Chapter 10: Counting

CS1231 Discrete Structures

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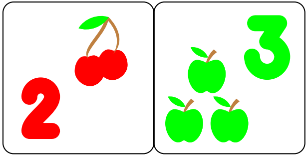
2022/23 Semester 2



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How to count without counting?

Kac (1965)

Counting the elements of a finite set

Counting for computer science

- to assess the resources (e.g., time and memory) used by algorithms

Technique 10.3.16 (counting argument)

One way to prove $k = \ell$, where $k, \ell \in \mathbb{N}$, is to find two ways of counting the number of elements of the same finite set, the first of which results in k and the second results in ℓ.

Plan

- sums and differences
- products and powers
- permutations and combinations

Unions of finite sets

Proposition 10.1.1

Let $A_0, A_1, A_2, \ldots$ be finite sets. Then $A_0 \cup A_1 \cup \cdots \cup A_n$ is finite for all $n \in \mathbb{N}$.

Proof

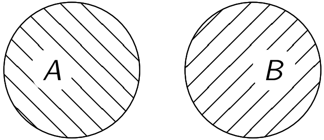
Proceed by induction on n as in Tutorial Exercise 9.3 using Tutorial Exercise 9.1. $\square$

combinatorics $\approx$ mathematics of counting

Combinatorial interpretation of Proposition 10.1.1

If x is a variable that can only take an element of one of finitely many finite sets, then there are finitely many ways to substitute objects into x.

The Addition Rule



Example 10.1.2

The sets $\{1,2\}$ and $\{3,4,5\}$ are disjoint. Note that

$$|\{1,2\}\cup\{3,4,5\}| = |\{1,2,3,4,5\}| = 5 = 2+3 = |\{1,2\}| + |\{3,4,5\}|.$$

Proposition 10.1.3 (Addition Rule)

Let A and B be disjoint finite sets. Then $|A\cup B| = |A| + |B|$.

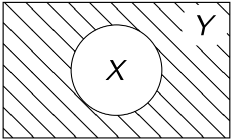
Proof

Proceed as in Tutorial Exercise 9.1. $\square$

Combinatorial interpretation of the Addition Rule

Let $m, n \in \mathbb{N}$. Suppose that the variable x can only take either one of m objects or one of n objects, and that these m objects are different from the n objects here. Then there are exactly $m+n$ ways to substitute objects into x.

The Difference Rule



Example 10.1.4

Note that $\{1,2\} \subseteq \{1,2,3,4,5\}$. Also

$$|\{1,2,3,4,5\} \setminus \{1,2\}| = |\{3,4,5\}| = 3 = 5 - 2 = |\{1,2,3,4,5\}| - |\{1,2\}|.$$

Corollary 10.1.5 (Difference Rule)

Let X and Y be finite sets. Then $Y \setminus X$ is finite, and if $X \subseteq Y$, then $|Y \setminus X| = |Y| - |X|$.

Proof sketch

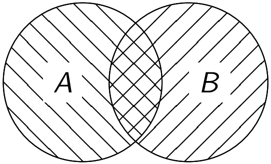
$$\begin{aligned} |Y| &= |(Y \setminus X) \cup X| && \text{as } (Y \setminus X) \cup X = X \cup Y = Y \text{ because } X \subseteq Y; \\ &= |Y \setminus X| + |X| && \text{by the Addition Rule, as } Y \setminus X \text{ and } X \text{ are disjoint.} \end{aligned}$$

$\therefore \quad |Y \setminus X| = |Y| - |X|.$ $\square$

Combinatorial interpretation of the Difference Rule

Let $m, n \in \mathbb{N}$. Suppose that the variable x can only take one of m objects except n of them. Then there are exactly $m - n$ ways to substitute objects into x.

The Inclusion–Exclusion Rule for two sets



Remark 10.1.6

The Addition Rule becomes false if one drops the disjointness condition. For example,

$$|\{1,2\} \cup \{2,3,4\}| = |\{1,2,3,4\}| = 4 \neq 5 = 2+3 = |\{1,2\}| + |\{2,3,4\}|.$$

Theorem 10.1.7 (Inclusion–Exclusion Rule for two sets)

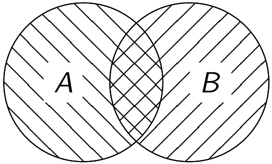
Let A, B be finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof sketch

$$\begin{aligned}
|A \cup B| &= |(A \setminus B) \cup B| && \text{as } A \cup B = (A \setminus B) \cup B; \\
&= |A \setminus B| + |B| && \text{by the Addition Rule, as } (A \setminus B) \cap B = \varnothing; \\
&= |A \setminus (A \cap B)| + |B| && \text{as } A \setminus B = A \setminus (A \cap B); \\
&= |A| - |A \cap B| + |B| && \text{by the Difference Rule, as } A \cap B \subseteq A.
\end{aligned}$$

$\square$

The Inclusion–Exclusion Rule for two sets: example



Example 10.1.8

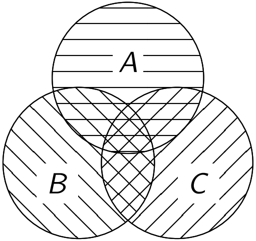
By the Inclusion–Exclusion Rule for two sets,

$$\begin{aligned} |\{1,2\} \cup \{2,3,4\}| &= |\{1,2\}| + |\{2,3,4\}| - |\{1,2\} \cap \{2,3,4\}| \\ &= |\{1,2\}| + |\{2,3,4\}| - |\{2\}| \\ &= 2 + 3 - 1 = 4. \end{aligned}$$

Combinatorial interpretation of the Inclusion–Exclusion Rule for two sets

Let $m, n \in \mathbb{N}$. Suppose that the variable x can only take either one of m objects or one of n objects, and that the m objects and the n objects here have exactly k objects in common. Then there are exactly $m + n - k$ ways to substitute objects into x.

The Inclusion–Exclusion Rule for three sets



Corollary 10.1.9 (Inclusion–Exclusion Rule for three sets)

Let A, B, C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Proof

$|A \cup B \cup C|$

$= |A| + |B \cup C| - |A \cap (B \cup C)|$ by the Inclusion–Exclusion Rule for two sets;

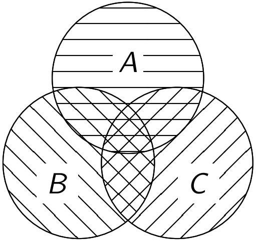
$= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)|$ by the Distributive Laws;

$= |A| + |B| + |C| - |B \cap C|$

$\quad - |A \cap B| - |A \cap C| + |A \cap B \cap A \cap C|$ by the Inclusion–Exclusion Rule for two sets;

$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$ $\square$

The Inclusion–Exclusion Rule for three sets: example



Example 10.1.10

By the Inclusion–Exclusion Rule for three sets,

$$
\begin{aligned}
&|\{-1,0\} \cup \{0,2,3\} \cup \{-1,0,1,2\}| \\
&= |\{-1,0\}| + |\{0,2,3\}| + |\{-1,0,1,2\}| - |\{-1,0\} \cap \{0,2,3\}| - |\{0,2,3\} \cap \{-1,0,1,2\}| \\
&\quad - |\{-1,0,1,2\} \cap \{-1,0\}| + |\{-1,0\} \cap \{0,2,3\} \cap \{-1,0,1,2\}| \\
&= |\{-1,0\}| + |\{0,2,3\}| + |\{-1,0,1,2\}| - |\{0\}| - |\{0,2\}| - |\{-1,0\}| + |\{0\}| \\
&= 2+3+4-1-2-2+1 = 5.
\end{aligned}
$$

Quick check

Exercise 10.1.11

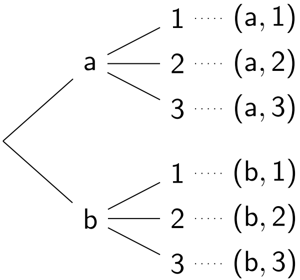
There are 40 sets, of which 12 are countable and 31 are infinite. How many of them are both countable and infinite?

Cartesian products: an example

Example 10.2.1

Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then

$$|A \times B| = |\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}| = 6 = 2 \times 3 = |A| \times |B|.$$



General Multiplication Rule

Proposition 10.2.2 (General Multiplication Rule)

Fix $m, n \in \mathbb{N}$. Let A be a set of size m, and for each $x \in A$, let B_x be a set of size n. Then $\{(x, y) : x \in A \text{ and } y \in B_x\}$ is finite and has size mn.

Proof sketch

Use our conditions on m and n to find bijections $f\colon \{1, 2, \dots, m\} \to A$ and $g_i\colon \{1, 2, \dots, n\} \to B_{f(i)}$ for each $i \in \{1, 2, \dots, m\}$. Then

$$h = \big\{\big(((i-1)n + j, (f(i), g_i(j))\big) : i \in \{1, 2, \dots, m\} \text{ and } j \in \{1, 2, \dots, n\}\big\}$$

is bijection $\{1, 2, \dots, mn\} \to \{(x, y) : x \in A \text{ and } y \in B_x\}$. $\square$

Combinatorial interpretation of the General Multiplication Rule

Let $m, n \in \mathbb{N}$. Suppose that in (x, y) the variable x can only take one of m objects, and possibly depending on what x takes the variable y can only take one of n objects. Then there are exactly mn ways to substitute objects into (x, y).

General Multiplication Rule for more sets

Corollary 10.2.3

Let $n \in \mathbb{Z}^+$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}$. Suppose that in $(x_1, x_2, \ldots, x_n)$,

- the variable x_1 can only take one of m_1 objects;
- possibly depending on what x_1 takes, the variable x_2 can only take one of m_2 objects;
- possibly depending on what x_1, x_2 take, the variable x_3 can only take one of m_3 objects;
- ...
- possibly depending on what $x_1, x_2, \ldots, x_{n-1}$ take, the variable x_n can only take one of m_n objects.

Then there are exactly $m_1 m_2 \ldots m_n$ ways to substitute objects into $(x_1, x_2, \ldots, x_n)$.

Proof sketch

Proceed by induction on n using the General Multiplication Rule. $\square$

Multiplication Rule

Corollary 10.2.4 (Multiplication Rule)

Let A, B be finite sets and $n \in \mathbb{Z}_{\geqslant 2}$.

(1) $A \times B$ is finite and $|A \times B| = |A| \times |B|$.

(2) $|A^n| = |A|^n$.

Proof

(1) Apply the General Multiplication Rule with $B_x = B$ for each $x \in A$.

(2) Applying part (1) repeatedly,

$$|A^n| = \underbrace{|A \times A \times \ldots \times A|}_{n\text{-many } A\text{'s}} = \underbrace{|A| \times |A| \times \cdots \times |A|}_{n\text{-many } |A|\text{'s}} = |A|^n. \qquad \square$$

Representing subsets using ordered tuples or strings

Convention 10.2.5

Let A be a set and $n \in \mathbb{N}$. Notationally, it is sometimes convenient to identify an ordered n-tuple $(a_1, a_2, \ldots, a_n) \in A^n$ with the string $a_1 a_2 \ldots a_n$ of length n.

Example 10.2.6

Let $A = \{1, 2, 3, 4, 5\}$. One can represent each subset $S \subseteq A$ by the string $d_1 d_2 d_3 d_4 d_5$ over $\{0, 1\}$ satisfying

$$i \in S \quad \Leftrightarrow \quad d_i = 1$$

for all $i \in \{1, 2, 3, 4, 5\}$. For instance,

$$\begin{array}{lll} \{\qquad\quad\} & \text{is represented by} & 00000; \\ \{1,\ \ 3,\ \ 5\} & \text{is represented by} & 10101; \\ \{1,2,\ \ 4\ \ \} & \text{is represented by} & 11010; \\ \{1,2,3,4,5\} & \text{is represented by} & 11111. \end{array}$$

As one can verify, this representation gives rise to a bijection $\mathcal{P}(A) \to \{0, 1\}^5$. So $|\mathcal{P}(A)| = |\{0, 1\}^5| = 2^5 = 32$ by Corollary 10.2.4(2).

Cardinality of the power set

Combinatorial interpretation: For all $n \in \mathbb{N}$, the number of ways to make n binary choices is 2^n.

Theorem 10.2.7

Let A be a finite set. Then $\mathcal{P}(A)$ is finite and $|\mathcal{P}(A)| = 2^{|A|}$.

Proof

Let $n = |A|$ and $A = \{a_1, a_2, \ldots, a_n\}$. As in Example 10.2.6, represent each subset $S \subseteq A$ by the string $d_1 d_2 \ldots d_n$ over $\{0, 1\}$ satisfying

$$a_i \in S \quad \Leftrightarrow \quad d_i = 1$$

for all $i \in \{1, 2, \ldots, n\}$. This representation gives rise to a bijection $\mathcal{P}(A) \to \{0, 1\}^n$. So $|\mathcal{P}(A)| = |\{0, 1\}^n| = 2^n$ by Corollary 10.2.4(2). $\square$

Remark 10.2.8

Like in the proof of Theorem 10.2.7, instead of counting the objects themselves, often it is more convenient to count suitably chosen representations of them instead. Implicitly, each such representation is a bijection. Proofs of bijectivity are usually omitted if they are straightforward and distracting. We will do the same. Nevertheless, the representations themselves should be clearly formulated.

Permutations of a set

Let $r, n \in \mathbb{N}$ and Γ be a finite set.

Definition 10.3.3 and Remark 10.3.5

(1) An *r-permutation* of Γ is a string of length r over Γ in which no symbol appears in two different positions.
(One can view an r-permutation of a set Γ as a way to pick r elements from Γ without replacement where order matters.)

(2) Let $P(n, r)$ denote the number of r-permutations of a set of size n.
(Some write ${}_nP_r$ or ${}^nP_r$ or P^n_r for $P(n, r)$.)

(3) A *permutation* of Γ is a $|\Gamma|$-permutation of Γ.
(One can view a permutation of a finite set Γ of size n as a way to arrange the n elements of Γ into n positions.)

Example 10.3.4

Let $\Gamma = \{\mathsf{E}, \mathsf{I}, \mathsf{L}, \mathsf{N}, \mathsf{S}, \mathsf{T}\}$.

(1) The 4-permutations of Γ include LNST, TELI, LINE, and SENT,
but *not* any of SIT, ABCD, SEEN, and NILET.

(2) The permutations of Γ include EILNST, TELSIN, LISTEN, and SILENT.

Number of r-permutations of a set (1/2)

Theorem 10.3.6

For all $r, n \in \mathbb{N}$,

$$P(n,r) = \begin{cases} \dfrac{n!}{(n-r)!}, & \text{if } r \leqslant n; \\ 0, & \text{if } r > n. \end{cases}$$

Proof for $r \leqslant n$

Fix a set Γ of size n. As no symbol appears in two different positions, the r-permutations are precisely those $x_1 x_2 \ldots x_r$ where

- x_1 can only take one of the n elements of Γ, say a_1;
- x_2 can only take one of the $n-1$ elements of $\Gamma \setminus \{a_1\}$, say a_2;
- ...
- x_r can only take one of the $n-(r-1)$ elements of $\Gamma \setminus \{a_1, a_2, \ldots, a_{r-1}\}$, say a_r.

So the General Multiplication Rule tells us that

$$P(n,r) = n \times (n-1) \times \cdots \times (n-(r-1)). \qquad \square$$

Number of r-permutations of a set (2/2)

Proof that there is no r-permutation of a set of size n when $r > n$

Fix a set Γ of size n. We prove that there is no r-permutation of Γ by contradiction.

- Suppose there is some r-permutation of Γ, say $a_1 a_2 \ldots a_r$.
- As no symbol appears in two different positions in an r-permutation, the function $f\colon \{1, 2, \ldots, r\} \to \Gamma$ where each $f(i) = a_i$ is an injection.
- So $r = |\{1, 2, \ldots, r\}| \leqslant |\Gamma| = n$ by the Pigeonhole Principle.
- The contradicts the condition that $r > n$. $\square$

Combinatorial interpretation of Theorem 10.3.6

Let $r, n \in \mathbb{N}$.

(1) If $r \leqslant n$, then there are exactly $\frac{n!}{(n-r)!}$ ways to pick r objects from n objects without replacement where order matters.

(2) If $r > n$, then there is no way to pick r objects from n objects without replacement.

Number of permutations of a set

Corollary 10.3.7

Let Γ be a set of size $n \in \mathbb{N}$. Then Γ has exactly $n!$ permutations.

Proof

By Theorem 10.3.6, the number of permutations of Γ is

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!. \qquad \square$$

Combinatorial interpretation of Corollary 10.3.7

Let $n \in \mathbb{N}$. Then there are exactly $n!$ ways to arrange n objects into n positions.

Example 10.3.8

Let $\Gamma = \{\mathsf{E}, \mathsf{I}, \mathsf{L}, \mathsf{N}, \mathsf{S}, \mathsf{T}\}$. Note $|\Gamma| = 6$. So, according to the results above,

(1) the number of 4-permutations of Γ is $P(6,4) = \frac{6!}{(6-4)!} = 6 \times 5 \times 4 \times 3 = 360$; and

(2) the number of permutations of Γ is $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Permutations of a string

Definition 10.3.9

Let Γ be a set and $a_1 a_2 \ldots a_n$ be a string over Γ. A *permutation* of $a_1 a_2 \ldots a_n$ is a string over Γ of the form

$$a_{f(1)} a_{f(2)} \cdots a_{f(n)}$$

where f is some bijection $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Remark 10.3.10

(1) One can view a permutation of a string as a way to rearrange the symbols in the string.

(2) Let $n \in \mathbb{N}$ and Γ be a set of size n, say $\Gamma = \{a_1, a_2, \ldots, a_n\}$. Then the permutations of the set Γ in the sense of Definition 10.3.3 are precisely the permutations of the string $a_1 a_2 \ldots a_n$ in the sense of Definition 10.3.9.

Counting the number of permutations of a string: Example 10.3.11

Here is a complete list of the permutations of the string EGG:

$$\text{EGG}, \qquad \text{GEG}, \qquad \text{GGE}.$$

Here is a complete list of the permutations of the set $\Gamma = \{E, G_1, G_2\}$:

$$\begin{array}{ccc} EG_1G_2, & G_1EG_2, & G_1G_2E, \\ EG_2G_1, & G_2EG_1, & G_2G_1E. \end{array}$$

Each permutation s of the string EGG corresponds to exactly 2! permutations of the set Γ because one can arrange the 2 objects G_1, G_2 into the 2 positions where G appears in s in exactly 2! ways. Therefore, by the General Multiplication Rule,

$$\begin{pmatrix}\text{number of}\\ \text{permutations}\\ \text{of } \Gamma\end{pmatrix} = \begin{pmatrix}\text{number of}\\ \text{permutations}\\ \text{of EGG}\end{pmatrix} \times \begin{pmatrix}\text{number of ways to}\\ \text{arrange } G_1, G_2 \text{ into}\\ \text{2 positions}\end{pmatrix}.$$

$$\therefore \qquad 3! = \begin{pmatrix}\text{number of}\\ \text{permutations}\\ \text{of EGG}\end{pmatrix} \times 2!.$$

From this, we see that the number of permutations of EGG is $\frac{3!}{2!} = 6/2 = 3$.

Counting the number of permutations of a string: Exercise 10.3.12

How many permutations of the string BICONDITIONAL are there?

P = {B, I1, C, O1, N1, D, I2, T, I3, O2, N2, A, L} |P| = 13!

$$\text{\# of permuations of biconditional} \quad \frac{\text{\# of permutations of P}}{\begin{array}{c}\text{\# of ways to} \\ \text{subs I1,I2,I3 into} \\ \text{3 positions}\end{array} \times \begin{array}{c}\text{\# of ways to} \\ \text{subs O1, O2 into} \\ \text{2 positions}\end{array} \times \begin{array}{c}\text{\# of ways to} \\ \text{subs N1, N2 into} \\ \text{2 positions}\end{array}}$$

Subsets of a fixed size

Definition 10.3.13

Let $r, n \in \mathbb{N}$ and A be a finite set.

(1) An *r-combination* of A is a subset of A of size r.

(2) Let $\binom{n}{r}$ denote the number of r-combinations of a set of size n. We read $\binom{n}{r}$ as "n choose r".

Remark 10.3.14

(1) One can view an r-combination of a set A as a way to pick r elements from Γ without replacement where order does not matter.

(2) Some write $C(n, r)$ or ${}_nC_r$ or ${}^nC_r$ or C_r^n for $\binom{n}{r}$.

Theorem 10.3.15

For all $r, n \in \mathbb{N}$,

$$\binom{n}{r} = \begin{cases} \dfrac{n!}{r!\,(n-r)!}, & \text{if } r \leqslant n; \\ 0, & \text{if } r > n. \end{cases}$$

Combinatorial interpretation:
If $r \leqslant n$, then there are exactly $\frac{n!}{(n-r)!\,r!}$ ways to pick r objects from n objects without replacement where order does not matter.

Proof that $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$ when $r \leqslant n$

Let A be a set of size n, say $\{a_1, a_2, \ldots, a_n\}$.
Represent each subset $S \subseteq A$ by the string $d_1 d_2 \ldots d_n$ over $\{0, 1\}$ satisfying

$$a_i \in S \quad \Leftrightarrow \quad d_i = 1$$

for all $i \in \{1, 2, \ldots, n\}$. Under this representation, subsets of A of size r are represented precisely by the permutations of the string

$$\underbrace{11\ldots1}_{r\text{-many 1's}}\overbrace{00\ldots0}^{(n-r)\text{-many 0's}}$$

The number of such permutations is

$$\frac{\begin{array}{c}\text{number of permutations of}\\ \{1_1, 1_2, \ldots, 1_r, 0_1, 0_2, \ldots, 0_{n-r}\}\end{array}}{\begin{pmatrix}\text{number of ways to arrange}\\ 1_1, 1_2, \ldots, 1_r \text{ into}\\ r \text{ positions}\end{pmatrix} \times \begin{pmatrix}\text{number of ways to arrange}\\ 0_1, 0_2, \ldots, 0_{n-r} \text{ into}\\ n-r \text{ positions}\end{pmatrix}} = \frac{n!}{r!\,(n-r)!}$$

by the General Multiplication Rule and Corollary 10.3.7. $\square$

Summary

Let A, B, X, Y, Γ be finite sets and $m, n, r \in \mathbb{N}$, where $r \leqslant n$.

Proposition 10.1.3 (Addition Rule). If A and B are disjoint, then $|A \cup B| = |A| + |B|$.

Corollary 10.1.5 (Difference Rule). If $X \subseteq Y$, then $|Y \setminus X| = |Y| - |X|$.

Theorem 10.1.7 (Inclusion–Exclusion Rule). $|A \cup B| = |A| + |B| - |A \cap B|$.

Proposition 10.2.2 (General Multiplication Rule). Suppose that in (x, y) the variable x can only take one of m objects, and possibly depending on what x takes the variable y can only take one of n objects. Then there are exactly mn ways to substitute objects into (x, y).

Theorem 10.2.7. $|\mathcal{P}(A)| = 2^{|A|}$.

Definition 10.3.3 and Theorem 10.3.6. An *r-permutation* of Γ is a string of length r over Γ in which no symbol appears in two different positions. The number of r-permutations of a set of size n, denoted $P(n, r)$, is equal to $\frac{n!}{(n-r)!}$.

Definition 10.3.3 and Corollary 10.3.7. A *permutation* of Γ is a $|\Gamma|$-permutation of Γ. The number of permutations of a set of size n is equal to $n!$.

Definition 10.3.13 and Theorem 10.3.15. An *r-combination* of A is a subset of A of size r. The number of r-combinations of a set of size n, denoted $\binom{n}{r}$, is equal to $\frac{n!}{r!\,(n-r)!}$.