CS1231 Chapter 10

Counting

10.1 Sums and differences

Proposition 10.1.1. Let $A_0, A_1, A_2, \ldots$ be finite sets. Then $A_0 \cup A_1 \cup \cdots \cup A_n$ is finite for all $n \in \mathbb{N}$.

Proof. Proceed by induction on n as in Tutorial Exercise 9.3 using Tutorial Exercise 9.1. $\square$

Combinatorial interpretation of Proposition 10.1.1. If x is a variable that can only take an element of one of finitely many finite sets, then there are finitely many ways to substitute objects into x.

Figure 10.1: Addition Rule and Difference Rule

Example 10.1.2. The sets $\{1, 2\}$ and $\{3, 4, 5\}$ are disjoint. Note that

$$|\{1, 2\} \cup \{3, 4, 5\}| = |\{1, 2, 3, 4, 5\}| = 5 = 2 + 3 = |\{1, 2\}| + |\{3, 4, 5\}|.$$

Proposition 10.1.3 (Addition Rule)**.** Let A and B be disjoint finite sets. Then $|A \cup B| = |A| + |B|$.

A n B = ∅

Proof. Proceed as in Tutorial Exercise 9.1. $\square$

Combinatorial interpretation of the Addition Rule. Let $m, n \in \mathbb{N}$. Suppose that the variable x can only take either one of m objects or one of n objects, and that these m objects are different from the n objects here. Then there are exactly $m+n$ ways to substitute objects into x.

Example 10.1.4. Note that $\{1,2\} \subseteq \{1,2,3,4,5\}$. Also

$$|\{1,2,3,4,5\} \setminus \{1,2\}| = |\{3,4,5\}| = 3 = 5 - 2 = |\{1,2,3,4,5\}| - |\{1,2\}|.$$

Corollary 10.1.5 (Difference Rule)**.** Let X and Y be finite sets. Then $Y \setminus X$ is finite, and if $X \subseteq Y$, then $|Y \setminus X| = |Y| - |X|$.

x must be a subset of y

Proof. First, note that $Y \setminus X = \{x \in Y : x \notin X\} \subseteq Y$. So $Y \setminus X$ is finite by Proposition 9.2.6(1) as Y is finite. Now, in view of Extra Exercise 4.9, we know $Y \setminus X$ and X are disjoint and $(Y \setminus X) \cup X = X \cup Y$. As $X \subseteq Y$, we also know from Tutorial Exercise 4.6 that $X \cup Y = Y$. Hence

$$\begin{aligned}
|Y| &= |(Y \setminus X) \cup X| && \text{as } (Y \setminus X) \cup X = X \cup Y = Y; \\
&= |Y \setminus X| + |X| && \text{by the Addition Rule, as } Y \setminus X \text{ and } X \text{ are disjoint.}
\end{aligned}$$

$\therefore \quad |Y \setminus X| = |Y| - |X|.$ □

Combinatorial interpretation of the Difference Rule. Let $m, n \in \mathbb{N}$. Suppose that the variable x can only take one of m objects except n of them. Then there are exactly $m - n$ ways to substitute objects into x.

Remark 10.1.6. The Addition Rule becomes false if one drops the disjointness condition. For example,

$$|\{1,2\} \cup \{2,3,4\}| = |\{1,2,3,4\}| = 4 \neq 5 = 2 + 3 = |\{1,2\}| + |\{2,3,4\}|.$$

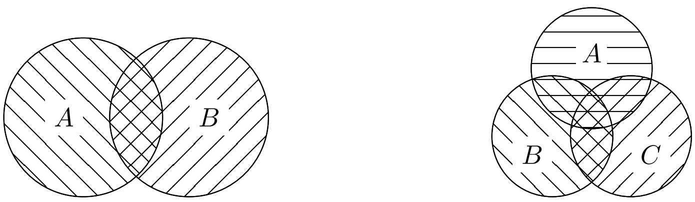


Figure 10.2: Inclusion–Exclusion Rule for two and three sets

Theorem 10.1.7 (Inclusion–Exclusion Rule for two sets)**.** Let A, B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

no need to be disjoint

Proof. Observe that $A \setminus B = A \setminus (A \cap B)$. (Exercise: prove this.) Thus 10a

$$\begin{aligned}
|A \cup B| &= |(A \setminus B) \cup B| && \text{by Extra Exercise 4.9;} \\
&= |A \setminus B| + |B| && \text{by the Addition Rule, as } (A \setminus B) \cap B = \varnothing; \\
&= |A \setminus (A \cap B)| + |B| && \text{by the observation above;} \\
&= |A| - |A \cap B| + |B| && \text{by the Difference Rule, as } A \cap B \subseteq A.
\end{aligned}$$

disjoint

□

Example 10.1.8. By the Inclusion–Exclusion Rule for two sets,

$$\begin{aligned}
|\{1,2\} \cup \{2,3,4\}| &= |\{1,2\}| + |\{2,3,4\}| - |\{1,2\} \cap \{2,3,4\}| \\
&= |\{1,2\}| + |\{2,3,4\}| - |\{2\}| \\
&= 2 + 3 - 1 = 4.
\end{aligned}$$

Combinatorial interpretation of the Inclusion–Exclusion Rule for two sets. Let $m, n \in \mathbb{N}$. Suppose that the variable x can only take either one of m objects or one of n objects, and that the m objects and the n objects here have exactly k objects in common. Then there are exactly $m + n - k$ ways to substitute objects into x.

Corollary 10.1.9 (Inclusion–Exclusion Rule for three sets)**.** Let A, B, C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Proof.

$$\begin{aligned}
&|A \cup B \cup C| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| && \text{by the Inclusion–Exclusion Rule for two sets;} \\
&= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| && \text{by the Distributive Laws;} \\
&= |A| + |B| + |C| - |B \cap C| \\
&\quad - |A \cap B| - |A \cap C| + |A \cap B \cap A \cap C| && \text{by the Inclusion–Exclusion Rule for two sets;} \\
&= |A| + |B| + |C| - |A \cap B| \\
&\quad - |B \cap C| - |C \cap A| + |A \cap B \cap C| && \text{by the set identity on idempotence.}
\end{aligned}$$

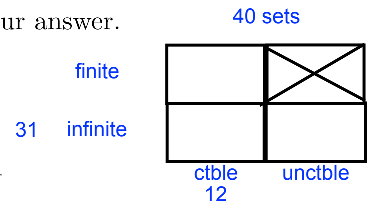
□

Example 10.1.10. By the Inclusion–Exclusion Rule for three sets,

$$\begin{aligned}
&|\{-1,0\} \cup \{0,2,3\} \cup \{-1,0,1,2\}| \\
&= |\{-1,0\}| + |\{0,2,3\}| + |\{-1,0,1,2\}| - |\{-1,0\} \cap \{0,2,3\}| - |\{0,2,3\} \cap \{-1,0,1,2\}| \\
&\quad - |\{-1,0,1,2\} \cap \{-1,0\}| + |\{-1,0\} \cap \{0,2,3\} \cap \{-1,0,1,2\}| \\
&= |\{-1,0\}| + |\{0,2,3\}| + |\{-1,0,1,2\}| - |\{0\}| - |\{0,2\}| - |\{-1,0\}| + |\{0\}| \\
&= 2 + 3 + 4 - 1 - 2 - 2 + 1 = 5.
\end{aligned}$$

if you are not countable, you cannot be finite

Exercise 10.1.11. There are 40 sets, of which 12 are countable and 31 are infinite. How many of them are both countable and infinite? Explain your answer. ✎ 10b



$|C| = 12$
$|I| = 31$
$|C \cup I| = 40$
$|C \cup I| = |C| + |I| - |C \cap I|$
$= 3$

10.2 Products and powers

Example 10.2.1. Let $A = \{\mathrm{a}, \mathrm{b}\}$ and $B = \{1, 2, 3\}$. Then

$$|A \times B| = |\{(\mathrm{a},1), (\mathrm{a},2), (\mathrm{a},3), (\mathrm{b},1), (\mathrm{b},2), (\mathrm{b},3)\}| = 6 = 2 \times 3 = |A| \times |B|.$$

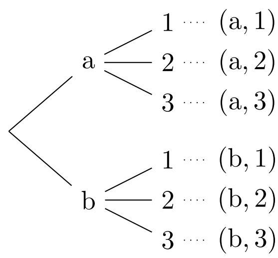


Figure 10.3: The Cartesian product $\{\mathrm{a}, \mathrm{b}\} \times \{1, 2, 3\}$

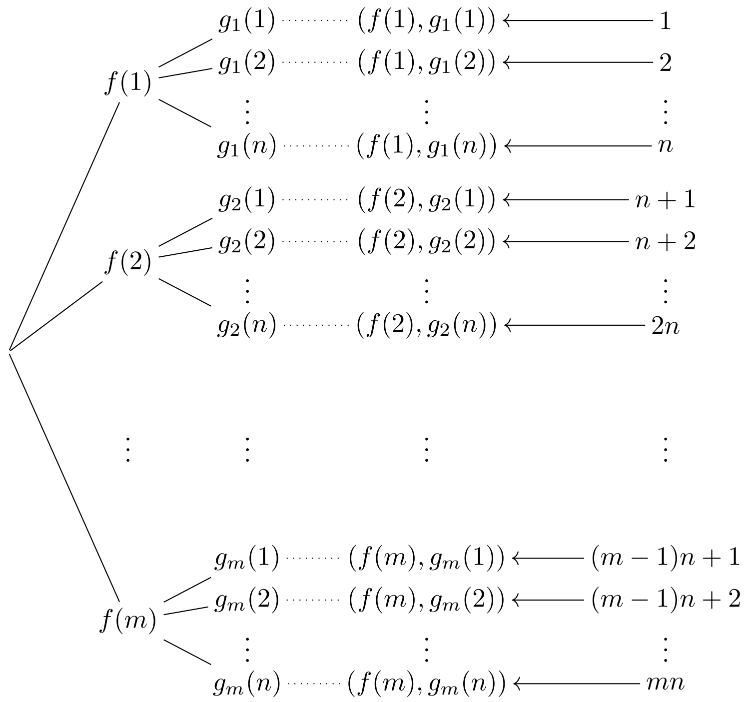


Figure 10.4: Proof of the General Multiplication Rule

Proposition 10.2.2 (General Multiplication Rule)**.** Fix $m, n \in \mathbb{N}$. Let A be a set of size m, and for each $x \in A$, let B_x be a set of size n. Then $\{(x, y) : x \in A \text{ and } y \in B_x\}$ is finite and has size mn.

Proof. Use our conditions on m and n to find bijections $f\colon \{1, 2, \ldots, m\} \to A$ and $g_i\colon \{1, 2, \ldots, n\} \to B_{f(i)}$ for each $i \in \{1, 2, \ldots, m\}$. Then

$$h = \big\{\big((i-1)n + j, (f(i), g_i(j))\big) : i \in \{1, 2, \ldots, m\} \text{ and } j \in \{1, 2, \ldots, n\}\big\}$$

is bijection $\{1, 2, \ldots, mn\} \to \{(x, y) : x \in A \text{ and } y \in B_x\}$. This shows what we want. $\square$

Explanation of why h is a bijection in the proof above (extra material). Before verifying bijectivity, let us check that h is indeed a function as claimed.

(F1) It suffices to prove that

$$\forall u \in \{1, 2, \ldots, mn\} \;\; \exists i \in \{1, 2, \ldots, m\} \;\; \exists j \in \{1, 2, \ldots, n\} \;\; \big(u = (i-1)m + j\big). \qquad (*)$$

If $m = 0$ or $n = 0$, then $mn = 0$ and thus $(*)$ is vacuously true. So suppose $m > 0$ and $n > 0$. We will proceed by induction on x.

(Base step) Note that $1 = (1-1)n + 1$; so $(*)$ is true for $x = 1$.

(Induction step) Let $v \in \mathbb{Z}^+$ such that $(*)$ is true for $u = v$. We want to prove $(*)$ for $u = v + 1$. If $v + 1 > mn$, then $(*)$ is vacuously true. So suppose $v + 1 \leqslant mn$, so that $v \leqslant mn - 1$ and thus $v \in \{1, 2, \ldots, mn\}$. Use the induction hypothesis to find $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ such that $v = (i-1)n + j$. Then either $i \leqslant m - 1$ or $j \leqslant n - 1$, because otherwise $v = (i-1)n + j = (m-1)n + n = mn > mn - 1 \geqslant v$, which is a contradiction.

Case 1: suppose $j = n$. Then $i \leqslant m - 1$. So

$$v + 1 = (i-1)n + j + 1 = (i-1)n + n + 1 = in + 1 = ((i+1) - 1)n + 1,$$

where $i + 1 \leqslant m$.

Case 2: suppose $j \leqslant n-1$. Then $v+1 = (i-1)n + j + 1$, where $j+1 \leqslant n$.

Hence $v+1 = (i_0 - 1)m + j_0$ for some $i_0 \in \{1, 2, \ldots, m\}$ and $j_0 \in \{1, 2, \ldots, n\}$.
This completes the induction.

(F2) Let $u \in \{1, 2, \ldots, mn\}$ and $i_1, i_2 \in \{1, 2, \ldots, m\}$ and $j_1, j_2 \in \{1, 2, \ldots, n\}$ such that $(u, (f(i_1), g_{i_1}(j_1))), (x, (f(i_2), g_{i_2}(j_2))) \in h$. Without loss of generality, assume $i_1 \geqslant i_2$. Now $(i_1 - 1)n + j_1 = u = (i_2 - 1)n + j_2$ according to the definition of h. As $i_1 \geqslant i_2$ and $j_1, j_2 \in \{1, 2, \ldots, n\}$, this implies

$$0 \leqslant i_1 - i_2 = \frac{j_2 - j_1}{n} \leqslant \frac{n-1}{n} < 1.$$

So $i_1 - i_2 = 0$ because $i_1 - i_2 \in \mathbb{Z}$. Hence $i_1 = i_2$, implying $(i_1 - 1)n + j_1 = (i_1 - 1)n + j_2$ and thus $j_1 = j_2$. It follows that $(f(i_1), g(j_1)) = (f(i_2), g(j_2))$.

(Surjectivity) Take $(x_0, y_0) \in \{(x, y) : x \in A \text{ and } y \in B_x\}$. Then $x_0 \in A$ and $y_0 \in B_{x_0}$. Use the surjectivity of f and the g_i's to find $i_0 \in \{1, 2, \ldots, n\}$ and $j_0 \in \{1, 2, \ldots, m\}$ such that $x_0 = f(i_0)$ and $y_0 = g_{i_0}(j_0)$. Then

$$1 = (1-1)n + 1 \leqslant (i_0 - 1)n + j_0 \leqslant (m-1)n + n = mn$$

and $h\big((i_0 - 1)n + j_0\big) = \big(f(i_0), g_{i_0}(j_0)\big) = (x_0, y_0)$.

(Injectivity) Take $i_1, i_2 \in \{1, 2, \ldots, m\}$ and $j_1, j_2 \in \{1, 2, \ldots, m\}$ such that

$$h\big((i_1 - 1)n + j_1\big) = h\big((i_2 - 1)n + j_2\big).$$

In view of the definition of h, this means $\big(f(i_1), g_{i_1}(j_1)\big) = \big(f(i_2), g_{i_2}(j_2)\big)$ Thus $f(i_1) = f(i_2)$ and $g_{i_1}(j_1) = g_{i_2}(j_2)$. The former implies $i_1 = i_2$ because f is injective. So the latter implies $j_1 = j_2$ as g_{i_1} is injective. ■

Combinatorial interpretation of the General Multiplication Rule. Let $m, n \in \mathbb{N}$. Suppose that in (x, y) the variable x can only take one of m objects, and possibly depending on what x takes the variable y can only take one of n objects. Then there are exactly mn ways to substitute objects into (x, y).

Corollary 10.2.3. Let $n \in \mathbb{Z}^+$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}$. Suppose that in $(x_1, x_2, \ldots, x_n)$,

- the variable x_1 can only take one of m_1 objects;
- possibly depending on what x_1 takes, the variable x_2 can only take one of m_2 objects;
- possibly depending on what x_1, x_2 take, the variable x_3 can only take one of m_3 objects;
- ...
- possibly depending on what $x_1, x_2, \ldots, x_{n-1}$ take, the variable x_n can only take one of m_n objects.

Then there are exactly $m_1 m_2 \ldots m_n$ ways to substitute objects into $(x_1, x_2, \ldots, x_n)$.

Proof. We proceed by induction on n.

(Base step) Clearly, if the variable x_1 can only take one of m_1 objects, then there are exactly m_1 ways to substitute objects into x_1.

(Induction step) Let $k \in \mathbb{Z}_{\geqslant 2}$ such that the corollary is true for $n = k$. Suppose that $m_1, m_2, \ldots, m_{k+1} \in \mathbb{N}$ and in $(x_1, x_2, \ldots, x_{k+1})$,

- the variable x_1 can only take one of m_1 objects;
- possibly depending on what x_1 takes, the variable x_2 can only take one of m_2 objects;
- possibly depending on what x_1, x_2 take, the variable x_3 can only take one of m_3 objects;
- ...
- possibly depending on what $x_1, x_2, \ldots, x_{k-1}$ take, the variable x_k can only take one of m_k objects; and
- possibly depending on what $x_1, x_2, \ldots, x_k$ take, the variable x_{k+1} can only take one of m_{k+1} objects.

Then, by the induction hypothesis, there are $m_1 m_2 \ldots m_k$ ways to substitute objects into $(x_1, x_2, \ldots, x_k)$. It follows from the General Multiplication Rule that there are $m_1 m_2 \ldots m_k m_{k+1}$ ways to substitute objects into $(x_1, x_2, \ldots, x_k, x_{k+1})$.

This completes the induction. □

Corollary 10.2.4 (Multiplication Rule)**.** Let A, B be finite sets and $n \in \mathbb{Z}_{\geqslant 2}$.

(1) $A \times B$ is finite and $|A \times B| = |A| \times |B|$.

(2) $|A^n| = |A|^n$.

Proof. (1) Apply the General Multiplication Rule with $B_x = B$ for each $x \in A$.

(2) Applying part (1) repeatedly,

$$|A^n| = |\underbrace{A \times A \times \cdots \times A}_{n\text{-many } A\text{'s}}| = \underbrace{|A| \times |A| \times \cdots \times |A|}_{n\text{-many } |A|\text{'s}} = |A|^n.$$ □

5. Let A be a set and $n \in \mathbb{N}$. Notationally, it is sometimes convenient to identify an ordered n-tuple $(a_1, a_2, \ldots, a_n) \in A^n$ with the string $a_1 a_2 \ldots a_n$ of length n. This identification exploits the bijectivity of the function from A^n to the set of all strings over A of length n mapping each $(a_1, a_2, \ldots, a_n) \in A^n$ with $a_1 a_2 \ldots a_n$.

Example 10.2.6. Let $A = \{1, 2, 3, 4, 5\}$. One can represent each subset $S \subseteq A$ by the string $d_1 d_2 d_3 d_4 d_5$ over $\{0, 1\}$ satisfying

$$i \in S \quad \Leftrightarrow \quad d_i = 1$$

for all $i \in \{1, 2, 3, 4, 5\}$. For instance,

$\{\ \}$	is represented by	00000;
$\{1, 3, 5\}$	is represented by	10101;
$\{1,2, 4\}$	is represented by	11010;
$\{1,2,3,4,5\}$	is represented by	11111.

if it is in the set, it is represented by 1, else 0

As one can verify, this representation gives rise to a bijection $\mathcal{P}(A) \to \{0, 1\}^5$. So $|\mathcal{P}(A)| = |\{0, 1\}^5| = 2^5 = 32$ by Corollary 10.2.4(2).

Theorem 10.2.7. Let A be a finite set. Then $\mathcal{P}(A)$ is finite and $|\mathcal{P}(A)| = 2^{|A|}$.

Proof. Let $n = |A|$ and $A = \{a_1, a_2, \ldots, a_n\}$. As in Example 10.2.6, represent each subset $S \subseteq A$ by the string $d_1 d_2 \ldots d_n$ over $\{0, 1\}$ satisfying

$$a_i \in S \quad \Leftrightarrow \quad d_i = 1$$

for all $i \in \{1, 2, \ldots, n\}$. This representation gives rise to a bijection $\mathcal{P}(A) \to \{0,1\}^n$. So $|\mathcal{P}(A)| = |\{0,1\}^n| = 2^n$ by Corollary 10.2.4(2). □

Combinatorial interpretation of Theorem 10.2.7. Let $n \in \mathbb{N}$. Then the number of ways to make n binary choices is 2^n.

Remark 10.2.8. Like in the proof of Theorem 10.2.7, instead of counting the objects themselves, often it is more convenient to count suitably chosen representations of them instead. Implicitly, each such representation is a bijection. Proofs of bijectivity are usually omitted if they are straightforward and distracting. We will do the same. Nevertheless, the representations themselves should be clearly formulated.

10.3 Permutations and combinations

Definition 10.3.1. Define $0! = 1$ and $(n + 1)! = (n + 1) \times n!$ for all $n \in \mathbb{N}$.

Remark 10.3.2. For every $n \in \mathbb{N}$,

$$\begin{aligned} n! &= n \times (n-1)! = n \times (n-1) \times (n-2)! = \cdots \\ &= n \times (n-1) \times \cdots \times 1 \times 0! = n \times (n-1) \times \cdots \times 1. \end{aligned}$$

Definition 10.3.3. Let $r, n \in \mathbb{N}$ and Γ be a finite set.

(1) An *r-permutation* of Γ is a string of length r over Γ in which no symbol appears in two different positions.

(2) Let $P(n, r)$ denote the number of r-permutations of a set of size n.

(3) A *permutation* of Γ is a $|\Gamma|$-permutation of Γ.

Example 10.3.4. Let $\Gamma = \{\mathrm{E}, \mathrm{I}, \mathrm{L}, \mathrm{N}, \mathrm{S}, \mathrm{T}\}$.

(1) The 4-permutations of Γ include LNST, TELI, LINE, and SENT, but *not* any of SIT, ABCD, SEEN, and NILET.

(2) The permutations of Γ include EILNST, TELSIN, LISTEN, and SILENT.

Remark 10.3.5. (1) One can view an r-permutation of a set Γ as a way to pick r elements from Γ without replacement where order matters.

(2) One can view a permutation of a finite set Γ of size n as a way to arrange the n elements of Γ into n positions.

(3) Some write ${}_nP_r$ or ${}^nP_r$ or P_r^n for $P(n, r)$.

Theorem 10.3.6. For all $r, n \in \mathbb{N}$,

$$P(n, r) = \begin{cases} \dfrac{n!}{(n-r)!}, & \text{if } r \leqslant n; \\ 0, & \text{if } r > n. \end{cases}$$

n*(n-1)…*(n-r+1)

Proof. Fix a set Γ of size n.

Suppose there is some r-permutation of Γ, say $a_1 a_2 \dots a_r$. As no symbol appears in two different positions in an r-permutation, the function $f\colon \{1, 2, \dots, r\} \to \Gamma$ where each $f(i) = a_i$ is an injection. So $r = |\{1, 2, \dots, r\}| \leqslant |\Gamma| = n$ by the Pigeonhole Principle.

The paragraph above shows that, when $r > n$, there is no r-permutation of Γ, and thus $P(n, r) = 0$. So let us concentrate on the case when $r \leqslant n$.

Proof for r ≤ n As no symbol appears in two different positions, the r-permutations are precisely those $x_1 x_2 \dots x_r$ where

- x_1 can only take one of the n elements of Γ, say a_1;
- x_2 can only take one of the $n - 1$ elements of $\Gamma \setminus \{a_1\}$, say a_2;
- $\dots$
- x_r can only take one of the $n - (r - 1)$ elements of $\Gamma \setminus \{a_1, a_2, \dots, a_{r-1}\}$, say a_r.

So the General Multiplication Rule tells us that

$$\begin{aligned} P(n, r) &= n \times (n - 1) \times \cdots \times (n - (r - 1)) \\ &= \frac{n(n-1)\dots(n-(r-1))(n-r)(n-(r+1))\dots 1}{(n-r)(n-(r+1))\dots 1} \\ &= \frac{n!}{(n-r)!} \qquad \text{by Remark 10.3.2.} \end{aligned}$$

□

Combinatorial interpretation of Theorem 10.3.6. Let $r, n \in \mathbb{N}$.

(1) If $r \leqslant n$, then there are exactly $\frac{n!}{(n-r)!}$ ways to pick r objects from n objects without replacement where order matters.

(2) If $r > n$, then there is no way to pick r objects from n objects without replacement.

Corollary 10.3.7. Let Γ be a set of size $n \in \mathbb{N}$. Then Γ has exactly $n!$ permutations.

Proof. By Theorem 10.3.6, the number of permutations of Γ is

$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!.$$

□

Combinatorial interpretation of Corollary 10.3.7. Let $n \in \mathbb{N}$. Then there are exactly $n!$ ways to arrange n objects into n positions.

Example 10.3.8. Let $\Gamma = \{\mathrm{E}, \mathrm{I}, \mathrm{L}, \mathrm{N}, \mathrm{S}, \mathrm{T}\}$. Note $|\Gamma| = 6$. So, according to Theorem 10.3.6 and Corollary 10.3.7,

(1) the number of 4-permutations of Γ is $P(6, 4) = \frac{6!}{(6-4)!} = 6 \times 5 \times 4 \times 3 = 360$; and

(2) the number of permutations of Γ is $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Definition 10.3.9. Let Γ be a set and $a_1 a_2 \dots a_n$ be a string over Γ. A *permutation* of $a_1 a_2 \dots a_n$ is a string over Γ of the form

$$a_{f(1)} a_{f(2)} \dots a_{f(n)}$$

where f is some bijection $\{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.

Remark 10.3.10. (1) One can view a permutation of a string as a way to rearrange the symbols in the string.

(2) Let $n \in \mathbb{N}$ and Γ be a set of size n, say $\Gamma = \{a_1, a_2, \ldots, a_n\}$. Then the permutations of the set Γ in the sense of Definition 10.3.3 are precisely the permutations of the string $a_1 a_2 \ldots a_n$ in the sense of Definition 10.3.9.

Example 10.3.11. Here is a complete list of the permutations of the string EGG:

$$\text{EGG}, \qquad \text{GEG}, \qquad \text{GGE}.$$

To calculate (as opposed to count) the number of permutations of EGG, consider the set $\Gamma = \{\mathrm{E}, \mathrm{G}_1, \mathrm{G}_2\}$. Here is a complete list of the permutations of the set Γ:

$$\begin{array}{lll} \mathrm{E}\mathrm{G}_1\mathrm{G}_2, & \mathrm{G}_1\mathrm{E}\mathrm{G}_2, & \mathrm{G}_1\mathrm{G}_2\mathrm{E}, \\ \mathrm{E}\mathrm{G}_2\mathrm{G}_1, & \mathrm{G}_2\mathrm{E}\mathrm{G}_1, & \mathrm{G}_2\mathrm{G}_1\mathrm{E}. \end{array}$$

Each permutation s of the string EGG corresponds to exactly $2!$ permutations of the set Γ because one can arrange the 2 objects $\mathrm{G}_1, \mathrm{G}_2$ into the 2 positions where G appears in s in exactly $2!$ ways by (the combinatorial interpretation of) Corollary 10.3.7. Therefore, by the General Multiplication Rule,

$$\begin{pmatrix}\text{number of}\\ \text{permutations}\\ \text{of } \Gamma\end{pmatrix} = \begin{pmatrix}\text{number of}\\ \text{permutations}\\ \text{of EGG}\end{pmatrix} \times \begin{pmatrix}\text{number of ways to}\\ \text{arrange } \mathrm{G}_1, \mathrm{G}_2 \text{ into}\\ \text{2 positions}\end{pmatrix}.$$

$$\therefore \qquad 3! = \begin{pmatrix}\text{number of}\\ \text{permutations}\\ \text{of EGG}\end{pmatrix} \times 2!.$$

From this, we see that the number of permutations of EGG is $\frac{3!}{2!} = 6/2 = 3$.

Exercise 10.3.12. How many permutations of the string BICONDITIONAL are there? Explain your answer. ✎ 10c

13! / (3! * 2! * 2!)

Definition 10.3.13. Let $r, n \in \mathbb{N}$ and A be a finite set.

(1) An *r-combination* of A is a subset of A of size r.

(2) Let $\binom{n}{r}$ denote the number of r-combinations of a set of size n. We read $\binom{n}{r}$ as "n choose r".

Remark 10.3.14. (1) One can view an r-combination of a set A as a way to pick r elements from Γ without replacement where order does not matter.

(2) Some write $C(n, r)$ or ${}_nC_r$ or ${}^nC_r$ or C^n_r for $\binom{n}{r}$.

Theorem 10.3.15. For all $r, n \in \mathbb{N}$,

$$\binom{n}{r} = \begin{cases} \dfrac{n!}{r!\,(n-r)!}, & \text{if } r \leqslant n; \\ 0, & \text{if } r > n. \end{cases}$$

Proof. Let A be a set of size n, say $\{a_1, a_2, \ldots, a_n\}$.

Let $S \subseteq A$. Then S is finite by Proposition 9.2.6(1). Also, the function $f\colon S \to A$ where each $f(x) = x$ is an injection. So $|S| \leqslant |A|$ by the Pigeonhole Principle.

The paragraph above shows that, when $r > n$, there is no subset of A of size r, and thus $\binom{n}{r} = 0$. So let us concentrate on the case when $r \leqslant n$.

As in the proof of Theorem 10.2.7, represent each subset $S \subseteq A$ by the string $d_1 d_2 \dots d_n$ over $\{0, 1\}$ satisfying

$$a_i \in S \quad \Leftrightarrow \quad d_i = 1$$

for all $i \in \{1, 2, \dots, n\}$. Under this representation, subsets of A of size r are represented precisely by strings over $\{0, 1\}$ of length n with exactly r-many 1's and $(n - r)$-many 0's, i.e., by the permutations of the string

$$\underbrace{11\dots1}_{r\text{-many 1's}}\overbrace{00\dots0}^{(n-r)\text{-many 0's}}$$

As in Example 10.3.11, the number of such permutations is

$$\frac{\begin{array}{c}\text{number of permutations of}\\ \{1_1, 1_2, \dots, 1_r, 0_1, 0_2, \dots, 0_{n-r}\}\end{array}}{\left(\begin{array}{l}\text{number of ways to arrange}\\ 1_1, 1_2, \dots, 1_r \text{ into}\\ r \text{ positions}\end{array}\right) \times \left(\begin{array}{l}\text{number of ways to arrange}\\ 0_1, 0_2, \dots, 0_{n-r} \text{ into}\\ n - r \text{ positions}\end{array}\right)} = \frac{n!}{r!\,(n-r)!}$$

by the General Multiplication Rule and Corollary 10.3.7. □

Combinatorial interpretation of Theorem 10.3.15. Let $r, n \in \mathbb{N}$.

(1) If $r \leqslant n$, then there are exactly $\frac{n!}{(n-r)!\,r!}$ ways to pick r objects from n objects without replacement where order does not matter.

(2) If $r > n$, then there is no way to pick r objects from n objects without replacement.

Technique 10.3.16 (counting argument)**.** One way to prove $k = \ell$, where $k, \ell \in \mathbb{N}$, is to find two ways of counting the number of elements of the same finite set, the first of which results in k and the second results in ℓ.

Tutorial exercises

An asterisk (*) indicates a more challenging exercise.

10.1. The Inclusion–Exclusion Rule tells us how one can calculate the size of the union of finitely many finite sets from the sizes of the intersections between these sets. We saw how this is done for two sets and for three sets in Theorem 10.1.7 and in Corollary 10.1.9.

State and prove an Inclusion–Exclusion Rule for four sets.

10.2. Fix $m, n \in \mathbb{Z}^+$. Let A be a set of size m and B be a set of size n.

(a) How many relations are there from A to B?

(b) How many functions are there from A to B?

(c)* In the case when $|B| = 3$, how many surjections are there from A to B?

(d) How many injections are there from A to B?

(e) How many bijections are there from A to B?

Briefly explain your answers.

10.3. Pascal's Formula states that: for all $r, n \in \mathbb{Z}^+$ with $r \leqslant n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

(a) Prove Pascal's Formula algebraically.

(b) Prove Pascal's Formula using a counting argument.

10.4. Consider the following proposition.

$$\forall r, n \in \mathbb{N} \qquad \binom{0}{r} + \binom{1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

(a) (Induction corner) Prove the proposition above algebraically by induction.

(b) Prove the proposition above using a counting argument.

Extra exercises

10.5. Fix $m \in \mathbb{Z}^+$. Let A be a set of size m.

(a) How many relations are there on A?

(b) How many reflexive relations are there on A?

(c) How many symmetric relations are there on A?

(d) How many antisymmetric relations are there on A?

Briefly explain your answers.