Tutorial solutions for Chapter 1

Sometimes there are other correct answers.

1.1. All of these are true, except the proposition in (c), which is false.

(a)
$$\begin{array}{ccccc} p & \wedge & q & \to & r \\ \mathrm{F} & & \mathrm{F} & & \mathrm{T} \\ & \mathrm{F} & & & \\ & & & \mathrm{T} & \end{array}$$

(b)
$$\begin{array}{cccccc} p & \leftrightarrow & q & \vee & \neg & r \\ \mathrm{F} & & \mathrm{F} & & & \mathrm{T} \\ & & & & \mathrm{F} & \\ & & & \mathrm{F} & & \\ & \mathrm{T} & & & & \end{array}$$

(c)
$$\begin{array}{cccccccc} p & \wedge & \big(q & \vee & (\neg & r & \to & p)\big) \\ \mathrm{F} & & \mathrm{F} & & & \mathrm{T} & & \mathrm{F} \\ \mathrm{F} & & & ? & & & & \\ & \mathrm{F} & & & & & & \end{array}$$

(d) Treating p, q, r as propositional variables,

$$\big(q \vee r \vee (\neg p \leftrightarrow q)\big) \wedge (q \vee r) \equiv q \vee r$$

by the absorption logical identities. Since r is a true proposition, the given proposition is also true.

1.2. (a) $t \leftrightarrow c \wedge a$.

(b) $t \to a$.

(c) $t \to c$.

1.3. (a) (i)

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

(ii)

p	$p \wedge p$
T	T
F	F

(iii)

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(iv)

t	c	$\neg t$
T	F	F

(b) (i)

$$\begin{aligned}
p \vee (q \wedge r) &\equiv \neg\neg p \vee (\neg\neg q \wedge \neg\neg r) && \text{by the Double Negative Law;}\\
&\equiv \neg\neg p \vee \neg(\neg q \vee \neg r) && \text{by De Morgan's Laws;}\\
&\equiv \neg\big(\neg p \wedge (\neg q \vee \neg r)\big) && \text{by De Morgan's Laws;}\\
&\equiv \neg\big((\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)\big) && \text{by (a)(i);}\\
&\equiv \neg\big(\neg(p \vee q) \vee \neg(p \vee r)\big) && \text{by De Morgan's Laws;}\\
&\equiv \neg\neg\big((p \vee q) \wedge (p \vee r)\big) && \text{by De Morgan's Laws;}\\
&\equiv (p \vee q) \wedge (p \vee r) && \text{by the Double Negative Law.}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\neg\big(p \vee (q \wedge r)\big) &\equiv \neg p \wedge \neg(q \wedge r) && \text{by De Morgan's Laws;}\\
&\equiv \neg p \wedge (\neg q \vee \neg r) && \text{by De Morgan's Laws;}\\
&\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r) && \text{by (a)(i);}\\
&\equiv \neg(p \vee q) \vee \neg(p \vee r) && \text{by De Morgan's Laws;}\\
&\equiv \neg\big((p \vee q) \wedge (p \vee r)\big) && \text{by De Morgan's Laws.}
\end{aligned}$$

Thus $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(ii)

$$\begin{aligned}
p \vee p &\equiv \neg\neg p \vee \neg\neg p && \text{by the Double Negative Law;}\\
&\equiv \neg(\neg p \wedge \neg p) && \text{by De Morgan's Laws;}\\
&\equiv \neg\neg p && \text{by (a)(ii);}\\
&\equiv p && \text{by the Double Negative Law.}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\neg(p \vee p) &\equiv \neg p \wedge \neg p && \text{by the Double Negative Law;}\\
&\equiv \neg p && \text{by (a)(ii).}
\end{aligned}$$

Thus $p \vee p \equiv p$.

(iii)

$$\begin{aligned}
p \vee (p \wedge q) &\equiv (p \vee p) \wedge (p \vee q) && \text{by (b)(i);}\\
&\equiv p \wedge (p \vee q) && \text{by (b)(ii);}\\
&\equiv p && \text{by (a)(iii).}
\end{aligned}$$

(iv)

$$\begin{aligned}
\neg c &\equiv \neg\neg t && \text{by (a)(iv);}\\
&\equiv t && \text{by the Double Negative Law.}
\end{aligned}$$

Additional explanations. Let us provide a slightly more intuitive explanation of why the Absorption Laws are true. Fix a substitution of propositions into the propositional variables p and q.

Consider the identity $p \vee (p \wedge q) \equiv p$; the argument for the other Absorption Law is similar. We split into two cases.

- Case 1: suppose p evaluates to T. Then $p \vee (p \wedge q)$ evaluates to T too by the definition of $\vee$.
- Case 2: suppose p evaluates to F. Then $p \wedge q$ evaluates to F by the definition of $\wedge$. In view of the definition of $\vee$, we conclude that $p \vee (p \wedge q)$ evaluates to F.

In all cases, the LHS and the RHS evaluate to the same truth value. So the equivalence holds.

1.4. Let us use a truth table:

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

As the column for $p \leftrightarrow q$ is exactly the same as the column for $(p \to q) \wedge (q \to p)$, the two compound expressions are equivalent.

Alternative solution. According to our definition of $\leftrightarrow$ in Definition 1.3.11, under a substitution of propositions into the propositional variables p and q, we know $p \leftrightarrow q$ evaluates to true if and only if p and q evaluate to the same truth value. As propositions by definition have exactly one truth value amongst "true" and "false", the latter in turn is equivalent to p and q both evaluating to true or both evaluating to false. This means the compound expression $(p \wedge q) \vee (\neg p \wedge \neg q)$ evaluates to true under this substitution. All these show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$. Hence

$$
\begin{aligned}
& p \leftrightarrow q \\
& \equiv (p \wedge q) \vee (\neg p \wedge \neg q) && \text{by the above;} \\
& \equiv \big((p \wedge q) \vee \neg p) \wedge (p \wedge q) \vee \neg q)\big) && \text{by the Distributive Laws;} \\
& \equiv \big((p \vee \neg p) \wedge (q \vee \neg p)\big) \wedge \big((p \vee \neg q) \wedge (q \vee \neg q)\big) && \text{by the Distributive Laws;} \\
& \equiv \big(T \wedge (q \vee \neg p)\big) \wedge \big((p \vee \neg q) \wedge t\big) && \text{by the logical identities on negation;} \\
& \equiv (q \vee \neg p) \wedge (p \vee \neg q) && \text{as } t \text{ is an identity for } \wedge; \\
& \equiv (p \to q) \wedge (q \to p) && \text{by the logical identity on the implication.}
\end{aligned}
$$

Here t denotes a tautology.

1.5. Let t be a tautology.

$$
\begin{aligned}
& (p \to q) \wedge (q \to r) \to (p \to r) \\
& \equiv \neg\big((\neg p \vee q) \wedge (\neg q \vee r)\big) \vee (\neg p \vee r) && \text{by the logical identity on the implication;} \\
& \equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee \neg p \vee r && \text{by De Morgan's Laws;} \\
& \equiv (p \wedge \neg q) \vee \neg p \vee (q \wedge \neg r) \vee r && \text{by De Morgan's Laws;} \\
& \equiv \big((p \vee \neg p) \wedge (\neg q \vee \neg p)\big) \vee \big((q \vee r) \wedge (\neg r \vee r)\big) && \text{by the Distributive Laws;} \\
& \equiv \big(t \wedge (\neg q \vee \neg p)\big) \vee \big((q \vee r) \wedge t\big) && \text{by the logical identities on negation;} \\
& \equiv \neg q \vee \neg p \vee q \vee r && \text{as } t \text{ is the identity for } \wedge; \\
& \equiv t \vee \neg p \vee r && \text{by the logical identities on negation;} \\
& \equiv t && \text{as } t \text{ is annihilating with respect to } \vee.
\end{aligned}
$$

Alternatively, one can use the following truth table:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

1.6. No. Let us substitute a false proposition into p and a true proposition into q. Then $p \to q$ evaluates to T, while $q \to p$ evaluates to F. So $(p \to q) \to (q \to p)$ evaluates to F. This shows the compound expression given is not a tautology. (There is no other substitution under which this expression evaluates to F.)

Additional comments. To determine whether the given expression is a tautology, one may want to draw a complete truth table. However, to prove that this expression is not a tautology, it suffices to identify one row of the truth table in which it evaluates to F. This is essentially what the solution above is doing.

One may use the logical identities to show that the expression given is equivalent to $p \vee \neg q$. While $p \vee \neg q$ does not look like a tautology, this does not logically imply it is not a tautology: we need to demonstrate a way to substitute propositions into p and q which makes $p \vee \neg q$ evaluate to F; cf. Note 1.4.24. One may find it easier to come up with such a substitution for the equivalent expression $p \vee \neg q$ than for the original expression.

1.7. By the definition of tautologies, the compound expression $P \leftrightarrow Q$ is a tautology means $P \leftrightarrow Q$ evaluates to true no matter what propositions we substitute into its propositional variables. According to the definition of $\leftrightarrow$, this in turn means P and Q evaluate to the same truth value no matter what propositions we substitute into their propositional variables. This is exactly what the definition of $P \equiv Q$ says.

Extra exercises

1.8. These are not equivalent. (Note that, since we are given two compound expressions, we are referring to equivalence in the sense of Definition 1.4.7 here.)

Let us substitute a false proposition into p and r, and a true proposition into q. Then $p \wedge q \rightarrow r$ evaluates to T, while $(p \rightarrow r) \wedge (q \rightarrow r)$ evaluates to F. So the two compound expressions are not equivalent.

1.9. (a)

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$	$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Since the column for $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$ consists entirely of T's, we see that $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$ is a tautology.

Alternatively, one can use the logical identities as follows.

$$
\begin{aligned}
& (p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r \\
& \equiv (p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee r) \rightarrow r && \text{by the logical identity on implication;} \\
& \equiv (p \vee q) \wedge \big((\neg p \wedge \neg q) \vee r\big) \rightarrow r && \text{by the Distributive Law;} \\
& \equiv (p \vee q) \wedge \big(\neg(p \vee q) \vee r\big) \rightarrow r && \text{by De Morgan's Laws;} \\
& \equiv \big((p \vee q) \wedge \neg(p \vee q)\big) \vee \big((p \vee q) \wedge r\big) \rightarrow r && \text{by the Distributive Law;} \\
& \equiv \big((p \vee q) \wedge r\big) \rightarrow r && \text{by the logical identities on negation and identities;} \\
& \equiv \neg\big((p \vee q) \wedge r\big) \vee r && \text{by the logical identity on implication;} \\
& \equiv \big(\neg(p \vee q) \vee \neg r\big) \vee r && \text{by De Morgan's Laws;} \\
& \equiv \neg(p \vee q) \vee (\neg r \vee r) && \text{by the Associative Laws;} \\
& \equiv t && \text{by the logical identities on negation and identities,}
\end{aligned}
$$

where t is a tautology.

(b)

p	c	$\neg p$	$\neg p \rightarrow c$	$(\neg p \rightarrow c) \rightarrow p$
T	F	F	T	T
F	F	T	F	T

Since the column for $(\neg p \to c) \to p$ consists entirely of T's, we see that $(\neg p \to c) \to p$ is a tautology.

Alternatively, we may use the logical identities as follows:

$$\begin{aligned}
(\neg p \to c) \to p &\equiv \neg(p \vee c) \vee p && \text{by the logical identity on implication;} \\
&\equiv \neg p \vee p && \text{by the logical identities on negation and identities;} \\
&\equiv t && \text{by the logical identity on negation,}
\end{aligned}$$

where t is a tautology.

1.10. Suppose $P \equiv Q$.

Consider any substitution of propositions into the propositional variables in P and Q. We split into two cases.

- Suppose P evaluates to T under this substitution. Then Q evaluates to T too because $P \equiv Q$. So both $\neg P$ and $\neg Q$ evaluate to F according to the definition of $\neg$.
- Suppose P evaluates to F under this substitution. Then Q evaluates to F too because $P \equiv Q$. So both $\neg P$ and $\neg Q$ evaluate to T according to the definition of $\neg$.

Thus $\neg P$ and $\neg Q$ evaluate to the same truth value in all cases.

Since this is true regardless of which substitution we pick, we conclude that $\neg P \equiv \neg Q$.