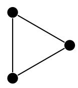
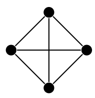
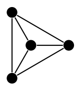
Tutorial solutions for Chapter 11

Sometimes there are other correct answers.

11.1. (a) K_1: K_2: K_3:  K_4:  or 

(b) Unordered pairs of vertices in $\mathrm{V}(K_n)$ are precisely the 2-combinations of $\mathrm{V}(K_n)$. Since there is an edge in K_n between any pair of distinct vertices (and there is no need to worry about the order of the vertices because our graph is undirected), there are at least $\binom{|\mathrm{V}(K_n)|}{2} = \binom{n}{2}$ edges in K_n. There is no other edge because K_n has no loop. So K_n has exactly $\binom{n}{2}$ edges.

11.2. Each such G is uniquely determined by the set $\mathrm{E}(G)$ of its edges. So it suffices to count the $\mathrm{E}(G)$'s. Define

$$C_1 = \big\{\{x\} : x \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}\big\},$$
$$C_2 = \big\{\{x, y\} : x, y \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\} \text{ and } x \neq y\big\}.$$

Note that these are respectively the set of all 1-combinations and the set of all 2-combinations of $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$.

(a) These $\mathrm{E}(G)$'s are precisely the subsets of $C_1 \cup C_2$. Therefore, as $C_1 \cap C_2 = \varnothing$, the Addition Rule tells us that the number of such graphs is

$$|\mathcal{P}(C_1 \cup C_2)| = 2^{|C_1|+|C_2|} = 2^{\binom{3}{1}+\binom{3}{2}} = 2^{3+3} = 2^6 = 64.$$

(b) The $\mathrm{E}(G)$'s where G has no loop are precisely the subsets of C_2. So there are $|\mathcal{P}(C_2)| = 2^{\binom{3}{2}} = 2^3 = 8$ such graphs with no loop.

(c) The only cycle whose vertex set is $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ is abca. So the $\mathrm{E}(G)$'s where G has a cycle are precisely those sets of the form

$$X \cup \big\{\{\mathrm{a}, \mathrm{b}\}, \{\mathrm{b}, \mathrm{c}\}, \{\mathrm{c}, \mathrm{a}\}\big\}$$

where $X \subseteq \mathcal{P}(C_1)$. Hence the number of such graphs that have a cycle is $|\mathcal{P}(C_1)| = 2^{\binom{3}{1}} = 2^3 = 8$.

(d) The number of such graphs that are acyclic is, by the Difference Rule and part (b),

$$\begin{aligned}
&\begin{pmatrix}\text{number of such graphs that}\\ \text{have no loop and no cycle}\end{pmatrix}\\
&= \begin{pmatrix}\text{number of such graphs}\\ \text{that have no loop}\end{pmatrix} - \begin{pmatrix}\text{number of such graphs that}\\ \text{have no loop but has a cycle}\end{pmatrix}\\
&= 8 - 1 = 7
\end{aligned}$$

because the only such graph that has no loop but has a cycle is abca. So, applying the Difference Rule again, the number of such graphs that are cyclic is, by part (a),

$$\begin{pmatrix}\text{number of}\\ \text{such graphs}\end{pmatrix} - \begin{pmatrix}\text{number of such graphs}\\ \text{that are acyclic}\end{pmatrix} = 64 - 7 = 57.$$

11.3. (a) Graph (i) is isomorphic to itself and to (iii), but not to any of the others.

Graph (ii) is isomorphic to itself and to (iv), but not to any of the others.

An isomorphism from any graph here to itself is the identity function on $\{1, 2, \ldots, 8\}$.

An isomorphism from (i) to (iii) is the function $f\colon \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ satisfying

$$
\begin{array}{llll}
f(1) = 1, & f(2) = 2, & f(3) = 6, & f(4) = 7, \\
f(5) = 8, & f(6) = 4, & f(7) = 3, & f(8) = 5.
\end{array}
$$

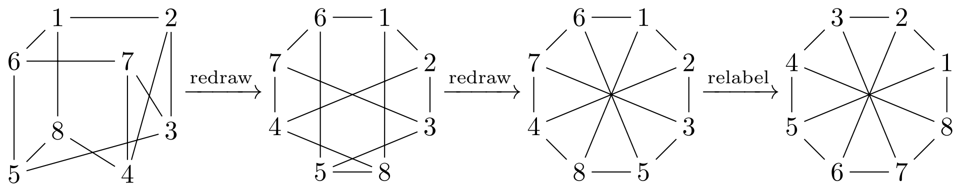
An isomorphism from (ii) to (iv) is the function $g\colon \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ satisfying

$$
\begin{array}{llll}
g(1) = 2, & g(2) = 1, & g(3) = 8, & g(4) = 5, \\
g(5) = 7, & g(6) = 3, & g(7) = 4, & g(8) = 6.
\end{array}
$$

Graph (ii) has a cycle of length 5, say 123761, but (i) does not. So (i) is not isomorphic to (ii).

Alternative explanation of why (i) is not isomorphic to (ii). In (i), every vertex is in exactly three cycles of length 4. In (ii), every vertex is in exactly two cycles of length 4. So (i) is not isomorphic to (ii).

Extra explanation.



(ii) (iv)

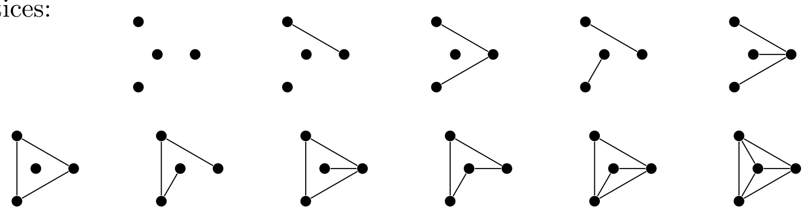
(b) For $n = 2, 3, 4$, there are $2, 4, 11$ respectively.

Explanation.

2 vertices:

3 vertices:

4 vertices:



Moral. Usually we want to count without counting, but sometimes it is easier to simply count.

Additional comment. It is a famous open question whether there is an efficient algorithm to determine whether two given graphs are isomorphic.

11.4. (a) Yes, as proved below.

By definition, if H is a connected component of G, then $\mathrm{V}(H) \subseteq \mathrm{V}(G)$. As G has at least one vertex, Proposition 11.3.6 tells us that G has a connected component, say H_0, with at least one vertex. This implies $(\varnothing, \{\})$ is not a connected component of G because it is a proper subgraph of H_0, and H_0 is connected. Thus $\mathrm{V}(H) \neq \varnothing$ for all connected components H of G. From Proposition 11.3.6, we already know that every element of $\mathrm{V}(G)$ is in $\mathrm{V}(H)$ for some connected component H of G. Finally, we verify that, for all connected components H_1, H_2 of G, if $\mathrm{V}(H_1) \cap \mathrm{V}(H_2) \neq \varnothing$, then $\mathrm{V}(H_1) = \mathrm{V}(H_2)$.

Let H_1, H_2 be connected components of G such that $\mathrm{V}(H_1) \cap \mathrm{V}(H_2) \neq \varnothing$. Define $H = \big(\mathrm{V}(H_1) \cup \mathrm{V}(H_2), \mathrm{E}(H_1) \cup \mathrm{E}(H_2)\big)$. We will show that H is connected. In view of the maximality of H_1 and H_2, this will imply $H_1 = H = H_2$, and thus $\mathrm{V}(H_1) = \mathrm{V}(H_2)$ in particular.

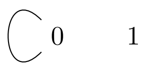
Pick any $u, v \in \mathrm{V}(H) = \mathrm{V}(H_1) \cup \mathrm{V}(H_2)$.

Case 1: suppose $u, v \in \mathrm{V}(H_1)$ or $u, v \in \mathrm{V}(H_2)$. Say $u, v \in \mathrm{V}(H_1)$. As H_1 is connected, there is a path between u and v in H_1, hence in G.

Case 2: suppose exactly one of u, v is in $\mathrm{V}(H_1)$. Say $u \in \mathrm{V}(H_1)$, so that $v \in \mathrm{V}(H_2)$. Take $x \in \mathrm{V}(H_1) \cap \mathrm{V}(H_2)$. As $u, x \in \mathrm{V}(H_1)$ and H_1 is connected, there is a path between u and x in H_1, hence in H. As $x, v \in \mathrm{V}(H_2)$ and H_2 is connected, there is a path between x and v in H_2, hence in H. So Lemma 11.1.10 tells us there is a path between u and v in H. □

(b) No, as proved below.

Consider the graph $G = (\{0,1\}, \{00\})$. Here is a drawing on G.



This graph G has two connected components:

$$H_1 = (\{0\}, \{00\}) \quad \text{and} \quad H_2 = (\{1\}, \{\}).$$

So $\{\mathrm{E}(H) : H \text{ is a connected component of } G\} = \{\{00\}, \{\}\}$. This is not a partition of $\mathrm{E}(G)$ because partitions by definition cannot contain the empty set as an element. □

Additional comment. The appearance of the empty set is the only possible obstacle for the edge sets of the connected components of an undirected graph to form a partition. To see this, consider any undirected graph G. Define

$$\mathscr{C} = \{\mathrm{E}(H) : H \text{ is a connected component of } G\} \setminus \{\varnothing\}.$$

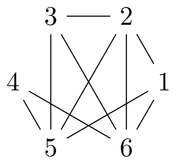
We verify that $\mathscr{C}$ is a partition of $\mathrm{E}(G)$.

As connected components H of G are subgraphs of G, each element $\mathrm{E}(H) \in \mathscr{C}$ is a subset of $\mathrm{E}(G)$. Every $\mathrm{E}(H) \in \mathscr{C}$ is nonempty by the definition of $\mathscr{C}$.

Take any edge xy in G. If $x = y$, then x is a path of length 0 between x and y in G. If $x \neq y$, then xy is a path of length 1 between x and y in G. So in all cases we have a path between x and y in G. Apply Theorem 11.3.7 to find a connected component H of G with both x and y in it. As H is connected, so is $(\mathrm{V}(H), \mathrm{E}(H) \cup \{xy\})$ because no vertex is added here. Thus $xy \in \mathrm{E}(H)$ by the maximality of the connected component H of G.

Let H_1, H_2 be connected components of G such that $\mathrm{E}(H_1) \cap \mathrm{E}(H_2) \neq \varnothing$. Pick any $xy \in \mathrm{E}(H_1) \cap \mathrm{E}(H_2)$. Then $x, y \in \mathrm{V}(H_1) \cap \mathrm{V}(H_2)$. So $\mathrm{V}(H_1) \cap \mathrm{V}(H_2) \neq \varnothing$, and thus $\mathrm{V}(H_1) = \mathrm{V}(H_2)$ by part (a). If $uv \in \mathrm{E}(H_1)$, then $(\mathrm{V}(H_2), \mathrm{E}(H_2) \cup \{uv\})$ is connected because H_2 is connected and no vertex is added here, so $uv \in \mathrm{E}(H_2)$ by the maximality of H_2 as a connected component of G. Similarly, we see that every element of $\mathrm{E}(H_2)$ is an element of $\mathrm{E}(H_1)$. These show $\mathrm{E}(H_1) = \mathrm{E}(H_2)$. $\square$

11.5. (a)



(b)* Let G be an undirected graph with at least one vertex but with no loop. If G is connected, then there is nothing to prove. So suppose G is not connected. Take $u, v \in \mathrm{V}(\overline{G})$. We want a path between u and v in $\overline{G}$. If $u = v$, then u is a path of length 0 between u and v in $\overline{G}$. Similarly, if $uv \in \mathrm{E}(\overline{G})$, then uv is a path of length 1 between u and v in $\overline{G}$. So suppose $u \neq v$ and $uv \notin \mathrm{E}(\overline{G})$.

Now $uv \in \mathrm{E}(G)$ by the definition of $\overline{G}$. So from Theorem 11.3.7 we obtain a connected component H of G which has both u and v in it. As G is not connected, we get vertices a, b in G with no path between them in G. Use Proposition 11.3.6 to find connected components H_a and H_b of G such that $a \in \mathrm{V}(H_a)$ and $b \in \mathrm{V}(H_b)$. We know $b \notin \mathrm{V}(H_a)$ because H_a is connected but there is no path between a and b in G. So $H_a \neq H_b$. Thus H cannot be equal to both H_a and H_b. Say $H \neq H_a$. If $ua \in \mathrm{E}(G)$, then Theorem 11.3.7 gives some connected component H^+ that has both u and a in it, but then

$$
\begin{array}{lll}
 & u \in \mathrm{V}(H^+) \cap \mathrm{V}(H) & \text{as } u \in \mathrm{V}(H); \\
\therefore & H^+ = H & \text{by Exercise 11.4(a);} \\
\therefore & a \in \mathrm{V}(H) & \text{as } a \in \mathrm{V}(H^+); \\
\therefore & a \in \mathrm{V}(H) \cap \mathrm{V}(H_a) & \text{as } a \in \mathrm{V}(H_a); \\
\therefore & H = H_a & \text{by Exercise 11.4(a),}
\end{array}
$$

which contradicts the fact that $H \neq H_a$. So $ua \notin \mathrm{E}(G)$. Similarly, we can show $va \notin \mathrm{E}(G)$. Thus $ua, va \in \mathrm{E}(\overline{G})$ by the definition of $\overline{G}$. This guarantees uav is a path between u and v in $\overline{G}$. $\square$

11.6. We proceed by induction on n.

(Base step) Pick any $x \in \mathrm{V}(H_0)$. The definition of H_0 gives a path P between v and x in G of length 0. Hence P has the form $(\{x_0\}, \{\})$ where $x_0 = v$ and $x_0 = x$. This P is a path between v and x also in H_0 because $\{x_0\} = \{x\} \subseteq \mathrm{V}(H_0)$ and $\{\} \subseteq \mathrm{E}(H_0)$.

(Induction step) Let $k \in \mathbb{N}$ such that the proposition is true for $n = k$. Pick any $x \in \mathrm{V}(H_{k+1})$. If $x \in \mathrm{V}(H_k)$, then the induction hypothesis directly gives us what we want. So suppose $x \notin \mathrm{V}(H_k)$. Then the definition of H_k and H_{k+1} gives us a path $x_0 x_1 \ldots x_k x_{k+1}$ in G where $x_0 = v$ and $x_{k+1} = x$. Then $x_0 x_1 \ldots x_k$ is a path between v and x_k in G of length k. So $x_k \in \mathrm{V}(H_k)$ by the definition of H_k. Apply the induction hypothesis to find a path $y_0 y_1 \ldots y_\ell$ in H_k where $y_0 = v$ and $y_\ell = x_k$. Note that $x_k x_{k+1} \in \mathrm{E}(H_{k+1})$ because $x_k \in \mathrm{V}(H_k) \subseteq \mathrm{V}(H_{k+1})$ and $x_{k+1} \in \mathrm{V}(H_{k+1})$. As $y_\ell = x_k$, this tells us $y_0 y_1 \ldots y_\ell x_{k+1}$ is a path between v and x in H_{k+1}, as required.

This completes the induction. $\square$

Proof without induction (extra material). Let $n \in \mathbb{N}$. Pick any $x \in \mathrm{V}(H_n)$. Use the definition of H_n to find a path $x_0 x_1 \ldots x_\ell$ of length at most n in G where $x_0 = v$

and $x_\ell = x$. Then each $x_i \in \mathrm{V}(H_n)$ because $x_0x_1 \ldots x_i$ is a path of length at most $\ell \leqslant n$ in G. Thus each $x_ix_{i+1} \in \mathrm{E}(H_n)$ by the definition of $\mathrm{E}(H_n)$. It follows that $x_0x_1 \ldots x_\ell$ is a path in H_n. □

Extra exercises

11.7. ($\subseteq$) Let $xy \in \mathrm{E}(H)$. As H is a graph, this implies $x, y \in \mathrm{V}(H)$. As H is a subgraph of G, this implies $xy \in \mathrm{E}(G)$.

($\supseteq$) Let $xy \in \mathrm{E}(G)$ such that $x, y \in \mathrm{E}(H)$. As H is connected, so is $(\mathrm{V}(H), \mathrm{E}(H) \cup \{xy\})$ because no vertex is added here. Thus $xy \in \mathrm{E}(H)$ by the maximality of the connected component H of G. □

Additional comment. We used this multiple times in the additional comment for Exercise 11.4.

11.8. (a)* We prove the contrapositive. Assume G is not connected. Use the definition of connectedness to find in G vertices u, v between which there is no path. Use Proposition 11.3.6 to find a connected component H_u of G that has u in it. Define H_v to be the subgraph of G where

$$\begin{aligned} \mathrm{V}(H_v) &= \mathrm{V}(G) \setminus \mathrm{V}(H_u), \quad \text{and} \\ \mathrm{E}(H_v) &= \{xy \in \mathrm{E}(G) : x, y \in \mathrm{V}(H_v)\}. \end{aligned}$$

First, we show $\mathrm{E}(G) = \mathrm{E}(H_u) \cup \mathrm{E}(H_v)$. The $\supseteq$ part is true because H_u and H_v are both subgraphs of G. For the $\subseteq$ part, take any $xy \in \mathrm{E}(G)$. If exactly one of the vertices x, y is in H_u, say $x \in \mathrm{V}(H_u)$ and $y \in \mathrm{V}(G) \setminus \mathrm{V}(H_u) = \mathrm{V}(H_v)$, then $H^+ = (V(H_u) \cup \{y\}, \mathrm{E}(H_u) \cup \{xy\})$ is again a connected subgraph of G, but H is a proper subgraph of H^+ because $y \notin \mathrm{V}(H_u)$, which contradicts the maximality of the connected component H of G. So either $x, y \in \mathrm{V}(H_u)$ or $x, y \in \mathrm{V}(G) \setminus \mathrm{V}(H_u) = \mathrm{V}(H_v)$. If $x, y \in \mathrm{V}(H_u)$, then $xy \in \mathrm{E}(H_u)$ by Exercise 11.7. If $x, y \in \mathrm{V}(H_v)$, then $xy \in \mathrm{E}(H_v)$ by the definition of $\mathrm{E}(H_v)$. So $xy \in \mathrm{E}(H_u) \cup \mathrm{E}(H_v)$ in all cases.

Second, if $xy \in \mathrm{E}(H_u)$, then $x \in \mathrm{V}(H_u)$ as H_u is an undirected graph, and so $x \notin \mathrm{V}(G) \setminus \mathrm{V}(H_u) = \mathrm{V}(H_v)$, implying $xy \notin \mathrm{E}(H_v)$ by the definition of $\mathrm{E}(H_v)$. This shows $\mathrm{E}(H_u) \cap \mathrm{E}(H_v) = \varnothing$.

Third, let $k = |\mathrm{V}(H_u)|$. Then $k \geqslant 1$ because $u \in \mathrm{V}(H_u)$. Also

$$\begin{aligned} n - k &= |\mathrm{V}(G)| - |\mathrm{V}(H_u)| \\ &= |\mathrm{V}(G) \setminus \mathrm{V}(H_u)| && \text{by the Difference Rule;} \\ &= |\mathrm{V}(H_v)| && \text{by the definition of } \mathrm{V}(H_v); \\ &\geqslant 1 && \text{as } v \in \mathrm{V}(G) \setminus \mathrm{V}(H_u) = \mathrm{V}(H_v) \text{ by Theorem 11.3.7.} \end{aligned}$$

Combining all these,

$$
\begin{aligned}
|\mathrm{E}(G)| &= |\mathrm{E}(H_u) \cup \mathrm{E}(H_v)| \\
&= |\mathrm{E}(H_u)| + |\mathrm{E}(H_v)| && \text{by the Addition Rule;} \\
&\leqslant \binom{|\mathrm{V}(H_u)|}{2} + \binom{|\mathrm{V}(H_v)|}{2} && \text{by Exercise 11.1;} \\
&= \binom{k}{2} + \binom{n-k}{2} \\
&= \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} && \text{by Theorem 10.3.15;} \\
&= \frac{1}{2}(n^2 - n - 2nk + 2k^2) \\
&= \frac{1}{2}\big((n^2 - 3n + 2) + (2n - 2 - 2nk + 2k^2)\big) \\
&= \frac{1}{2}\big((n-1)(n-2) - 2(k-1)(n-k-1)\big) \\
&= \binom{n-1}{2} - (k-1)\big((n-k) - 1\big) \\
&\leqslant \binom{n-1}{2} && \text{as } k \geqslant 1 \text{ and } n-k \geqslant 1. \qquad \square
\end{aligned}
$$

Why is H^+ connected in the proof above (extra explanations). Take any vertices a, b in H^+.

Suppose $a, b \in \mathrm{V}(H_u)$. As H_u is connected, there is a path between a and b in H_u, and hence in H^+.

Suppose exactly one of a, b is in H_u. Say $a \in \mathrm{V}(H_u)$ and $b = y$. As $x \in \mathrm{V}(H_u)$ and H_u is connected, there is a path, say $x_0x_1 \ldots x_\ell$, between a and x in H_u. Then $x_0x_1 \ldots x_\ell y$ is a path between a and b in H^+ because $y \notin \mathrm{V}(H_u)$.

Suppose $a = y = b$. Then y is a path of length 0 between a and b in H^+.

So there is a path between a and b in H^+ in all cases.

(b) There are 2.

Explanation. Here is a complete list: