Chapter 6: Equivalence relations and partial orders

CS1231 Discrete Structures

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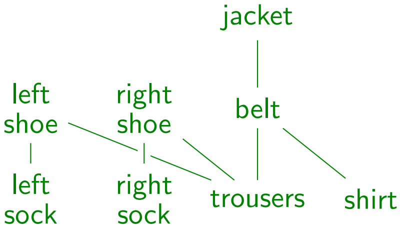
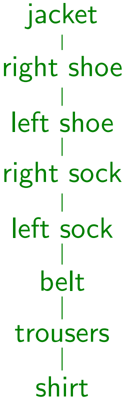
Mathematics is the art of giving the same name to different things.

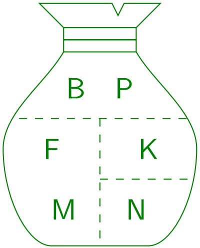
Poincaré (1908)

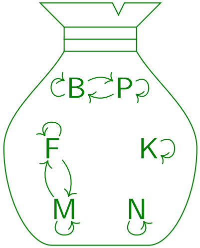
Plan

- equivalence relations $=$
  - reflexivity, symmetry, transitivity
  - equivalence classes
- partial orders and total orders $\leqslant$
  - antisymmetry and totality
  - Well-Ordering Principle









What does the equality relation satisfy?

(1) Every object is equal to itself.

(2) If x is equal to y, then y is in equal to x.

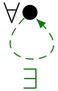
(3) If x is equal to y, and y is equal to z, then x is equal to z.

Definition 6.1.1

Let A be a set and R be a relation on A.

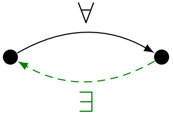
(1) R is *reflexive* if every element of A is R-related to itself, i.e.,

$$\forall x \in A \; (x \, R \, x).$$



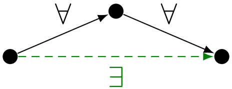
(2) R is *symmetric* if x is R-related to y implies y is R-related to x, for all $x, y \in A$, i.e.,

$$\forall x, y \in A \; (x \, R \, y \Rightarrow y \, R \, x).$$



(3) R is *transitive* if x is R-related to y and y is R-related to z imply x is R-related to z, for all $x, y, z \in A$, i.e.,

$$\forall x, y, z \in A \; (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z).$$



A finite relation

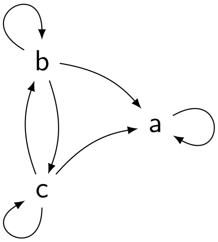
Example 6.1.2

Let R be the relation represented by the following arrow diagram.

- Then R is reflexive.
- It is not symmetric because $\mathrm{b}\ R\ \mathrm{a}$ but $\mathrm{a} \not R\ \mathrm{b}$.
- It is transitive, as one can show by exhaustion:

$$
\begin{aligned}
&\mathrm{a}\ R\ \mathrm{a} \wedge \mathrm{a}\ R\ \mathrm{a} \Rightarrow \mathrm{a}\ R\ \mathrm{a}; &\qquad &\mathrm{c}\ R\ \mathrm{a} \wedge \mathrm{a}\ R\ \mathrm{a} \Rightarrow \mathrm{c}\ R\ \mathrm{a};\\
&\mathrm{b}\ R\ \mathrm{a} \wedge \mathrm{a}\ R\ \mathrm{a} \Rightarrow \mathrm{b}\ R\ \mathrm{a}; &\qquad &\mathrm{c}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{a} \Rightarrow \mathrm{c}\ R\ \mathrm{a};\\
&\mathrm{b}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{a} \Rightarrow \mathrm{b}\ R\ \mathrm{a}; &\qquad &\mathrm{c}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{b} \Rightarrow \mathrm{c}\ R\ \mathrm{b};\\
&\mathrm{b}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{b} \Rightarrow \mathrm{b}\ R\ \mathrm{b}; &\qquad &\mathrm{c}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{c} \Rightarrow \mathrm{c}\ R\ \mathrm{c};\\
&\mathrm{b}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{c} \Rightarrow \mathrm{b}\ R\ \mathrm{c}; &\qquad &\mathrm{c}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{a} \Rightarrow \mathrm{c}\ R\ \mathrm{a};\\
&\mathrm{b}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{a} \Rightarrow \mathrm{b}\ R\ \mathrm{a}; &\qquad &\mathrm{c}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{b} \Rightarrow \mathrm{c}\ R\ \mathrm{b};\\
&\mathrm{b}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{b} \Rightarrow \mathrm{b}\ R\ \mathrm{b}; &\qquad &\mathrm{c}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{c} \Rightarrow \mathrm{c}\ R\ \mathrm{c}\\
&\mathrm{b}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{c} \Rightarrow \mathrm{b}\ R\ \mathrm{c}; & &
\end{aligned}
$$

and all the others instances are vacuously true.



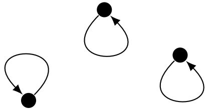
Equality and inclusion

Example 6.1.3

Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x = y.$$

Then R is reflexive, symmetric, and transitive.

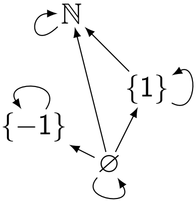


Example 6.1.4

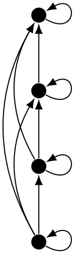
Let R_0 denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \mathrel{R_0} y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R_0 is reflexive, may not be symmetric, but is transitive.



Inequalities

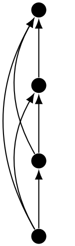


Exercise 6.1.5

Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \, R \, y \quad \Leftrightarrow \quad x \leqslant y.$$

Is R reflexive? Is R symmetric? Is R transitive? 6a



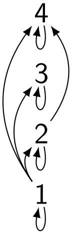
Exercise 6.1.6

Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \, R' \, y \quad \Leftrightarrow \quad x < y.$$

Is R reflexive? Is R symmetric? Is R transitive? 6b

Divisibility



Example 6.1.8

Let R denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x \, R \, y \quad \Leftrightarrow \quad x \mid y.$$

$\exists k \in \mathbb{Z} \ (kx = y)$

Then R is reflexive, not symmetric, but transitive.

Proof

(reflexivity) For each $a \in \mathbb{Z}^+$, we know $a = a \times 1$ and so $a \mid a$ by the definition of divisibility.

(non-symmetry) Note $1 \mid 2$ but $2 \nmid 1$.

(transitivity) Let $a, b, c \in \mathbb{Z}^+$ such that $a \mid b$ and $b \mid c$. Use the definition of divisibility to find $k, \ell \in \mathbb{Z}$ such that $b = ak$ and $c = b\ell$. Then

$$c = b\ell = (ak)\ell = a(k\ell)$$

where $k\ell \in \mathbb{Z}$. Thus $a \mid c$ by the definition of divisibility. $\square$

Exercises on reflexivity, symmetry, and transitivity

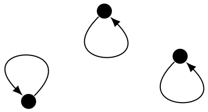
Exercise 6.1.9

Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$. View R as a relation on A. Is R reflexive? Is R symmetric? Is R transitive? ✎ 6c

Exercise 6.1.10

Let R be a relation on a set A. Prove that R is transitive if and only if $R \circ R \subseteq R$. ✎ 6d

Equivalence relations



Definition 6.1.11

An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Example 6.1.12

The equality relation on a set, as defined in Example 6.1.3, is an equivalence relation.

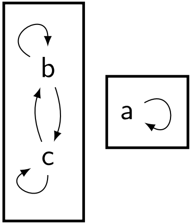
Convention 6.1.13

People usually use equality-like symbols such as $\sim$, $\approx$, $\simeq$, $\cong$, and $\equiv$ to denote equivalence relations. These symbols are often defined and redefined to mean different equivalence relations in different situations. We may read $\sim$ as "is equivalent to".

A finite equivalence relation

Example 6.1.14

Let R be the relation represented by the arrow diagram below.



- Then R is reflexive, symmetric and transitive.
- So it is an equivalence relation.

Equivalence classes

When there is no risk of confusion, we may drop the subscript.

Definition 6.2.1

Let $\sim$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x with respect to $\sim$, denoted $[x]_\sim$, is defined by

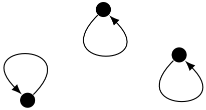
$$[x]_\sim = \{y \in A : x \sim y\}.$$

the set of all elements of A that x is $\sim$-related to

Example 6.2.2

Let A be a set. The equivalence classes with respect to the equality relation on A are of the form

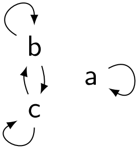
$$[x] = \{y \in A : x = y\} = \{x\}, \qquad \text{where } x \in A.$$



Example 6.2.3

If R is the equivalence relation represented by the arrow diagram on the right, then

$$[\mathrm{a}] = \{\mathrm{a}\} \quad \text{and} \quad [\mathrm{b}] = \{\mathrm{b}, \mathrm{c}\} = [\mathrm{c}].$$



Question

Is every element contained in a unique equivalence class?

Partitions

$S_1, S_2, \ldots$ are nonempty subsets of A such that every element of A is in exactly one S_i

Definition 6.3.1 and Remark 6.3.2 (special version)

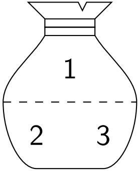
Call $\mathscr{C} = \{S_1, S_2, \ldots\}$ a *partition* of a set A if

(0) each S_i is a *nonempty* subset of A, i.e., $\forall i \; (\varnothing \neq S_i \subseteq A)$;

(1) every element of A is in some S_i, i.e., $\forall x \in A \; \exists i \; (x \in S_i)$; and

(2) if some S_i and S_j have a nonempty intersection, then they are equal, i.e., $\forall i, j \; (S_i \cap S_j \neq \varnothing \Rightarrow S_i = S_j)$.

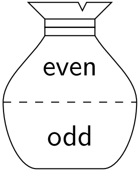
The S_i's are called the *components* of the partition.



Example 6.3.3

One partition of the set $A = \{1, 2, 3\}$ is $\{\{1\}, \{2, 3\}\}$. The others are

$$\{\{1\}, \{2\}, \{3\}\}, \quad \{\{2\}, \{1, 3\}\}, \quad \{\{3\}, \{1, 2\}\}, \quad \{\{1, 2, 3\}\}.$$



Example 6.3.4

One partition of $\mathbb{Z}$ is $\{\{2k : k \in \mathbb{Z}\}, \{2k + 1 : k \in \mathbb{Z}\}\}$.

Equivalence classes are nonempty

Lemma 6.3.5

Let $\sim$ be an equivalence relation on a set A.

(1) $x \in [x]$ for all $x \in A$.

(2) Any equivalence class is nonempty.

Proof

(1) Let $x \in A$. Then $x \sim x$ by reflexivity. So $x \in [x]$ by the definition of $[x]$.

(2) Any equivalence class is of the form $[x]$ for some $x \in A$, and so it must be nonempty by (1). $\square$

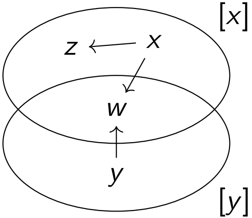
The intersection of two distinct equivalence classes is empty

Lemma 6.3.6

Let $\sim$ be an equivalence relation on a set A.
For all $x, y \in A$, if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$.

Proof

- Assume $[x] \cap [y] \neq \varnothing$.
- Say, we have $w \in [x] \cap [y]$, so that $w \in [x]$ and $w \in [y]$ by the definition of $\cap$.
- This means $x \sim w$ and $y \sim w$ by the definition of $[x]$ and $[y]$. $(*)$
- We show $[x] \subseteq [y]$.
  - Take $z \in [x]$.
  - Then $x \sim z$ by the definition of $[x]$.
  - By symmetry, we know from $(*)$ that $w \sim x$.
  - Altogether we have $y \sim w \sim x \sim z$.
  - So transitivity tells us $y \sim z$.
  - Thus $z \in [y]$ by the definition of $[y]$.
- Similarly, one can show $[y] \subseteq [x]$.
- Thus $[x] = [y]$. $\square$

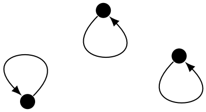


The equivalence classes form a partition

Definition 6.3.8

Let A be a set and $\sim$ be an equivalence relation on A. Denote by $A/\sim$ the set of all equivalence classes with respect to $\sim$, i.e.,

$$A/\sim = \{[x]_\sim : x \in A\}.$$



We may read $A/\sim$ as "the quotient of A by $\sim$".

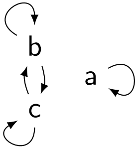
Example 6.3.9

Let A be a set. Then $A/=$ is equal to $\{\{x\} : x \in A\}$.

Example 6.3.10

If R is the equivalence relation on the set $A = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ represented by the arrow diagram on the right, then

$$A/\sim = \{[\mathrm{a}], [\mathrm{b}], [\mathrm{c}]\} = \big\{\{\mathrm{a}\}, \{\mathrm{b}, \mathrm{c}\}, \{\mathrm{b}, \mathrm{c}\}\big\} = \big\{\{\mathrm{a}\}, \{\mathrm{b}, \mathrm{c}\}\big\}.$$

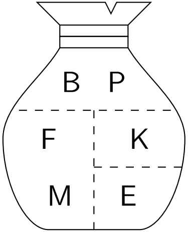


Theorem 6.3.11

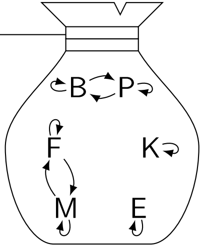
Let $\sim$ be an equivalence relation on a set A. Then $A/\sim$ is a partition of A.

Informal descriptions of the terms

1.	underlying set	A	the set to be "partitioned"
2.	components	S	subsets of A, mutually disjoint, together union to A
3.	partition	$\mathscr{C}$	the set of all components
4.	same-component relation	$\sim$	equivalence relation



1.	underlying set	A	the set of all vertices
2.	relation	R	the set of all arrows
3.	equivalence relation	$\sim$	If ignoring directions of arrows one can walk from x to y, then there is an arrow from x to y.
4.	equivalence classes	$[x]$, where $x \in A$	connected components
5.	quotient	$A/\sim$	the set of all connected components



Quick check

Question 6.3.7

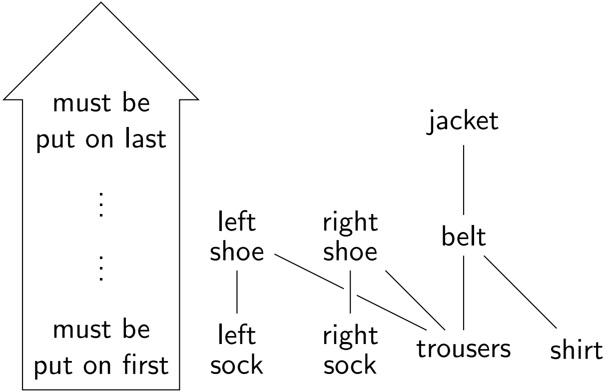
Consider an equivalence relation. Is it true that if x is an element of an equivalence class S, then $S = [x]$?

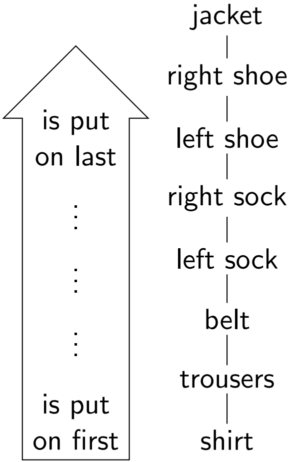
6e

Motivating examples of partial orders

(1) the "*must be* done before (or at the same time as)" relation on the set of all tasks

(2) the "*is* done before (or at the same time as)" relation on the set of all tasks





Motivating examples of partial orders: a closer look

(1) the "*must be* done before (or at the same time as)" relation on the set of all tasks

(2) the "*is* done before (or at the same time as)" relation on the set of all tasks

- Each such relation has two versions: one with the parenthetical phrase, and one without. They have the same mathematical content. We focus on the former.
- So all such relations are reflexive and transitive.
- No multi-tasking is allowed, i.e., if R is one of the relations above, then

$$\forall x, y\ (\, x\, R\, y \wedge y\, R\, x \quad \Rightarrow \quad x = y\,). \qquad \text{(antisymmetry)}$$

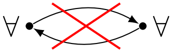
- There may be x, y such that x need not be done before y, and y need not be done before x, i.e., maybe $\exists x, y\ (x \not R\ y \wedge y \not R\ x)$ if R is the relation in (1). (partiality)
- However, as time is linear and there is no multi-tasking, for all tasks x, y, either x is done before or at the same time as y, or y is done before or at the same time as x, i.e., $\forall x, y\ (x\, R\, y \vee y\, R\, x)$ if R is the relation in (2). (totality)
- Here "partiality" means "possibly partial", while "total" means "necessarily total".

Partial orders

Definition 6.4.1

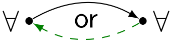
Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A\ (x\,R\,y \land y\,R\,x \Rightarrow x = y)$.



(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

(4) R is a *(non-strict)* $\underbrace{\textit{total order}}_{\textit{linear order}}$ if R is a partial order and $\underbrace{\forall x, y \in A\ \underbrace{(x\,R\,y \lor y\,R\,x)}_{x \text{ and } y \text{ are } \textit{comparable}}}_{\text{any two elements are comparable}}$.



Note 6.4.2

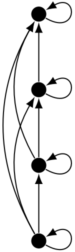
A total order is always a partial order.

Example 6.4.3

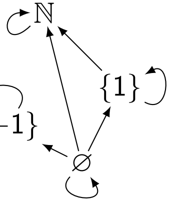
Let R denote the non-strict less-than relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x\,R\,y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is antisymmetric. In fact, it is a total order.



Further examples of partial orders



Example 6.4.4

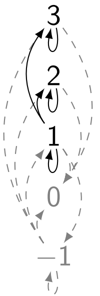
Let R_0 denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \, R_0 \, y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R is antisymmetric. It is always a partial order, but it may not be a total order.

{-1} is not a subset of {1}
neither is {1} a subset of {-1}

Example 6.4.5



Let R_1 denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

there exists a k : kx = y

$$x \, R_1 \, y \quad \Leftrightarrow \quad x \mid y.$$

NOT antisymmertic as -1 | 1 and 1 | -1 but 1 =/= -1

Is R_1 antisymmetric? Is R_1 a partial order? Is R_1 a total order? ✎ 6f

Example 6.4.6

Let R_2 denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x \, R_2 \, y \quad \Leftrightarrow \quad x \mid y.$$

2 does not divide 3 &
3 does not divide 2

Is R_2 antisymmetric? Is R_2 a partial order? Is R_2 a total order? ✎ 6g

Well-Ordering Principle

$1 \in S$ and $1 \leqslant x$ for all $x \in S$

Example 6.4.8

(1) $S = \{x \in \mathbb{Z}_{\geqslant 0} : 0 < x < 5\}$ has smallest element 1.

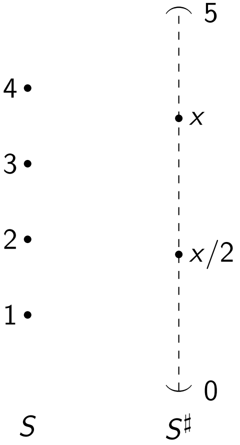
(2) $S^\sharp = \{x \in \mathbb{Q}_{\geqslant 0} : 0 < x < 5\}$ has no smallest element because if $x \in S^\sharp$, then $x/2 \in S^\sharp$ and $x/2 < x$.

Theorem 6.4.9 (Well-Ordering Principle)

Let $b \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}_{\geqslant b}$. If $S \neq \varnothing$, then S has a smallest element.

Why the Well-Ordering Principle

- It is useful in proving the termination of algorithms.
- It can be used as an alternative to induction.



Well-Ordering Principle — proof

(Extra) Combine the two steps. ✐ 6h

Theorem 6.4.9 (Well-Ordering Principle)

Let $b \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}_{\geqslant b}$. If $S \neq \varnothing$, then S has a smallest element.

$$\begin{array}{rl} & \vdots \\ k+1 \bullet & \notin S \\ k \bullet & \notin S \\ & \vdots \\ b+1 \bullet & \notin S \\ b \bullet & \notin S \end{array}$$

Proof

We prove the contrapositive. Assume that S has no smallest element. As $S \subseteq \mathbb{Z}_{\geqslant b}$, it suffices to show that $\underbrace{n \notin S}_{P(n)}$ for all $n \in \mathbb{Z}_{\geqslant b}$. We prove this by Strong MI on n. (Here $c = b$.)

(Base step: show that $\overbrace{P(b), P(b+1), \ldots, P(c)}^{P(b)}$ are true.) If $b \in S$, then b is the smallest element of S because $S \subseteq \mathbb{Z}_{\geqslant b}$, which contradicts our assumption. So $b \notin S$.

(Induction step: show that $\overbrace{\forall k \in \mathbb{Z}_{\geqslant c} \ \big(P(b) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)\big)}^{\forall k \in \mathbb{Z}_{\geqslant b} \ (P(b) \wedge \cdots \wedge P(k) \Rightarrow P(k+1))}$ is true.) Let $k \in \mathbb{Z}_{\geqslant b}$ such that $b, \ldots, k \notin S$. If $k+1 \in S$, then $k+1$ is the smallest element of S because $S \subseteq \mathbb{Z}_{\geqslant b}$, which contradicts our assumption. So $k+1 \notin S$.

Hence $\forall n \in \mathbb{Z}_{\geqslant b} \ P(n)$ by Strong MI. $\square$

Summary

Let A be a set and R be a relation on A.

Definition 6.3.1. A *partition* of A is a set $\mathscr{C}$ of *nonempty* subsets of A such that

$$\forall x \in A \ \ \exists! S \in \mathscr{C} \ \ (x \in S).$$

Definitions 6.1.1 and 6.1.11. Call R an *equivalence relation* on A if

- (reflexivity) $\forall x \in A \quad (x\, R\, x)$;
- (symmetry) $\forall x, y \in A \quad (x\, R\, y \Rightarrow y\, R\, x)$; and
- (transitivity) $\forall x, y, z \in A \quad (x\, R\, y \wedge y\, R\, z \Rightarrow x\, R\, z)$.

Theorem 6.3.11. The equivalence classes with respect to an equivalence relation on A form a partition of A.

Definition 6.4.1(1,2). Call R a *partial order* on A if R is reflexive, transitive, and

- (antisymmetry) $\forall x, y \in A \quad (x\, R\, y \wedge y\, R\, x \Rightarrow x = y)$.

Definition 6.4.1(4). A partial order R on A is a *total order* if $\forall x, y \in A \ \ (x\, R\, y \vee y\, R\, x)$.

Well-Ordering Principle. Let $b \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}_{\geqslant b}$. If $S \neq \varnothing$, then S has a smallest element.