CS1231S — DISCRETE STRUCTURES
ANSWER SHEETS

INSTRUCTIONS

1. Write all answers onto these answer sheets. You are to submit only these answer sheets and not the question paper.
2. These answer sheets consist of 7 pages.
3. You may write in pen or pencil.
4. Do NOT start writing until you are told to do so.
5. Do not write your name. Write your Student Number (eg: A0123456X) below and on pages 3, 5 and 7:

Student number

A	0							

For internal use only

MCQs (24 marks)	MRQs (21 marks)	Q20 (4 marks)	Q21 (4 marks)	Q22 (7 marks)	Q23 (20 marks)	Q24 (20 marks)	Total (100 marks)

Write your answers for the MCQs and MRQs below, in CAPITAL LETTERS:

1 B	2 A	3 B	4 B	5 C	6 D
7 A	8 B	9 A	10 A	11 E	12 E

Explanation/working for MCQs/MRQs on page 8 onwards.

13 AB	14 AD	15 AC	16 ABCD
17 ABC	18 AD	19 BCE	

20. Prove by induction. [4 marks]

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition " $a_n = \frac{(n+4)!}{5!(n-1)!}$ ".
2. (Base step) $P(1)$ is true because $\frac{(1+4)!}{5!(1-1)!} = \frac{5!}{5!0!} = 1 = a_1$.
3. (Induction step)

3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., such that
$$a_k = \frac{(k+4)!}{5!\,(k-1)!}.$$
3.2. Then $a_{k+1} = a_k + \frac{(k+1)(k+2)(k+3)(k+4)}{4!}$ by the definition of $a_1, a_2, a_3, \ldots$;

3.3. $= \frac{(k+4)!}{5!(k-1)!} + \frac{(k+1)(k+2)(k+3)(k+4)}{4!}$ by the induction hypothesis;

3.4. $= \frac{(k+4)!+5(k-1)!(k+1)(k+2)(k+3)(k+4)}{5!(k-1)!}$

3.5. $= \frac{(k+4)!}{5!(k-1)!}\left(1 + \frac{5}{k}\right)$

3.6. $= \frac{(k+4)!}{5!(k-1)!}\left(\frac{k+5}{k}\right)$

3.7. $= \frac{(k+5)!}{5!k!}$

3.8. $= \frac{((k+4)+1)!}{5!((k+1)-1)!}$.

3.9. So $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI.

Alternative way to complete the induction step.

3. (Induction step)

3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., such that
$$a_k = \frac{(k+4)!}{5!\,(k-1)!}.$$
3.2. Then $a_{k+1} = a_k + \frac{(k+1)(k+2)(k+3)(k+4)}{4!}$ by the definition of $a_1, a_2, a_3, \ldots$;

3.3. $= \frac{(k+4)!}{5!(k-1)!} + \frac{(k+1)(k+2)(k+3)(k+4)}{4!}$ by the induction hypothesis;

3.4. $= \binom{k+4}{5} + \binom{k+4}{4}$ by Theorem 9.5.1 in Epp;

3.5. $= \binom{k+5}{5}$ by Pascal's Formula, as $k + 5 \geqslant$
$1 + 5 > 5$;

3.6. $= \frac{(k+5)!}{5!((k+5)-5)!}$ by Theorem 9.5.1 in Epp;

3.8. $= \frac{((k+4)+1)!}{5!((k+1)-1)!}$.

3.9. So $P(k+1)$ is true.

Extra exercise.
This question essentially tells one to prove that, for each $m \in \mathbb{Z}_{\geqslant 4}$,
$$\binom{4}{4} + \binom{5}{4} + \cdots + \binom{m}{4} = \binom{m+1}{5}.$$
Give a combinatorial proof of this statement.

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21. Let $f: A \to A$ such that $X \subseteq f(X)$ for all $X \subseteq A$. Prove that $f = \mathrm{id}_A$, the identity function on A. [4 marks]

1.	The domain and the codomain of f and id_A are all A.		
2.	2.1.	Let $x \in A$ and $X = \{x\}$.	
	2.2.	Then $X \subseteq A$	by the definition of $\subseteq$.
	2.3.	So $X \subseteq f(X)$	by assumption.
	2.4.	This implies $x \in f(X)$	as $X = \{x\}$.
	2.5.	But $f(X) = f(\{x\}) = \{f(x)\}$	by the definition of $f(X)$.
	2.6.	So $f(x) = x = \mathrm{id}_A(x)$	by the definition of id_A.
3.	So $f = \mathrm{id}_A$		by line 1 and block 2.

22. [7 marks]

(a) No.

1. Let $A_1 = A_2 = B_1 = \mathbb{R}$ and $B_2 = \emptyset$.
2. Then A_1, A_2, B_1 are uncountable by Tutorial 8 Question 10.
3. Also, as B_2 is finite, it is countable.
4. Note $A_1 \cup A_2 = \mathbb{R} \cup \mathbb{R}$ by the definition of A_1 and A_2;
5. $= \mathbb{R}$ by the Idempotent Law;
6. $= \mathbb{R} \cup \emptyset$ by the Identity Law;
7. $= B_1 \cup B_2$ by the definition of B_1 and B_2.
8. So $|A_1 \cup A_2| = |B_1 \cup B_2|$ by Proposition 10.2.3(1). ☐

(b) No.

1. Let $C_1 = C_2 = \{1\}$ and $D_1 = D_2 = \{1,2\}$.
2. As A_1, A_2, B_1, B_2 are finite, they are countable.
3. Note $|C_1 \times C_2| = |C_1| \times |C_2|$ by the Product Rule;
4. $= 1 \times 1 = 1$
5. $\neq 4 = 2 \times 2$
6. $= |D_1| \times |D_2|$
7. $= |D_1 \times D_2|$ by the Product Rule. ☐

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23. [Total: 20 marks]

(a) (i) [2]

23,426

$x_1 + x_2 + x_3 + x_4 = 50$ such that $x_i \geq 0$ for $i = 1,2,3,4$: $\binom{53}{50}$

(a)(ii) [2]

316,251

$x_1 + x_2 + x_3 + x_4 + x_5 = 50$ such that $x_i \geq 0$ for $i = 1,2,3,4,5$: $\binom{54}{50}$

(b) (i) [2]

0.0345

(b)(ii) [2]

0.406

$P(defective)$
$= P(A) \cdot P(defective|A) + P(B) \cdot P(defective|B) + P(C) \cdot P(defective|C)$
$= (0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)$

$$P(B|car\ is\ defective) = \frac{P(B\ \cap\ defective)}{P(defective)} = \frac{P(B) \cdot P(defective|B)}{P(defective)} = \frac{0.35 \times 0.04}{0.0345}$$

(c) [3]

0.5

(d) [3]

0.158

$$P(Bus\ stop \mid {\sim}Exam\ hall) = \frac{P(Bus\ stop\ \cap {\sim}Exam\ hall)}{P({\sim}Exam\ hall)} = \frac{P(Bus\ stop)}{1 - P(Exam\ hall)} = \frac{0.2}{0.4}$$

X = number of pages with error.

	Probability	X
1st page only	$0.09 \times (1 - 0.25) = 0.0675$	1
1st and 2nd	$0.09 \times 0.25 = 0.0225$	2
2nd page only	$(1 - 0.09) \times (0.05) = 0.0455$	1
None	$(1 - 0.09) \times (1 - 0.05) = 0.8645$	0

$E[X] = 0.0675 + (2 \times 0.0225) + 0.0455 = \mathbf{0.158}$

(e) [3]

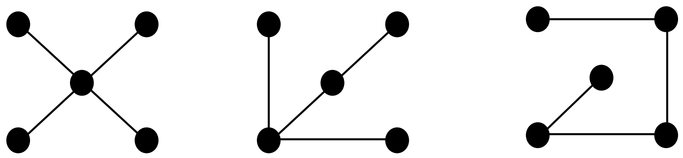
$P(1) = P(A) = p$
$P(0) = P(\bar{A}) = 1 - P(A) = 1 - p$
$E[I] = 0(1 - p) + p = \boldsymbol{p}$

(f) [3]

1. Let the 4 pigeonholes be the equivalence classes [0],[1],[2] and [3] of the congruence-mod-4 relation $\sim_4$.
2. Given any 5 distinct non-negative integers, two of them will be in [i], where i = 0, 1, 2, or 3, by the Pigeonhole Principle.
3. Let these 2 numbers be x and y. Hence, $x = 4k + i$ and $y = 4l + i$, for some integers k and l, by example 6.4.3.
4. Then $x - y = (4k + i) - (4l + i) = 4(k - l)$.

24. [Total: 20 marks]

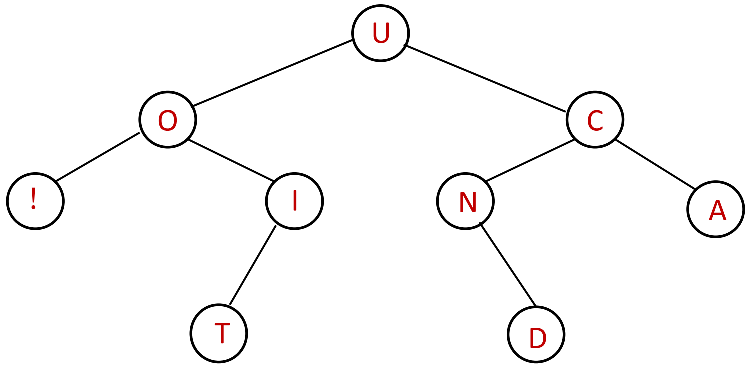
(a)
[3]



Degrees: (1,1,1,1,4), (1,1,1,2,3), (1,1,2,2,2)

A spanning tree of 5 vertices has 4 edges (theorem 10.5.2), hence total degree is 8 (Handshake theorem). There are at least two vertices with degree 1 (Lecture 14 slide 12 exercise), therefore the remaining 6 degrees are distributed among the remaining 3 vertices: (1,1,4), (1,2,3) and (2,2,2). These are the only 3 spanning trees possible. One mark is deducted if student draws trees that are isomorphic to one another.

(b)
[3]



Alternative answers: T is right child of I; D is left child of N.

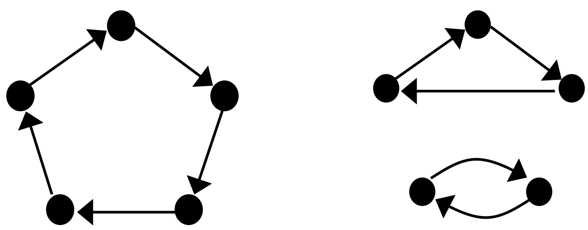
(c)(i)
[3]

$IsFunction(G) \equiv \forall v \in A\ deg^{+}(v) = 1.$

$IsSurjective(G) \equiv \forall v \in A\ (deg^{+}(v) = 1 \wedge deg^{-}(v) \geq 1)$
or, $IsSurjective(G) \equiv \forall v \in A\ (deg^{+}(v) = 1 \wedge deg^{-}(v) = 1).$

$IsInjective(G) \equiv \forall v \in A\ (deg^{+}(v) = 1 \wedge deg^{-}(v) \leq 1)$
or, $IsInjective(G) \equiv \forall v \in A\ (deg^{+}(v) = 1 \wedge deg^{-}(v) = 1).$

(c)(ii)
[2]



Note: The question is supposed to ask for bijections, but we made a mistake and asked for functions. So we accept any directed graphs that represent functions.

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24. (continue…)

(d) [3]

20

Use adjacency matrix M to compute M^3.

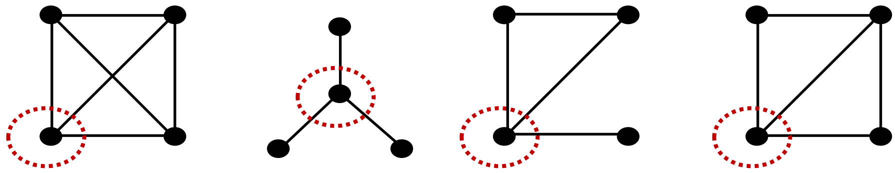
$$M = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix}.\ M^2 = \begin{bmatrix} 10 & 2 & 6 & 6 \\ 2 & 14 & 6 & 6 \\ 6 & 6 & 14 & 2 \\ 6 & 6 & 2 & 10 \end{bmatrix}$$

Or, a → b = 3 ways, b → c = 2 ways, c → d = 3 ways, therefore $3 \times 2 \times 3 = 18$ ways.

a → c → b → d = 2 ways. Hence, total = $18 + 2 = \mathbf{20}$ ways.

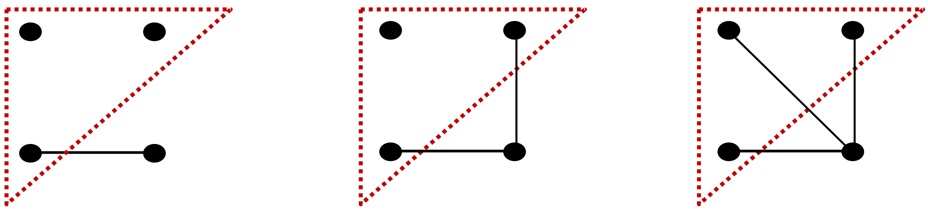
(e)(i) [2]

A number of possible answers; only 4 are shown below. We accept any two.
Dominating set shown in dotted circle.



(e)(ii) [2]

Dominating set shown in dotted triangle. We accept any two.
Question should have specified simple graphs, but we didn't, so we accept correct graphs with loops/parallel edges as well.



(e)(iii) [2]

$\{c, h\}$ or $\{d, g\}$: the total cost is 4 + 3 = **7** units.

=== END ===

2. As we defined it, recursion is a method of construction, not a method of proof.

3. One can prove (i) by an argument similar to that in Assignment 2 Question 1(b).
   For (ii), let $P(n)$ be "$n \neq 3$"; cf. Tutorial 6 Question D5(b).

4. Given $n \in \mathbb{Z}^+$, apply the Well-Ordering Principle to $\{m \in \mathbb{Z}^+ : n \leqslant m^3\}$.

5. For (i), let $X_1 = \{1\}$ and $X_2 = \{-1\}$, so that $f(X_1 \cap X_2) = f(\emptyset) = \emptyset \neq \{1\} = f(X_1) \cap f(X_2)$. We can prove (ii) as follows:
   1. Let $Y_1, Y_2 \subseteq \mathbb{Q}$.
   2. ($\subseteq$)
      2.1. Let $x \in f^{-1}(Y_1 \cap Y_2)$.
      2.2. Use the definition of $f^{-1}(Y_1 \cap Y_2)$ to find $y \in Y_1 \cap Y_2$ such that $f(x) = y$.
      2.3. Then $y \in Y_1$ and $y \in Y_2$.
      2.4. So $x \in f^{-1}(Y_1)$ and $x \in f^{-1}(Y_2)$ by the definition of $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$.
      2.5. Hence $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$.
   3. ($\supseteq$)
      3.1. Let $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$.
      3.2. Then $x \in f^{-1}(Y_1)$ and $x \in f^{-1}(Y_2)$.
      3.3. Use the definition of $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ to find $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $f(x) = y_1$ and $f(x) = y_2$.
      3.4. Then $y_1 = y_2 \in Y_1 \cap Y_2$.
      3.5. So $x \in f^{-1}(Y_1 \cap Y_2)$ by the definition of $f^{-1}(Y_1 \cap Y_2)$.

6. The right dot at the bottom row has no arrow pointing to it. So f is not surjective.
   The middle dot at the bottom row has two arrows pointing to it. So f is not injective.

7. Every dot in the diagram has exactly one arrow pointing to it. So g is surjective and injective.

8.

For (i), note that if $k \in \mathbb{Z}^+$ such that $\underbrace{f \circ f \circ \cdots \circ f}_{k\text{-many } f\text{'s}} = \mathrm{id}_A$, then Tutorial 7 Question 8 implies

$$\underbrace{f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}}_{k\text{-many } f\text{'s}} = \underbrace{(f \circ f \circ \cdots \circ f)^{-1}}_{k\text{-many } f\text{'s}} = \mathrm{id}_A^{-1} = \mathrm{id}_A.$$

For (ii), let $A = \mathbb{Q} \times \mathbb{Q}$. Define $f, g: A \to A$ by setting, for all $x_1, x_2 \in \mathbb{Q}$,

$$f(x_1, x_2) = (x_2, x_1) \quad \text{and} \quad g(x_1, x_2) = \left(2x_2, \frac{x_1}{2}\right).$$

Then it can be verified directly that $f \circ f = \mathrm{id}_A = g \circ g$. However, for all $x_1, x_2 \in \mathbb{Q}$,

$$(g \circ f)(x_1, x_2) = g\big(f(x_1, x_2)\big) = g(x_2, x_1) = \left(2x_1, \frac{x_2}{2}\right),$$

and thus $\big(\underbrace{(g \circ f) \circ (g \circ f) \circ \cdots \circ (g \circ f)}_{k\text{-many } (g \circ f)\text{'s}}\big)(x_1, x_2) = \left(2^k x_1, \frac{x_2}{2^k}\right) \neq (x_1, x_2)$ whenever $k \in \mathbb{Z}^+$, unless $x_1 = 0 = x_2$.

<u>Alternative explanation for (ii) from the Telegram chat</u>

Let $A = \mathbb{Z}$. Define $f, g: A \to A$ by setting, for all $n \in \mathbb{Z}$,

$$f(n) = \begin{cases} n+1, & \text{if } n \text{ is even;} \\ n-1, & \text{if } n \text{ is odd,} \end{cases} \qquad g(n) = \begin{cases} n+1, & \text{if } n \text{ is odd;} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

It can be verified directly that both f and g are bijections of order 2. If n is an even integer, then $(g \circ f)(n) = g(f(n)) = g(n+1) = n+2$ as $n+1$ is odd. Therefore, whenever $k \in \mathbb{Z}^+$,

$$\big(\underbrace{(g \circ f) \circ (g \circ f) \circ \cdots \circ (g \circ f)}_{k\text{-many } (g \circ f)\text{'s}}\big)(0) = 2k \neq 0 = \mathrm{id}_A(0).$$

9. For (i), consider an injection $f: A \to B$ that is not surjective. Use non-surjectivity to find $b \in B$ such that $\forall x \in A\ f(x) \neq b$. Define $B_0 = B \setminus \{b\}$ and $f_0: A \to B_0$ by setting $f_0(x) = f(x)$ for all $x \in A$. By the choice of b, this f_0 is well defined. Moreover, it is an injection because f is. So the Pigeonhole Principle tells us $|A| \leqslant |B_0| = |B| - 1 < |B|$.

   For (ii), consider a surjection $g: A \to B$ that is not injective. Use non-injectivity to find $a_1, a_2 \in A$ such that $g(a_1) = g(a_2)$. Define $A_0 = A \setminus \{a_2\}$ and $g_0: A_0 \to B$ by setting $g_0(x) = g(x)$ for all $x \in A_0$. Then g_0 remains a surjection by the choice of a_1 and a_2. So $|A| = |A_0| + 1 > |A_0| \geqslant |B|$ by the Dual Pigeonhole Principle.

10. (i) is true according to Cantor's definition of same-cardinality. One can prove the nontrivial direction of (ii) as follows.

Let A, B be infinite sets with the same cardinality. Denote by $\mathrm{Bij}(A, B)$ the set of all bijections $A \to B$. In view of Cantor's definition of same-cardinality, we know $\mathrm{Bij}(A, B) \neq \emptyset$. Take any $f \in \mathrm{Bij}(A, B)$. Apply Proposition 10.3.7 to obtain a countable infinite subset $A_0 \subseteq A$. Use Note 10.3.4 to find a sequence $a_0, a_1, a_2, \ldots \in A_0$ in which every element of A_0 appears exactly once.

For each $S \in \mathcal{P}(\mathbb{Z}_{\geqslant 0})$, define φ_S to be the function $\mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfying, for each $m \in \mathbb{Z}_{\geqslant 0}$,

$$\varphi_S(2m) = \begin{cases} 2m+1, & \text{if } m \in S; \\ 2m, & \text{if } m \notin S, \end{cases} \qquad \text{and} \qquad \varphi_S(2m+1) = \begin{cases} 2m, & \text{if } m \in S; \\ 2m+1, & \text{if } m \notin S. \end{cases}$$

It can be verified directly that each φ_S is a bijection, and if S_1, S_2 are different subsets of $\mathbb{Z}_{\geqslant 0}$, then $\varphi_{S_1} \neq \varphi_{S_2}$. From these, we may define a function $\psi \colon \mathcal{P}(\mathbb{Z}_{\geqslant 0}) \to \mathrm{Bij}(A, B)$ by setting $\psi(S)$ to be the function $\psi_S \colon A \to B$ which satisfies

- $\psi_S(a_i) = f(a_{\varphi_S(i)})$ for all $i \in \mathbb{Z}_{\geqslant 0}$, and
- $\psi_S(x) = f(x)$ for all $x \in A \setminus A_0$,

for each $S \in \mathcal{P}(\mathbb{Z}_{\geqslant 0})$. It can be verified directly that each ψ_S is a bijection, and thus ψ is well defined. In addition, one can verify that ψ is an injection.

Therefore, the function $\tilde{\psi} \colon \mathcal{P}(\mathbb{Z}_{\geqslant 0}) \to \mathrm{range}(\psi)$ satisfying $\tilde{\psi}(S) = \psi(S)$ for all $S \in \mathcal{P}(\mathbb{Z}_{\geqslant 0})$ is a bijection. This shows $|\mathcal{P}(\mathbb{Z}_{\geqslant 0})| = |\mathrm{range}(\psi)|$. Recall from Theorem 10.4.3 that $\mathcal{P}(\mathbb{Z}_{\geqslant 0})$ is not countable. So $\mathrm{range}(\psi)$ is also not countable by Tutorial 8 Question D3. An application of Proposition 10.3.6 then gives us the uncountability of $\mathrm{Bij}(A, B)$, as $\mathrm{range}(\psi) \subseteq \mathrm{Bij}(A, B)$.

11. Solution 1:
    - Let the 2 enemies be A and B.
    - 13 people can be arranged around a circle in $(13-1)! = 12!$ ways.
    - There are 13 places in between the people where A and B can be separately placed hence we can place them in 13×12 positions.
    - Total number of ways of arrangement = $12! \times 13 \times 12 = 12 \times 13!$

    Solution 2:
    - Place A at any seat.
    - 13 people can be arranged in 13! ways.
    - The seats to the left and right of A cannot be occupied by B, hence B can be placed at any of the remaining 12 positions
    - $A\ p_1, p_2, \ldots, p_{13}, \Rightarrow B$ can be placed in any gap between p_i and p_{i+1} for $i \in 1 \ldots 12$
    - Total number of ways of arrangement $= 12 \times 13!$

13. If $a_1, a_2, \ldots, a_k \in \mathbb{Z}$, then $\{a_1, a_2, \ldots, a_k\} = \bigcup_{i=1}^{k} \{x \in \mathbb{Z} : a_i \leqslant x \leqslant a_i\} \in S$. So A and B are both true.

    It then follows from the third part of the recursion clause that S contains all *cofinite* subsets of $\mathbb{Z}$, i.e., those $X \subseteq \mathbb{Z}$ which make $\mathbb{Z} \setminus X$ finite. So D is not true.

    Now, as one can show by structural induction over S, all the elements of S are either finite or cofinite. So C is not true.

14. The domain and the codomain of f are both $\{\mathrm{s}, \mathrm{u}\}^*$. So A is true.
    The codomain of f contains u, but the range of f does not. So B is not true.
    As B is not true, we know f is not surjective. So C is not true according to Theorem 9.3.19.
    Removing all occurrences of u once is the same as removing them twice. So D is true.

15. Note $f(\mathbf{true}, \mathbf{true}, \mathbf{true}) = \mathbf{true} = f(\mathbf{false}, \mathbf{false}, \mathbf{false})$ and $f(\mathbf{false}, \mathbf{false}, \mathbf{true}) = \mathbf{false}$.

    So f is surjective but not injective. These also show that D is not true.

    If $q, r \in \mathrm{Bool}$ such that $f(\mathbf{false}, q, r) = \mathbf{true}$,then it must be the case that $r = \mathbf{false}$, and thus $f(\mathbf{true}, q, r) = \mathbf{true}$ as well. This shows C.

16. Here is how f looks like.

    $5^0 + 1 = 2 \to 3 \to 4 \to 5$ $\qquad$ $5^1 + 1 = 6 \to 7 \to 8 \to \cdots \to 5^2$

    $5^2 + 1 \to 5^2 + 2 \to \cdots \to 5^3$ $\qquad$ $5^3 + 1 \to 5^3 + 2 \to \cdots \to 5^4$ $\qquad$ etc.

17. A is true by Example 8.1.5.

    B is true because the set of all functions $\mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ is a domset.

    C can be proved by a diagonalization argument as follows.

    1. Let $\mathcal{D}$ be a countable set satisfying (i).
    2. Adding all the constant functions $\mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ if needed, we may assume $\mathcal{D}$ is infinite.
    2. Use Note 10.3.4 to find a sequence $g_0, g_1, g_2, \ldots \in \mathcal{D}$ in which every element of $\mathcal{D}$ appears.
    3. Define $f: \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by setting $f(n) = \max\{g_0(n), g_1(n), \ldots, g_n(n)\} + 1$ for each $n \in \mathbb{Z}_{\geqslant 0}$.
    4. For all $i, m \in \mathbb{Z}_{\geqslant 0}$, if $n = m + i$, then $n \geqslant m$ and $f(n) > g_i(n)$ by the definition of f.
    5. This shows $\forall g \in \mathcal{D} \sim\exists m \in \mathbb{Z}_{\geqslant 0}\ \forall n \in \mathbb{Z}_{\geqslant m}\ f(n) \leqslant g(n)$.
    6. Thus (ii) is false.

    Extra information: Domsets are usually called *dominating families*.

18. $A \subseteq B \Rightarrow P(A) \leq P(B)$

    $A \subseteq B \Rightarrow \bar{B} \subseteq \bar{A} \Rightarrow P(\bar{A}) \geq P(\bar{B})$

19. Statement IV is false. Therefore, we can rule out option A.

    Statement C is vacuously true.

    Statement D is false. If G contains four vertices of odd degree, it does not contain an Eulerian circuit.

    Statement E is a catch. Converse of the Statement E is not true. A graph with Eulerian circuit has 0 (an even number) odd-degree vertices. Therefore, Statement E is true.