NATIONAL UNIVERSITY OF SINGAPORE

CS1231/CS1231S – DISCRETE STRUCTURES

(Semester 2: AY2020/21)

(ANSWERS)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY ONE (21)** questions (excluding question 0) in **THREE (3)** parts and comprises **TEN (10)** printed pages.
2. Answer ALL questions.
3. This is an **OPEN BOOK** assessment.
4. The maximum mark of this assessment is 100.

Question	Max. mark
Q0	3
Part A: Q1 – 10	20
Part B: Q11 – 16	18
Part C: Q17	5
Part C: Q18	8
Part C: Q19	6
Part C: Q20	20
Part C: Q21	20
Total	**100**

5. You are to submit a single pdf file (size $\leq$ 20MB) to your submission folder on LumiNUS.
6. The number of pages in your file should not exceed 6.
7. Your submitted file should be named after your **Student Number** (eg: A1234567X.pdf) and your Student Number should be written on the first page of your file.
8. Do NOT write your name anywhere in your submitted file.

——— END OF INSTRUCTIONS ———

1	**B**	2	**C**	3	**A**	4	**B**	5	**D**				
6	**D**	7	**C**	8	**C**	9	**D**	10	**D**				
11	**A**	12	**C**	13	**B,C,D,E**	14	**D,E**	15	**B,D**	16	**A,B,C**		

0. **Check that you have done the following:**
   (a) Submission folder consists of a **single pdf file** and no other files. [1 mark]
   (b) File named correctly with **Student Number** (eg: A1234567X.pdf). [1 mark]
   (c) Student Number written **on top of the first page** of submitted file. [1 mark]

Part A: Multiple Choice Questions [Total: 10×2 = 20 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer. You are advised to write your answers on a **single line or two lines** to conserve space. For example:

1. A 2. B 3. C 4. D ...

Please write in **CAPITAL LETTERS**.

1. Which of the following is/are true?
   (i) To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to
   - show that $P(0)$, $P(1)$ are true; and
   - show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

   true

   (ii) To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to
   - show that $P(0)$ is true; and
   - show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(k) \Rightarrow P(k+2)\big)$ is true.

   false

   A. (i) and (ii).

   B. (i) only.

   C. (ii) only.

   D. None.

Answer: B.
(i) This is a weakening of the usual MI.
(ii) Let $P(n)$ be "n is even".

2. Let $A = \{1,2,3\}$. Define $g, h : A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

   What is the order of the function $g \circ (g \circ h)^{-1} \circ h \circ h \circ g \circ h \circ h$?

   A. 1.

   B. 2.

   C. 3.

   D. 4.

Answer: C.
Recall from Tutorial 7 Question 7 that the order of h is 2. In particular, we know $h \circ h = \mathrm{id}_A$. So

$$\begin{aligned} g \circ (g \circ h)^{-1} \circ h \circ h \circ g \circ h \circ h &= g \circ (g \circ h)^{-1} \circ \mathrm{id}_A \circ g \circ h \circ h \\ &= g \circ (g \circ h)^{-1} \circ (g \circ h) \circ h = g \circ \mathrm{id}_A \circ h = g \circ h, \end{aligned}$$

which has order 3 by Tutorial 7 Question 7 again.

3. For a set A and a function $f: A \to A$, define

$$C_f = \{f^{-1}(\{y\}) : y \in A \text{ and } f(y) = y\}.$$

Which of the following is/are true for all sets A and all functions $f: A \to A$?

(i) If $f \circ f = f$, then C_f is a partition of A.

(ii) If C_f is a partition of A, then $f \circ f = f$.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.

Answer: A.

(i) Suppose $f \circ f = f$. (Let's show that every $S \in C_f$ is nonempty.) Let $y \in A$ such that $f(y) = y$. Then $y \in f^{-1}(\{y\})$ and so $f^{-1}(\{y\})$ must be nonempty. (Let's show that every $x \in A$ is in at least one $S \in C_f$.) Let $x \in A$. Define $y = f(x)$. Then $f(y) = f(f(x)) = (f \circ f)(x) = f(x) = y$. So $x \in f^{-1}(\{y\}) \in C_f$. (Let's show that every $x \in A$ is in at most one $S \in C_f$.) Let $x, y_1, y_2 \in A$ such that $x \in f^{-1}(\{y_1\})$ and $x \in f^{-1}(\{y_2\})$. Then $y_1 = f(x) = y_2$ and thus $f^{-1}(\{y_1\}) = f^{-1}(\{y_2\})$.

(ii) Suppose C_f is a partition of A. Let $x \in A$. Find $y \in A$ such that $f(y) = y$ and $x \in f^{-1}(\{y\})$. Then $f(x) = y = f(y) = f(f(x)) = (f \circ f)(x)$.

4. Which of the following is/are true for all functions $f: A \to B$ and all finite subsets $X \subseteq A$, assuming $f(X)$ is finite?

(i) $|f(X)| \leqslant |X|$.

(ii) $|f(X)| \geqslant |X|$.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.

Answer: B.

(i) If $X = \{x_1, x_2, \ldots, x_n\}$ has cardinality n, then $f(X) = \{f(x_1), f(x_2), \ldots, f(x_n)\}$ has cardinality at most n.

(ii) Consider $A = B = \{0,1\}$ and $f(0) = 0 = f(1)$, with $X = \{0,1\}$. Then $|f(X)| = |f(\{0,1\})| = |\{f(0), f(1)\}| = |\{0,0\}| = |\{0\}| = 1 < 2 = |\{0,1\}| = |X|$.

5. Which of the following is/are true for all functions $f: A \to B$ and all finite subsets $Y \subseteq B$, assuming $f^{-1}(Y)$ is finite?

(i) $|f^{-1}(Y)| \leqslant |Y|$.

(ii) $|f^{-1}(Y)| \geqslant |Y|$.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.

Answer: D.

Consider $A = B = \{0,1\}$ and $f(0) = 0 = f(1)$.

(i) Let $Y_1 = \{0\}$. Then $|f^{-1}(Y_1)| = |f^{-1}(\{0\})| = |\{0,1\}| = 2 > 1 = |\{0\}| = |Y_1|$.

(ii) Let $Y_2 = \{1\}$. Then $|f^{-1}(Y_2)| = |f^{-1}(\{1\})| = |\emptyset| = 0 < 1 = |\{1\}| = |Y_2|$.

6. Let $\mathrm{Bool} = \{\mathbf{true}, \mathbf{false}\}$. Define $f\colon \mathrm{Bool}^2 \to \mathrm{Bool}^2$ by setting, for all $p, q \in \mathrm{Bool}$,

$$f(p,q) = ({\sim}p \wedge q, p \wedge {\sim}q).$$

Which of the following is true?

A. f is injective and surjective.

B. f is injective but not surjective.

C. f is surjective but not injective.

D. f is neither injective nor surjective.

Answer: D.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{{\sim}p \wedge q}$	$\boldsymbol{p \wedge {\sim}q}$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	F	F

7. Which of the following is/are true?

(i) If A is a set of sets that has an element that is infinite, then A is infinite.

(ii) If A is a set of sets that has a subset that is infinite, then A is infinite.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.

Answer: C.

(i) The set $\{\mathbb{Z}\}$ has only 1 element, namely $\mathbb{Z}$, although $\mathbb{Z}$ is an infinite set.

(ii) If A has an infinite subset B, then all the infinitely many elements of B are elements of A too.

8. Define $f\colon \mathcal{P}(\mathbb{Z}_{\geqslant 0}) \setminus \{\emptyset\} \to \mathbb{Z}_{\geqslant 0}$ by setting $f(S)$ to be the smallest element of S whenever $S \in \mathcal{P}(\mathbb{Z}_{\geqslant 0}) \setminus \{\emptyset\}$. Which of the following is/are true?

(i) The function f has an inverse.

(ii) $f^{-1}(\{n\})$ is uncountable for some $n \in \mathbb{Z}_{\geqslant 0}$.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.

Answer: C.

(i) Note that $f(\{0\}) = 0 = f(\{0,1\})$, although $\{0\} \neq \{0,1\}$. So f is not injective. It follows that f does not have an inverse.

(ii) Note that $f^{-1}(\{0\}) = \left\{\{0\} \cup S : S \in \mathcal{P}\left(\mathbb{Z}_{\geqslant 1}\right)\right\}$. If S_1, S_2 and different elements of $\mathcal{P}(\mathbb{Z}_{\geqslant 1})$, then $\{0\} \cup S_1 \neq \{0\} \cup S_2$. As $\mathcal{P}(\mathbb{Z}_{\geqslant 1})$ is uncountable by Theorem 10.4.3, we deduce that $f^{-1}(\{0\})$ is also uncountable.

9. How many permutations of **AprilFools!** are there?

A. $9!$

B. $11!$

C. $\frac{11!}{2}$

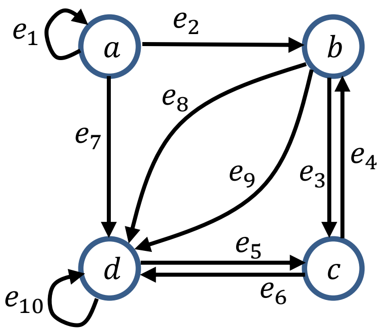
D. $\frac{11!}{4}$

E. $\frac{11!}{24}$

Answer: D.

Lecture 12 slide 15: $\frac{11!}{2!2!} = \frac{11!}{4}$.

10. How many **walks of length 3** are there from vertex a to vertex d in the directed graph given below?



A. 6

B. 7

C. 8

D. 9

E. 10

Answer: D.

There are **9** walks of length 3: $ae_1ae_2be_8d$; $ae_1ae_2be_9d$; $ae_2be_8de_{10}d$; $ae_2be_9de_{10}d$; $ae_7de_5ce_6d$; $ae_1ae_7de_{10}d$; $ae_2be_3ce_6d$; $ae_1ae_1ae_7d$; $ae_7de_{10}de_{10}d$.

Alternatively, compute A^3 and obtain its A^3_{14} value.

$$A^2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 4 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 1 & 2 & 4 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} ? & ? & ? & \mathbf{9} \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

Part B: Multiple Response Questions [Total: 6×3 = 18 marks]

Each multiple response question (MRQ) is worth **THREE marks** and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C.

Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

You are advised to write your answers on **a single line** to conserve space. For example:

11. A,B   12. B,D   13. C   14. A,B,C,D   …

Please write in **CAPITAL LETTERS**.

11. Recall that $\{\mathrm{s}, \mathrm{u}\}^*$ denotes the set of all strings over $\{\mathrm{s}, \mathrm{u}\}$. Define a subset $T \subseteq \{\mathrm{s}, \mathrm{u}\}^*$ recursively as follows.

- $\mathrm{su} \in T$. (base clause)
- If $\sigma, \tau \in T$, then $\mathrm{s}\sigma\mathrm{u} \in T$ and $\mathrm{s}\sigma\tau\mathrm{u} \in T$. (recursion clause)
- Membership for T can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the following is/are in T?

A. $\mathrm{ssssusuuusuu}$

B. $\mathrm{ssussusuuu}$

C. $\mathrm{ssussuusssuu}$

D. $\mathrm{ussusussuu}$

Answer: A.

A. $\mathrm{su} \to \mathrm{ssusuu} \to \mathrm{sssusuuu} \to \mathrm{ssssusuuusuu}$.

One can show the following statements by structural induction over T.

(a) All elements of T start with s and end with u.

(b) All elements of T of length at least 3 start with ss and end with uu.

(c) In no element of T can there be an occurrence of usus.

(d) In every element of T, the number of s's is equal to the number of u's.

We can use (c) to eliminate B, use (d) to eliminate C, and use (a) to eliminate D.

12. Define a subset $S \subseteq \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ recursively as follows.

- $(1,0) \in S$. (base clause)
- If $(m_1, n_1), (m_2, n_2) \in S$, then (recursion clause)

$$(m_1 + 1, n_1 + 1) \in S \quad \text{and} \quad (m_1 + m_2 + 1, n_1 + n_2 + 2) \in S.$$

- Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the following is/are in S?

A. $(12{,}31)$.

B. $(31{,}12)$.

C. $(1231{,}1230)$.

D. $(2021{,}1231)$.

Answer: C.

One can start from the base clause and apply the first recursion clause 1230 times to show that $(1231{,}1230) \in S$.

As one can show by structural induction over S, every $(m, n) \in S$ satisfies $m = n + 1$. This eliminates A, B, and D.

13. Which of the following is/are true for all sets A and all functions $f: A \to A$?

    A. $f = \mathrm{id}_A$ if and only if $g \circ f = g$ for some function g with domain A.

    B. $f = \mathrm{id}_A$ if and only if $g \circ f = g$ for all injective functions g with domain A.

    C. $f = \mathrm{id}_A$ if and only if $g \circ f = g$ for all surjective functions g with domain A.

    D. $f = \mathrm{id}_A$ if and only if $g \circ f = g$ for all bijective functions g with domain A.

    E. $f = \mathrm{id}_A$ if and only if $\mathrm{id}_A \circ f = \mathrm{id}_A$.

Answer: B, C, D, E.

E is true by the alternative solution to Tutorial 7 Question 4. This implies B, C, D are all true because id_A is bijective.

For A, let $A = \{-1,1\}$ and $f: A \to A$ satisfying $f(1) = -1$ and $f(-1) = 1$. Define $g: A \to \{0\}$ by setting $g(x) = 0$ for all $x \in A$. Then $g \circ f = g$ but $f \neq \mathrm{id}_A$.

14. Which of the following sets is/are countable?

    A. The set of all partitions of $\mathbb{Z}$.

    B. The set of all partial orders on $\mathbb{Z}$.

    C. The set of all functions $\mathbb{Z} \to \mathbb{Z}$.

    D. The set $\mathbb{Z}^*$ of all strings over $\mathbb{Z}$.

    E. The set of all simple undirected graphs whose vertex set is a finite subset of $\mathbb{Z}$.

Answer: D, E.

A. Each nonempty subset $S \subseteq \mathbb{Z}$ gives rise to a partition $\mathcal{C}_S = \{S\} \cup \{\{x\} : x \in \mathbb{Z} \setminus S\}$ of $\mathbb{Z}$. In this sense, different subsets of $\mathbb{Z}$ of cardinality at least 2 give rise to different partitions of $\mathbb{Z}$. From Theorem 10.4.3 and Exercise 10.4.4(5), we know there are uncountably many subsets of $\mathbb{Z}$ but only countably many of them have cardinality less than 2. So there must be uncountably many partitions of $\mathbb{Z}$ by Proposition 10.3.5.

B. Each $S \subseteq \mathbb{Z}$ gives rise to a partial order $\preccurlyeq_S$ on $\mathbb{Z}$ satisfying, for all $x, y \in \mathbb{Z}$,

$$x \preccurlyeq_S y \quad \Leftrightarrow \quad x = y \lor (x \in S \land y \notin S).$$

In this sense, different nonempty proper subsets of $\mathbb{Z}$ give rise to different partial orders on $\mathbb{Z}$. From Theorem 10.4.3, we know there are uncountably many subsets of $\mathbb{Z}$, but only 2 of them are empty or improper. So there must be uncountably many partial orders on $\mathbb{Z}$.

C. Every sequence $a_0, a_1, a_2, \ldots$ over $\{0,1\}$ gives rise to a function $f_a\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f_a(x) = \begin{cases} a_x, & \text{if } x \geqslant 0; \\ 2, & \text{if } x < 0. \end{cases}$$

As one can readily verify, different sequences over $\{0,1\}$ give rise to different functions $\mathbb{Z} \to \mathbb{Z}$ in this sense. From Exercise 10.4.4(7), we know there are uncountably many sequences over $\{0,1\}$. So there must be uncountably many functions $\mathbb{Z} \to \mathbb{Z}$ by Proposition 10.3.5.

D. Every string $a_0 a_1 \ldots a_{\ell-1}$ over $\mathbb{Z}$ gives rise to the string $\sigma_0 \sigma_1 \ldots \sigma_{\ell-1}$ over $\{\mathrm{s}, \mathrm{u}\}$, where

$$\sigma_i = \begin{cases} \mathrm{s}\,\overbrace{\mathrm{uuuuu} \ldots \mathrm{u}}^{a_i+1 \text{ many}}, & \text{if } a_i \geqslant 0; \\ \mathrm{ss}\,\underbrace{\mathrm{uuuuu} \ldots \mathrm{u}}_{-a_i \text{ many}}, & \text{if } a_i < 0. \end{cases}$$

Note that different strings over $\mathbb{Z}$ give rise to different strings over $\{\mathrm{s}, \mathrm{u}\}$ in this sense. From Exercise 10.4.4(6), we know there are countably many strings over $\{\mathrm{s}, \mathrm{u}\}$. So there must be countably many strings over $\mathbb{Z}$ by Proposition 10.3.5.

E. If G is a simple undirected graph whose vertex set V_G is a finite subset of $\mathbb{Z}$, then we can identify its edge set E_G as a subset of the finite subsets of $\mathbb{Z}$. There are countably many choices for V_G and countably many choices for E_G by Exercise 10.4.4(5). So altogether there are countably many choices by Tutorial 8 Question 5.

15. Given the following statement:

$$\exists x \in \mathbb{Z}\,\big(P(x) \wedge Q(x)\big) \equiv \big(\exists x \in \mathbb{Z}\, P(x)\big) \wedge \big(\exists x \in \mathbb{Z}\, Q(x)\big)$$

what are the predicates $P(x)$ and $Q(x)$ that can make the above statement false?

A. $P(x) = Q(x)$.

B. $P(x) =$ "x is prime"; $Q(x) =$ "x is composite".

C. $P(x) =$ "x is divisible by 3"; $Q(x) =$ "x is divisible by 5".

D. $P(x) =$ "x is even"; $Q(x) =$ "x is odd".

E. $P(x) =$ "x is even"; $Q(x) =$ "x is an even prime".

Answer: B, D.

16. Given a complete graph K_{99} and a complete bipartite graph $K_{80,60}$, which of the following statements is/are true?

A. K_{99} is an Eulerian graph.

B. K_{99} is a Hamiltonian graph.

C. $K_{80,60}$ is an Eulerian graph.

D. $K_{80,60}$ is a Hamiltonian graph.

Answer: A, B, C.

The degree at each vertex in K_{99} is an even number 98, therefore K_{99} is an Eulerian graph. Since K_{99} is complete, it is a Hamiltonian graph.

The degree at each vertex in $K_{80.60}$ is either 60 or 80, both even numbers, therefore $K_{80.60}$ is an Eulerian graph. A Hamiltonian circuit in a complete bipartite graph must alternate between the vertices on each side, and since $80 \neq 60$, by the pigeonhole principle a vertex will repeat in the circuit, therefore $K_{80.60}$ is not a Hamiltonian graph.

Part C: There are 5 questions in this part [Total: 59 marks]

17. [5 marks]

Define a sequence $a_1, a_2, a_3, \ldots$ by setting, for each $n \in \mathbb{Z}^+$,

$$a_1 = \frac{1}{10} \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{(3n+2)(3n+5)}.$$

Prove using (usual or strong) induction that, for all $n \in \mathbb{Z}^+$,

$$a_n = \frac{n}{2(3n+2)}.$$

Marks will not be given for proofs that do not involve induction.

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition "$a_n = \frac{n}{2(3n+2)}$".
2. (Base step) $P(1)$ is true because
$$\frac{1}{2(3 \times 1 + 2)} = \frac{1}{10} = a_1.$$
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e.,
$$a_k = \frac{k}{2(3k+2)}.$$
   3.2. Then $a_{k+1} = a_k + \frac{1}{(3k+2)(3k+5)}$ by the definition of a_{k+1};
   3.3. $= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$ by the induction hypothesis;
   3.4. $= \frac{1}{3k+2}\left(\frac{k}{2} + \frac{1}{3k+5}\right)$
   3.5. $= \frac{1}{3k+2} \cdot \frac{3k^2+5k+2}{2(3k+5)}$
   3.6. $= \frac{1}{3k+2} \cdot \frac{(3k+2)(k+1)}{2(3k+5)}$
   3.7. $= \frac{k+1}{2(3(k+1)+2)}$.
   3.8. So $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}^+\ P(n)$ is true by MI.

18. [8 marks]

Let A be a set and $\mathcal{C}$ be a partition of A. Prove that there exists a function $f: A \to A$ such that

$$f \circ f = f \quad \text{and} \quad \mathcal{C} = \{f^{-1}(\{y\}) : y \in A \text{ and } f(y) = y\}.$$

Answer:

1. For each $S \in \mathcal{C}$, pick an element $y_S \in S$ that will be fixed throughout. (This is possible because $\mathcal{C}$ is a partition, and every element of a partition is a nonempty set.)
2. Define $f: A \to A$ by setting, for all $x \in A$ and all $S \in \mathcal{C}$,
$$f(x) = y_S \quad \Leftrightarrow \quad x \in S.$$
3. Then f is a well defined because every element of A is in exactly one component of $\mathcal{C}$.
4. Note that for all $S \in \mathcal{C}$,
   - 4.1. $y_S \in S$ by the choice of y_S on line 1;
   - 4.2. $\therefore$ $f(y_S) = y_S$ by the definition of f.
5. We verify $f \circ f = f$.
   - 5.1. Let $x \in A$.
   - 5.2. Use the fact that $\mathcal{C}$ is a partition of A to find $S \in \mathcal{C}$ such that $x \in S$.
   - 5.3. Then $f(x) = y_S$ by the definition of f.
   - 5.4. So $(f \circ f)(x) = f\big(f(x)\big) = f(y_S) = y_S = f(x)$ by line 4.
6. We verify $\mathcal{C} = \{f^{-1}(\{y\}) : y \in A \text{ and } f(y) = y\}$.
   - 6.1. $(\subseteq)$
     - 6.1.1. Let $S \in \mathcal{C}$.
     - 6.1.2. From line 4, we know $f(y_S) = y_S$.
     - 6.1.3. For all x,
       - 6.1.3.1. $x \in S \quad \Leftrightarrow \quad f(x) = y_S$ by the definition of f;
       - 6.1.3.2. $\Leftrightarrow \quad x \in f^{-1}(\{y_S\})$ by the definition of $f^{-1}(\{y_S\})$.
     - 6.1.4. So $S = f^{-1}(\{y_S\})$.
   - 6.2. $(\supseteq)$
     - 6.2.1. Let $y \in A$ such that $f(y) = y$.
     - 6.2.2. Use the fact that $\mathcal{C}$ is a partition of A to find $S \in \mathcal{C}$ such that $y \in S$.
     - 6.2.3. Then the definition of f implies $y_S = f(y) = y$.
     - 6.2.4. For all x,
       - 6.2.4.1. $x \in f^{-1}(\{y\}) \quad \Leftrightarrow \quad f(x) = y$
       - 6.2.4.2. $\Leftrightarrow \quad f(x) = y_S$ by line 6.2.3;
       - 6.2.4.3. $\Leftrightarrow \quad x \in S$ by the definition of f.
     - 6.2.5. So $f^{-1}(\{y\}) = S$.

19. [6 marks]

Let A be a set. Let S be the set of all functions $\{0,1\} \to A$, i.e.,

$$S = \{\alpha \mid \alpha: \{0,1\} \to A\}.$$

Prove that $|S| = |A^2|$ according to Cantor's definition of same-cardinality.

Answer:

1. Define $f: S \to A^2$ by setting $f(\alpha) = (\alpha(0), \alpha(1))$ for all $\alpha \in S$.
2. Define $g: A^2 \to S$ by setting $g(a, b)$ to be the function $\alpha: \{0,1\} \to A$ satisfying
$$\alpha(0) = a \quad \text{and} \quad \alpha(1) = b,$$
for all $a, b \in A$.
3. For all $(a, b) \in A$ and all $\alpha \in S$,
   3.1. $f(\alpha) = (a, b) \iff \alpha(0) = a$ and $\alpha(1) = b$ by the definition of f;
   3.2. $\iff g(a, b) = \alpha$ by the definition of g.
4. So g is an inverse of f.
5. Thus f is bijective by Theorem 9.3.19.
6. This shows $|S| = |A^2|$.

Direct proof that f is bijective:

1. (Injectivity)
   1.1. Let $\alpha, \beta \in S$ such that $f(\alpha) = f(\beta)$.
   1.2. Then $(\alpha(0), \alpha(1)) = (\beta(0), \beta(1))$ by the definition of f.
   1.3. So $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$.
   1.4. Since both α and β have domain $\{0,1\}$ and codomain A, this shows $\alpha = \beta$.
2. (Surjectivity)
   2.1. Let $(a, b) \in A^2$.
   2.2. Define $\alpha: \{0,1\} \to A$ by setting $\alpha(0) = a$ and $\alpha(1) = b$.
   2.3. Then $f(\alpha) = (\alpha(0), \alpha(1)) = (a, b)$ by the definition of f.

Direct proof that g is bijective:

1. (Injectivity)
   1.1. Let $(a_1, b_1), (a_2, b_2) \in A^2$ such that $g(a_1, b_1) = g(a_2, b_2)$.
   1.2. Take $\alpha \in S$ satisfying $g(a_1, b_1) = \alpha = g(a_2, b_2)$.
   1.3. Then the definition of g tells us
$$a_1 = \alpha(0) = a_2 \quad \text{and} \quad b_1 = \alpha(1) = b_2.$$
   1.4. So $(a_1, b_1) = (a_2, b_2)$.
2. (Surjectivity)
   2.1. Pick any $\alpha \in S$.
   2.2. Let $\beta = g(\alpha(0), \alpha(1))$.
   2.3. Then $\beta(0) = \alpha(0)$ and $\beta(1) = \alpha(1)$ by the definition of g.
   2.4. As both α and β have domain $\{0,1\}$ and codomain A, this shows $\alpha = \beta = g(\alpha(0), \alpha(1))$.

20. Counting and Probability [Total: 20 marks]

Note that working is not required for parts (a) to (d).

(a) How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

$$\boldsymbol{x_1 + x_2 + x_3 + x_4 = 56.}$$

Write your answer as a single number. [3 marks]

Answer:

$x_1 \geq 2^1 + 1 = 3;\ x_2 \geq 2^2 + 2 = 6;\ x_3 \geq 2^3 + 3 = 11; x_4 \geq 2^4 + 4 = 20.$

$3 + 6 + 11 + 20 = 40.$

Let $y_i = x_i - (2^i + i)$, then $y_1 + y_2 + y_3 + y_4 = 16$, where $y_i \geq 0$.

This becomes a multiset problem with $n = 4, r = 16$.

$\binom{r+n-1}{r} = \binom{16+4-1}{16} = \binom{19}{16} = \mathbf{969}.$

(b) On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

(i) What is the probability of rolling a 6? Write your answer as a single fraction. [1 mark]

(ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction. [3 marks]

Answer:

Probability of $\frac{1}{9}$ to roll a 1, 2 or 3; probability of $\mathbf{\frac{2}{9}}$ to roll a 4, 5 or 6.

Maximum of 1: 1 way → (1,1) with probability $\frac{1}{81}$.

Maximum of 2: 3 ways → (1,2), (2,1), (2,2) with probability $\frac{3}{81}$.

Maximum of 3: 5 ways → (1,3)x2, (2,3)x2, (3,3) with probability $\frac{5}{81}$.

Maximum of 4: 7 ways → (1,4)x2, (2,4)x2, (3,4)x2, (4,4) with probability $\frac{12}{81} + \frac{4}{81} = \frac{16}{81}$.

Maximum of 5: 9 ways → (1,5)x2, (2,5)x2, (3,5)x2, (4,5)x2, (5,5) with probability $\frac{12}{81} + \frac{12}{81} = \frac{24}{81}$.

Maximum of 6: 11 ways → (1,6)x2, (2,6)x2, (3,6)x2, (4,6)x2, (5,6)x2, (6,6) with probability $\frac{12}{81} + \frac{20}{81} = \frac{32}{81}$.

Expected value = $\left(\frac{1}{81} \times 1\right) + \left(\frac{3}{81} \times 2\right) + \left(\frac{5}{81} \times 3\right) + \left(\frac{16}{81} \times 4\right) + \left(\frac{24}{81} \times 5\right) + \left(\frac{32}{81} \times 6\right) = \mathbf{\frac{398}{81}}$.

(c) There are three urns U_1, U_2 and U_3. Urn U_k $(1 \le k \le 3)$ contains k red balls and $k+1$ blue balls.

(i) If you draw 2 balls at random from U_2 without replacement, what is the probability of drawing **at least one red ball**? Write your answer as a single fraction. [2 marks]

(ii) Four words "I", "CAN", "DO" and "IT" are separately written on 4 slips of paper and concealed, each having an equal chance of being selected.

You select a slip of paper at random and reveal the word written on it. The length of the word, k, directs you to urn U_k to pick a ball. If the ball picked is **blue**, what is the probability that it comes from U_2? Write your answer as a single fraction or a percentage rounded to 4 significant figures. [3 marks]

Answer:

(i) There are 2 red balls and 3 blue balls in U_2.

Number of ways to draw 2 balls = $\binom{5}{2} = 10$.

Number of ways to draw 1 red ball and 1 blue ball = $\binom{2}{1} \times \binom{3}{1} = 6$.

Number of ways to draw 2 red balls = $\binom{2}{2} \times \binom{3}{0} = 1$.

Therefore, P(at least one red ball) = $\frac{6+1}{10} = \mathbf{\frac{7}{10}}$.

Alternatively, number of ways to draw 2 blue balls = $\binom{3}{2} \times \binom{2}{0} = 3$.

Therefore, P(at least one red ball) = $\frac{10-3}{10} = \mathbf{\frac{7}{10}}$.

(ii) $P(U_1) = \frac{1}{4}, P(U_2) = \frac{1}{2}, P(U_3) = \frac{1}{4}$.

$P(U_1 \cap Blue) = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}, P(U_2 \cap Blue) = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10}, P(U_3 \cap Blue) = \frac{1}{4} \times \frac{4}{7} = \frac{1}{7}$.

$$P(U_2 \mid Blue) = \frac{\frac{3}{10}}{\frac{1}{6} + \frac{3}{10} + \frac{1}{7}} = \frac{\frac{3}{10}}{\frac{256}{420}} = \mathbf{\frac{63}{128}} \quad \text{or } \mathbf{49.22\%}.$$

(d) Aaron loves Hawaii and is always looking forward to spending his holiday there. He is intrigued to hear that new bridges will be constructed on the islands of Maui, Moloka'i and Lana'i, making a total of 13 bridges between these islands. Each bridge connects two islands.

If there are 11 bridges with one end on Maui and 6 with one end on Lana'i, (i) how many bridges are there connecting Maui and Moloka'i, and (ii) how many bridges are there connecting Moloka'i and Lana'i? [4 marks]

Answer:

Let Ma, Mo and La be the set of bridges on Maui, Moloka'i and Lana'i respectively.

Short method:

1. $|Ma \cap Mo| = |Ma \cup Mo \cup La| - |La|$ (bridges connecting Maui and Moloka'i do not touch Lana'i) $= 13 - 6 = \mathbf{7}$.
2. $|Mo \cap La| = |Ma \cup Mo \cup La| - |Ma|$ (bridges connecting Moloka'i and Lana'i do not touch Maui) $= 13 - 11 = \mathbf{2}$.

Long method:

1. $|Ma \cup Mo \cup La| = 13$. (given)
2. Since every bridge connects two islands, $|Ma \cap Mo \cap La| = 0$.
3. Similarly, $|Ma \cap Mo| + |Mo \cap La| + |La \cap Ma| = |Ma \cup Mo \cup La|$.
4. By inclusion-exclusion rule,

   $$|Ma \cup Mo \cup La|$$
   $$= |Ma| + |Mo| + |La| - (|Ma \cap Mo| + |Mo \cap La| + |La \cap Ma|) + |Ma \cap Mo \cap La|$$
   $$= |Ma| + |Mo| + |La| - |Ma \cup Mo \cup La| + 0 \text{ (by 3 and 4)}$$

   So $|Ma| + |Mo| + |La| = 2 \times |Ma \cup Mo \cup La| = 2 \times 13 = 26$.

   (Note: The value 26 may also be obtained by using the Handshake theorem.)
5. Hence, $|Mo| = 26 - |Ma| + |La| = 26 - 11 - 6 = 9$.
6. $|Ma \cap Mo| = |Ma \cup Mo \cup La| - (|Mo \cap La| + |La \cap Ma|)$ (by 4)

   $= |Ma \cup Mo \cup La| - |La| = 13 - 6 = \mathbf{7}$.
7. $|Mo \cap La| = |Ma \cup Mo \cup La| - (|Ma \cap Mo| + |La \cap Ma|)$ (by 4)

   $= |Ma \cup Mo \cup La| - |Ma| = 13 - 11 = \mathbf{2}$.
8. Therefore, there are **7** bridges connecting Maui and Moloka'i, and **2** bridges connecting Moloka'i and Lana'i.

(e) You are to pick 14 numbers from 1 through 20. Is it true that no matter how you pick the 14 numbers, there is always a pair of numbers such that one is three times the other? Explain your answer. [4 marks]

Answer: False

Short answer: Use a counterexample; many to choose from, eg: 1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, or 7 through 20, etc.

Alternative: Partition the 20 numbers into these 14 sets: {1,3,9}, {2,6,18}, {4,12}, {5,15}, {7}, {8}, {10}, {11}, {13}, {14}, {16}, {17}, {19}, {20}. You can pick one number from each of the 14 sets and you will not get any pair of numbers such that one is three times the other. Many other alternative partitions work.

21. Graphs and Trees [Total: 20 marks]

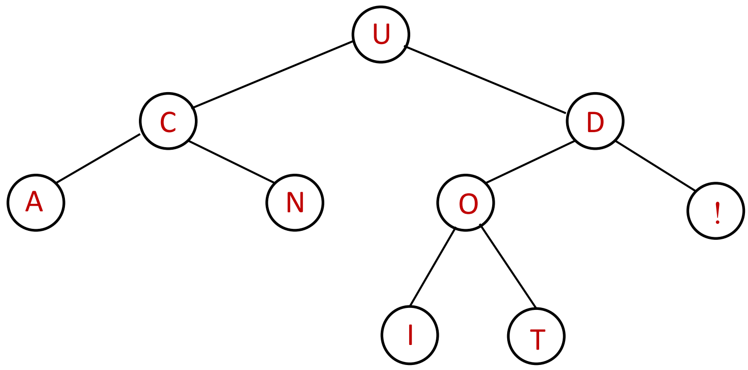
(a) The **pre-order traversal** and **post-order traversal** of a full binary tree with 9 vertices are given below:

Pre-order: **U C A N D O I T !**

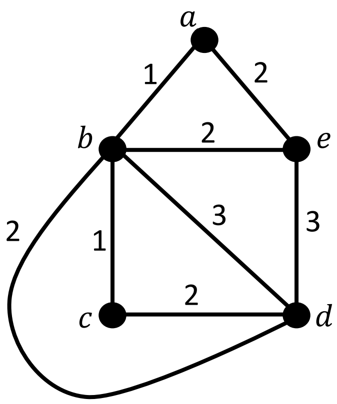
Post-order: **A N C I T O ! D U**

Draw the full binary tree (clearly!). [3 marks]

Answer:

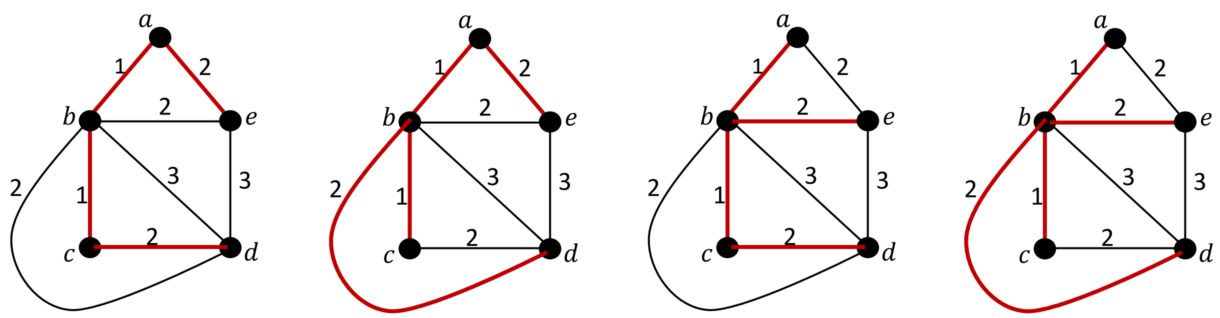


(b) In the following weighted graph, how many minimum spanning trees are there? You do not need to provide working or diagrams. [3 marks]



Answer: **4**

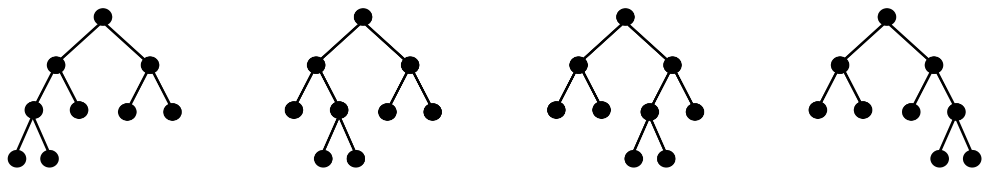
The MST consists of 4 edges, 2 of which have weight 1. The remaining 2 have weight 2. There are $\binom{4}{2} = 6$ choices, but 2 of them result in a circuit.



(c) A *height-balanced binary tree* (or simply *balanced binary tree*) is a binary tree in which the heights of the left and right subtrees under any vertex differ by not more than one.

Draw all balanced <u>full</u> binary trees with 9 vertices. [4 marks]

Answer: **4**



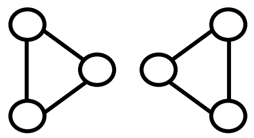
(d) A *regular graph* is a simple undirected graph where every vertex has the same degree. A *2-regular graph* is a regular graph where every vertex has degree 2.

Prove or disprove the following statement:

All 2-regular graphs are connected graphs. [3 marks]

Answer: False

Counterexample:



(e) Assume that all graphs in this question are simple undirected graphs.

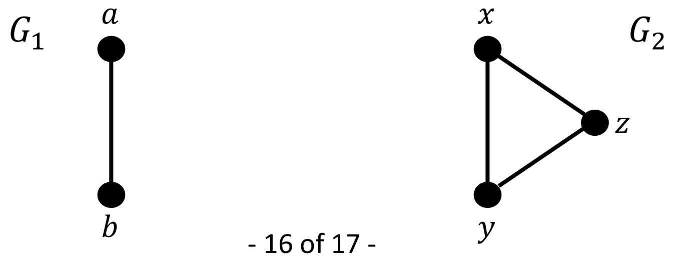
Given two simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *graph union*, *graph join* and *graph product* are defined as follows:

Graph union: The union $G_\cup = G_1 \cup G_2$ has vertex set $V_\cup = V_1 \cup V_2$ and edge set $E_\cup = E_1 \cup E_2$.

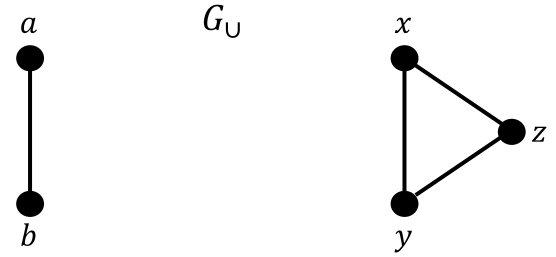
Graph join: The join $G_+ = G_1 + G_2$ has vertex set $V_+ = V_1 \cup V_2$ and edge set $E_+ = E_1 \cup E_2 \cup$ {all edges connecting every vertex in V_1 with every vertex in V_2}.

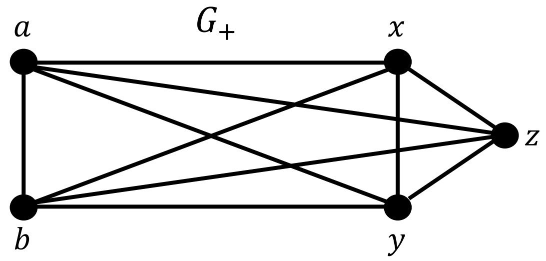
Graph product: The product $G_\times = G_1 \times G_2$ has vertex set $V_\times = V_1 \times V_2$ (Cartesian product of V_1 and V_2) and two vertices $(\alpha, \beta), (\gamma, \delta) \in V_\times$ are connected by an edge if and only if the vertices $\alpha, \beta, \gamma, \delta$ satisfy the following (with $\sim$ denoting "is adjacent to"): $(\alpha = \gamma \ \wedge \ \beta \sim \delta) \vee (\beta = \delta \ \wedge \ \alpha \sim \gamma)$

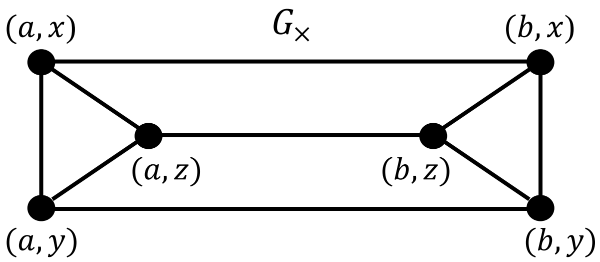
Given the following graphs G_1 and G_2, draw their union graph (1 mark), join graph (2 marks) and product graph (4 marks). You should label the vertices clearly on your graphs. [7 marks]



Answer:







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