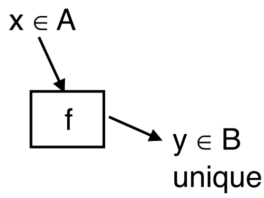
CS1231 Chapter 7



Functions

7.1 Basics

every element of A is related to exactly one B

Let A, B be sets. A *function* or a *map* from A to B is a relation f from A to B such that any element of A is f-related to a unique element of B, i.e.,

(F1) every element of A is f-related to at least one element of B, or in symbols,

$$\forall x \in A \ \ \exists y \in B \ \ (x, y) \in f;$$

y = f(x)

every A has at least one arrow going out of it

(F2) every element of A is f-related to at most one element of B, or in symbols,

$$\forall x \in A \ \ \forall y_1, y_2 \in B \ \ \big((x, y_1) \in f \wedge (x, y_2) \in f \Rightarrow y_1 = y_2\big).$$

every A has at most one arrow going out of it

We write $f\colon A \to B$ for "f is a function from A to B". Here A is called the *domain* of f, and B is called the *codomain* of f.

codomain contains set of all outputs AND can have others

The negations of (F1) and (F2) can be expressed respectively as

($\neg$F1) $\exists x \in A \ \ \forall y \in B \ \ (x, y) \notin f$; and

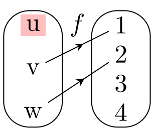
($\neg$F2) $\exists x \in A \ \ \exists y_1, y_2 \in B \ \ \big((x, y_1) \in f \wedge (x, y_2) \in f \wedge y_1 \neq y_2\big)$.

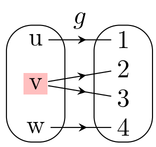
Let $A = \{\mathrm{u}, \mathrm{v}, \mathrm{w}\}$ and $B = \{1, 2, 3, 4\}$.

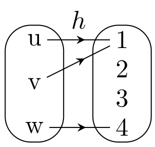
(1) $f = \{(\mathrm{v}, 1), (\mathrm{w}, 2)\}$ is *not* a function $A \to B$ because $\mathrm{u} \in A$ such that no $y \in B$ makes $(\mathrm{u}, y) \in f$, violating (F1).

(2) $g = \{(\mathrm{u}, 1), (\mathrm{v}, 2), (\mathrm{v}, 3), (\mathrm{w}, 4)\}$ is *not* a function $A \to B$ because $\mathrm{v} \in A$ and $2, 3 \in B$ such that $(\mathrm{v}, 2), (\mathrm{v}, 3) \in g$ but $2 \neq 3$, violating (F2).

(3) $h = \{(\mathrm{u}, 1), (\mathrm{v}, 1), (\mathrm{w}, 4)\}$ *is* a function $A \to B$ because both (F1) and (F2) are satisfied.







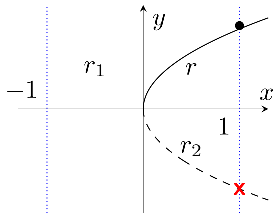
(1) $r = \{(x, y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} : x = y^2\}$ *is* a function $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ because for every $x \in \mathbb{R}_{\geqslant 0}$, there is a unique $y \in \mathbb{R}_{\geqslant 0}$ such that $(x, y) \in r$, namely $y = \sqrt{x}$.

(2) $r_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_{\geqslant 0} : x = y^2\}$ is *not* a function $\mathbb{R} \to \mathbb{R}_{\geqslant 0}$ because $-1 \in \mathbb{R}$ that is not equal to y^2 for any $y \in \mathbb{R}_{\geqslant 0}$, violating (F1).

when x = -1, there is no y such that y^2 = -1 violates (F1)

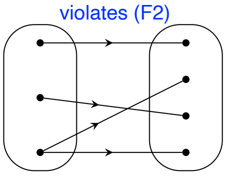
(3) $r_2 = \{(x,y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R} : x = y^2\}$ is *not* a function $\mathbb{R}_{\geqslant 0} \to \mathbb{R}$ because $1 \in \mathbb{R}_{\geqslant 0}$ and $-1, 1 \in \mathbb{R}$ such that $1 = (-1)^2$ and $1 = 1^2$ but $-1 \neq 1$, violating (F2).

when x = 1, y can be both 1 or -1
violates (F2)
(1,-1) ∈ r2,
(1,1) ∈ r2

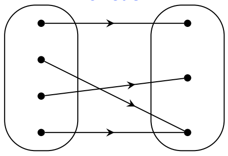


Which of the arrow diagrams below represent a function from the LHS set to the RHS set? ✐ 7a

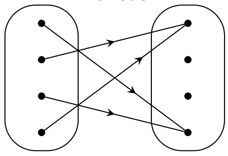
(a) violates (F2)



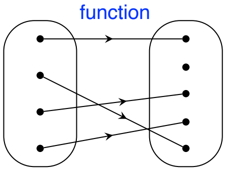
(b) function



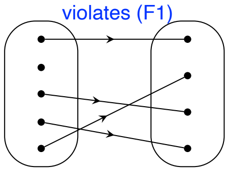
(c) function



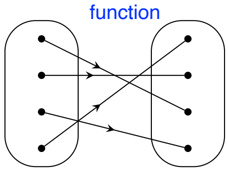
(d) function



(e) violates (F1)



(f) function



7.2 Images

Let $f \colon A \to B$.

y = f(x)

(1) If $x \in A$, then $f(x)$ denotes the unique element $y \in B$ such that $(x,y) \in f$. We call $f(x)$ *the image of x* under f.

(2) The *range of f*, denoted $\mathrm{range}(f)$, is defined by

$$\mathrm{range}(f) = \{f(x) : x \in A\}.$$

It follows from the definition of images that if $f \colon A \to B$ and $x \in A$, then for all $y \in B$,

$$\underbrace{(\overset{\text{input}}{x}, \overset{\text{output}}{y}) \in f}_{\text{function}} \quad \Leftrightarrow \quad \underbrace{\overset{\text{output}}{y} = f(\overset{\text{input}}{x})}_{\text{function}}.$$

IMPORTANT!

(1) The range of a function is the set that contains all the outputs of the function *and nothing else*, while the codomain is the set associated to the function as part of its specification that contains all the outputs *but maybe also other objects*.

(2) For any function, the range is a subset of the codomain.

The function $r \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ in Example 7.1.4(1) satisfies

$$\forall x, y \in \mathbb{R}_{\geqslant 0} \quad \big(y = r(x) \Leftrightarrow x = y^2\big).$$

Note that $\mathrm{range}(r) \supseteq \mathbb{R}_{\geqslant 0}$, because for every $y \in \mathbb{R}_{\geqslant 0}$, there is $x \in \mathbb{R}_{\geqslant 0}$ such that $y = r(x)$, namely $x = y^2$. So $\mathrm{range}(r) = \mathbb{R}_{\geqslant 0}$ by Remark 7.2.3(2).

A *Boolean function* is a function $\{\mathrm{T}, \mathrm{F}\}^n \to \{\mathrm{T}, \mathrm{F}\}$ where $n \in \mathbb{Z}^+$.

We can represent the compound expression $p \vee q$, where p, q are propositional variables, using the Boolean function $d\colon \{\mathrm{T},\mathrm{F}\}^2 \to \{\mathrm{T},\mathrm{F}\}$ where, for all $p_0, q_0 \in \{\mathrm{T},\mathrm{F}\}$,

> $d(p_0, q_0)$ is the truth value that $p \vee q$ evaluates to when one substitutes propositions of truth values p_0 and q_0 into the propositional variables p and q respectively.

For instance, we have $d(\mathrm{T},\mathrm{T}) = \mathrm{T}$ and $d(\mathrm{F},\mathrm{F}) = \mathrm{F}$. Hence $\mathrm{range}(d) = \{\mathrm{T},\mathrm{F}\}$ by Remark 7.2.3(2).

Let $f, g\colon A \to B$. Then $f = g$ if and only if $f(x) = g(x)$ for all $x \in A$.

($\Rightarrow$) Assume $f = g$. Let $x \in A$. Then

$$\begin{array}{lll} & (\overset{\text{input}}{x}, \overset{\text{output}}{f(x)}) \in f & \text{by the } \Leftarrow \text{ part of Remark 7.2.2.} \\ \therefore & (x, f(x)) \in g & \text{as } f = g. \\ \therefore & f(x) = g(x) & \text{by the } \Rightarrow \text{ part of Remark 7.2.2.} \end{array}$$

($\Leftarrow$) Assume $f(x) = g(x)$ for all $x \in A$. For each $x \in A$ and each $y \in B$,

$$\begin{array}{llll} (x,y) \in f & \Leftrightarrow & y = f(x) & \text{by Remark 7.2.2;} \\ & \Leftrightarrow & y = g(x) & \text{by our assumption;} \\ & \Leftrightarrow & (x,y) \in g & \text{by Remark 7.2.2.} \end{array}$$

So $f = g$. □

The descriptions of r and d in Examples 7.2.4 and 7.2.6 in terms of $r(x)$ and $d(p,q)$ uniquely characterize these functions by Proposition 7.2.7, and can thus serve as definitions of r and d.

Let $f\colon \{0,2\} \to \mathbb{Z}$ and $g\colon \{0,2\} \to \mathbb{Z}$ defined by setting, for all $x \in \{0,2\}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2.$$

Then $f = g$ by Proposition 7.2.7, because $f(x) = g(x)$ for every $x \in \{0,2\}$.

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Q} \to \mathbb{Q}$ defined by

$$\forall x \in \mathbb{Z} \ \ (f(x) = x^3) \quad \text{and} \quad \forall x \in \mathbb{Q} \ \ (g(x) = x^3).$$

Then $f \neq g$ because $(1/2, 1/8)$ is an element of g but not of f.

7.3 Composition

Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f$ is a function $A \to C$. Moreover, for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$

(F1) Let $x \in A$. Use (F1) for f to find $y \in B$ such that $(x,y) \in f$. Use (F1) for g to find $z \in C$ such that $(y,z) \in g$. Then $(x,z) \in g \circ f$ by the definition of $g \circ f$.

(F2) Let $x \in A$ and $z_1, z_2 \in C$ such that $(x, z_1), (x, z_2) \in g \circ f$. Use the definition of $g \circ f$ to find $y_1, y_2 \in B$ such that $(x, y_1), (x, y_2) \in f$ and $(y_1, z_1), (y_2, z_2) \in g$. Then (F2) for f implies $y_1 = y_2$. So $z_1 = z_2$ as g satisfies (F2).

(F2)
there is at most one way to go from A to B ,
one at most one way to go from B to C
Therefore, at most one way to go from A to C

These show $g \circ f$ is a function $A \to C$. Now, for every $x \in A$,

(image)

$$\begin{array}{lll} & (x, f(x)) \in f \ \text{ and } \ \big(f(x), g(f(x))\big) \in g & \text{by the } \Leftarrow \text{ part of Remark 7.2.2;} \\ \therefore & (x, g(f(x))) \in g \circ f & \text{by the definition of } g \circ f; \\ \therefore & g(f(x)) = (g \circ f)(x) & \text{by the } \Rightarrow \text{ part of Remark 7.2.2.} \end{array}$$

□

Noncommutativity of function composition

Let $f, g \colon \mathbb{Z} \to \mathbb{Z}$ such that for every $x \in \mathbb{Z}$,

$$f(x) = 3x \quad \text{and} \quad g(x) = x + 1.$$

By Proposition 7.3.1, for every $x \in \mathbb{Z}$,

$$(g \circ f)(x) = g(f(x)) = g(3x) = 3x + 1 \quad \text{and} \quad (f \circ g)(x) = f(g(x)) = f(x+1) = 3(x+1).$$

Note $(g \circ f)(0) = 1 \neq 3 = (f \circ g)(0)$. So $g \circ f \neq f \circ g$ by Proposition 7.2.7.

Let A be a set. Then the *identity function* on A, denoted id_A, is the function $A \to A$ which satisfies, for all $x \in A$,

$$\mathrm{id}_A(x) = x.$$

$$x \in A \to \boxed{\mathrm{id}_A} \to x \in A$$

Let $f \colon A \to B$. Then $f \circ \mathrm{id}_A = f$ by Proposition 7.2.7, because Proposition 7.3.1 implies

- $f \circ \mathrm{id}_A$ is a function $A \to B$; and
- $(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x)$ for all $x \in A$.

Prove that $\mathrm{id}_B \circ f = f$ for all functions $f \colon A \to B$. 7b

Which of the following define a function $f \colon \mathbb{Z} \to \mathbb{Z}$ that satisfies $f \circ f = f$? 7c

idempotent

(1) $f(x) = 1231$ for all $x \in \mathbb{Z}$.

(2) $f(x) = x$ for all $x \in \mathbb{Z}$.

(3) $f(x) = -x$ for all $x \in \mathbb{Z}$.

(4) $f(x) = 3x + 1$ for all $x \in \mathbb{Z}$.

(5) $f(x) = x^2$ for all $x \in \mathbb{Z}$.

7.4 Inverse and bijectivity

Let $f \colon A \to B$.

(1) *f is surjective* or *onto* if

for every dot on the right, it must have an arrow pointing towards it

$$\forall y \in B \ \ \exists x \in A \ \ y = f(x). \tag{$F^{-1}1$}$$

A *surjection* is a surjective function. surjective & function!

(2) f is *injective* or *one-to-one* if

can't have two different dots on the left pointing to same dot on the right

$$\forall x_1, x_2 \in A \ \ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2). \tag{$F^{-1}2$}$$

An *injection* is an injective function.

(3) f is *bijective* if it is both surjective and injective. A *bijection* is a bijective function.

In view of Remark 7.2.2, one can formulate ($F^{-1}1$) and ($F^{-1}2$) for a general relation f from A to B as follows:

($F^{-1}1$) $\forall y \in B \ \ \exists x \in A \ \ (x, y) \in f$;

($F^{-1}2$) $\forall x_1, x_2 \in A \ \ \forall y \in B \ \ \big((x_1, y) \in f \wedge (x_2, y) \in f \Rightarrow x_1 = x_2\big)$.

By the definition of f^{-1}, these are equivalent respectively to (F1) and (F2) for f^{-1}, i.e.,

Figure 7.1: Surjectivity (left) and injectivity (right)

- $\forall y \in B \ \ \exists x \in A \ \ (y, x) \in f^{-1}$; and
- $\forall x_1, x_2 \in A \ \ \forall y \in B \ \ \big((y, x_1) \in f^{-1} \wedge (y, x_2) \in f^{-1} \Rightarrow x_1 = x_2\big)$.

So f^{-1} is a function $B \to A$ if and only if f satisfies the relational version of ($\mathrm{F}^{-1}1$) and ($\mathrm{F}^{-1}2$). Similarly, the conditions (F1) and (F2) are equivalent to ($\mathrm{F}^{-1}1$) and ($\mathrm{F}^{-1}2$) for f^{-1}.

Proposition 7.4.3 If f is a bijection $A \to B$, then f^{-1} is a bijection $B \to A$.

In view of the discussion in Remark 7.4.2, conditions (F1), (F2), (1), and (2) for f are equivalent respectively to conditions (1), (2), (F1), and (F2) for f^{-1}. $\square$

The function $f\colon \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is surjective.

usual proof of surjectivity : Take any $y \in \mathbb{Q}$. Let $x = (y - 1)/3$. Then $x \in \mathbb{Q}$ and $f(x) = 3x + 1 = y$. $\square$

Remark: A function $f\colon A \to B$ is *not* surjective if and only if

$$\exists y \in B \ \ \forall x \in A \ \ (y \neq f(x)).$$ negation of surjective

Example: Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not surjective.

Proof: Note $g(x) = x^2 \geqslant 0 > -1$ for all $x \in \mathbb{Z}$. So $g(x) \neq -1$ for all $x \in \mathbb{Z}$, although $-1 \in \mathbb{Z}$. $\square$

Injectivity

As in Example 7.4.4, define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Then f is injective.

Proof: Let $x_1, x_2 \in \mathbb{Q}$ such that $f(x_1) = f(x_2)$. Then $3x_1 + 1 = 3x_2 + 1$. So $x_1 = x_2$. $\square$

Remark: A function $f\colon A \to B$ is *not* injective if and only if

$$\exists x_1, x_2 \in A \ \ (f(x_1) = f(x_2) \wedge x_1 \neq x_2).$$

Example: As in Example 7.4.6, define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not injective.

Note $g(1) = 1^2 = 1 = (-1)^2 = g(-1)$, although $1 \neq -1$. $\square$

Amongst the arrow diagrams in Question 7.1.5 that represent a function, which ones represent injections, which ones represent surjections, and which ones represent bijections? ✎ 7d

Operational Inverse

Let $f\colon A \to B$ and $g\colon B \to A$. Then

$$g = f^{-1} \quad \Leftrightarrow \quad \forall x \in A \ \ \forall y \in B \ \ \big(g(y) = x \Leftrightarrow y = f(x)\big).$$

$$
\begin{aligned}
g = f^{-1} \quad &\Leftrightarrow \quad \forall y \in B \;\; \forall x \in A \;\; \big((y,x) \in g \Leftrightarrow (y,x) \in f^{-1}\big) && \text{as } g, f^{-1} \subseteq B \times A;\\
&\Leftrightarrow \quad \forall x \in A \;\; \forall y \in B \;\; \big((y,x) \in g \Leftrightarrow (x,y) \in f\big) && \text{by the definition of } f^{-1};\\
&\Leftrightarrow \quad \forall x \in A \;\; \forall y \in B \;\; \big(g(y) = x \Leftrightarrow y = f(x)\big) && \text{by Remark 7.2.2.} \qquad \square
\end{aligned}
$$

As in Example 7.4.7, define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Note that for all $x, y \in \mathbb{Q}$,

$$\underline{y = 3x + 1 \quad \Leftrightarrow \quad x = (y-1)/3.}$$

Let $g\colon \mathbb{Q} \to \mathbb{Q}$ such that $g(y) = (y-1)/3$ for all $y \in \mathbb{Q}$. The equivalence above implies

$$\forall x, y \in \mathbb{Q} \;\; \big(y = f(x) \Leftrightarrow x = g(y)\big).$$

So Proposition 7.4.11 tells us $\underline{g = f^{-1}}$.

Note 7.4.13: Unlike in Example 7.4.12, in general we are *not* guaranteed a description of the inverse of a bijection f that is significantly different from the trivial description that it is the inverse of f.

Let f be a bijection $A \to B$. Then $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

We know f^{-1} is a function by Proposition 7.4.3, because f is bijection.
For the first part, let $x \in A$. Define $y = f(x)$. Then

$$
\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}(f(x)) && \text{by Proposition 7.3.1;}\\
&= f^{-1}(y) && \text{by the definition of } y;\\
&= x && \text{by Proposition 7.4.11, as } y = f(x);\\
&= \mathrm{id}_A(x) && \text{by the definition of } \mathrm{id}_A.
\end{aligned}
$$

So $f^{-1} \circ f = \mathrm{id}_A$ by Proposition 7.2.7.
The proof of the second part is similar, and is left as an exercise. $\square$ ✎ 7e

Tutorial exercises

7.1. Which of the following conditions are equivalent to the surjectivity of a function $f\colon A \to B$? For each positive answer, give a brief explanation. For each negative answer, give a function $f\colon \{\mathrm{a}, \mathrm{b}\} \to \{0, 1\}$ that demonstrates the non-equivalence.

(a) For any $x \in A$, there is $y \in B$ such that $y = f(x)$.
(b) Some $y \in B$ is equal to $f(x)$ for some $x \in A$.
(c) All $f(x)$'s, where $x \in A$, are equal to the same $y \in B$.
(d) No $y \in B$ is different from all the $f(x)$'s where $x \in A$.

7.2. Which of the following conditions are equivalent to the injectivity of a function $f\colon A \to B$? For each positive answer, give a brief explanation. For each negative answer, give a function $f\colon \{\mathrm{a}, \mathrm{b}\} \to \{0, 1\}$ that demonstrates the non-equivalence.

(a) For all $x_1, x_2 \in A$, whenever $x_1 = x_2$, we have $f(x_1) = f(x_2)$.
(b) For all $x_1, x_2, \in A$, whenever $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.
(c) There are no elements $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

(d) There are distinct $x_1, x_2 \in A$ that make $f(x_1) = f(x_2)$.

7.3. For all $A, B \subseteq \mathbb{Z}$, define

$$R_{A,B} = \{(x, y) \in A \times B : y = x^2 - 1\}$$

considered as a relation from A to B.

(a) Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3\}$. Is $R_{A,B}$ a function $A \to B$? Briefly explain your answer.

(b) Let $A = \{-2, -1, 0, 1, 2\}$. Give an example of $B \subseteq \{-4, -3, \ldots, 3, 4\}$ such that $R_{A,B}$ is a surjection $A \to B$.

(c) Let $B = \{0, 1, 2, 3\}$. Give an example of $A \subseteq \{-4, -3, \ldots, 3, 4\}$ such that $R_{A,B}$ is an injection $A \to B$.

7.4. Let $f = \{(n, x) \in \mathbb{Z}^2 : n = 2x \text{ or } n = 2x + 1\}$, viewed as a relation from $\mathbb{Z}$ to $\mathbb{Z}$.

(a) Prove that f is a function $\mathbb{Z} \to \mathbb{Z}$.

(b) Is f surjective? Prove that your answer is correct.

(c) Is f injective? Prove that your answer is correct.

(d) Determine the range of f.

7.5. Define a Boolean function $g \colon \{\mathrm{T}, \mathrm{F}\}^3 \to \{\mathrm{T}, \mathrm{F}\}$ by setting $g(p_0, q_0, r_0)$ to be the truth value that the compound expression $p \vee q \to \neg r$ evaluates to when one substitutes propositions of truth values p_0, q_0 and r_0 into the propositional variables p, q and r respectively, for all $p_0, q_0, r_0 \in \{\mathrm{T}, \mathrm{F}\}$.

(a) Is g surjective? Prove that your answer is correct.

(b) Is g injective? Prove that your answer is correct.

(c) Give an example of a function $f \colon \{\mathrm{T}, \mathrm{F}\} \to \{\mathrm{T}, \mathrm{F}\}^3$ such that $g \circ f = \mathrm{id}_{\{\mathrm{T},\mathrm{F}\}}$.

7.6. We encountered several situations, e.g., in Proposition 7.4.14, where we can compose one function with another to give an identity function. When we can compose one function f with another function g to get an identity function, we can find out a lot of information about f from g. This question gives one example of this phenomenon.

Let $f \colon A \to B$ and $g \colon B \to A$. Prove that if f is surjective and $g \circ f = \mathrm{id}_A$, then g is injective.

7.7. (Induction corner) Someone attempts to prove that all people on the NUS campus have the same birthday as follows.

> Let $P(n)$ be the sentence
>
> > if the NUS campus has exactly n people in it, then all these n people have the same birthday.
>
> > $P(1)$ is true because if the NUS campus has exactly 1 person in it, then clearly all people on the NUS campus have the same birthday.
> >
> > Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true. Suppose the NUS campus now has exactly $k+1$ people in it. Pick two different people a, b on the NUS campus. Ask a to leave the campus. Since there are k people left on the NUS campus, by the induction hypothesis, all the remaining people have the same birthday, including b. Tell a to come back to campus, and then ask b to leave the campus. Since there are k people left on the NUS campus, by the induction hypothesis, all the remaining people have the same birthday, including a. The people who stayed on the campus throughout have the same birthday as both a and b. So a and b have the same birthday. This shows $P(k+1)$ is true.

Hence $\forall n \in \mathbb{Z}^+ \quad P(n)$ is true by MI.

What is wrong with this proof?

Extra exercises

7.8. Let $f = \{(x, y) \in \mathbb{Q}^2 : y = \pm x\}$ and $g = \{(x, y) \in \mathbb{Q}^2 : y(x^2 + 1) = 1\}$, viewed as relations from $\mathbb{Q}$ to $\mathbb{Q}$. Is f a function $\mathbb{Q} \to \mathbb{Q}$? Is g a function $\mathbb{Q} \to \mathbb{Q}$?

7.9. In this exercise, we prove a strong converse to Proposition 7.4.3.

Let $f \colon A \to B$. Prove that, if the relation f^{-1} is a function $B \to A$, then f is bijective.

7.10. Let f be as defined in Tutorial Exercise 7.4. Define $g = \{(x, n) \in \mathbb{Z}^2 : n = 2x\}$, viewed as a relation from $\mathbb{Z}$ to $\mathbb{Z}$.

(a) Prove that g is a function $\mathbb{Z} \to \mathbb{Z}$.

(b) Is g surjective? Prove that your answer is correct.

(c) Is g injective? Prove that your answer is correct.

(d) Prove that $f \circ g = \mathrm{id}_{\mathbb{Z}}$ but $g \circ f \neq \mathrm{id}_{\mathbb{Z}}$.

(e) Is $g = f^{-1}$? Explain your answer.