neq f^{-1}$. □