CS1231 Chapter 8

Cardinality

8.1 Pigeonhole principles

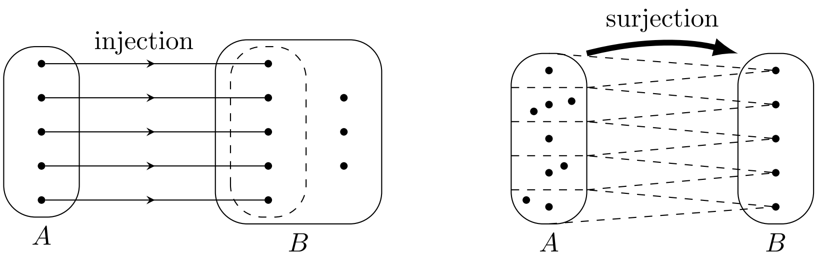

Figure 8.1: Injections, surjections, and the number of elements in the domain and the codomain

Theorem 8.1.1 (Pigeonhole Principle)**.** Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. If there is an injection $A \to B$, then $n \leqslant m$.

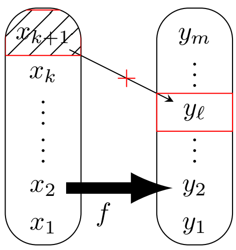

Figure 8.2: Induction proofs for the Pigeonhole Principles

Proof (extra material). We prove this by induction on n.

(Base step) If $n = 0$, then $m \geqslant 0 = n$ for all $m \in \mathbb{N}$. So the theorem is true for $n = 0$.

(Induction step) Let $k \in \mathbb{N}$ such that the theorem is true for $n = k$. We want to prove the theorem for $n = k + 1$. Consider $A = \{x_1, x_2, \ldots, x_{k+1}\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $m \in \mathbb{N}$, such that the x's are different, and the y's are different. Suppose we have an injection $f\colon A \to B$. Let $y_\ell = f(x_{k+1})$. By the injectivity of f, as the x's are all different, no $i \in \{1, 2, \ldots, k\}$ can make $f(x_i) = f(x_{k+1}) = y_\ell$. All such $f(x_i)$'s must appear in the list

$$y_1, y_2, \ldots, y_{\ell-1}, y_{\ell+1}, \ldots, y_m.$$

Let $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{m-1}$ denote the elements of this list. Define $\hat{f}\colon \{x_1, x_2, \ldots, x_k\} \to \{\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{m-1}\}$ by setting $\hat{f}(x_i) = f(x_i)$ for each $i \in \{1, 2, \ldots, k\}$. Then $\hat{f}$ is injective because if $i, j \in \{1, 2, \ldots, k\}$ such that $\hat{f}(x_i) = \hat{f}(x_j)$, then $f(x_i) = f(x_j)$ by the definition of $\hat{f}$, and so the injectivity of f implies $x_i = x_j$. As the x's are all different and the $\hat{y}$'s are all different, the induction hypothesis tells us $k \leqslant m - 1$. Hence $k + 1 \leqslant m$.

This completes the induction. □

Theorem 8.1.2 (Dual Pigeonhole Principle)**.** Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. If there is a surjection $A \to B$, then $n \geqslant m$.

Proof. See Tutorial Exercise 8.6. □

Theorem 8.1.3. Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. Then $n = m$ if and only if there is a bijection $A \to B$.

Proof. ($\Rightarrow$) Suppose $n = m$. Define $f\colon A \to B$ by setting $f(x_i) = y_i$ for each $i \in \{1, 2, \ldots, n\}$. This definition is unambiguous because the x's are different.

Surjectivity follows from the observation that for every $y_i \in B$, we have $x_i \in A$ such that $f(x_i) = y_i$.

To show injectivity, suppose $i, j \in \{1, 2, \ldots, n\}$ such that $f(x_i) = f(x_j)$. The definition of f tells us $f(x_i) = y_i$ and $f(x_j) = y_j$. Then $y_i = f(x_i) = f(x_j) = y_j$. So $i = j$ because the y's are different. This implies $x_i = x_j$.

($\Leftarrow$) This follows directly from Theorem 8.1.1 and Theorem 8.1.2. □

Exercise 8.1.4. Prove the converse to Theorem 8.1.1. Prove also the converse to Theorem 8.1.2 when $B \neq \varnothing$. ✎ 8a

8.2 Same cardinality

Definition 8.2.1 (Cantor)**.** A set A is said to have the *same cardinality* as a set B if there is a bijection $A \to B$.

Note 8.2.2. We defined what it means for a set to have the same cardinality as another set without defining what the cardinality of a set is.

Example 8.2.3. (1) Let $n \in \mathbb{N}$. Then $\{0, 1, \ldots, n-1\}$ has the same cardinality as $\{1, 2, \ldots, n\}$ because the function $f\colon \{0, 1, \ldots, n-1\} \to \{1, 2, \ldots, n\}$ satisfying $f(x) = x + 1$ for all $x \in \{0, 1, \ldots, n-1\}$ is a bijection.

(2) $\mathbb{N}$ has the same cardinality as $\mathbb{N} \setminus \{0\}$ because the function $g\colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$ satisfying $g(x) = x + 1$ for all $x \in \mathbb{N}$ is a bijection.

(3) $\mathbb{N}$ has the same cardinality as $\mathbb{N} \setminus \{1, 3, 5, \ldots\}$ because the function $h\colon \mathbb{N} \to \mathbb{N} \setminus \{1, 3, 5, \ldots\}$ satisfying $h(x) = 2x$ for all $x \in \mathbb{N}$ is a bijection.

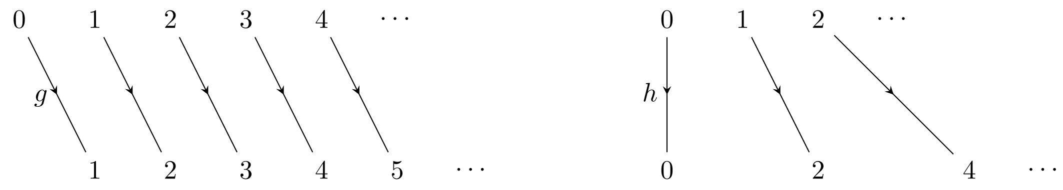

Figure 8.3: Removing 1 or half of the elements from $\mathbb{N}$

Proposition 8.2.4. Let A, B, C be sets.

(1) A has the same cardinality as A. (reflexivity)

(2) If A has the same cardinality as B, then B has the same cardinality as A. (symmetry)

(3) If A has the same cardinality as B, and B has the same cardinality as C, then A has the same cardinality as C. (transitivity)

Proof. See Tutorial Exercise 8.3. □

Exercise 8.2.5. Prove that, if f is an injection $A \to B$, then A has the same cardinality as range(f). 8b

Definition 8.2.6. A set A is *finite* if it has the same cardinality as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. In this case, we call n *the cardinality* or the *size* of A, and we denote it by $|A|$. A set is *infinite* if it is not finite.

Exercise 8.2.7. Prove that no function $\mathbb{N} \to \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, can be injective. Deduce that $\mathbb{N}$ is infinite. 8c

Lemma 8.2.8. Let A and B be sets of the same cardinality. Then A is finite if and only if B is finite.

Proof. See Tutorial Exercise 8.5. □

Question 8.2.9. Which of the following is/are true for all sets A, B? 8d

(1) If there is a bijection $A \to B$, then A has the same cardinality as B.

(2) If there is a surjection $A \to B$ that is not an injection, then A does not have the same cardinality as B.

(3) If there is an injection $A \to B$ that is not a surjection, then A does not have the same cardinality as B.

(4) If there is a function $A \to B$ that is neither a surjection nor an injection, then A does not have the same cardinality as B.

Tutorial exercises

An asterisk (*) indicates a more challenging question.

8.1. In view of the Pigeonhole Principle, for (possibly infinite) sets A and B, one may define "the cardinality of A is at most that of B" to mean the existence of an injection $A \to B$. One desirable property of this definition is transitivity, i.e., if A, B, C are sets such that the cardinality of A is at most that of B and the cardinality of B is at most that of C, then the cardinality of A is at most that of C. To prove this, one needs to find an injection $A \to C$ given injections $A \to B$ and $B \to C$. The aim of this exercise is to show that we can get the required injection by composing together the given injections. We also aim to establish the analogous result for surjections.

Let $f\colon A \to B$ and $g\colon B \to C$. Prove the following propositions.

(a) If f and g are surjective, then so is $g \circ f$.

(b) If f and g are injective, then so is $g \circ f$.

(Hint: mimic the proof of Proposition 7.3.1.)

8.2. Here we consider the converses to the propositions proved in Exercise 8.1.

(a) Give an example of sets A, B, C and functions $f\colon A \to B$ and $g\colon B \to C$ such that $g \circ f$ is surjective but f is not surjective.

(b) Give an example of sets A, B, C and functions $f\colon A \to B$ and $g\colon B \to C$ such that $g \circ f$ is injective but g is not injective.

(c) Give an example of sets A, B, C and functions $f\colon A \to B$ and $g\colon B \to C$ such that $g \circ f$ is bijective but both f and g are not bijective.

(Hint: some diagrams may help.)

8.3. Prove Proposition 8.2.4, i.e., the following propositions.

(a) (Reflexivity) Every set A has the same cardinality as itself.

(b) (Symmetry) For all sets A and B, if A has the same cardinality as B, then B has the same cardinality as A.

(c) (Transitivity) For all sets A, B and C, if A has the same cardinality as B, and B has the same cardinality as C, then A has the same cardinality as C.

hard qn

can prob skip this

8.4. Consider the equivalence relation R from Tutorial Exercise 6.3, i.e., the equivalence relation R on $\mathbb{R}^2$ satisfying, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$(x_1, y_1)\ R\ (x_2, y_2) \quad \Leftrightarrow \quad 3(x_1 - x_2) = y_1 - y_2.$$

Recall from Tutorial Exercise 6.3(b) that the equivalence classes have the form

$$[(u, v)] = \{(x, y) \in \mathbb{R}^2 : y = 3x + (v - 3u)\},$$

where $(u, v) \in \mathbb{R}^2$. Prove that all these equivalence classes have the same cardinality.

(Hint: thinking geometrically may help.)

8.5. Prove Lemma 8.2.8, i.e., the following proposition.

> Let A and B be sets of the same cardinality. Then A is finite if and only if B is finite.

8.6.* (Induction corner) Recall from Theorem 8.1.2 that the Dual Pigeonhole Principle states the following.

importance of drawing a picture

> Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. If there is a surjection $A \to B$, then $n \geqslant m$.

Someone attempts to prove this principle as follows.

proving that n ⩾ m
base: when n = 0, m = 0

> We prove this by induction on n.
>
> **(Base step)** Consider the theorem for $n = 0$. Let f be a surjection $\{\} \to \{y_1, y_2, \ldots, y_m\}$, where $m \in \mathbb{N}$, such that the y's are different. Suppose $m \geqslant 1$. Consider y_1. The surjectivity of f gives $x \in \{\}$ such that $f(x) = y_1$. However, no x can be in $\{\}$. This is a contradiction. So $m = 0 = n$.
>
> **(Induction step)** Let $k \in \mathbb{N}$ such that the theorem is true for $n = k$. We want to prove the theorem for $n = k+1$. Let $A = \{x_1, x_2, \ldots, x_{k+1}\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $m \in \mathbb{N}$, such that the x's are different, and the y's are different. Suppose we have a surjection $f\colon A \to B$. Let $y_\ell = f(x_{k+1})$, so that no $i \in \{1, 2, \ldots, k\}$ makes $f(x_i) = y_\ell$. Then all such $f(x_i)$'s must appear in the list
>
> WRONG: does not apply for surjection, only injection
>
> $$y_1, y_2, \ldots, y_{\ell-1}, y_{\ell+1}, \ldots, y_m.$$
>
> Let $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{m-1}$ denote the elements of this list. Define
>
> $$\hat{f}\colon \{x_1, x_2, \ldots, x_k\} \to \{\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{m-1}\}$$
>
> by setting $\hat{f}(x_i) = f(x_i)$ for each $i \in \{1, 2, \ldots, k\}$. We claim that $\hat{f}$ is surjective. To prove this, consider any $\hat{y}$. It must equal y_h where $h \in \{1, 2, \ldots, m\} \setminus \{\ell\}$. By the surjectivity of f, we have $i \in \{1, 2, \ldots, k+1\}$ such that $y_h = f(x_i)$. As $\ell \neq h$ and the y's are all different, we know $y_\ell \neq y_h = f(x_i)$. Since $y_\ell = f(x_{k+1})$, we deduce that $i \neq k+1$. Hence $y_h = f(x_i) = \hat{f}(x_i)$. As the x's are all different and the $\hat{y}$'s are all different, the induction hypothesis tells us $k \geqslant m - 1$. So $k + 1 \geqslant m$.
>
> This completes the induction.

(a) What is wrong with this attempt?

(b) Turn this attempt into a correct proof.

Extra exercises

8.7. We demonstrated in Exercise 8.2 that the converses to the propositions proved in Exercise 8.1 are false. However, some partial converses are true, as we show in this exercise.

Let $f\colon A \to B$ and $g\colon B \to C$. Prove the following propositions.

(a) If $g \circ f$ is surjective, then g is surjective.

(b) If $g \circ f$ is injective, then f is injective.

8.8. In this exercise, we verify that any finite set has at most one cardinality according to Definition 8.2.6, as one would expect.

Let A be a set, and $m, n \in \mathbb{N}$. Prove that, if A has the same cardinality as both $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$, then $m = n$.

8.9.* In this exercise, we consider the special case of the propositions in Question 8.2.9(2) and (3) when the sets involved are finite.

Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. Prove the following propositions.

(a) If there is a surjection $f\colon A \to B$ that is not an injection, then $n > m$.

(b) If there is an injection $f\colon A \to B$ that is not a surjection, then $n < m$.