Chapter 9: Countability

CS1231 Discrete Structures

Wong Tin Lok

National University of Singapore

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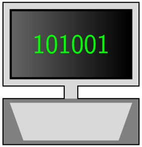
> The following are advantages of digital representation of numerical values compared to analog representation:
>
> 1. Digital representation is more accurate.
> 2. Digital information are easier to store.
> 3. Digital systems are easier to design.
> 4. Noise has less effect.
> 5. Digital systems can easily be fabricated in an integrated circuit.
>
> Elahi (2019)

Plan



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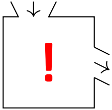
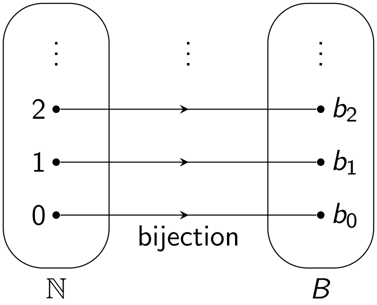
- countable sets $\approx$ sets whose sizes (digital) computers can handle
- countability
- uncountable sets
- non-computability



Countability



Definition 9.1.1 (Cantor)

A set is *countable* if it is finite or it has the same cardinality as $\mathbb{N}$.

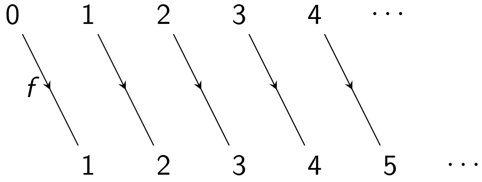
Note 9.1.2

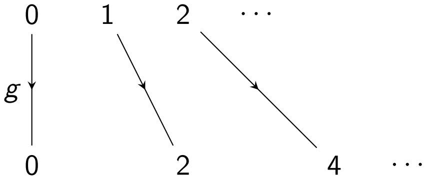
Some authors allow only infinite sets to be countable.

Example 9.1.3

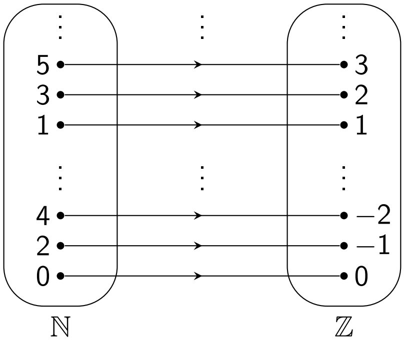
(1) $\mathbb{N}$ is countable by Proposition 8.2.4(1).

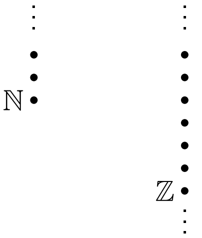
(2) Both $\mathbb{N} \setminus \{0\}$ and $\mathbb{N} \setminus \{1, 3, 5, \ldots,\}$ are countable.





Bi-infinite sequence





Proposition 9.1.4

$\mathbb{Z}$ is countable.

Proof sketch

Let $f\colon \mathbb{N} \to \mathbb{Z}$ such that $f(0), f(1), f(2), f(3), f(4), f(5), \ldots$ are respectively

$$0, +1, -1, +2, -2, +3, -3, +4, -4, +5, -5, +6, -6, \ldots \qquad (*)$$

- f is surjective because every element of $\mathbb{Z}$ appears in $(*)$ at some position $n \in \mathbb{N}$.
- f is injective because numbers in different positions of $(*)$ are different.

So f is a bijection. This shows $\mathbb{Z}$ is countable. $\square$

A bijection $f\colon \mathbb{Z} \to \mathbb{N}$

For each $x \in \mathbb{Z}$, set

$$f(x) = \begin{cases} 2x, & \text{if } x \geqslant 0; \\ 2(-x-1)+1, & \text{if } x < 0. \end{cases}$$

This definition indeed assigns to each element $x \in \mathbb{Z}$ an element $f(x) \in \mathbb{N}$ because if $x \geqslant 0$, then $2x \geqslant 0$ as well; and if $x < 0$, then $x \leqslant -1$ as $x \in \mathbb{Z}$, and so $2(-x-1)+1 \geqslant 2(-(-1)-1)+1 = 1 > 0$. It suffices to show that f is bijective.

To show surjectivity, pick any $y \in \mathbb{N}$. Note that y is either even or odd. If y is even, say $y = 2x$ where $x \in \mathbb{Z}$, then $x = y/2 \geqslant 0$, and so $f(x) = 2x = y$. If y is odd, say $y = 2x+1$ where $x \in \mathbb{Z}$, then

$$x + 1 = \frac{y-1}{2} + 1 \geqslant \frac{0-1}{2} + 1 = \frac{1}{2} > 0,$$

and so $f(-x-1) = 2(-(-x-1)-1)+1 = 2x+1 = y$. Thus some $x \in \mathbb{Z}$ makes $f(x) = y$ in all cases.

To show injectivity, pick $x_1, x_2 \in \mathbb{Z}$ such that $f(x_1) = f(x_2)$. If $f(x_1)$ is even, then $f(x_1) = 2x_1$ and $f(x_2) = 2x_2$ because no integer is both even and odd, and so $x_1 = x_2$. If $f(x_1)$ is odd, then $f(x_1) = 2(-x_1-1)+1$ and $f(x_2) = 2(-x_2-1)+1$ for a similar reason, and so $x_1 = x_2$. Thus $x_1 = x_2$ in all cases.

Semi-infinite grid

Theorem 9.1.5 (Cantor 1877)

$\mathbb{N} \times \mathbb{N}$ is countable.

Proof sketch

The function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that $f(0), f(1), f(2), \ldots$ are respectively

$$(0,0), (1,0), (0,1),$$
$$(2,0), (1,1), (0,2),$$
$$(3,0), (2,1), (1,2), (0,3), \ldots$$

following the arrows in the right diagram is a bijection. This shows $\mathbb{N} \times \mathbb{N}$ is countable. $\square$

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & \cdots \\
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}
$$

Strings of finite lengths

1	0	$\cdots$	1

Proposition 9.1.6

$\{0,1\}^*$ is countable.

Proof sketch

Let $f : \mathbb{N} \to \{0,1\}^*$ such that $f(0), f(1), f(2), \ldots$ are respectively

$$\underbrace{\varepsilon}_{\text{length } 0},\ \underbrace{0, 1}_{\text{length } 1},\ \underbrace{00, 01, 10, 11}_{\text{length } 2},\ \underbrace{000, 001, 010, 011, 100, 101, 110, 111}_{\text{length } 3},\ \ldots.$$

Then f is a bijection. This shows $\{0,1\}^*$ is countable. $\square$

Guess: which of the following sets is/are countable?

countable means computer programs can run

Not all of these are within the scope of this module.

(1) $\mathbb{Z}$ countable

(2) $\mathbb{Q}$ countable fractions can be represented as (x,y)

(3) $\mathbb{R}$ not countable

(4) $\mathbb{C}$ not countable

(5) the set of all finite sets of integers countable e.g. arrays

(6) the set of all strings over $\{0, 1\}$ countable

(7) the set of all infinite sequences over $\{0, 1\}$ not countable

(8) the set of all functions $A \to B$ where A, B are finite sets of integers countable

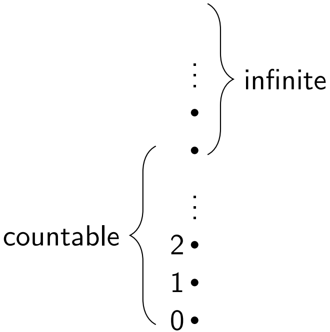
(9) the set of all computer programs countable

Lemma 9.2.1

Let A and B be sets of the same cardinality. Then A is countable if and only if B is countable.

✐ 9a

Countable infinity is the smallest infinity



Proposition 9.2.4

Every infinite set B has a countable infinite subset.

Proof sketch

Run the following procedure.

1. Initialize $i = 0$.
2. While $B \setminus \{g_1, g_2, \ldots, g_i\} \neq \varnothing$ do:
   - 2.1. Pick any $g_{i+1} \in B \setminus \{g_1, g_2, \ldots, g_i\}$.
   - 2.2. Increment i to $i + 1$.

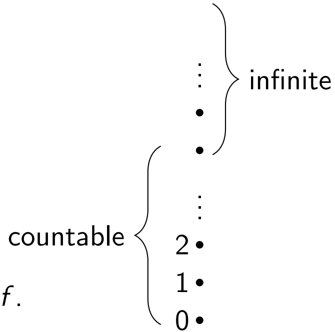
This procedure cannot stop because otherwise $B = \{g_1, g_2, \ldots, g_\ell\}$ for some $\ell \in \mathbb{N}$, and so it is not infinite.

Define $A = \{g_i : i \in \mathbb{Z}^+\}$, and $g\colon \mathbb{N} \to A$ by setting $g(i) = g_{i+1}$ for each $i \in \mathbb{N}$. As g is a bijection $\mathbb{N} \to A$, we deduce that A is countable. □

Example 9.2.5

Knowing that $\mathbb{R} \setminus \mathbb{Q}$ is infinite tells us $\mathbb{R} \setminus \mathbb{Q}$ has a countable infinite subset.

Countable cardinalities are the smallest cardinalities



Proposition 9.2.6

(1) Any subset A of a finite set B is finite. 9b

(2) Any subset A of a countable set B is countable.

Proof sketch of (2) when B is infinite

As B is countable and infinite, there is a bijection $\mathbb{N} \to B$, say f.
Run the following procedure.

1. Initialize $i = 0$.
2. While $A \setminus \{g_1, g_2, \ldots, g_i\} \neq \varnothing$ do:
   - 2.1. Let m_{i+1} be the smallest element in $\{m \in \mathbb{N} : f(m) \in A \setminus \{g_1, g_2, \ldots, g_i\}\}$.
   - 2.2. Set $g_{i+1} = f(m_{i+1})$.
   - 2.3. Increment i to $i + 1$.

the procedure stops

Then the function mapping each i to g_i is a bijection $\mathbb{Z}^+ \to A$ or $\{1, 2, \ldots, \ell\} \to A$ for some $\ell \in \mathbb{N}$. This shows A is countable. $\square$

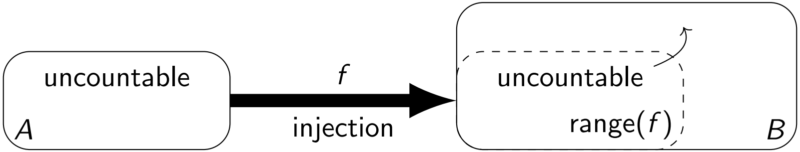
the procedure does not stop

Injecting an uncountable set into another set

Corollary 9.2.7

(1) A set B is infinite if there is an injection f from some infinite set A to B. ✐ 9c

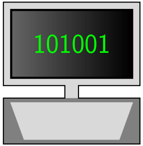
(2) A set B is uncountable if there is an injection f from some uncountable set A to B.



Proof of (2)

As f is an injection $A \to B$, Exercise 8.2.5 implies that A has the same cardinality as $\text{range}(f)$. Since A is uncountable, Lemma 9.2.1 tells us $\text{range}(f)$ is also uncountable. Hence B is uncountable by Proposition 9.2.6(2) because $\text{range}(f) \subseteq B$. □

There are countably many programs



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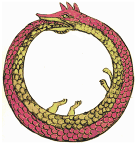
Example 9.2.8

The set of all programs is countable.

Proof

- Every program is stored in a computer as a string over $\{0, 1\}$.
- So the set of all programs is a subset of the set $\{0, 1\}^*$ of all strings over $\{0, 1\}$, which we know is countable from Proposition 9.1.6.
- So Proposition 9.2.6(2) tells us that the set of all programs is countable. $\square$

A powerful set operation



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Theorem 9.3.1 (when $A = \mathbb{N}$, Cantor 1891)

$\mathbb{N}$ does not have the same cardinality as $\mathcal{P}(\mathbb{N})$.

Proof

Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Define $R = \{x \in \mathbb{N} : x \notin f(x)\}$.

Suppose we have $a \in \mathbb{N}$ such that $R = f(a)$. From the definition of R, we know

$$\forall x \in \mathbb{N} \ \big(x \in R \quad \Leftrightarrow \quad x \notin f(x)\big). \qquad (*)$$

As $R = f(a)$, applying $(*)$ to $x = a$ gives

$$a \in f(a) \quad \Leftrightarrow \quad a \notin f(a).$$

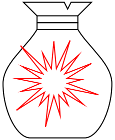
This is a contradiction.

Hence $R \in \mathcal{P}(\mathbb{N})$ such that $R \neq f(x)$ for any $x \in \mathbb{N}$. This shows f is not surjective. $\square$

Exercise

Imitate this proof to show that no set A has the same cardinality as $\mathcal{P}(A)$. 9d

An uncountable set



Corollary 9.3.2 (when $A = \mathbb{N}$)

$\mathcal{P}(\mathbb{N})$ is uncountable.

Proof

According to the definition of countability, we need to show that $\mathcal{P}(\mathbb{N})$ is infinite, and that $\mathcal{P}(\mathbb{N})$ does not have the same cardinality as $\mathbb{N}$. We already have the latter from Theorem 9.3.1. For the former, we proceed as follows.

- Let $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ defined by setting $f(n) = \{n\}$ for each $n \in \mathbb{N}$.
- Then f is injective because if $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$, then $\{n_1\} = \{n_2\}$, and thus $n_1 = n_2$ by the definition of set equality.
- Recall that $\mathbb{N}$ is infinite from Exercise 8.2.7.
- So Corollary 9.2.7(1) tells us $\mathcal{P}(\mathbb{N})$ is infinite too, as required. $\square$

Exercise

Imitate this proof to show that $\mathcal{P}(A)$ is uncountable for all countable infinite sets A. 9e

Noncomputable sets

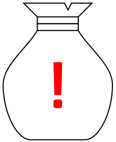
uncountably many

Assumption 9.4.1. Our programs have no time and memory limitation.

Corollary 9.4.2

There is a subset S of $\mathbb{N}$ that is not computed by any program, countably many
i.e., no program can, when given any input $n \in \mathbb{N}$,

$$\text{output T if } n \in S \quad \text{and} \quad \text{output F if } n \notin S.$$



Proof

- Suppose that every subset $S \subseteq \mathbb{N}$ is computed by a program.
- For each $S \in \mathcal{P}(\mathbb{N})$, let $f(S)$ be the smallest program that computes S.
- This defines a function $f\colon \mathcal{P}(\mathbb{N}) \to \{0,1\}^*$, because each program has a unique representation by an element of $\{0,1\}^*$ within a computer.
- This function f is injective because if $S_1, S_2 \in \mathcal{P}(\mathbb{N})$ such that $f(S_1) = f(S_2)$, then S_1 and S_2 are computed by the same program, and thus $S_1 = S_2$.
- Recall from Corollary 9.3.2 that $\mathcal{P}(\mathbb{N})$ is uncountable.
- So Corollary 9.2.7(2) implies $\{0,1\}^*$ is uncountable as well.
- This contradicts the countability of $\{0,1\}^*$ from Proposition 9.1.6. $\square$

The Halting Problem

EXTRA MATERIAL

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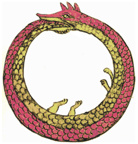

Theorem 9.4.3 (Turing 1936)

There is no program that computes

$$H = \{\sigma \in \{0,1\}^* : \sigma \text{ is a program that does not stop on the empty input}\},$$

i.e., no program can, when given any input $\sigma \in \{0,1\}^*$,

$$\text{output T if } \sigma \in H \quad \text{and} \quad \text{output F if } \sigma \notin H.$$

Proof

Suppose not. Use a program that computes H to devise a program R satisfying

$$\forall \sigma \in \{0,1\}^* \ \left(R \text{ stops on input } \sigma \quad \Leftrightarrow \quad \begin{array}{l} \sigma \text{ is a program that does} \\ \text{not stop on input } \sigma \end{array} \right). \qquad (*)$$

Applying $(*)$ to $\sigma = R$ gives

$$R \text{ stops on input } R \quad \Leftrightarrow \quad \begin{array}{l} R \text{ is a program that does} \\ \text{not stop on input } R. \end{array}$$

We have a contradiction. $\square$

Summary

Definition 9.1.1 A set is *countable* if it is finite or it has the same cardinality as $\mathbb{N}$.

Proposition 9.2.4 Every infinite set has a countable infinite subset.

Proposition 9.2.6(2) Any subset of a countable set is countable.

Corollary 9.2.7(2) A set B is uncountable if there is an injection from some uncountable set to B.

Examples of countable sets: finite sets, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, $\{0,1\}^*$, the set of all computer programs

Corollary 9.3.2 $\mathcal{P}(\mathbb{N})$ is uncountable.

Corollary 9.4.2 There is a subset of $\mathbb{N}$ that is not computed by any program.