CS1231 Chapter 6

Equivalence relations and partial orders

6.1 Equivalence relations

Definition 6.1.1. Let A be a set and R be a relation on A.

(1) R is *reflexive* if every element of A is R-related to itself, i.e.,

$$\forall x \in A \quad (x \mathrel{R} x).$$

does it refer to all distinct elements?

(2) R is *symmetric* if x is R-related to y implies y is R-related to x, for all $x, y \in A$, i.e.,

$$\forall x, y \in A \quad (x \mathrel{R} y \Rightarrow y \mathrel{R} x).$$

or if x not related to y AND y not related to x

(3) R is *transitive* if x is R-related to y and y is R-related to z imply x is R-related to z, for all $x, y, z \in A$, i.e.,

$$\forall x, y, z \in A \quad (x \mathrel{R} y \wedge y \mathrel{R} z \Rightarrow x \mathrel{R} z).$$

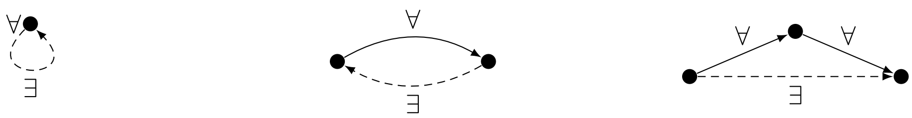
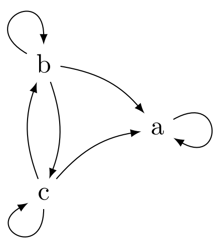


Figure 6.1: Reflexivity, symmetry, and transitivity

Example 6.1.2. Let R be the relation represented by the following arrow diagram.



Then R is reflexive. It is not symmetric because b R a but a $\not R$ b. It is transitive, as one can show by exhaustion:

$$
\begin{array}{ll}
\mathrm{a}\ R\ \mathrm{a} \wedge \mathrm{a}\ R\ \mathrm{a} \Rightarrow \mathrm{a}\ R\ \mathrm{a}; & \mathrm{c}\ R\ \mathrm{a} \wedge \mathrm{a}\ R\ \mathrm{a} \Rightarrow \mathrm{c}\ R\ \mathrm{a}; \\
\mathrm{b}\ R\ \mathrm{a} \wedge \mathrm{a}\ R\ \mathrm{a} \Rightarrow \mathrm{b}\ R\ \mathrm{a}; & \mathrm{c}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{a} \Rightarrow \mathrm{c}\ R\ \mathrm{a}; \\
\mathrm{b}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{a} \Rightarrow \mathrm{b}\ R\ \mathrm{a}; & \mathrm{c}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{b} \Rightarrow \mathrm{c}\ R\ \mathrm{b}; \\
\mathrm{b}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{b} \Rightarrow \mathrm{b}\ R\ \mathrm{b}; & \mathrm{c}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{c} \Rightarrow \mathrm{c}\ R\ \mathrm{c}; \\
\mathrm{b}\ R\ \mathrm{b} \wedge \mathrm{b}\ R\ \mathrm{c} \Rightarrow \mathrm{b}\ R\ \mathrm{c}; & \mathrm{c}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{a} \Rightarrow \mathrm{c}\ R\ \mathrm{a}; \\
\mathrm{b}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{a} \Rightarrow \mathrm{b}\ R\ \mathrm{a}; & \mathrm{c}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{b} \Rightarrow \mathrm{c}\ R\ \mathrm{b}; \\
\mathrm{b}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{b} \Rightarrow \mathrm{b}\ R\ \mathrm{b}; & \mathrm{c}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{c} \Rightarrow \mathrm{c}\ R\ \mathrm{c} \\
\mathrm{b}\ R\ \mathrm{c} \wedge \mathrm{c}\ R\ \mathrm{c} \Rightarrow \mathrm{b}\ R\ \mathrm{c}; &
\end{array}
$$

and all the others instances are vacuously true.

Example 6.1.3. Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x\ R\ y \quad \Leftrightarrow \quad x = y.$$

Then R is reflexive, symmetric, and transitive.

Example 6.1.4. Let R_0 denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x\ R_0\ y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R_0 is reflexive, may not be symmetric (when U contains x, y such that $x \subsetneq y$), but is transitive.

Exercise 6.1.5. Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

reflexive as x ⩽ y for all x,y∈Q;
Not symmetric as 0⩽1but 1≰0;
transitive as x⩽y and y⩽z imply x⩽z

$$x\ R\ y \quad \Leftrightarrow \quad x \leqslant y.$$

Is R reflexive? Is R symmetric? Is R transitive? Briefly explain your answers. ✎ 6a

Exercise 6.1.6. Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x\ R'\ y \quad \Leftrightarrow \quad x < y.$$

Is R reflexive? Is R symmetric? Is R transitive? Briefly explain your answers. ✎ 6b

Definition 6.1.7. Let $n, d \in \mathbb{Z}$. Then d is said to *divide* n if

$$n = dk \quad \text{for some } k \in \mathbb{Z}.$$

We write $d \mid n$ for "d divides n", and $d \nmid n$ for "d does not divide n". We also say

"n is *divisible* by d" or "n is a *multiple* of d" or "d is a *factor/divisor* of n"

for "d divides n".

Example 6.1.8. Let R denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x\ R\ y \quad \Leftrightarrow \quad x \mid y.$$

Then R is reflexive, not symmetric, but transitive.

Proof. (reflexivity) For each $a \in \mathbb{Z}^+$, we know $a = a \times 1$ and so $a \mid a$ by the definition of divisibility.

This relation is not reflexive because 2 !R 2. It is not symmetric because 3 R 2 but 2 !R 3. It is not transitive because 2 R 1 and 1 R 2 but 2 !R 2.

(non-symmetry) Note $1 \mid 2$ but $2 \nmid 1$.

(transitivity) Let $a, b, c \in \mathbb{Z}^+$ such that $a \mid b$ and $b \mid c$. Use the definition of divisibility to find $k, \ell \in \mathbb{Z}$ such that $b = ak$ and $c = b\ell$. Then $c = b\ell = (ak)\ell = a(k\ell)$, where $k\ell \in \mathbb{Z}$. Thus $a \mid c$ by the definition of divisibility. □

Exercise 6.1.9. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$. View R as a relation on A. Is R reflexive? Is R symmetric? Is R transitive? ✎ 6c

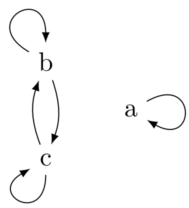
Exercise 6.1.10. Let R be a relation on a set A. Prove that R is transitive if and only if $R \circ R \subseteq R$. ✎ 6d

Definition 6.1.11. An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Example 6.1.12. The equality relation on a set, as defined in Example 6.1.3, is an equivalence relation.

Convention 6.1.13. People usually use equality-like symbols such as $\sim$, $\approx$, $\simeq$, $\cong$, and $\equiv$ to denote equivalence relations. These symbols are often defined and redefined to mean different equivalence relations in different situations. We may read $\sim$ as "is equivalent to".

Example 6.1.14. Let R be the relation represented by the following arrow diagram.



Then R is reflexive, symmetric, and transitive. So it is an equivalence relation on $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$.

6.2 Equivalence classes

Definition 6.2.1. Let $\sim$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x with respect to $\sim$, denoted $[x]_\sim$, is defined to be the set of all elements of A that x is $\sim$-related to, i.e.,

$$[x]_\sim = \{y \in A : x \sim y\}.$$

When there is no risk of confusion, we may drop the subscript and write simply $[x]$.

Example 6.2.2. Let A be a set. The equivalence classes with respect to the equality relation on A are of the form

$$[x] = \{y \in A : x = y\} = \{x\},$$

where $x \in A$.

Example 6.2.3. If R is the equivalence relation represented by the arrow diagram in Example 6.1.14, then

$$[\mathrm{a}] = \{\mathrm{a}\} \quad \text{and} \quad [\mathrm{b}] = \{\mathrm{b}, \mathrm{c}\} = [\mathrm{c}].$$

6.3 Partitions

Definition 6.3.1. Call $\mathscr{C}$ *a partition* of a set A if

(0) $\mathscr{C}$ is a set of *nonempty* subsets of A;

(1) every element of A is in some element of $\mathscr{C}$; and

(2) if two elements of $\mathscr{C}$ have a nonempty intersection, then they are equal.

Elements of a partition are called *components* of the partition.

Remark 6.3.2. One can rewrite the three conditions in the definition of partitions respectively as follows:

(0) $\forall S \in \mathscr{C} \quad (\varnothing \neq S \subseteq A)$;

(1) $\forall x \in A \quad \exists S \in \mathscr{C} \quad (x \in S)$;

(2) $\forall S_1, S_2 \in \mathscr{C} \quad (S_1 \cap S_2 \neq \varnothing \Rightarrow S_1 = S_2)$.

Here are two alternative ways to put this definition in words:

- $\mathscr{C}$ is a set of nonempty subsets $S \subseteq A$ such that every element of A is in exactly one $S \in \mathscr{C}$;
- $\mathscr{C}$ is a set of mutually disjoint nonempty subsets of A whose union is A.

Example 6.3.3. One partition of the set $A = \{1, 2, 3\}$ is $\{\{1\}, \{2, 3\}\}$. The others are

$$\{\{1\}, \{2\}, \{3\}\}, \quad \{\{2\}, \{1, 3\}\}, \quad \{\{3\}, \{1, 2\}\}, \quad \{\{1, 2, 3\}\}.$$

Example 6.3.4. One partition of $\mathbb{Z}$ is

$$\{\{2k : k \in \mathbb{Z}\}, \{2k + 1 : k \in \mathbb{Z}\}\}.$$

Lemma 6.3.5. Let $\sim$ be an equivalence relation on a set A.

(1) $x \in [x]$ for all $x \in A$.

(2) Any equivalence class is nonempty.

Proof. (1) Let $x \in A$. Then $x \sim x$ by reflexivity. So $x \in [x]$ by the definition of $[x]$.

(2) Any equivalence class is of the form $[x]$ for some $x \in A$, and so it must be nonempty by (1). □

Lemma 6.3.6. Let $\sim$ be an equivalence relation on a set A. For all $x, y \in A$, if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$.

Proof. Assume $[x] \cap [y] \neq \varnothing$. Say, we have $w \in [x] \cap [y]$. This means $x \sim w$ and $y \sim w$ by the definition of $[x]$ and $[y]$.

To show $[x] = [y]$, we need to prove both $[x] \subseteq [y]$ and $[y] \subseteq [x]$. We will concentrate on the former; the latter is similar.

Take $z \in [x]$. Then $x \sim z$ by the definition of $[x]$. By symmetry, we know from the first paragraph that $w \sim x$. Altogether we have $y \sim w \sim x \sim z$. So transitivity tells us $y \sim z$. Thus $z \in [y]$ by the definition of $[y]$. □

Yes. Proof;
We know x ∈ [x] by Lemma 6.3.5(1). So x ∈ S ∩ [x] by the hypothesis. This implies S ∩ [x] != ∅. Hence S = [x] by Lemma 6.3.6.

Question 6.3.7. Consider an equivalence relation. Is it true that if x is an element of an equivalence class S, then $S = [x]$? ✎ 6e

Definition 6.3.8. Let A be a set and $\sim$ be an equivalence relation on A. Denote by $A/{\sim}$ the set of all equivalence classes with respect to $\sim$, i.e.,

$$A/{\sim} = \{[x]_\sim : x \in A\}.$$

A / ~ is the set of all connected components

We may read $A/{\sim}$ as "the quotient of A by $\sim$".

Example 6.3.9. Let A be a set. Then from Example 6.2.2 we know $A/{=}$ is equal to $\{\{x\} : x \in A\}$.

Example 6.3.10. If R is the equivalence relation on the set $A = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ represented by the arrow diagram in Example 6.1.14, then from Example 6.2.3 we know

$$A/{\sim} = \{[\mathrm{a}], [\mathrm{b}], [\mathrm{c}]\} = \{\{\mathrm{a}\}, \{\mathrm{b}, \mathrm{c}\}, \{\mathrm{b}, \mathrm{c}\}\} = \{\{\mathrm{a}\}, \{\mathrm{b}, \mathrm{c}\}\}.$$

Theorem 6.3.11. Let $\sim$ be an equivalence relation on a set A. Then $A/{\sim}$ is a partition of A.

Proof. Conditions (0) and (1) in the definition of partitions are guaranteed by the definition of equivalence classes and Lemma 6.3.5. Condition (2) is given by Lemma 6.3.6. □

6.4 Partial orders

Definition 6.4.1. Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A \ \ (x \mathrel{R} y \wedge y \mathrel{R} x \Rightarrow x = y)$.

(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

(3) Suppose R is a partial order. Let $x, y \in A$. Then x, y are *comparable (under R)* if

$$x \mathrel{R} y \quad \text{or} \quad y \mathrel{R} x.$$

(4) R is a *(non-strict) total order* or a *(non-strict) linear order* if R is a partial order and every pair of elements is comparable, i.e.,

$$\forall x, y \in A \ \ (x \mathrel{R} y \vee y \mathrel{R} x).$$

Note 6.4.2. A total order is always a partial order.

Example 6.4.3. Let R denote the non-strict less-than relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is antisymmetric. In fact, it is a total order.

Example 6.4.4. Let R_0 denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \mathrel{R_0} y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R_0 is antisymmetric. It is always a partial order, but it may not be a total order.

Example 6.4.5. Let R_1 denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \; R_1 \; y \quad \Leftrightarrow \quad x \mid y.$$

The divisibility relation on Z is not antisymmetric because $1 \mid -1$ and $-1 \mid 1$, but $1 \neq -1$

Is R_1 antisymmetric? Is R_1 a partial order? Is R_1 a total order? 6f

Example 6.4.6. Let R_2 denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x \; R_2 \; y \quad \Leftrightarrow \quad x \mid y.$$

The divisibility relation on Z+ is antisymmetric, as shown below. So it is a partial order by Example 6.1.8. It is not total because $2 \nmid 3$ and $3 \nmid 2$.

Is R_2 antisymmetric? Is R_2 a partial order? Is R_2 a total order? 6g

Definition 6.4.7. Let R be a (non-strict) partial order on a set A. A ***smallest element*** of A (with respect to the partial order R) is an element $m \in A$ such that $m \; R \; x$ for all $x \in A$.

Example 6.4.8. (1) $S = \{x \in \mathbb{Z}_{\geqslant 0} : 0 < x < 5\}$ has smallest element 1.

(2) $S^\sharp = \{x \in \mathbb{Q}_{\geqslant 0} : 0 < x < 5\}$ has no smallest element because if $x \in S^\sharp$, then $x/2 \in S^\sharp$ and $x/2 < x$.

Theorem 6.4.9 (Well-Ordering Principle)**.** Let $b \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}_{\geqslant b}$. If $S \neq \varnothing$, then S has a smallest element.

Proof. We prove the contrapositive. Assume that S has no smallest element. As $S \subseteq \mathbb{Z}_{\geqslant b}$, it suffices to show that $n \notin S$ for all $n \in \mathbb{Z}_{\geqslant b}$. We prove this by Strong MI on n. Let $P(n)$ be the predicate "$n \notin S$" over $\mathbb{Z}_{\geqslant b}$.

(Base step) If $b \in S$, then b is the smallest element of S because $S \subseteq \mathbb{Z}_{\geqslant b}$, which contradicts our assumption. So $b \notin S$.

(Induction step) Let $k \in \mathbb{Z}_{\geqslant b}$ such that $P(b), P(b+1), \ldots, P(k)$ are true, i.e., that $b, b+1, \ldots, k \notin S$. If $k+1 \in S$, then $k+1$ is the smallest element of S because $S \subseteq \mathbb{Z}_{\geqslant b}$, which contradicts our assumption. So $k+1 \notin S$. This means $P(k+1)$ is true.

Hence $\forall n \in \mathbb{Z}_{\geqslant b} \;\; P(n)$ is true by Strong MI. $\square$

Exercise 6.4.10 (extra)**.** The base step and the induction step in our proof of the Well-Ordering Principle above look very similar. Combine the two into one. (Hint: use Exercise 3.2.25.) 6h

Tutorial exercises

An asterisk (*) indicates a more challenging question.

6.1. Let $A = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}\}$. Consider the following relation on A:

$$R = \{(\mathrm{a},\mathrm{a}), (\mathrm{a},\mathrm{b}), (\mathrm{a},\mathrm{c}), (\mathrm{b},\mathrm{a}), (\mathrm{b},\mathrm{b}), (\mathrm{b},\mathrm{c}), (\mathrm{c},\mathrm{a}), (\mathrm{c},\mathrm{b}), (\mathrm{c},\mathrm{c}), (\mathrm{d},\mathrm{d}), (\mathrm{e},\mathrm{e})\}.$$

(a) Draw an arrow diagram for R.

(b) Is R an equivalence relation? Briefly explain your answer. If R is an equivalence relation, then also write down its equivalence classes in roster notation.

(c) Is R a partial order? Is R a total order? Briefly explain your answers.

6.2. For each of the relations defined below, determine whether it is

Here are the arrow diagrams of 8 of the binary relations on the set $A = \{a, b, c\}$. Set inclusion $\subseteq$ is a partial order on the set consisting of these 8 relations. This is indicated by the lines between the arrow diagrams.

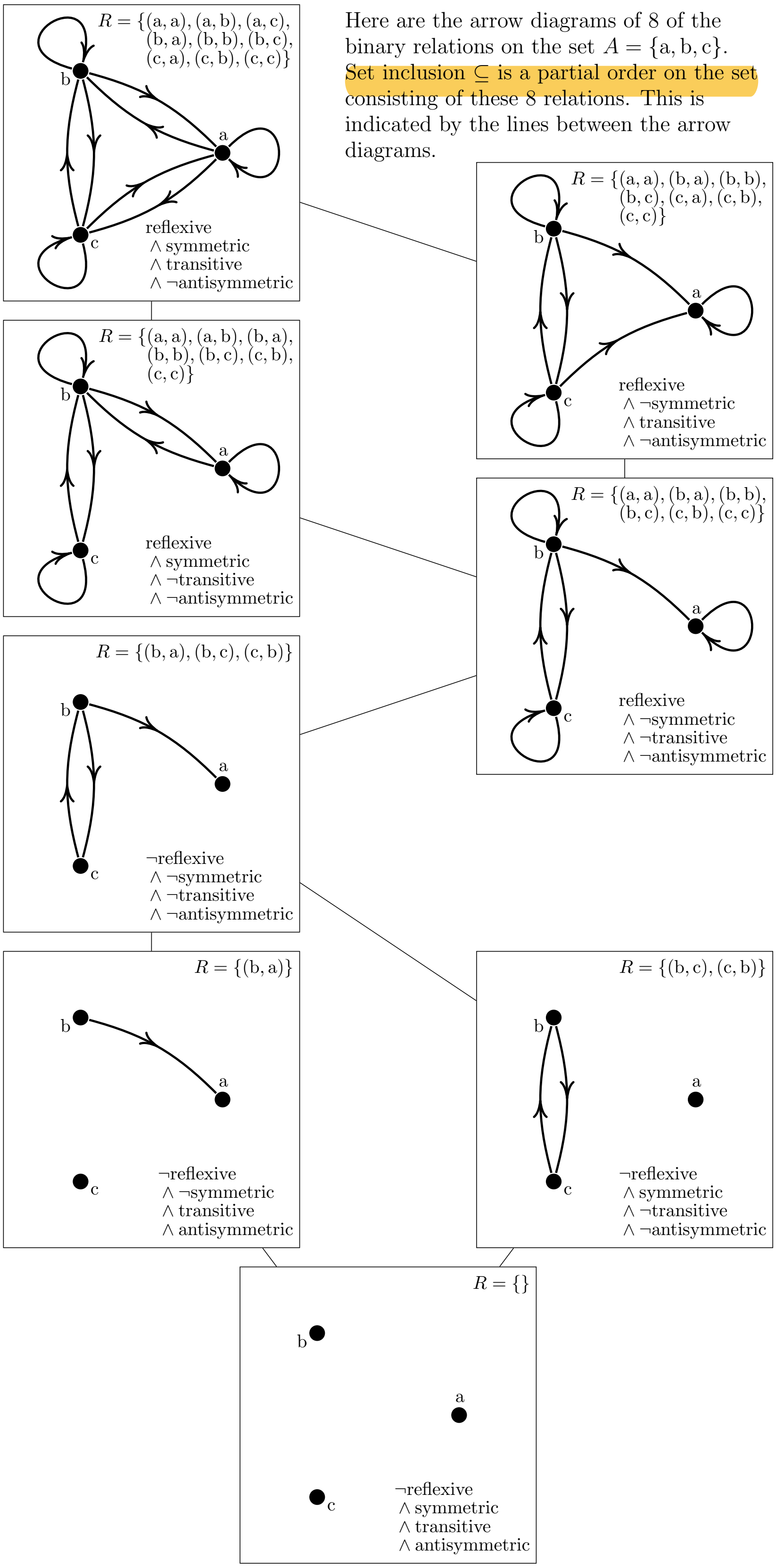


Figure 6.2: A partial order on a set of relations

(i) reflexive, (ii) symmetric, (iii) antisymmetric, (iv) transitive.

Briefly explain your answers.

(a) Define $R = \{(x, y) \in \mathbb{Q}^2 : xy \geqslant 0\}$, considered as a relation on $\mathbb{Q}$.

(b) Define the relation S on $\mathbb{R}^2$ by setting, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$(x_1, y_1)\ S\ (x_2, y_2) \quad \Leftrightarrow \quad x_1 \leqslant x_2 \text{ and } y_1 \leqslant y_2.$$

6.3. Define a relation R on $\mathbb{R}^2$ by setting, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$(x_1, y_1)\ R\ (x_2, y_2) \quad \Leftrightarrow \quad 3(x_1 - x_2) = y_1 - y_2.$$

(a) Prove that R is an equivalence relation.

(b) Let $(u, v) \in \mathbb{R}^2$. Describe the equivalence class $[(u, v)]$ geometrically as a set of points in the plane $\mathbb{R}^2$.
(Here we are assuming some prior knowledge in coordinate geometry.)

(c) Prove that, for every $(u, v) \in \mathbb{R}^2$, there exists $c \in \mathbb{R}$ such that $(0, c) \in [(u, v)]$.

6.4. Let $\sim$ be an equivalence relation on a set A. Prove that for all $x, y \in A$,

$$x \sim y \quad \Leftrightarrow \quad [x] = [y].$$

to prove equality on sets :
prove x ⊆ y and y ⊆ x

6.5. Consider the following proposition.

Every relation that is both symmetric and transitive must be reflexive.

(a) Someone tries to prove this proposition as follows.

> Let R be a relation on a set A that is both symmetric and transitive. As R is symmetric, we know $x\ R\ y$ and $y\ R\ x$ for all $x, y \in A$. So $x\ R\ x$ for all $x \in A$ by transitivity. This shows the reflexivity of R.

What is wrong with this attempt?

(b) Give a counterexample to this proposition.

6.6. Prove or disprove each of the following propositions.

(a) Any relation that is symmetric cannot be antisymmetric.

(b) Any relation that is not symmetric must be antisymmetric.

6.7.* (Induction corner) The aim of this exercise is to prove that there is no closed walk involving 2 or more vertices in the directed graph corresponding to a partial order.

Let R be a partial order on a set A. Prove by induction on n that, for every integer $n \geqslant 2$, there exist no $x_1, x_2, \ldots, x_n \in A$ satisfying $x_1 \neq x_2$ and

$$x_1\ R\ x_2 \quad \text{and} \quad x_2\ R\ x_3 \quad \text{and} \quad \ldots \quad \text{and} \quad x_{n-1}\ R\ x_n \quad \text{and} \quad x_n\ R\ x_1.$$

Extra exercises

6.8. For each of the relations defined below, determine whether it is

(i) reflexive, (ii) symmetric, (iii) antisymmetric, (iv) transitive.

Briefly explain your answers.

(a) Define $R = \{(x, y) \in \mathbb{Q}^2 : xy > 0\}$, considered as a relation on $\mathbb{Q}$.

(b) Define the relation S on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}$,

$$x \; S \; y \quad \Leftrightarrow \quad y = x + 1.$$

6.9. We saw in Exercise 6.1.10 how one can characterize transitivity in terms of relation composition. In this exercise, we characterize symmetry in terms of relation inverse.

Let R be a relation on a set A. Prove that R is symmetric if and only if $R^{-1} = R$.

6.10. Consider again the congruence-mod-2 relation R from Tutorial Exercise 5.4, i.e., the relation R on $\mathbb{Z}$ which satisfies

$$a \; R \; b \quad \Leftrightarrow \quad a - b \text{ is even}$$

for all $a, b \in \mathbb{Z}$.

(a) Prove that R is an equivalence relation.

(b) What are the equivalence classes with respect to R?