Tutorial solutions for Chapter 4

Sometimes there are other correct answers.

4.1. (a) **False.** (The empty set $\varnothing$ is by definition a set with no element. In particular, the empty set $\varnothing$ cannot be its element.)

(b) **True.** (The empty set $\varnothing$ is a subset of any set by Proposition 4.2.7. In particular, it is a subset of itself.)

(c) **True.** (According to the definition of roster notation, the set $\{\varnothing\}$ contains exactly one element $\varnothing$.)

(d) **True.** (The empty set $\varnothing$ is a subset of any set by Proposition 4.2.7. In particular, it is a subset of $\{\varnothing\}$.)

(e) **False.** (The empty set $\varnothing$ is an element of $\{\varnothing, 1\}$, but it is not an element of $\{1\}$.)

(f) **False.** (According to the definition of roster notation, the set $\{\{1,2\},\{2,3\},4\}$ contains exactly 3 elements: the set $\{1,2\}$, the set $\{2,3\}$, and the number 4; none of these is equal to 1.)

(g) **True.** (The set $\{1,2\}$ contains exactly 2 elements: the number 1 and the number 2; both of these are elements of $\{3,2,1\}$.)

(h) **True.** (The set $\{3,3,2\}$ contains exactly 2 elements: the number 3 and the number 2; both of these are elements of $\{3,2,1\}$. However, the number 1 is an element of $\{3,2,1\}$ but not an element of $\{3,3,2\}$.)

4.2. **Yes**, as shown below.

($\subseteq$) Let $a \in A$. Use the definition of A to find $n \in \mathbb{Z}$ such that $a = 2n+1$. Then $a = 2(n+1)-1$. As $n \in \mathbb{Z}$, we know $n+1 \in \mathbb{Z}$. So $a \in B$ by the definition of B.

($\supseteq$) Let $b \in B$. Use the definition of B to find $n \in \mathbb{Z}$ such that $a = 2n-1$. Then $b = 2(n-1)+1$. As $n \in \mathbb{Z}$, we know $n-1 \in \mathbb{Z}$. So $b \in A$ by the definition of A.

Hence $A = B$ by the definition of set equality. $\square$

4.3. $\mathcal{P}(\mathcal{P}(\varnothing)) = \{\varnothing, \{\varnothing\}\}$.

Explanations. We know $\varnothing$ is a subset of $\varnothing$ by Proposition 4.2.7. If A is a nonempty set, then A has an element and thus A cannot be a subset of $\varnothing$ which by definition has no element. So the only subset of $\varnothing$ is $\varnothing$. This tells us $\mathcal{P}(\varnothing) = \{\varnothing\}$.

Now $\{\varnothing\}$ has exactly 1 element, namely the empty set $\varnothing$. In constructing a subset of $\{\varnothing\}$, one can choose to take this element of $\{\varnothing\}$ or not, giving two results $\varnothing$ and $\{\varnothing\}$. No other set can be a subset of $\{\varnothing\}$. So

$$\mathcal{P}(\mathcal{P}(\varnothing)) = \mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}.$$

4.4. (a) $\{6, 8, 10, 12\}$.

Explanation. The elements of this set are exactly the elements of U that are even.

(b) $\{-7, -6, -5, \ldots, 7\}$.

Explanation.

$\{m - n : m, n \in U\}$
$= \{5-5, 5-6, \ldots, 5-12, 6-5, 6-6, \ldots, 6-12, \ldots, 12-5, 12-6, \ldots, 12-12\}$
$= \{0, -1, \ldots, -7, 1, 0, \ldots, -6, \ldots, 7, 6, \ldots, 0\}$
$= \{-7, -6, -5, \ldots, 7\}$.

(c) $\{-5, -4, -3, -2, -1, 0\}$.

Explanation. The elements of this set are exactly the integers n for which $-5 \leqslant n \leqslant 5$ is true but $1 \leqslant n \leqslant 10$ is false.

(d) $\{6, 8, 10, 12\}$.

Explanation. $\overline{\{5,7,9\} \cup \{9,11\}} = \overline{\{5,7,9,11\}} = \{6,8,10,12\}$, in a context where U is the universal set.

4.5. (a) $A \setminus B = \{1, 9\}$ and $B \setminus A = \{2, 6, 8, 10, 12, 14\}$. So $A \triangle B = \{1, 2, 6, 8, 9, 10, 12, 14\}$.

(b) Compare the following truth tables.

$z \in A$	$z \in B$	$z \in A \setminus B$	$z \in B \setminus A$	$z \in A \triangle B$
T	T	F	F	F
T	F	T	F	T
F	T	F	T	T
F	F	F	F	F

$z \in A$	$z \in B$	$z \in A \cup B$	$z \in A \cap B$	$z \in (A \cup B) \setminus (A \cap B)$
T	T	T	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

Since the last columns of the two tables are the same, we conclude that $A \triangle B = (A \cup B) \setminus (A \cap B)$. $\square$

Alternatively, one may use the set identities in the context where we have a universal set U.

$A \triangle B$	
$= (A \setminus B) \cup (B \setminus A)$	by the definition of $\triangle$;
$= (A \cap \overline{B}) \cup (B \cap \overline{A})$	by the set identity on set difference;
$= \big((A \cap \overline{B}) \cup B\big) \cap \big((A \cap \overline{B}) \cup \overline{A}\big)$	by distributivity;
$= (A \cup B) \cap (\overline{B} \cup B) \cap (A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})$	by distributivity;
$= (A \cup B) \cap U \cap U \cap (\overline{B} \cup \overline{A})$	by the set identity on complement;
$= (A \cup B) \cap U \cap U \cap (\overline{B \cap A})$	by De Morgan's Laws;
$= (A \cup B) \cap (\overline{B \cap A})$	as U is the identity for $\cap$;
$= (A \cup B) \cap (\overline{A \cap B})$	as $\cap$ is commutative;
$= (A \cup B) \setminus (A \cap B)$	by the set identity on set difference.

$\square$

Another way to proceed in propositional logic as follows: for every object z,

$$\begin{aligned}
& z \in A \triangle B \\
\Leftrightarrow \quad & z \in (A \setminus B) \cup (B \setminus A) && \text{by the definition of } \triangle; \\
\Leftrightarrow \quad & (z \in A \wedge z \notin B) \vee (z \in B \wedge z \notin A) && \text{by the definition of } \cup, \cap, \setminus; \\
\Leftrightarrow \quad & \big((z \in A \wedge z \notin B) \vee z \in B\big) \wedge \big((z \in A \wedge z \notin B) \vee z \notin A\big) && \\
& && \text{by distributivity;} \\
\Leftrightarrow \quad & (z \in A \vee z \in B) \wedge (z \notin B \vee z \in B) \wedge (z \in A \vee z \notin A) \wedge (z \notin B \vee z \notin A) && \\
& && \text{by distributivity;} \\
\Leftrightarrow \quad & (z \in A \vee z \in B) \wedge (z \notin B \vee z \notin A) && \text{by the logical identities} \\
& && \text{on negation and identities;} \\
\Leftrightarrow \quad & (z \in A \vee z \in B) \wedge \neg(z \in B \wedge z \in A) && \text{by De Morgan's Laws;} \\
\Leftrightarrow \quad & z \in (A \cup B) \setminus (B \cap A) && \text{by the definition of } \cup, \cap, \setminus.
\end{aligned}$$

□

4.6. (⇒) Assume $A \subseteq B$. We want to show $A \cup B = B$. We already know $B \subseteq A \cup B$ from Example 4.3.9(2). So it remains to show $A \cup B \cup B$.

Take $z \in A \cup B$. Then $z \in A$ or $z \in B$ by the definition of $\cup$.

- If $z \in A$, then $z \in B$ by our assumption that $A \subseteq B$.
- If $z \in B$, then of course $z \in B$.

So $z \in B$ in all cases, as required.

(⇐) Assume $A \cup B = B$. We want to show $A \subseteq B$. Take any $z \in A$. This implies $z \in A \cup B$ in view of Example 4.3.9(2). Thus $z \in B$ since $A \cup B = B$ by assumption, as required. □

Alternative proof for ⇐. Assume $A \cup B = B$. Then Example 4.3.9(1) tells us $A \subseteq A \cup B = B$. □

4.7.
- $P_1(A, \mathscr{C}_1)$ is **false** because $9 \in A$ that is not in any $S \in \mathscr{C}_1$.
- $P_2(A, \mathscr{C}_1)$ is **true** because no distinct elements $S_1, S_2 \in \mathscr{C}_1$ have a nonempty intersection, as one can verify exhaustively.
- $P_1(A, \mathscr{C}_2)$ is **true** because every $x \in A$ is in some $S \in \mathscr{C}_2$, as one can verify exhaustively.
- $P_2(A, \mathscr{C}_2)$ is **false** because $\{6, 7, 8\}$ and $\{8, 9\}$ are distinct elements of $\mathscr{C}_2$ that have a nonempty intersection $\{8\}$.

Additional comment. Note that $P_2(A, \mathscr{C})$ is equivalent to

$$\forall x \in A \ \ \forall S_1, S_2 \in \mathscr{C} \ \ (x \in S_1 \wedge x \in S_2 \Rightarrow S_1 = S_2).$$

So the conjunction of $P_1(A, \mathscr{C})$ and $P_2(A, \mathscr{C})$ can be expressed as

$$\forall x \in A \ \ \exists! S \in \mathscr{C} \ \ x \in S.$$

4.8. Fix sets $A_0, A_1, A_2, \ldots$. Let $P(n)$ be the predicate "$\overline{A_0 \cup A_1 \cup \ldots A_n} = \overline{A}_0 \cap \overline{A}_1 \cap \cdots \cap \overline{A}_n$" over $\mathbb{Z}^+$.

(Base step) $P(1)$ is true because $\overline{A_0 \cup A_1} = \overline{A}_0 \cap \overline{A}_1$ by De Morgan's Laws.

(Induction step) Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e., that $\overline{A_0 \cup A_1 \cup \cdots \cup A_k} = \overline{A}_0 \cap \overline{A}_1 \cap \cdots \cap \overline{A}_k$. Then

$$\begin{aligned}
&\overline{A_0 \cup A_1 \cup \cdots \cup A_k \cup A_{k+1}} \\
&= \overline{A_0 \cup A_1 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} && \text{by De Morgan's Laws;} \\
&= \overline{A_0} \cap \overline{A_1} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}} && \text{by the induction hypothesis.}
\end{aligned}$$

This shows $P(k+1)$ is true.

Hence $\forall n \in \mathbb{Z}^+ \quad P(n)$ is true by MI. □

Alternative solution. Let $P(n)$ be the predicate

$$\overline{A_0 \cup A_1 \cup \ldots A_n} = \overline{A}_0 \cap \overline{A}_1 \cap \cdots \cap \overline{A}_n \text{ for all sets } A_0, A_1, \ldots, A_n$$

over $\mathbb{Z}^+$.

(Base step) $P(1)$ is true because $\overline{A_0 \cup A_1} = \overline{A}_0 \cap \overline{A}_1$ by De Morgan's Laws.

(Induction step) Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e., that $\overline{A_0 \cup A_1 \cup \cdots \cup A_k} = \overline{A}_0 \cap \overline{A}_1 \cap \cdots \cap \overline{A}_k$ for all sets $A_0, A_1, \ldots, A_k$. For all sets $B_0, B_1, \ldots, B_{k+1}$,

$$\begin{aligned}
&\overline{B_0 \cup B_1 \cup \cdots \cup B_k \cup B_{k+1}} \\
&= \overline{B_0} \cap \overline{B_1} \cap \cdots \cap \overline{B_k \cup B_{k+1}} && \text{by the induction hypothesis;} \\
&= \overline{B_0} \cap \overline{B_1} \cap \cdots \cap \overline{B_k} \cap \overline{B_{k+1}} && \text{by De Morgan's Laws.}
\end{aligned}$$

This shows $P(k+1)$ is true.

Hence $\forall n \in \mathbb{Z}^+ \quad P(n)$ is true by MI. □

Additional comment. In the first proof above, it does not matter much whether we put the universal quantifier for $A_0, A_1, A_2, \ldots$ into $P(n)$ or not. This is *not* true in the alternative proof.

Extra exercises

4.9. If $x \in A_0 \cap B_0 = (A \setminus B) \cap B$, then $x \in A \setminus B$ and $x \in B$ by the definition of $\cap$, and so $x \notin B$ and $x \in B$, which is not possible. Therefore, the set $A_0 \cap B_0$ cannot have any element, and hence it must be equal to $\varnothing$ by Theorem 4.1.18.

We know $A_0 \cup B_0 = A \cup B$ because for every object z,

$$\begin{aligned}
& z \in A_0 \cup B_0 \\
\Leftrightarrow\ & z \in (A \setminus B) \cup B && \text{by the definitions of } A_0 \text{ and } B_0; \\
\Leftrightarrow\ & (z \in A \wedge z \notin B) \vee z \in B && \text{by the definition of } \setminus \text{ and } \cup; \\
\Leftrightarrow\ & (z \in A \vee z \in B) \wedge (z \notin B \vee z \in B) && \text{by the Distributive Laws;} \\
\Leftrightarrow\ & z \in A \vee z \in B && \text{by the logical identities on negation and identities;} \\
\Leftrightarrow\ & z \in A \cup B && \text{by the definition of } \cup.
\end{aligned}$$

Additional comment. If we are in a context with a universal set U, then we can proceed alternatively as follows:

$$\begin{aligned}
A_0 \cap B_0 &= (A \setminus B) \cap B && \text{by the definition of } A_0 \text{ and } B_0;\\
&= (A \cap \overline{B}) \cap B && \text{by the set identity on set difference;}\\
&= A \cap \varnothing && \text{by the set identity on the complement;}\\
&= \varnothing && \text{as } \varnothing \text{ is an identity for } \cap;\\
A_0 \cup B_0 &= (A \setminus B) \cup B && \text{by the definition of } A_0 \text{ and } B_0;\\
&= (A \cap \overline{B}) \cup B && \text{by the set identity on set difference;}\\
&= (A \cup B) \cap (\overline{B} \cup B) && \text{by the Distributive Laws;}\\
&= (A \cup B) \cup U && \text{by the set identities on the complements;}\\
&= A \cup B && \text{as } U \text{ is an identity for } \cup.
\end{aligned}$$

4.10. ($\Rightarrow$) Assume $A \subseteq B$. We want to show $A \cap B = A$. We already know $A \cap B \subseteq A$ from Example 4.3.9(1). So it remains to show $A \cap B \supseteq A$.

Take $z \in A$. Then $z \in B$ by our assumption that $A \subseteq B$. So $z \in A \cap B$ by the definition of $\cap$, as required.

($\Leftarrow$) Assume $A \cap B = A$. We want to show $A \subseteq B$. Take any $z \in A$. This implies $z \in A \cap B$ as $A \cap B = A$ by assumption. Recall from Example 4.3.9(1) that $A \cap B \subseteq B$. So $z \in B$, as required. □

Alternative proof for $\Leftarrow$. Assume $A \cap B = A$. Then $A = A \cap B \subseteq B$ by Example 4.3.9(1). □

Another alternative proof. In view of Tutorial Exercise 4.6, it suffices to show that $A \cup B = B$ if and only if $A \cap B = A$.

($\Rightarrow$) Assume $A \cup B = B$. Then

$$\begin{aligned}
A \cap B &= A \cap (A \cup B) && \text{by assumption;}\\
&= A && \text{by the Absorption Law.}
\end{aligned}$$

($\Leftarrow$) Assume $A \cap B = A$. Then

$$\begin{aligned}
A \cup B &= (A \cap B) \cup B && \text{by assumption;}\\
&= B && \text{by the Absorption Law.}
\end{aligned}$$ □

4.11. • $\bigcap_{i=0}^{n} A_i = \{1\} = A_0$.

Explanation. Note that $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$. So

$$\begin{aligned}
\bigcap_{i=0}^{n} A_i &= A_0 \cap A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n && \text{by the definition of } \bigcap_{i=0}^{n} A_i;\\
&= A_0 \cap A_2 \cap A_3 \cap \cdots \cap A_n && \text{by Extra Exercise 4.10, as } A_0 \subseteq A_1;\\
&= A_0 \cap A_3 \cap \cdots \cap A_n && \text{by Extra Exercise 4.10, as } A_0 \subseteq A_2;\\
&= A_0 \cap \cdots \cap A_n && \text{by Extra Exercise 4.10, as } A_0 \subseteq A_3;\\
&= \cdots = A_0 \cap A_n &&\\
&= A_0 && \text{by Extra Exercise 4.10, as } A_0 \subseteq A_n.
\end{aligned}$$

• $\bigcap_{i=0}^{\infty} A_i = \{1\} = A_0$.

Explanation. ($\subseteq$) Let $z \in \bigcap_{i=0}^{\infty} A_i$. Then $z \in A_i$ for each $i \in \mathbb{N}$ by the definition of $\bigcap_{i=0}^{\infty} A_i$. In particular, we know $z \in A_0$.

($\supseteq$) Note that every element of A_0 is an element of A_i, for every $i \in \mathbb{N}$. This means every element of A_0 is an element of $\bigcap_{i=0}^{\infty} A_i$ according to the definition of $\bigcap_{i=0}^{\infty} A_i$. So $A_0 \subseteq \bigcap_{i=0}^{\infty} A_i$.

- $\bigcup_{i=0}^{n} A_i = \{x \in \mathbb{Q} : \frac{1}{n+1} \leqslant x \leqslant n+1\} = A_n$.

  Explanation. Note that $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$. So

$$\begin{aligned}
\bigcup_{i=0}^{n} A_i &= A_0 \cup A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n && \text{by the definition of } \bigcup_{i=0}^{n} A_i; \\
&= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n && \text{by Extra Exercise 4.6, as } A_0 \subseteq A_1; \\
&= A_2 \cup A_3 \cup \cdots \cup A_n && \text{by Extra Exercise 4.6, as } A_0 \subseteq A_2; \\
&= A_3 \cup \cdots \cup A_n && \text{by Extra Exercise 4.6, as } A_0 \subseteq A_3; \\
&= \cdots = A_{n-1} \cup A_n \\
&= A_n && \text{by Extra Exercise 4.6, as } A_0 \subseteq A_n.
\end{aligned}$$

- $\bigcup_{i=0}^{\infty} A_i = \{x \in \mathbb{Q} : x > 0\}$.

  Explanation. ($\subseteq$) Let $z \in \bigcup_{i=0}^{\infty} A_i$. Use the definition of $\bigcup_{i=0}^{\infty} A_i$ to find $i \in \mathbb{N}$ such that $z \in A_i$. According to the definition of A_i, this means $\frac{1}{i+1} \leqslant z \leqslant i+1$. Thus $z \geqslant \frac{1}{i+1} > 0$ as $i+1 \geqslant 0+1 > 0$. So $z \in \{x \in \mathbb{Q} : x > 0\}$.

  ($\supseteq$) Let $z \in \{x \in \mathbb{Q} : x > 0\}$. Then $x > 0$. Use the Archimedean property of $\mathbb{R}$ to find $j \in \mathbb{N}$ such that $j > z - 1$ and $j > \frac{1}{z} - 1$. Now $\frac{1}{j+1} \leqslant z \leqslant j+1$, and thus $z \in A_j$. This implies $z \in \bigcup_{i=0}^{\infty} A_i$.

  Note. Our definition of the A_i's does not cover the A_∞ case: if we try to substitute ∞ into i, then we face expressions such as $\frac{1}{\infty+1}$ and $\infty + 1$, which do not have commonly accepted meanings.

Moral. It is possible that an infinite intersection or union reaches its true value *only* after infinitely many steps; thus *no* finite number of steps suffices to reveal the true answer in general. There is a gap between "arbitrarily large finite" and "infinite".