Tutorial solutions for Chapter 3

Sometimes there are other correct answers.

3.1. One can rewrite the proposition to be proved symbolically as

$$\forall x \in \mathbb{R} \ \ x^2 \geqslant 0.$$

To prove this by contradiction (i.e., using Technique 3.2.16), one starts with the negation, which is equivalent to

$$\exists x \in \mathbb{R} \ \ x^2 < 0$$

by Theorem 2.3.1. However, the author started instead with

$$\forall x \in \mathbb{R} \ \ x^2 < 0.$$

This is not justified.

Moral 1. Proofs of true propositions need not be correct.

Moral 2. Negate propositions carefully when proving by contradiction.

3.2. (a) The argument given proves $\forall x \in \mathbb{R} \ \ (x^2 + 9 = 6x \to x = 3)$. However, what we want to prove is $\exists x \in \mathbb{R} \ \ (x^2 + 9 = 6x)$. The author did not explain how one can derive what we want to prove from what is proved.

The situation is different if the author proves $\forall x \in \mathbb{R} \ \ (x = 3 \to x^2 + 9 = 6x)$ instead, because instantiating x to 3 here (in the sense of Example 3.1.13) immediately gives a real number x satisfying $x^2 + 9 = 6x$, namely 3.

(b) Note that 3 is real number and $3^2 + 9 = 18 = 6 \times 3$. □

Moral. Do not mix up a conditional proposition and its converse: as shown in Theorem 1.4.12(3), the two may not be equivalent.

Additional information. Strictly speaking, proofs do not need to contain information about how one came up with them. However, sometimes authors still include this information because this can be useful to the reader.

3.3. **Proof.** ((a) $\to$ (b)) Assume (a) is true. Pick elements z and w of A. As $P(x, y)$ is a predicate on A, the sentence $P(z, w)$ is a proposition. So it is either true or false.

Case 1: suppose $P(z, w)$ **is true.** Then $P(w, z)$ is also true by (a).

Case 2: suppose $P(z, w)$ **is false.** If $P(w, z)$ is true, then $P(z, w)$ is also true by (a), which contradicts our supposition. So $P(w, z)$ is false.

Hence $P(z, w)$ and $P(w, z)$ are either both true or both false in all cases. As the choice of the elements z and w from A was arbitrary, we have shown (b).

((b) $\to$ (a)) Assume (b) is true. Pick elements z and w of A. Then $P(z, w)$ and $P(w, z)$ are either both true or both false by (b).

Case 1: suppose $P(z, w)$ and $P(w, z)$ are both true. Then $P(z, w) \to P(w, z)$ is true by the definition of $\to$.

Case 2: suppose $P(z, w)$ and $P(w, z)$ are both false. Then $P(z, w) \to P(w, z)$ is (vacuously) true by the definition of $\to$.

Hence $P(z, w) \to P(w, z)$ is true in all cases. As the choice of the elements z and w from A was arbitrary, we have shown (a). $\square$

- To prove (a) $\leftrightarrow$ (b), we split into (a) $\to$ (b) and (b) $\to$ (a) according to Technique 3.2.9.
- To each of the two directions, we split into two cases according to Technique 3.2.11.
- We applied universal instantiation and *modus ponens* from Example 3.1.11 and Example 3.1.13 in Case 1 of (a) $\to$ (b).
- We used a subproof by contradiction (i.e., Technique 3.2.16) in Case 2 of (a) $\to$ (b).

Alternative proof of (b) $\to$ (a). Assume (b) is true. Pick elements z and w of A such that $P(z, w)$ is true. Then $P(z, w)$ and $P(w, z)$ are either both true or both false by (b). Since $P(z, w)$ is true, it cannot be the case that $P(z, w)$ and $P(w, z)$ are both false. So they must be both true. In particular, this says $P(w, z)$ is true. As the choice of the elements z and w from A satisfying $P(z, w)$ was arbitrary, we have shown (a). $\square$

Moral. One can instantiate a universal proposition (in the sense of Example 3.1.13) in more than one way.

Additional remark. The sentence "As the choice of ... was arbitrary, ..." is sometimes omitted in proofs, as this should be clear to the (experienced) reader.

Extra exercises

3.4. The argument considers only three real numbers: -1, 0, and 1. It does not explain why $x^2 - 6x + 7 > 0$ is true for other real numbers, for example, for $x = 2$.

3.5. Suppose the proposition is not true. Let m, n, k be integers such that $m^2 + n^2 = k^2$, but m and n are both odd. Use Proposition 3.2.27 to find an integer a making $m^2 + n^2 = 4a + 2$. This tells us $k^2 = m^2 + n^2 = 4a + 2 = 2(2a + 1)$, where $2a + 1$ is an integer. So k^2 is even by the definition of even integers. Thus, in view Proposition 3.2.8 we know k is even. Use the definition of even integers to find an integer x such that $k = 2x$. Then $4a + 2 = m^2 + n^2 = k^2 = (2x)^2 = 4x^2$. Hence

$$x^2 - a = \frac{4x^2 - 4a}{4} = \frac{4a + 2 - 4a}{4} = \frac{1}{2},$$

which is not an integer. However, we know that $x^2 - a$ is an integer because x and a are. This is a contradiction. Therefore, the proposition must be true. $\square$