

## NATIONAL UNIVERSITY OF SINGAPORE

## CS1231 DISCRETE STRUCTURES

(Semester 1: 2022/2023)

[In the notation and the terminology for 2022/23 Semester 2]

Time Allowed: 2 Hours

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**INSTRUCTIONS TO STUDENTS**

1. Write your Student Number only. Do not write your name.
2. This assessment paper contains **FOUR** questions and comprises **NINE** printed pages.
3. Answer **ALL** questions. The marks for each question are indicated in brackets.
4. Write your solutions in the spaces provided.
5. This is an **OPEN** book examination.

STUDENT NUMBER: \_\_\_\_\_

EXAMINER'S USE ONLY		
Question	Marks	Score
Q1	20	
Q2	5	
Q3	19	
Q4	6	
Total	50	

1. Consider the relation  $f$  from  $\mathbb{Z}$  to  $\mathbb{Z}$  defined by

$$f = \{(m + 12, 31m) : m \in \mathbb{Z}\}.$$

- (i) Prove that  $f$  is a function  $\mathbb{Z} \rightarrow \mathbb{Z}$ . [4 marks]

- (ii) What is the codomain of  $f$ ? [1 mark]

- (iii) What is the range of  $f$ ? [1 mark]

- (iv) Is  $f$  surjective? Justify your answer. [3 marks]

1. (continued from the previous page)

(v) Is  $f$  injective? Justify your answer.

[3 marks]

(vi) Define  $f^1 = f$  and  $f^{n+1} = f^n \circ f$  for each  $n \in \mathbb{Z}^+$ . Prove that  $f^n$  is injective for any  $n \in \mathbb{Z}^+$ .  
[3 marks]

1. (continued from the previous page)

- (vii) Does the relation  $f$ , considered as a set, have the same cardinality as  $\mathbb{Z} \times \mathbb{Z}$ ? Justify your answer. [5 marks]

2. Let  $A$  be an uncountable subset of  $\mathbb{R}$ .

(i) Define  $A^- = A \cap \mathbb{R}^-$  and  $A_{\geq 0} = A \cap \mathbb{R}_{\geq 0}$ . Explain why  $A^-$  and  $A_{\geq 0}$  cannot both be countable. [1 mark]

(ii) Define  $B = \{x^2 : x \in A\}$ . Using (i), or otherwise, prove that  $B$  is uncountable. [4 marks]

3. Consider an undirected graph  $G = (V, E)$  where  $V = \{1, 2, 3, 4, 5\}$  and  $E = \{12, 13, 14, 23, 34, 45\}$ .
- (i) Draw  $G$ . [1 mark]
- (ii) How many connected components are there in  $G$ ? [1 mark]
- (iii) How many cycles are there in  $G$ ? [1 mark]
- (iv) How many paths are there between the vertices 1 and 5 in  $G$ ? [1 mark]
- (v) Draw a subgraph  $(V', E')$  of  $G$  that is not a tree but satisfies  $|E'| = |V'| - 1$ . [1 mark]
- (vi) Identify an edge  $xy$  such that  $(V, E \setminus \{xy\})$  is not connected. [1 mark]

3. (continued from the previous page)

(vii) Draw all spanning trees of  $G$ .

[3 marks]

(viii) Which of the spanning trees from (vii) are isomorphic to each other?

[3 marks]

(ix) Pick two non-isomorphic trees  $T$  and  $T'$  from (viii). Explain why  $T$  and  $T'$  are not isomorphic.

[2 marks]

3. (continued from the previous page)

(x) Determine the number of graphs with vertex set  $\{1, 2, 3, 4, 5\}$  that are isomorphic to  $G$ . Briefly explain your answer. [2 marks]

(xi) Suppose we add a vertex 6 to  $V$  and an edge 26 to  $E$ . Determine the number of graphs with vertex set  $\{1, 2, 3, 4, 5, 6\}$  that are isomorphic to  $(V \cup \{6\}, E \cup \{26\})$ . Briefly explain your answer. [3 marks]



4. Let  $H$  be a subgraph of a finite connected undirected graph  $G$ . Prove that  $H$  is a subgraph of a spanning tree of  $G$  if and only if  $H$  is acyclic. [6 marks]