CS1231 Discrete Structures
Semester 2, 2022/2023
Question 9.4

Question 1. Suppose $A_0, A_1, \ldots$ are countable sets. We make two further assumptions.

(a) For each i, A_i is infinite.

(b) For each $i \neq j$, $A_i \cap A_j = \emptyset$.

Then $\bigcup_{i=0}^{\infty} A_i$ is countable.

Proof. For each A_i, there exists a bijection $f_i : \mathbb{N} \to A_i$ by

$$f_i(j) = a_{ij}.$$

Define $g : \bigcup_{i=0}^{\infty} A_i \to \mathbb{N} \times \mathbb{N}$ by

$$g(a_{ij}) = (i, j).$$

We make the following claims:

(i) $\underline{g \text{ is well-defined:}}$ We need to establish (F1) and (F2) for the relation

$$g := \{(a_{ij}, (i, j)) : (i, j) \in \mathbb{N} \times \mathbb{N}\}.$$

For (F1), fix $x \in \bigcup_{i=0}^{\infty} A_i, (i, j)$. Thus, $x \in A_i$ for some $i \in \mathbb{N}$. Since $x \in A_i$ and $f_i : \mathbb{N} \to A_i$ is a bijection, there exists $j \in \mathbb{N}$ such that

$$x = f_i(j) = a_{ij}.$$

Then

$$(x, (i, j)) = (a_{ij}, (i, j)) \in g.$$

For (F2), suppose $(x, (i, j)), (x, (r, s)) \in g$. Then $x = a_{ij} = a_{rs}$. We claim that $i = r, j = s$.

- Suppose $i \neq r$. Then $a_{ij} \in A_i$ and $A_{rs} \in A_r$. Thus,

$$x \in A_i \cap A_r = \emptyset,$$

a contradiction. Therefore $i = r$. Hence, $a_{rs} = a_{is} \in A_i$.

- Since $f_i(j) = a_{ij} = a_{is} = f_i(s)$ and f_i is injective, $j = s$, as required.

Hence, $(i, j) = (r, s)$.

(ii) $\underline{g \text{ is injective:}}$ Suppose $g(a_{ij}) = g(a_{rs})$. Then $(i, j) = g(a_{ij}) = g(a_{rs}) = (r, s)$.

□

Question 2. Suppose $A_0, A_1, \ldots$ are countable sets. We make one assumption.

(a) For each i, A_i is infinite.

Then $\bigcup_{i=0}^{\infty} A_i$ is countable.

Proof. For each i, define the sets

$$B_i = A_i \backslash \left(\bigcup_{j=0}^{i-1} A_j \right).$$

We can check that

$$\bigcup_{i=0}^{\infty} A_i = \bigcup_{i=0}^{\infty} B_i,$$

and that $B_i \cap B_j \neq \emptyset$ whenever $i \neq j$. Therefore, $\bigcup_{i=0}^{\infty} B_i$ is countable by the previous result. Therefore, $\bigcup_{i=0}^{\infty} A_i$ is countable. □

Question 3. Suppose $A_0, A_1, \ldots$ are countable sets. We make no further assumption. Then $\bigcup_{i=0}^{\infty} A_i$ is countable.

Proof. For each i, there exists a countable infinite set A_i^* such that $A_i \subseteq A_i^*$. Taking unions, we can verify that

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} A_i^*.$$

By the previous result, the RHS is countable. Therefore, the LHS $\bigcup_{i=1}^{\infty} A_i$ is countable. □