Chapter 5: Relations

CS1231 Discrete Structures

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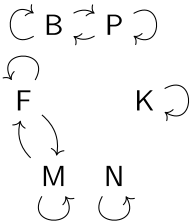
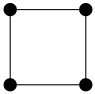
> Professionals in any discipline need to know the foundations of their field. So if you're a database professional, you need to know the relational model, because the relational model is the foundation (or a huge part of the foundation, at any rate) of the database field in particular. Date (2009)

Plan

Start using sets to represent mathematical objects.

- relations   representing *predicates*
  - ordered tuples and Cartesian products
  - composition
  - inverse
- graphs   representing *graphs*
  - directed graphs
  - undirected graphs





How one can represent a predicate by a set

belongs to	
Student ID	name
001R	Gates
012B	Brin
062E	Bezos
126N	Ma
254E	Zuckerberg

$$\begin{aligned}\{\ &(\text{001R},\ \text{Gates}),\\ &(\text{012B},\ \text{Brin}),\\ &(\text{062E},\ \text{Bezos}),\\ &(\text{126N},\ \text{Ma}),\\ &(\text{254E},\ \text{Zuckerberg})\ \}\end{aligned}$$

Ordered pairs and Cartesian products

Definition 5.1.1

An *ordered pair* is an expression of the form (x, y). Let (x_1, y_1) and (x_2, y_2) be ordered pairs. Then $(x_1, y_1) = (x_2, y_2)$ if

$$x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Example 5.1.2

(1) $(1, 2) \neq (2, 1)$, although $\{1, 2\} = \{2, 1\}$.

(2) $(3, 0.5) = (\sqrt{9}, \frac{1}{2})$.

read as "A cross B"

Definition 5.1.3

Let A, B be sets. The *Cartesian product* of A and B, denoted $A \times B$, is defined to be

$$\{(x, y) : x \in A \text{ and } y \in B\}.$$

Example 5.1.4

$$\{\mathrm{a}, \mathrm{b}\} \times \{1, 2, 3\} = \{(\mathrm{a}, 1), (\mathrm{a}, 2), (\mathrm{a}, 3), (\mathrm{b}, 1), (\mathrm{b}, 2), (\mathrm{b}, 3)\}.$$

$$\begin{array}{ccc}
(\mathrm{a}, 3) & — & (\mathrm{b}, 3) \\
| & & | \\
(\mathrm{a}, 2) & — & (\mathrm{b}, 2) \\
| & & | \\
(\mathrm{a}, 1) & — & (\mathrm{b}, 1)
\end{array}$$

Predicates as sets

Definition 5.1.5

Let A, B be sets.

(1) A *relation* from A to B is a subset of $A \times B$.

(2) Let R be a relation from A to B and $(x, y) \in A \times B$. Then we may write

$$x\ R\ y \text{ for } (x, y) \in R \quad \text{and} \quad x \not R\ y \text{ for } (x, y) \notin R.$$

We read "$x\ R\ y$" as "x is *R-related* to y" or simply "x is *related* to y".

Example 5.1.6

Let $\Gamma = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, 0, 1, 2, \ldots, 9\}$ and $\Phi = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, \mathrm{a}, \mathrm{b}, \ldots, \mathrm{z}\}$. Then

$$\begin{aligned}\{\ &(\text{001R}, \text{ Gates}),\\ &(\text{012B}, \text{ Brin}),\\ &(\text{062E}, \text{ Bezos}),\\ &(\text{126N}, \text{ Ma}),\\ &(\text{254E}, \text{ Zuckerberg})\ \}\end{aligned}$$

is a relation from Γ^* to Φ^*.

Arrow diagrams for binary relations

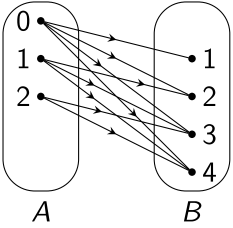
Example 5.1.7

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4\}$. Define the relation R from A to B by setting

$$x \, R \, y \quad \Leftrightarrow \quad x < y.$$

Then $0 \; R \; 1$ and $0 \; R \; 2$, but $2 \not R \; 1$. Thus

$$\begin{aligned} R = \{&(0,1), (0,2), (0,3), (0,4), \\ &(1,2), (1,3), (1,4), (2,3), (2,4)\}. \end{aligned}$$



Definition 5.1.8 (arrow diagrams)

An arrow from x to y indicates x is related to y.

How one can represent a predicate of higher arity by a set

teaching			
module	department	faculty	instructor
CS3234	CS	Computing	Turing
MA2001	Mathematics	Science	Gauss
MU2109	Music	Arts	Mozart
PC2130	Physics	Science	Newton
PL3103	Psychology	Arts	Freud

{ (CS3234, CS, Computing, Turing),
(MA2001, Mathematics, Science, Gauss),
(MU2109, Music, Arts, Mozart),
(PC2130, Physics, Science, Newton),
(PL3103, Psychology, Arts, Freud) }

Ordered n-tuples and Cartesian products

Let $n \in \mathbb{Z}_{\geqslant 2}$.

Definition 5.1.10

(1) An expression of the form $(x_1, x_2, \dots, x_n)$ is called an *ordered n-tuple*.

(2) Ordered n-tuples are defined recursively by setting, for all objects $x_1, x_2, \dots, x_{n+1}$,

$$(x_1, x_2, \dots, x_{n+1}) = \big((x_1, x_2, \dots, x_n), x_{n+1}\big).$$

Exercise 5.1.11

Prove by induction on n that for all $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$, 5a

$$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \quad \Leftrightarrow \quad x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } \dots \text{ and } x_n = y_n.$$

Definition 5.1.13

Let $A_1, A_2, \dots, A_n$ be sets. The *Cartesian product* of $A_1, A_2, \dots, A_n$, denoted $A_1 \times A_2 \times \dots \times A_n$, is defined to be

$$\{(x_1, x_2, \dots, x_n) : x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \dots \text{ and } x_n \in A_n\}.$$

If A is a set, then $A^n = \underbrace{A \times A \times \dots \times A}_{n\text{-many } A\text{'s}}$.

Example 5.1.14

$\{0, 1\} \times \{0, 1\} \times \{a, b\} = \{(0, 0, a), (0, 0, b), (0, 1, a), (0, 1, b), (1, 0, a), (1, 0, b), (1, 1, a), (1, 1,$

n-ary predicates as sets

Let $n \in \mathbb{Z}_{\geqslant 2}$.

Let $A_1, A_2, \ldots, A_n$ be sets.

Definition 5.1.15

A *n-ary relation over $A_1, A_2, \ldots, A_n$* is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

Example 5.1.16

Let $\Gamma = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, 0, 1, 2, \ldots, 9\}$ and $\Phi = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, \mathrm{a}, \mathrm{b}, \ldots, \mathrm{z}\}$. Then

$$\begin{aligned}&\{(\text{CS3234}, \text{CS}, \text{Computing}, \text{Turing}), (\text{MA2001}, \text{Mathematics}, \text{Science}, \text{Gauss}),\\ &\ (\text{MU2109}, \text{Music}, \text{Arts}, \text{Mozart}), (\text{PC2130}, \text{Physics}, \text{Science}, \text{Newton}),\\ &\ (\text{PL3103}, \text{Psychology}, \text{Arts}, \text{Freud})\}.\end{aligned}$$

is a 4-ary relation over $\Gamma^*, \Phi^*, \Phi^*, \Phi^*$.

A fictitious miniature university database and its representation (Figure 5.1)

identity	
Student ID	name
001R	Gates
012B	Brin
062E	Bezos
126N	Ma
254E	Zuckerberg

$$SN = \{ (001R, Gates), (012B, Brin), (062E, Bezos), (126N, Ma), (254E, Zuckerberg) \}$$

is enrolled in	
Student ID	module
126N	CS3234
254E	CS3234
001R	MA2001
012B	MA2001
062E	MA2001
126N	MA2001
012B	MU2109
001R	PC2130
062E	PL3103
254E	PL3103

$$SM = \{ (126N, CS3234), (254E, CS3234), (001R, MA2001), (012B, MA2001), (062E, MA2001), (126N, MA2001), (012B, MU2109), (001R, PC2130), (062E, PL3103), (254E, PL3103) \}$$

progress		
Student ID	faculty	year
062E	Arts	1
254E	Arts	2
012B	Science	2
001R	Science	1
126N	Science	3

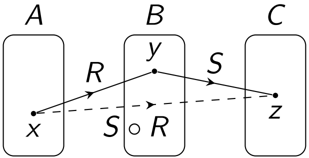
$$SFY = \{ (062E, Arts, 1), (254E, Arts, 2), (012B, Science, 2), (001R, Science, 1), (126N, Science, 3) \}$$

teaching			
module	department	faculty	instructor
CS3234	CS	Computing	Turing
MA2001	Mathematics	Science	Gauss
MU2109	Music	Arts	Mozart
PC2130	Physics	Science	Newton
PL3103	Psychology	Arts	Freud

$$MDFI = \{ (CS3234, CS, Computing, Turing), (MA2001, Mathematics, Science, Gauss), (MU2109, Music, Arts, Mozart), (PC2130, Physics, Science, Newton), (PL3103, Psychology, Arts, Freud) \}$$

The set $\{SM, SN, SFY, MDFI\}$ represents the relational database.

Composition: the teacher of the father of ...



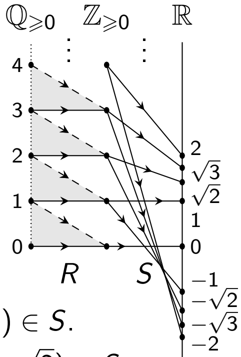
Definition 5.2.1

These must be equal.

Let R be a relation from A to B, and S be a relation from B to C. Then $S \circ R$ is the relation from A to C defined by

$$S \circ R = \{(x, z) \in A \times C : (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y \in B\}.$$

We read $S \circ R$ as "S composed with R" or "S circle R".



Example 5.2.4

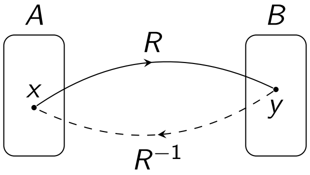
Define relations R from $\mathbb{Q}_{\geqslant 0}$ to $\mathbb{Z}_{\geqslant 0}$ and S from $\mathbb{Z}_{\geqslant 0}$ to $\mathbb{R}$ by:

$$R = \{(x, y) \in \mathbb{Q}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} : \lfloor x \rfloor = y\},$$
$$S = \{(y, z) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{R} : y = z^2\}.$$

- $(4.8, 2) \in S \circ R$ because $4 \in \mathbb{Z}_{\geqslant 0}$ such that $(4.8, 4) \in R$ and $(4, 2) \in S$.
- $(5/2, -\sqrt{2}) \in S \circ R$ because $2 \in \mathbb{Z}_{\geqslant 0}$ such that $(5/2, 2) \in R$ and $(2, -\sqrt{2}) \in S$.

In general, we have $S \circ R = \{(x, z) \in \mathbb{Q}_{\geqslant 0} \times \mathbb{R} : \lfloor x \rfloor = z^2\}$.

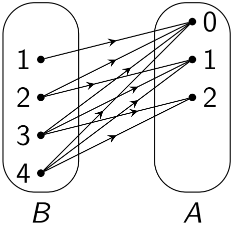
Inverse: from parents to children, and back



Definition 5.2.5

Let R be a relation from A to B. Then the *inverse of R* is the relation R^{-1} from B to A defined by

$$R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}.$$



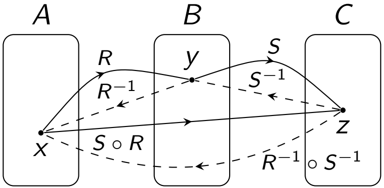
Example 5.2.6

Let R be a relation from A to B where

$$A = \{0, 1, 2\}, \qquad B = \{1, 2, 3, 4\},$$
$$R = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Then $R^{-1} = \{(1, 0), (2, 0), (3, 0), (4, 0), (2, 1), (3, 1), (4, 1), (3, 2), (4, 2)\}$.

Composition interacts with inversion



Proposition 5.2.7

Let R be a relation from A to B, and S be a relation from B to C. Then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof

Both $(S \circ R)^{-1}$ and $R^{-1} \circ S^{-1}$ are relations from C to A. For all $(z, x) \in C \times A$,

$$\begin{aligned} (z,x) \in (S \circ R)^{-1} \quad &\Leftrightarrow \quad (x,z) \in S \circ R && \text{by the definition of inverses;} \\ \Leftrightarrow \quad (x,y) \in R \quad &\text{and } (y,z) \in S \quad \text{for some } y \in B && \text{by the definition of composition;} \\ \Leftrightarrow \quad (y,x) \in R^{-1} \ &\text{and } (z,y) \in S^{-1} \ \text{for some } y \in B && \text{by the definition of inverses;} \\ \Leftrightarrow \quad (z,x) \in R^{-1} \circ S^{-1} & && \text{by the definition of composition.} \end{aligned}$$

So $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. □

Real-life example. Peter is Mary's parent's elder sibling.

$\Leftrightarrow$ Mary is Peter's younger sibling's child.

Commutativity of relation composition

Exercise 5.2.8

Let $A = \{0, 1, 2\}$. Define two relations R, S from A to A by:

$$R = \{(x, y) \in A^2 : x < y\} \quad \text{and} \quad S = \{(0, 1), (1, 2), (2, 0)\}.$$

Is $R \circ S = S \circ R$? Prove that your answer is correct. ✎ 5b

Summary

Let A, B, C be sets.

Definition 5.1.5. R is a *relation* from A to B $\Leftrightarrow$ $R \subseteq A \times B$.

Sometimes we write $x\,R\,y$ for $(x, y) \in R$.

Let R be a relation from A to B and S be a relation from B to C.

Definitions 5.2.1 and 5.2.5. For all $x \in A$, all $y \in B$ and all $z \in C$,

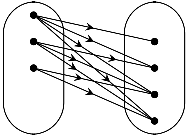
$$(x, z) \in S \circ R \quad \Leftrightarrow \quad \exists y \in B\ \big((x, y) \in R \land (y, z) \in S\big)$$

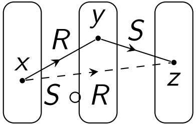
or $$x\,(S \circ R)\,z \quad \Leftrightarrow \quad \exists y \in B\ (x\,R\,y \land y\,S\,z);$$

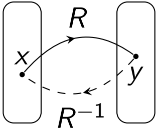
$$(y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R$$

or $$y\,R^{-1}\,x \quad \Leftrightarrow \quad x\,R\,y.$$

Proposition 5.2.7. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.







Representing directed graphs using sets

Arrow diagram. An arrow from x to y indicates x is related to y.

Definition 5.3.1 and Remark 5.3.2

A *(binary) relation on a set A* is a relation from A to A, i.e., a subset of $A \times A$.

Definition 5.3.5

A *directed graph* is an ordered pair (V, D) where V is a set and D is a binary relation on V. In the case when (V, D) is a directed graph,

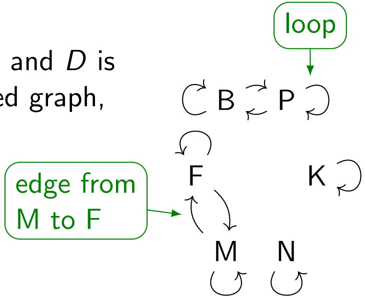
(1) the *vertices* or the *nodes* are the elements of V;

(2) the *edges* are the elements of D;

(3) an edge *from x to y* is the element $(x, y) \in D$;

(4) a *loop* is an edge from a vertex to itself;

(5) a *drawing* of (V, D) is an arrow diagram for D as a relation on A.



Examples 5.3.6 and 5.3.4

Let $V = \{\mathrm{B}, \mathrm{P}, \mathrm{F}, \mathrm{M}, \mathrm{K}, \mathrm{N}\}$, and

$$D = \{(\mathrm{B}, \mathrm{P}), (\mathrm{P}, \mathrm{B}), (\mathrm{F}, \mathrm{M}), (\mathrm{M}, \mathrm{F}), (\mathrm{B}, \mathrm{B}), (\mathrm{P}, \mathrm{P}), (\mathrm{F}, \mathrm{F}), (\mathrm{M}, \mathrm{M}), (\mathrm{K}, \mathrm{K}), (\mathrm{N}, \mathrm{N})\}.$$

Then (V, D) is a directed graph. The figure above is a drawing of (V, D).

Representing undirected graphs using sets

Drawing. A line between x and y indicates an edge between x and y.

Definition 5.3.7

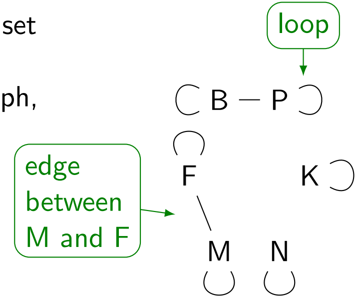
An *undirected graph* is an ordered pair (V, E) where V is a set and E is a set all of whose elements are of the form $\{x, y\}$ with $x, y \in V$. In the case when (V, E) is an undirected graph,

(1) the *vertices* or the *nodes* are the elements of V;

(2) the *edges* are the elements of E;

(3) an edge *between x and y* is the element $\{x, y\} \in E$;

(4) a *loop* is an edge between a vertex and itself.



Examples 5.3.6 and 5.3.4

Let $V = \{\mathrm{B}, \mathrm{P}, \mathrm{F}, \mathrm{M}, \mathrm{K}, \mathrm{N}\}$, and

$$E = \{\{\mathrm{B}, \mathrm{P}\}, \{\mathrm{P}, \mathrm{B}\}, \{\mathrm{F}, \mathrm{M}\}, \{\mathrm{M}, \mathrm{F}\}, \{\mathrm{B}, \mathrm{B}\}, \{\mathrm{P}, \mathrm{P}\}, \{\mathrm{F}, \mathrm{F}\}, \{\mathrm{M}, \mathrm{M}\}, \{\mathrm{K}, \mathrm{K}\}, \{\mathrm{N}, \mathrm{N}\}\}.$$

Then (V, E) is an undirected graph. The figure above is a drawing of (V, E).