NATIONAL UNIVERSITY OF SINGAPORE

CS1231 - DISCRETE STRUCTURES

(Semester 1: AY2018/19)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This assessment paper contains **EIGHTEEN (18)** questions in **TWO (2)** parts and comprises **TEN (10)** printed pages, including this page.

2. Answer **ALL** questions.

3. This is an **OPEN BOOK** assessment.

4. You are allowed to use **NUS APPROVED CALCULATORS**.

5. You are to submit two documents: The **OAS form** and the **Answer Sheets**. You may keep this question paper.

6. On the OAS form, **shade** and **write** your **Student Number** completely and accurately.

7. You must use 2B pencil for the OAS form.

8. On the first page of your Answer Sheets, **write** your **Student Number**. **Do not write your name.**

9. You may use pen or pencil to write your answers, but please erase cleanly, and write legibly. Marks may be deducted for illegible handwriting.

Part B: (40 marks) Structured questions. Write your answer in the Answer Sheet.

Marks may be deducted for illegible handwriting and unnecessary statements in proofs.

Q16. Counting and Probability (14 marks)

Answer the following parts. Working is not required.

(a) (2 marks) What is the probability that on three rolls of a fair six-sided die, at least one 6 shows up? Leave your answer as a fraction or marks will be deducted.

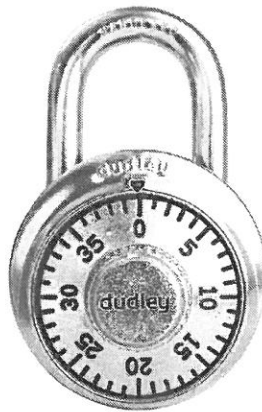(b) (2 marks) Figure 1 below shows a combination lock with 40 positions.

Figure 1: Combination Lock.

To open the lock, you rotate to a number in a clockwise direction, then to a second number in the counterclockwise direction, and finally to a third number in the clockwise direction. If consecutive numbers in the combination cannot be the same, how many combinations of three-number codes are there?

(c) (2 marks) There are 12 slips of paper in a bag. Some of the slips have a 2 written on them, and the rest have a 7 written on them. If the expected value of the number shown on a slip randomly drawn from the bag is 3.25, how many slips have a 2 written on them?

(d) A bowl contains three coins. Two of them are normal coins and one of them is a two-headed coin.

    i. (2 marks) You pick one coin at random and toss it. What is the probability that you get a head? Write your answer as a fraction.

    ii. (2 marks) You pick one coin at random, toss it and get a head. What is the probability that the coin is the two-headed coin? Write your answer as a fraction.

(e) (4 marks) A row of houses are randomly assigned distinct numbers between 1 and 50 inclusive. What is the minimum number of houses to ensure that there are 5 houses numbered consecutively?

To receive full credit, you must define the pigeon and pigeonholes.

Q17. Graphs (14 marks)

The *lazy caterer's sequence* describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

For example, with three straight cuts, you get seven pieces as shown in Figure 2 below.
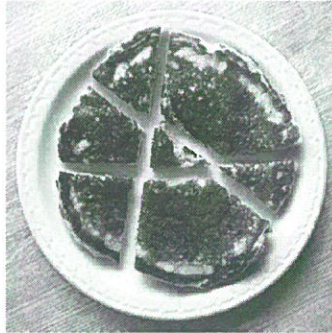

Figure 2: Pancake (Photo credit: Wikipedia).

Figure 3 below shows the first few values in the lazy caterer's sequence starting with $n = 0$ where n is the number of straight cuts. The sequence is 1, 2, 4, 7, 11, 16, ...
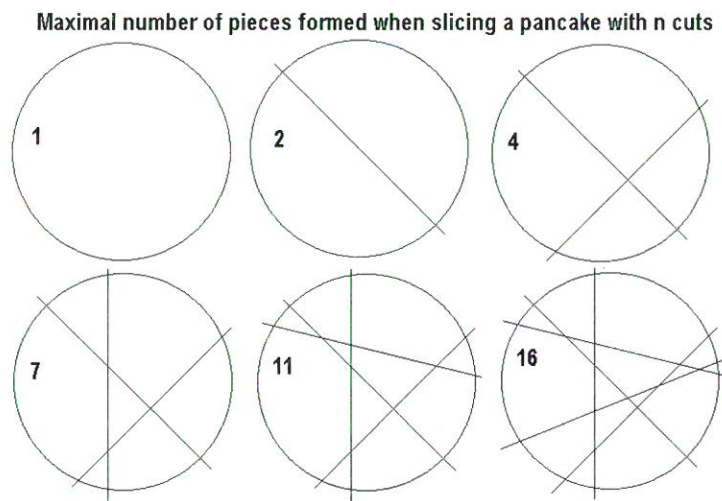

Figure 3: Lazy Caterer's Sequence.

We may model this problem by using a graph. Figure 4 shows the graph corresponding to $n = 3$, where the vertices are the intersections among the cuts and the boundary of the pancake.

We may define the following functions:

$P(n)$: number of pieces of pancakes with n cuts

$V(n)$: number of vertices of a graph corresponding to a pancake with n cuts

$E(n)$: number of edges of a graph corresponding to a pancake with n cuts

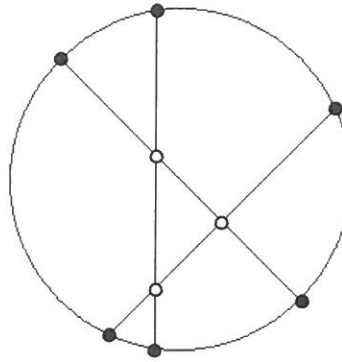In Figure 4, $P(3) = 7, V(3) = 9$ and $E(3) = 15$.

Figure 4: Graph representation.

The vertices of the graph are of two types: those with degree of three (which lie on the boundary of the circle, shown as black dots) and those with degree of four (which lie inside the circle, shown as white dots). Let's define two more functions:

$V_3(n)$: number of vertices with degree three

$V_4(n)$: number of vertices with degree four

In Figure 4, $V_3(3) = 6$ and $V_4(3) = 3$.

Answer the following parts. Working is not required.

(a) (2 marks) Express $E(n)$ in terms of $V_3(n)$ and $V_4(n)$.

(b) (2 marks) Write the recurrence relation for $V(n)$. The base case is $V(0) = 0$.

(c) (2 marks) Write the closed form formula for $V(n)$.

(d) (2 marks) Write the recurrence relation for $E(n)$. The base case is $E(0) = 0$.

(e) (2 marks) Write the closed form formula for $E(n)$.

(f) (2 marks) Euler's formula is given as $v - e + f = 2$. Relate v, e and f with the functions defined in this question.

(g) (2 marks) From part (f), or otherwise, derive the closed form formula for $P(n)$.

Q18. Functions (12 marks)

Private cars in Singapore have license plates (see Figure 5) in the format: $S\alpha_1\alpha_2\ x_1x_2x_3x_4\ c$, where each α_1 and α_2 is a single letter taken from the usual English alphabet (excluding I and O), and each $x_1, \ldots, x_4$ is a single digit taken from $\{0, 1, \ldots, 9\}$. The last letter c is a checksum letter, ie. a function of the preceding letters and numbers. Its purpose is to serve as a quick check on the validity of the license plate.

Figure 5: A typical Singapore car license plate. (Photo Credit: Wikipedia)

Let $\mathcal{L}$ denote the set of all possible strings of the form: $\alpha_1\alpha_2x_1x_2x_3x_4$. The possible values of α_i and x_j are as described above. Also, let $\mathcal{K} = \{A, Z, Y, X, U, T, S, R, P, M, L, K, J, H, G, E, D, C, B\}$. Then the checksum function may be defined as $f : \mathcal{L} \to \mathcal{K}$, where $f(\alpha_1\alpha_2x_1x_2x_3x_4)$ is calculated in three steps:

F1. Let n_1 be the positional value of α_1 in the English alphabet, ie.
$A = 1$, $B = 2$, $C = 3, \ldots$, $Z = 26$. And let n_2 be the positional value of α_2. (Note that since I and O are not allowed, $n_1, n_2 \notin \{9, 15\}$.)

F2. Compute $t = 9n_1 + 4n_2 + 5x_1 + 4x_2 + 3x_3 + 2x_4$, and $r = t\%19$. That is, r is the remainder of t modulo 19, which means $0 \le r < 19$.

F3. The checksum letter $c =$ the letter in $\mathcal{K}$ indexed by r, where
$0 = A, 1 = Z, 2 = Y, \ldots, 18 = B$. (Here, we are treating $\mathcal{K}$ as an ordered set, in which its elements are indexed by position, starting from 0.)

Using the example in Figure 5:

F1. $\alpha_1 = D$, $\alpha_2 = N$, $x_1 = 7$, $x_2 = 4$, $x_3 = 8$, $x_4 = 4$; and so $n_1 = 4$, $n_2 = 14$.

F2. Then $t = 9 \cdot 4 + 4 \cdot 14 + 5 \cdot 7 + 4 \cdot 4 + 3 \cdot 8 + 2 \cdot 4 = 175$, and so $r = 175\%19 = 4$.

F3. Hence $c = U$.

(a) (2 marks) Determine the checksum letter for $CS1231$. (No working needed.)

(b) (2 marks) Show that f is not one-to-one by finding another $y \in \mathcal{L}$ such that $f(y)$ is the same checksum letter as that in Figure 5. (No working needed. Just state a suitable y.)

(c) (8 marks) (Difficult) Is f onto? Prove or disprove.

_____ **END OF PAPER** _____