NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 1: AY2022/23)

Final Assessment Answer Sheet

Time Allowed: 2 Hours

INSTRUCTIONS

STUDENT NUMBER

A

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A
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NT

1. Write your **Student Number** on the right AND, using pen or pencil, shade the corresponding circle **completely** in the grid for each digit or letter. DO NOT WRITE YOUR NAME!
2. Zero mark will be given if you write/shade your Student Number incompletely or incorrectly.
3. Write your Student Number at the top of pages 3 and 5.
4. This answer sheet comprises **SIX (6) pages**.
5. All questions must be answered in the space provided; no extra sheets will be accepted as answers.
6. You must submit only this **ANSWER SHEET** and no other documents.
7. An excerpt of the question may be provided to aid you in answering in the correct box. It is not the exact question. You should still refer to the original question in the question paper.
8. You may write your answers using pencil (at least 2B) or pen as long as it is legible (no red ink, please).
9. The maximum mark for this paper is 100.
10. **Marks may be deducted** for (i) illegible handwriting, and/or (ii) excessively long explanations.
11. Each multiple choice question is intended to have only one answer. Shade the appropriate bubbles <u>using pencil only</u>.

For Examiner's Use Only		
Question	**Marks**	**Remarks**
Q1-20	/ 40	
Q21	/ 4	
Q22	/ 20	
Q23	/ 20	
Q24	/ 10	
Q25	/ 6	
Total	**/ 100**	

Part A: Multiple Choice Questions (Total: 40 marks)

Please shade only ONE bubble for each question. Please use ONLY pencil to shade.

	(A)	(B)	(C)	(D)	(E)
1.	●	○	○	○	○
2.	●	○	○	○	○
3.	○	○	○	●	○
4.	○	○	●	○	○
5.	●	○	○	○	○
6.	○	○	○	○	○
7.	○	●	○	○	○
8.	○	○	●	○	○
9.	○	●	○	○	○
10.	○	○	●	○	○
11.	○	○	○	●	○
12.	○	●	○	○	○
13.	○	○	○	●	○
14.	○	○	○	●	○
15.	●	○	○	○	○
16.	○	○	●	○	○
17.	○	○	○	○	●
18.	○	●	○	○	○
19.	○	○	●	○	○
20.	○	○	○	●	○
	(A)	(B)	(C)	(D)	(E)

Part B (Total: 60 marks)

21. **Mathematical induction.** [4 marks]

1. For each $n \in \mathbb{N}$, let $P(n)$ be the proposition $\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}$.

2. (Basis step)

   2.1. When $n = 0$, $f_0^2 = 0^2 = 0 = 0 \cdot 1 = f_0 f_1$.

   2.2. So $P(0)$ is true.

3. (Induction step)

   3.1. Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e. $\sum_{i=0}^{k} f_i^2 = f_k f_{k+1}$.

   3.2. Then $\sum_{i=0}^{k+1} f_i^2 = \sum_{i=0}^{k} f_i^2 + f_{k+1}^2$

   $= f_k f_{k+1} + f_{k+1}^2$ (by the induction hypothesis)

   3.3. $= f_{k+1}(f_k + f_{k+1})$ (by basic algebra)

   3.4. $= f_{k+1} f_{k+2}$ (by the definition of Fibonacci sequence)

   3.5. Thus $P(k+1)$ is true.

Therefore, $\forall\, n \in \mathbb{N}\ P(n)$ is true by Mathematical Induction.

22. **Counting and probability.** [20 marks]

(a) [2] **15**

(b) [2] **0.5**

(c) [2] **5/12**

(d) [2] **70**

(e) [2] **14**

(f) (i) [3] **0.375**

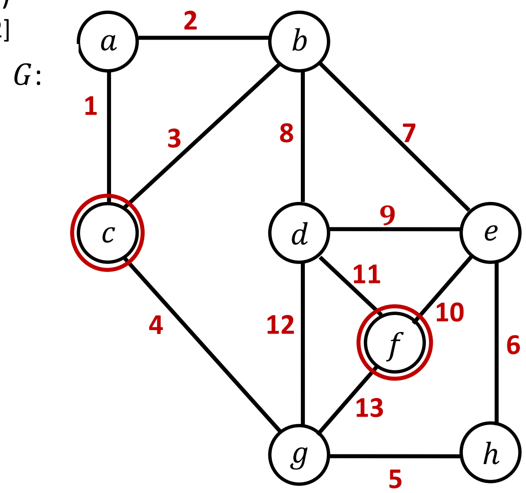
(ii) [2] **0.4**

(g) (i) [2] **6**

(ii) [3] $k!\binom{n}{k}$ or $\boldsymbol{n(n-1)(n-2)\cdots(n-k+1)}$

23. **Graphs and trees.** [20 marks]

(a)
[2]



(b)
[2]

Vertex c or f.

(c)
[2]

True

(d)
[2]

True

(e)
[2]

E D B F G K H A C

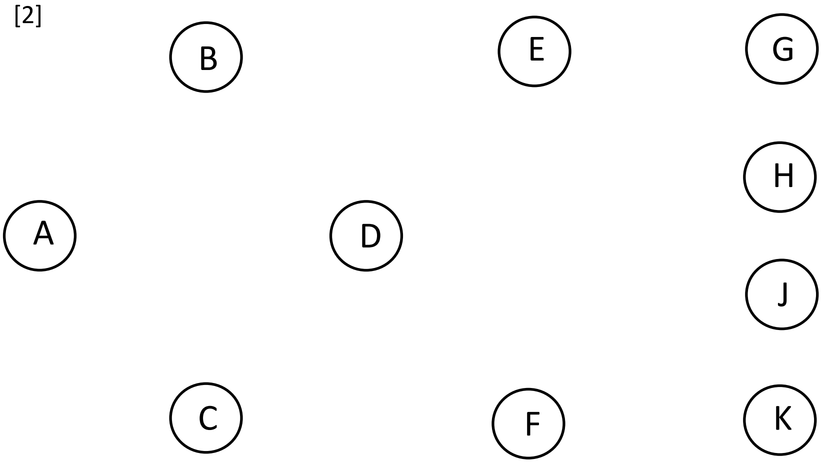
(f)
[3]

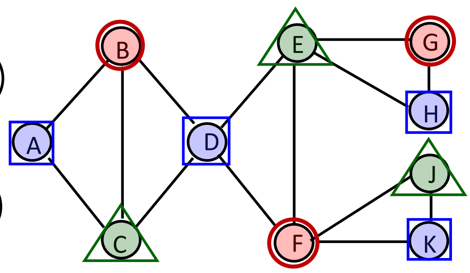
The vertices represent **the (volatile) items.**

Two vertices are adjacent when **the items they represent are conflicting.**

Vertices of the same colour **means that the items they represent can be put into the same container.**

(g) & (h)
[3] [2]





(i)
[2]

Packing solution: {B,F,G}, {A,D,H,K}, and {C,E,J}. 3 colours are sufficient.

24. **Functions and relations.** [10 marks]

(a)
[2]

$\boldsymbol{f(x) = x; g(x) = -x; h(x) = f(x) + g(x) = x - x = 0.}$

Therefore $h(x)$ is not injective (as $h(1) = h(2) = 0$).

(b)
[3]

True.

1. Let $x_1, x_2 \in \mathbb{R}$.
2. Suppose $x_2 > x_1$.
   2.1. Since g is a function, $\exists y_1, y_2 \in \mathbb{R}\ (y_1 = g(x_1) \wedge y_2 = g(x_2))$.
   2.2. Since g is an increasing function, $y_2 > y_1$.
   2.3. Since h is a function, $\exists z_1, z_2 \in \mathbb{R}\ (z_1 = h(y_1) \wedge z_2 = h(y_2))$.
   2.4. Since h is a decreasing function, $z_2 < z_1$.
   2.5. Therefore, $h \circ g(x_2) = h(g(x_2)) = h(y_2) = z_2 < z_1 = h(y_1)$
   $= h(g(x_1)) = h \circ g(x_1)$.
3. Thus $h \circ g$ is a decreasing function.

(c)
[5]

1. Define a function $f: \mathbb{R} \to \mathbb{R}$ by setting, for all $x \in \mathbb{R}$, $f(x) = -x$.
2. (f is injective)
   2.1. Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$.
   2.2. Then $-x = -y$ (by the definition of f).
   2.3. Then $x = y$ (by basic algebra), and hence f is injective.
3. (f is surjective)
   3.1. Let $y \in \mathbb{R}$.
   3.2. Choose $x = -y$, then we have $f(x) = -(-y) = y$, and hence f is surjective.
4. Hence from lines 2 and 3, f is a bijection.
5. ($\Rightarrow$) Let $x, y \in \mathbb{R}$ such that $x \leq y$.
   5.1. Then $-x \geq -y$ (by basic algebra).
   5.2. Then $f(x) \geq f(y)$ (by the definition of f).
6. ($\Leftarrow$) Let $x, y \in \mathbb{R}$ such that $f(x) \geq f(y)$.
   6.1. Then $-x \geq -y$ (by the definition of f).
   6.2. Then $x \leq y$ (by basic algebra).
7. Hence from lines 5 and 6, $(\mathbb{R}, \leq) \sim (\mathbb{R}, \geq)$.

(Continue on to next page if necessary.)

25. **Cardinality.** [6 marks]

(a)
[2]

False.

1. A is not finite because B is an infinite subset of A.
   (The argument that "if A is finite, then $|A| < |A \times A|$" is not correct as this misses out the cases where $|A| = 0$ and $|A| = 1$.)
2. If A is countably infinite, then by Theorem 7.4.3 (Any subset of a countable set is countable), then B is also countably infinite since B is infinite and $B \subseteq A$.
   2.1. Then $|A| = |B|$ so there is a bijection between A and B, which contradicts the given statement.

(b)
[4]

1. Consider the function $p: Z \to A$ given by $p(x, y) = x$, for $x, y \in A$.
2. Note that p is not surjective.
   2.1. Suppose not, that is, p is surjective, then $p(Z) = A$.
   2.2. By part (a), A is uncountable, so $p(Z)$ is also uncountable.
   2.3. Since Z is countable, by Lemma 9.2, there is a sequence $z_0, z_1, z_2, \cdots$ in which every element of Z appears.
   2.4. Now, $p(z_0), p(z_1), p(z_2), \cdots$ is a sequence in which every element of $p(Z)$ appears.
   2.5. Therefore, $p(Z)$ is countable (by Lemma 9.2), which contradicts 2.2.
3. One then can find $x \in A$ such that x is not in the image of p, i.e., $(x, y) \notin Z$ for all $y \in A$.
4. By line 3, it follows that $\{x\} \times A \subseteq (A \times A) \setminus Z$.

=== END OF PAPER ===