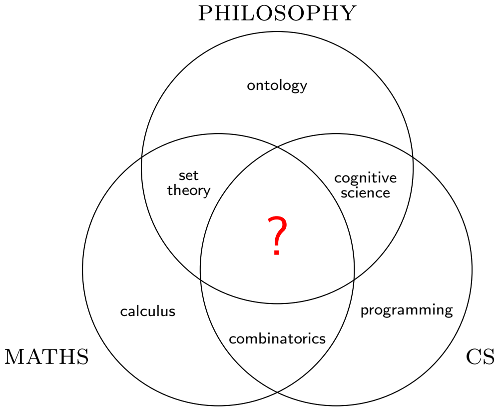
Chapter 4: Sets

CS1231 Discrete Structures

Wong Tin Lok

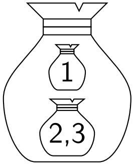
National University of Singapore

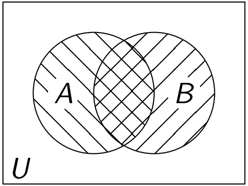
2022/23 Semester 1



Plan

- membership $\in$
- ways to specify sets
  - the set of all ...
  - roster notation
  - set-builder notation
  - replacement notation
  - the unique set that satisfies a property
- set equality $=$
- inclusion $\subseteq$
- power sets $\mathcal{P}$
- unions $\cup$, intersections $\cap$, complements $\overline{\cdot}$
- set identities and their proofs
- Venn diagrams
- (extra) Russell's Paradox





Sets

Why sets?

- The language of sets is an important part of modern mathematical discourse.
- Sets are interesting mathematical objects.
- For this module, they provide a topic on which we practise writing and understanding proofs.

Definition 4.1.1

(1) A *set* is an unordered collection of objects.

(2) These objects are called the *members* or *elements* of the set.

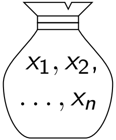
(3) Write

$x \in A$	for	x is an element of A;
$x \notin A$	for	x is not an element of A;
$x, y \in A$	for	x, y are elements of A;
$x, y \notin A$	for	x, y are not elements of A; etc.

(4) We may read $x \in A$ also as "x is in A" or "A *contains* x (as an element)".

Warning 4.1.2. Some use "contains" for the subset relation, but we do *not*.

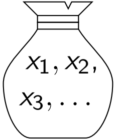
Specifying a set by listing out all its elements



Definition 4.1.3 (roster notation)

(1) The set whose only elements are $x_1, x_2, \dots, x_n$ is denoted $\{x_1, x_2, \dots, x_n\}$.

(2) The set whose only elements are $x_1, x_2, x_3, \dots$ is denoted $\{x_1, x_2, x_3, \dots\}$.

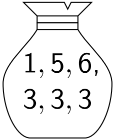


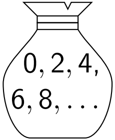
Note 4.1.4

For all objects $x_1, x_2, \dots, x_n, z$,

$$z \in \{x_1, x_2, \dots, x_n\} \quad \Leftrightarrow \quad z \text{ appears in the list } x_1, x_2, \dots, x_n.$$

Example 4.1.5





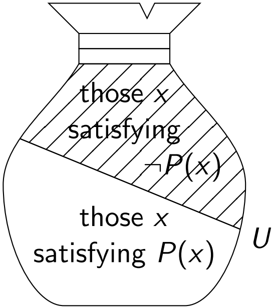
(1) The only elements of $A = \{1, 5, 6, 3, 3, 3\}$ are 1, 5, 6 and 3.
So $6 \in A$ but $7 \notin A$.

(2) The only elements of $B = \{0, 2, 4, 6, 8, \dots\}$ are the non-negative even integers.
So $4 \in B$ but $5 \notin B$.

Question

What are the elements of $\{2, 3, \dots\}$? All integers $x \geqslant 2$?

Specifying a set by describing its elements



Definition 4.1.6 (set-builder notation)

Let U be a set and $P(x)$ be a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted

$$\{x \in U : P(x)\} \quad \text{or} \quad \{x \in U \mid P(x)\}.$$

This is read as "the set of all x in U such that $P(x)$".

Note 4.1.7

Let U be a set and $P(x)$ be a predicate over U. For all objects z,

$$z \in \{x \in U : P(x)\} \quad \Leftrightarrow \quad z \in U \text{ and } P(z) \text{ is true.}$$

Example 4.1.8

(1) The elements of $C = \{x \in \mathbb{Z}_{\geqslant 0} : x \text{ is even}\}$ are precisely the elements of $\mathbb{Z}_{\geqslant 0}$ that are even, i.e., the non-negative even integers. So $6 \in C$ but $7 \notin C$.

(2) The elements of $D = \{x \in \mathbb{Z} : x \text{ is a prime number}\}$ are precisely the elements of $\mathbb{Z}$ that are prime numbers, i.e., the prime integers. So $7 \in D$ but $9 \notin D$.

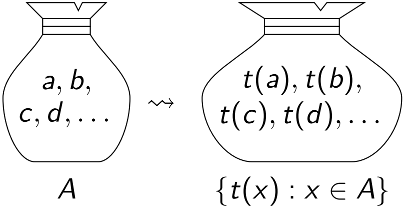
Specifying a set by replacement

Definition 4.1.9 (replacement notation)

Let A be a set and $t(x)$ be (the name of) an object for each element x of A. Then the set of all objects of the form $t(x)$ where x ranges over the elements of A is denoted

$$\{t(x) : x \in A\} \quad \text{or} \quad \{t(x) \mid x \in A\}.$$

This is read as "the set of all $t(x)$ where $x \in A$".



Note 4.1.10

Let A be a set and $t(x)$ be an object for each element x of A. For all objects z,

$$z \in \{t(x) : x \in A\} \quad \Leftrightarrow \quad \exists x \in A \;\; z = t(x).$$

Example 4.1.11

(1) The elements of $E = \{x + 1 : x \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x + 1$ where $x \in \mathbb{Z}_{\geqslant 0}$, i.e., the positive integers. 4a So $1 = 0 + 1 \in E$ but $0 \notin E$.

(2) The elements of $F = \{x - y : x, y \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x - y$ where $x, y \in \mathbb{Z}_{\geqslant 0}$, i.e., the integers. 4a So $-1 = 1 - 2 \in F$ but $\sqrt{2} \notin F$.

Set specification

	elements	conditions on the elements
roster notation	$\{x_1, x_2, \ldots$	$\}$
set-builder notation	$\{x$	$\in U : P(x) \}$
replacement notation	$\{t(x)$	$: x \in A\}$

Question

- Is any of these notations ambiguous?
- In other words, does each of these specify a unique set?

Definition 4.1.12

Two sets are *equal* if they have the same elements, i.e., for all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z \, (z \in A \Leftrightarrow z \in B).$$

Equality of sets: examples

Slogan 4.1.14. Order and repetition do not matter.

Example 4.1.13

$\{1, 5, 6, 3, 3, 3\} = \{1, 5, 6, 3\} = \{1, 3, 5, 6\}$.

Example 4.1.15

$\{y^2 : y \text{ is an odd integer}\} = \{x \in \mathbb{Z} : x = y^2 \text{ for some odd integer } y\} = \{1^2, 3^2, 5^2, \ldots\}$.

Example 4.1.16

$\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Proof

($\Rightarrow$) Take any $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. Then $z \in \mathbb{Z}$ and $z^2 = 1$. So

$$\begin{aligned} & z^2 - 1 = (z-1)(z+1) = 0. \\ \therefore \quad & z - 1 = 0 \quad \text{or} \quad z + 1 = 0. \\ \therefore \quad & z = 1 \quad \text{or} \quad z = -1. \end{aligned}$$

This means $z \in \{1, -1\}$.

($\Leftarrow$) Take any $z \in \{1, -1\}$. Then $z = 1$ or $z = -1$. In either case, we have $z \in \mathbb{Z}$ and $z^2 = 1$. So $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. $\square$

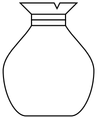
The empty set

Theorem 4.1.18

There exists a unique set with no element, i.e.,

- there is a set with no element; and (existence part)
- for all sets A, B, if both A and B have no element, then $A = B$. (uniqueness part)

Proof



- (existence part) The set $\{\}$ has no element.
- (uniqueness part) Let A, B be sets with no element. Then vacuously,

$$\forall z\ (z \in A \Rightarrow z \in B) \quad \text{and} \quad \forall z\ (z \in B \Rightarrow z \in A)$$

because the hypotheses in the implications are never true. So $A = B$. $\square$

Definition 4.1.19

The set with no element is called the *empty set*. It is denoted by $\varnothing$.

Inclusion of sets

Let A, B be sets.

Definition 4.2.1

Call A a *subset* of B, and write $A \subseteq B$, if

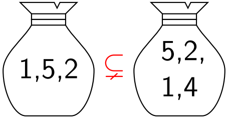
$$\forall z \, (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that B *includes* A, and write $B \supseteq A$ in this case.

Example 4.2.3 and Example 4.2.6

(1) $\{1, 5, 2\} \subsetneq \{5, 2, 1, 4\}$ but $\{1, 5, 2\} \not\subseteq \{2, 1, 4\}$.

(2) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. All these inclusions are proper.



Note 4.2.2. We avoid using the symbol $\subset$ because it may have different meanings to different people.

Remark 4.2.4

(1) $$A \not\subseteq B \quad \Leftrightarrow \quad \exists z \, (z \in A \text{ and } z \notin B).$$

(2) $$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

(3) $$A \subseteq A.$$

Definition 4.2.5

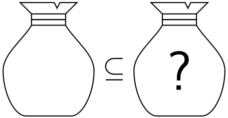
Call A a *proper subset* of B, write $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is *proper* or *strict*.

Vacuous inclusion

Proposition 4.2.7

The empty set is a subset of any set, i.e., for any set A,

$$\varnothing \subseteq A.$$



Proof

Vacuously,

$$\forall z \; (z \in \varnothing \Rightarrow z \in A)$$

because the hypothesis in the implication is never true. So $\varnothing \subseteq A$ by the definition of $\subseteq$. $\square$

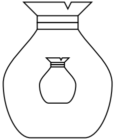
Sets of sets

Note 4.2.8

Sets can be elements of sets.

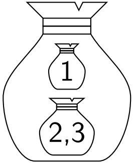
Example 4.2.9

(1) The set $A = \{\varnothing\}$ has exactly 1 element, namely the empty set.



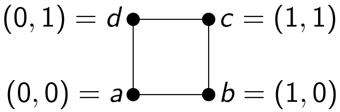
So A is not empty.

(2) The set $B = \{\{1\}, \{2, 3\}\}$ has exactly 2 elements, namely $\{1\}$, $\{2, 3\}$.



So $\{1\} \in B$, but $1 \notin B$.

Representation



How can one use a set to represent the square above?

- If one only wants to represent the connectivity between the points, then use

$$\big\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\big\}.$$

- If one also wants to represent the positions of the lines, then use

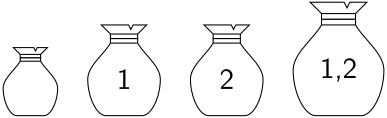
$$\{(x, y) : (x = 0 \text{ and } y \in [0, 1]) \text{ or } (x = 1 \text{ and } y \in [0, 1])$$
$$\text{or } (y = 0 \text{ and } x \in [0, 1]) \text{ or } (y = 1 \text{ and } x \in [0, 1])\}.$$

Power set

Definition 4.2.12

Let A be a set. The set of all subsets of A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example 4.2.13



(1) $\mathcal{P}(\varnothing) = \{\varnothing\}$

(2) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.

(3) $\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

(4) The following are subsets of $\mathbb{N}$ and thus are elements of $\mathcal{P}(\mathbb{N})$.

$$\varnothing, \{0\}, \{1\}, \{2\}, \ldots \{0, 1\}, \{0, 2\}, \{0, 3\} \ldots \{1, 2\}, \{1, 3\}, \{1, 4\} \ldots$$
$$\{2, 3\}, \{2, 4\}, \{2, 5\} \ldots \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \ldots$$
$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \ldots \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \ldots \ldots$$
$$\mathbb{N}, \mathbb{N}_{\geqslant 1}, \mathbb{N}_{\geqslant 2}, \ldots \{0, 2, 4, \ldots\}, \{1, 3, 5, \ldots\}, \{2, 4, 6, \ldots\}, \{3, 5, 7, \ldots\}, \ldots$$
$$\{x \in \mathbb{N} : (x-1)(x-2) < 0\}, \{x \in \mathbb{N} : (x-2)(x-3) < 0\}, \ldots$$
$$\{3x + 2 : x \in \mathbb{N}\}, \{4x + 3 : x \in \mathbb{N}\}, \{5x + 4 : x \in \mathbb{N}\}, \ldots \ldots$$

Membership vs inclusion

Note 4.2.10

Membership and inclusion can be different.

Question 4.2.11

Let $C = \{\{1\}, 2, \{3\}, 3, \{\{4\}\}\}$. Which of the following are true? 4b

- $\{1\} \in C$.
- $\{2\} \in C$.
- $\{3\} \in C$.
- $\{4\} \in C$.
- $\{1\} \subseteq C$.
- $\{2\} \subseteq C$.
- $\{3\} \subseteq C$.
- $\{4\} \subseteq C$.

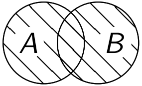
Boolean operations

Let A, B be sets.

Definition 4.3.1

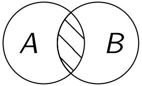
(1) The *union* of A and B, denoted $A \cup B$, is defined by

read as 'A union B' $\longrightarrow$ $A \cup B = \{x : x \in A \text{ or } x \in B\}$.



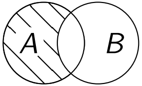
(2) The *intersection* of A and B, denoted $A \cap B$, is defined by

read as 'A intersect B' $\longrightarrow$ $A \cap B = \{x : x \in A \text{ and } x \in B\}$.



(3) The *complement* of B in A, denoted $A - B$ or $A \setminus B$, is defined by

read as 'A minus B' $\longrightarrow$ $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

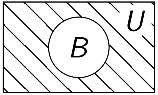


Convention and terminology 4.3.2

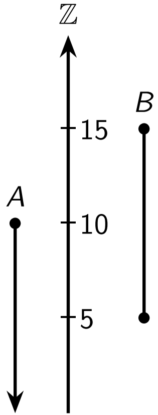
When working in a particular context, one usually works within a fixed set U which includes all the sets one may talk about, so that one only needs to consider the elements of U when proving set equality and inclusion. This U is called a *universal set*.

Definition 4.3.3

In a context where U is the universal set (so that implicitly $U \supseteq B$), the *complement* of B, denoted $\overline{B}$ or B^{c}, is defined by $\overline{B} = U \setminus B$.



Example 4.3.4 on Boolean operations



Let $A = \{x \in \mathbb{Z} : x \leqslant 10\}$ and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 15\}$. Then

$$\begin{aligned}
A \cup B &= \{x \in \mathbb{Z} : (x \leqslant 10) \vee (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x \leqslant 15\};\\
A \cap B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 10\};\\
A \setminus B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge \neg(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x < 5\};\\
\overline{B} &= \{x \in \mathbb{Z} : \neg(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : (x < 5) \vee (x > 15)\},
\end{aligned}$$

in a context where $\mathbb{Z}$ is the universal set. To show the first equation, one shows

$$\forall x \in \mathbb{Z}\ \big((x \leqslant 10) \vee (5 \leqslant x \leqslant 15) \Leftrightarrow (x \leqslant 15)\big), \qquad \text{etc.}$$

Set identities (Theorem 4.3.5)

For all set A, B, C in a context where U is the universal set, the following hold.

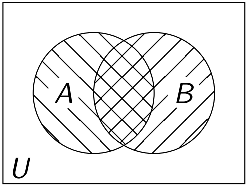
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotence	$A \cup A = A$	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Identities	$A \cup \varnothing = A$	$A \cap U = A$
Annihilators	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Complement	$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$
Double Complement Law	$\overline{(\overline{A})} = A$	
Top and bottom	$\overline{\varnothing} = U$	$\overline{U} = \varnothing$
Set difference	$A \setminus B = A \cap \overline{B}$	

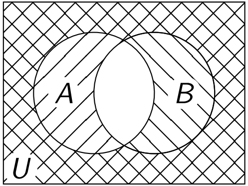
Venn diagrams

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

In the left diagram, hatch the regions representing A and B with ⧅ and ⧄ respectively.
In the right diagram, hatch the regions representing $\overline{A}$ and $\overline{B}$ with ⧅ and ⧄ respectively.





Then the □ region represents $\overline{A \cup B}$ on the left diagram, and the ⊠ region represents $\overline{A} \cap \overline{B}$ on the right diagram. Since these regions occupy the same region in the respective diagrams, we infer that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Note 4.3.6. This argument depends on the fact that each possibility for membership in A and B is represented by a region in the diagram.

Proving set identities using truth tables

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Proof #1

The rows in the following table list all the possibilities for an element $x \in U$:

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in \overline{A \cup B}$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in \overline{A} \cap \overline{B}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns under "$x \in \overline{A \cup B}$" and "$x \in \overline{A} \cap \overline{B}$" are the same, for any $x \in U$,

$$x \in \overline{A \cup B} \quad \Leftrightarrow \quad x \in \overline{A} \cap \overline{B}$$

no matter in which case we are. So $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$

Proving set identities directly

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Proof #2

Let $z \in U$. Then

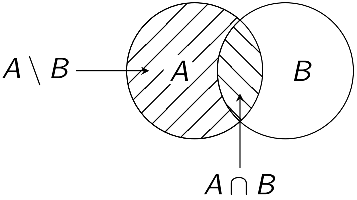
$$\begin{array}{lcl}
 & z \in \overline{A \cup B} & \\
\Leftrightarrow & z \notin A \cup B & \text{by the definition of } \overline{\,\cdot\,}; \\
\Leftrightarrow & \neg\big((z \in A) \vee (z \in B)\big) & \text{by the definition of } \cup; \\
\Leftrightarrow & (z \notin A) \wedge (z \notin B) & \text{by De Morgan's Laws for propositions;} \\
\Leftrightarrow & (z \in \overline{A}) \wedge (z \in \overline{B}) & \text{by the definition of } \overline{\,\cdot\,}; \\
\Leftrightarrow & z \in \overline{A} \cap \overline{B} & \text{by the definition of } \cap.
\end{array}$$

$\square$

Applications of the set identities

Example 4.3.7

Under the universal set U, show that $(A \cap B) \cup (A \setminus B) = A$ for all sets A, B.



Proof

$$
\begin{aligned}
(A \cap B) \cup (A \setminus B) &= (A \cap B) \cup (A \cap \overline{B}) && \text{by the set identity on set difference;}\\
&= A \cap (B \cup \overline{B}) && \text{by distributivity;}\\
&= A \cap U && \text{by the set identity on complement;}\\
&= A && \text{as } U \text{ is an identity for } \cap.
\end{aligned}
$$

□

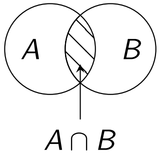
Definition 4.3.8

Two set A and B are *disjoint* if $A \cap B = \varnothing$.

Boolean operations and inclusion

Example 4.3.9(1)

Show that $A \cap B \subseteq A$ for all sets A, B.



Proof

By the definition of $\subseteq$, we need to show that

$$\forall z \; (z \in A \cap B \Rightarrow z \in A).$$

Let $z \in A \cap B$. Then $z \in A$ and $z \in B$ by the definition of $\cap$. In particular, we know $z \in A$, as required. $\square$

Example 4.3.9(2)

Show that $A \subseteq A \cup B$ for all sets A, B. 4c

Exercise 4.3.10

Show that for all sets A, B, C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$. 4d

Example 4.3.11: Is the following true?

4e

$$\boxed{(A \setminus B) \cup (B \setminus C) = A \setminus C \quad \text{for all sets } A, B, C.}$$

EXTRA MATERIAL

A set that is an element of itself?

Note 4.2.8 (recall)

Sets can be elements of sets.

Example 4.4.1

(1) $\varnothing \notin \varnothing$.

(2) $\mathbb{Z} \notin \mathbb{Z}$.

(3) $\{\varnothing\} \notin \{\varnothing\}$.

Question 4.4.2

Is there a set x such that $x \in x$? ✎ 4f

Hogarth (1754)

EXTRA MATERIAL

Consternation

There is just one point where I have encountered a difficulty. Russell (1902)

Theorem 4.4.3 (Russell 1901)

There is *no* set R such that

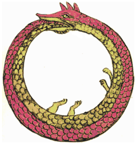
$$\forall x\ (x \in R \quad \Leftrightarrow \quad x \notin x). \qquad (*)$$

$\{x : x \notin x\}$?

Proof (by contradiction)

Suppose R is a set satisfying $(*)$. Applying $(*)$ to $x = R$ gives

$$R \in R \quad \Leftrightarrow \quad R \notin R. \qquad (\dagger)$$



Split into two cases.

- Case 1: assume $R \in R$. Then $R \notin R$ by the $\Rightarrow$ part of $(\dagger)$. This contradicts our assumption that $R \in R$.
- Case 2: assume $R \notin R$. Then $R \in R$ by the $\Leftarrow$ part of $(\dagger)$. This contradicts our assumption that $R \notin R$.

In either case, we get a contradiction. So the proof is finished. $\square$

Question 4.4.4. Can you write a proof that does not mention contradiction? ✎ 4g

EXTRA MATERIAL

Consternation?

There is just one point where I have encountered a difficulty. Russell (1902)

$\{x : x \notin x\}$?

Theorem 4.4.3 (Russell 1901)

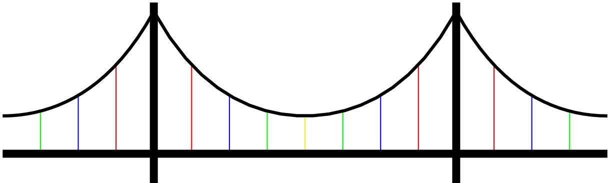
There is *no* set R such that

$$\forall x\ (x \in R \quad \Leftrightarrow \quad x \notin x).$$

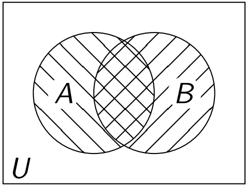
Morals

- Some predicates do not correspond to any set.
- The set of all sets, if it exists, needs to be handled with *extreme* care.

Suppose a contradiction were to be found in the axioms of set theory. Do you seriously believe that that bridge would fall down? (reportedly) Ramsey



Summary



Let A, B be sets.

Definition 4.1.1(3). Write $x \in A$ for "x is an element of A".

Roster notation. $\{x_1, x_2, \ldots \}$

Set-builder notation. $\{x \in U : P(x)\}$ where $P(x)$ is a predicate over a set U

Replacement notation. $\{t(x) : x \in A\}$ where $t(x)$ is an object for each $x \in A$

Definition 4.1.12. $A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B)$.

Definition 4.2.1. $A \subseteq B \quad \Leftrightarrow \quad \forall z\ (z \in A \Rightarrow z \in B)$.

Definition 4.2.5. $A \subsetneq B \quad \Leftrightarrow \quad A \subseteq B$ and $A \neq B$.

Slogan 4.1.14. Order and repetition do not matter.

Definition 4.1.19. The *empty set* $\varnothing$ is the unique set with no element.

Definition 4.2.12. The *power set* $\mathcal{P}(A)$ of a set A is the set of all subsets of A.

Definitions 4.3.1 and 4.3.3. In a context where U is the universal set,

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}, \qquad A \cap B = \{x : (x \in A) \wedge (x \in B)\},$$

$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\}, \qquad \overline{B} = \{x \in U : x \notin B\}.$$