Chapter 1: Propositional logic

CS1231 Discrete Structures

Wong Tin Lok

National University of Singapore

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> Overall, logic provides computer science with both a unifying foundational framework and a powerful tool for modeling and reasoning about aspects of computation.
>
> Halpern, Harper, Immerman, Kolaitis, Vardi, Vianu (2001)

Logic

- *Logic* is the study of the principles of valid inference.
- We concentrate on the deductive logic used in mathematics.
- We are concerned with the *form* of a deduction, as opposed to the subject matter.

Why logic?

- It is fundamental in the design of computers.
- It permeates many areas of computer science.
- It is the basis of proofs.

Plan

(1) propositions
(2) Boolean connectives
(3) conditionals
(4) equivalence

Propositions

We consider *only* sentences whose truth or falsity depends neither on time nor on anyone's knowledge.

Definition 1.1.1

(1) A *proposition* (or a *statement*) is a sentence that is either true or false but not both.

(2) The *truth value* of a proposition is true if the proposition is true; it is false otherwise.

(3) Often we abbreviate the truth values true and false by T and F respectively.

Example 1.1.2

Each of the following is a proposition.

(1) $1 + 2 = 3$. T

(2) $2 + 3 < 4$. F

(3) Every even integer is the sum of two odd integers. T

(4) Every even integer that is greater than four is the sum of two prime numbers. ?

Remark 1.1.3

One does not need to know the truth value of a sentence to know that the sentence has a unique truth value.

Propositions: non-examples

Example 1.1.5

None of the following is a proposition when x and y are variables.

(1) $1 + 2$

(2) Good morning!

(3) Please explain this in more detail.

(4) Why is this true?

(5) $x + y = 0$.

Theorem 1.1.6 (Liar Paradox, extra material)

The sentence below is not a proposition:

$$\text{The sentence } (*) \text{ is not true.} \tag{$*$}$$

Boolean connectives

Definitions 1.2.1, 1.2.2 and 1.2.3

Let p, q be propositions.

- We denote by $\neg p$ (or $\sim p$) the proposition "it is not the case that p". Often we read $\neg p$ as "not p", and call it the *negation* of p.
  - The proposition $\neg p$ is true if p is false; it is false otherwise.
- We denote by $p \wedge q$ the proposition "p and q". We call $p \wedge q$ the *conjunction* of p and q.
  - The conjunction $p \wedge q$ is true if p and q are both true; it is false otherwise.
- We denote by $p \vee q$ the proposition "p or q". We call $p \vee q$ the *disjunction* of p and q.
  - The disjunction $p \vee q$ is false if p and q are both false; it is true otherwise.

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Boolean connectives: examples

Example 1.2.6

Consider a fixed (not a variable) real number x. Let p and q be the propositions "$x > 1$" and "$x^2 > 1$" respectively.

(1) $p \vee q$ is "$x > 1$ or $x^2 > 1$", or equivalently "$x^2 > 1$".

(2) $\neg(p \vee q)$ is "it is not the case that $x > 1$ or $x^2 > 1$", or equivalently "$x^2 \leqslant 1$".

(a) Suppose $x = 12.31$.
Then p is true and q is true. So $p \vee q$ is true and $\neg(p \vee q)$ is false.

(b) Suppose $x = 0$.
Then p is false and q is false. So $p \vee q$ is false and $\neg(p \vee q)$ is true.

(c) Suppose $x = -2$.
Then p is false and q is true. So $p \vee q$ is true and $\neg(p \vee q)$ is false.

The conditional as a guarantee

Note 1.3.1

Let p, q be propositions.

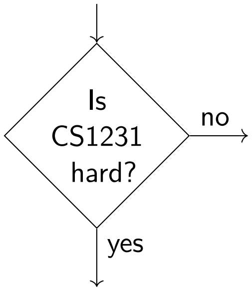
- In mathematics, we interpret the proposition

  "if p then q"

  as a *guarantee* that

  whenever the proposition p is true,
  the proposition q must also be true.

- This guarantee is false (or, more grammatically, fails) only when p is true but q is false; otherwise it is true.
- In this interpretation, the truth value of "if p then q" depends *only* on the truth values of p and q; there is no requirement on whether p and q have related subject matters, unlike in everyday English.



The conditional

Let p, q be propositions.

Definition 1.3.2

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

(1) We denote by $p \to q$ the proposition "if p then q". Often we read this as "p implies q", and call it a *conditional proposition* or an *implication*.

(2) The conditional proposition $p \to q$ is false if p is true and q is false; it is true otherwise.

(3) In the conditional proposition $p \to q$, we call p the *hypothesis* (or the *antecedent*) and q the *conclusion* (or the *consequent*).

Terminology 1.3.3

Alternative ways to express "if p then q" in mathematics:

- q if p.
- p is sufficient for q.
- q is necessary for p.
- q whenever p.
- p only if q.
- p is a sufficient condition for q.
- q is a necessary condition for p.

The conditional: an example

Example 1.3.4

Consider a fixed (not a variable) real number x. Let p and q be the propositions "$x > 1$" and "$x^2 > 1$" respectively.

(1) $p \to q$ is "if $x > 1$, then $x^2 > 1$".

(2) $q \to p$ is "if $x^2 > 1$, then $x > 1$".

(a) Suppose $x = 3$. Then p is true and q is true. So $p \to q$ is true and $q \to p$ is true.

(b) Suppose $x = 0$. Then p is false and q is false. So $p \to q$ is true and $q \to p$ is true.

(c) Suppose $x = -2$. Then p is false and q is true. So $p \to q$ is true but $q \to p$ is false.

vacuously

Terminology 1.3.5

A *vacuously true* conditional proposition is one in which the hypothesis is false.

Exercise 1.3.7

Why are all vacuously true conditional propositions true? 1a

Swapping the hypothesis and the conclusion

Let p, q be propositions.

Definition 1.3.8

(1) The *converse* of $p \to q$ is $q \to p$.

(2) The *inverse* of $p \to q$ is $(\neg p) \to (\neg q)$.

(3) The *contrapositive* of $p \to q$ is $(\neg q) \to (\neg p)$.

Note 1.3.9

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$(\neg p) \to (\neg q)$	$(\neg q) \to (\neg p)$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

- A conditional proposition and its converse may have a different truth values.
- A conditional proposition and its inverse may have a different truth values.
- A conditional proposition and its contrapositive must have the same truth value.

Quick check: converses, inverses, and contrapositives

Exercise 1.3.10

Let a, b, c, d be propositions. What are the converse, the inverse, and the contrapositive of $(a \wedge b) \rightarrow (c \vee d)$?

✏ 1b

The biconditional

Let p, q be propositions.

Definition 1.3.11

(1) We denote by $p \leftrightarrow q$ the proposition "p if and only if q". Sometimes we read this as "p is equivalent to q" and call it a *biconditional proposition* or an *equivalence*. Some abbreviate "if and only if" to "iff".

(2) The biconditional proposition $p \leftrightarrow q$ is true if p and q have the same truth value; it is false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Terminology 1.3.12

Alternative ways to express "p if and only if q" in mathematics:

- p is necessary and sufficient for q.
- p exactly if q.
- p precisely if q.
- p is a necessary and sufficient condition for q.
- p exactly when q.
- p precisely when q.

The biconditional: examples

Example 1.3.13

Consider a fixed (not a variable) real number x. Let p and q be the propositions "$x > 1$" and "$x^2 > 1$" respectively. Then $p \leftrightarrow q$ is

$$\text{“}x > 1 \text{ if and only if } x^2 > 1\text{”}.$$

(a) Suppose $x = 3$. Then p is true and q is true. So $p \leftrightarrow q$ is true.

(b) Suppose $x = -2$. Then p is false and q is true. So $p \leftrightarrow q$ is false.

Exercise 1.3.14

Explain why a proposition is true if and only if it is not false. ✎ 1c

Mind your if's and only-if's

Remark 1.3.15

In everyday English, "You will get ice cream if you behave."
sometimes actually means "You will get ice cream if and only if you behave."

- Here the "only if" part is left implicit.
- In mathematical contexts, we leave no part implicit:

  "if" means "if", and "only if" means "only if",

  except when following Convention 1.4.8.

Compound expressions

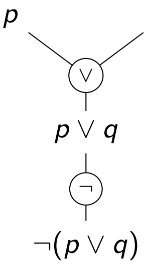
Definition 1.4.1

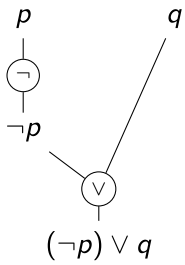
(1) A *propositional variable* is a variable for substituting in an arbitrary proposition.

(2) A *compound expression* is an expression constructed (grammatically) from propositional variables using $\neg$, $\wedge$, $\vee$, $\to$ and $\leftrightarrow$.

Example 1.4.2

Let p, q be propositional variables. Then $\neg(p \vee q)$ and $(\neg p) \vee q$ are compound expressions.





Order of precedence

$$\begin{array}{cc} & \neg \\ \wedge & \vee \\ \rightarrow & \leftrightarrow \end{array}$$

Convention 1.4.3

(1) When there is a choice, one always performs first the propositional connective nearer to the top in the figure on the right.

(2) We do not set up a rule to decide whether to perform $\wedge$ or $\vee$ first when there is a choice *except* in $p \wedge q \wedge r$, $p \vee q \vee r$, $\ldots$. Do *not* leave it to the reader to choose which of these to perform first. The same goes for $\rightarrow$ and $\leftrightarrow$.

(3) Use parentheses $(\ldots)$ to prevent ambiguities.

Example 1.4.4

Let p, q, r be propositional variables.

(1) One can write $(\neg p) \vee q$ alternatively as $\neg p \vee q$.

(2) One can write $(\neg q) \rightarrow (\neg p)$ alternatively as $\neg q \rightarrow \neg p$.

(3) Do *not* write $p \wedge q \vee r$ because our convention does not specify whether one should perform $\wedge$ or $\vee$ first.

Evaluation of compound expressions

Terminology 1.4.5

When one substitutes propositions into all the propositional variables occurring in a compound expression, one obtains a proposition. If the proposition obtained is true, then we say that the compound expression *evaluates to* T under this substitution; else we say that it *evaluates to* F.

Exercise 1.4.6

Let p and q be propositional variables. When one substitutes a true proposition into p and a false proposition into q, what does the compound expression $\neg p \to \neg q$ evaluates to? 1d

Equivalence

Definition 1.4.7

Two compound expressions P, Q are *equivalent* if (and only if) they evaluate to the same truth value under any substitution of propositions into the propositional variables. In this case, we write $P \equiv Q$.

Convention 1.4.8

In mathematical definitions, people often use "if" between the term being defined and the phrase being used to define the term. This is the *only* situation in mathematics when "if" should be understood as a (special) "if and only if".

Remark 1.4.9

We will look into the relationship between $\leftrightarrow$ and $\equiv$ in Tutorial Question 1.7.

Exercise 1.4.10

Convince yourself that the following are true for all compound expressions P, Q, R. 1e

(1) $P \equiv P$.

(2) If $P \equiv Q$, then $Q \equiv P$.

(3) If $P \equiv Q$ and $Q \equiv R$, then $P \equiv R$.

Checking equivalence using truth tables

Technique 1.4.11

One way to check whether two compound expressions are equivalent is to draw a truth table for the two expressions: if the columns for the two expressions are exactly the same, then the expressions are equivalent, else they are not.

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$(\neg p) \to (\neg q)$	$(\neg q) \to (\neg p)$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Theorem 1.4.12

Let p, q be propositional variables. Consider the conditional proposition $p \to q$.

(1) The conditional proposition $p \to q$ is equivalent to its contrapositive $\neg q \to \neg p$.

(2) The converse $q \to p$ is equivalent to the inverse $\neg p \to \neg q$.

(3) The conditional proposition $p \to q$ is not equivalent to its converse $q \to p$.

Conditional as Boolean

Theorem 1.4.14

Let p, q be propositional variables. Then $p \to q$ is equivalent to $\neg p \vee q$.

Proof

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The columns for $p \to q$ and $\neg p \vee q$ are exactly the same. So the two compound expressions are equivalent. $\square$

Non-equivalence

Technique 1.4.15

To prove that two compound expressions are not equivalent, it suffices to find *one* way to substitute true and false propositions into the propositional variables to make the expressions evaluate to different truth values.

Example 1.4.16

Let p, q be propositional variables. Then $\neg(p \vee q)$ and $\neg p \vee q$ are not equivalent.

Proof

- Let us substitute true propositions into both p and q.
- Under this substitution, the expression $p \vee q$ evaluates to T, and thus $\neg(p \vee q)$ evaluates to F.
- However, under the same substitution, the expression $\neg p \vee q$ evaluates to T.
- So these two compound expressions are not equivalent. $\square$

Equivalence: quick check

Exercise 1.4.17

Let p, q, r be propositional variables. Is $(p \wedge q) \vee r$ equivalent to $p \wedge (q \vee r)$? ✎ 1f

Tautologies and contradictions

Que sera, sera.

All animals are equal.

Definition 1.4.18

(1) A *tautology* is a compound expression that evaluates to true no matter what propositions are substituted into all its propositional variables.

(2) A *contradiction* is a compound expression that evaluates to false no matter what propositions are substituted into all its propositional variables.

Example 1.4.19

Let p be a propositional variable. Here is the truth table for $p \vee \neg p$ and $p \wedge \neg p$.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

- As all the entries in the column for $p \vee \neg p$ are T, we see that $p \vee \neg p$ is a tautology.
- As all the entries in the column for $p \wedge \neg p$ are F, we see that $p \wedge \neg p$ is a contradiction.

Logical identities

Theorem 1.4.20

Let t be a tautology and c be a contradiction. For all propositional variables p, q, r, the following equivalences hold.

Commutativity	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associativity	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributivity	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Idempotence	$p \vee p \equiv p$	$p \wedge p \equiv p$
Absorption	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Identities	$p \vee c \equiv p$	$p \wedge t \equiv p$
Annihilators	$p \vee t \equiv t$	$p \wedge c \equiv c$
Negation ✓	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
Double Negative Law	$\neg\neg p \equiv p$	
Top and bottom	$\neg c \equiv t$	$\neg t \equiv c$
Implication ✓	$p \to q \equiv \neg p \vee q$	

The Double Negative Law and one of De Morgan's Laws

Let p, q be propositional variables.

Double Negative Law	$\neg\neg p \equiv p$	
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proof

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

- The columns for $\neg\neg p$ and p are exactly the same in the left truth table. So these expressions are equivalent.
- Similarly, the columns for $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are exactly the same in the right truth table. So these expressions are also equivalent. $\square$

The other De Morgan Law

Let p, q be propositional variables.

Double Negative Law	$\neg\neg p \equiv p$	
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proof

$$
\begin{aligned}
\neg(p \wedge q) &\equiv \neg(\neg\neg p \wedge \neg\neg q) && \text{by the Double Negative Law;} \\
&\equiv \neg\neg(\neg p \vee \neg q) && \text{by the left De Morgan Law;} \\
&\equiv \neg p \vee \neg q && \text{by the Double Negative Law.}
\end{aligned}
$$

$\square$

Remark 1.4.21

Our proof of the right De Morgan Law above used the fact that if one substitutes equivalent compound expressions into the same propositional variable in equivalent compound expressions, then one obtains again equivalent compound expressions.

Real example

p $\quad$ r for "B is countable" $\quad$ q for "A is countable"

Example 1.4.25

Let A, B be sets. Consider the following propositions from Chapter 8.

(1) If A is a subset of B, then the countability of B implies the countability of A.

(2) If A is not countable and A is a subset of B, then B is not countable.

One can rewrite these two propositions symbolically as

$$p \to (r \to q) \quad \text{and} \quad (\neg q \wedge p) \to \neg r$$

respectively. Treating p, q, r as propositional variables, we see that

$$\begin{aligned}
p \to (r \to q) &\equiv \neg p \vee (\neg r \vee q) && \text{by the logical identity on the implication;} \\
&\equiv (q \vee \neg p) \vee \neg r && \text{by the commutativity and the associativity of } \vee; \\
&\equiv (\neg\neg q \vee \neg p) \vee \neg r && \text{by the Double Negative Law;} \\
&\equiv \neg(\neg q \wedge p) \vee \neg r && \text{by De Morgan's Laws;} \\
&\equiv (\neg q \wedge p) \to \neg r && \text{by the logical identity on the implication.}
\end{aligned}$$

So propositions (1) and (2) have the same truth value.

Proving that a compound expression is a tautology

Example 1.4.26

Let p, q be propositional variables. Then $\big((p \to q) \wedge p\big) \to q$ is a tautology.

Proof

$$\begin{aligned} \big((p \to q) \wedge p\big) \to q &\equiv \neg\big((p \to q) \wedge p\big) \vee q && \text{by the logical identity on the implication.} \\ &\equiv \neg(p \to q) \vee \neg p \vee q && \text{by De Morgan's Laws;} \\ &\equiv \neg(p \to q) \vee (p \to q) && \text{by the logical identity on the implication.} \end{aligned}$$

So $\big((p \to q) \wedge p\big) \to q$ is a tautology by Example 1.4.19. $\square$

Note 1.4.24

- Different (looking) compound expressions may be equivalent.
- Therefore, showing that a compound expression P is equivalent (via the logical identities, say) to one that is different from another compound expression Q alone is not sufficient to imply that P and Q are not equivalent.

Summary

p	q	$\neg p$ (not)	$p \wedge q$ (and)	$p \vee q$ (or)	$p \rightarrow q$ (implies)	$p \leftrightarrow q$ (if and only if)
T	T	F	T	T	T	T
T	F		F	T	F	F
F	T	T	F	T	T	F
F	F		F	F	T	T

original	$p \rightarrow q$	
converse	$q \rightarrow p$	
inverse	$(\neg p) \rightarrow (\neg q)$	
contrapositive	$(\neg q) \rightarrow (\neg p)$	

same truth value

may have different truth values

- Two compound expressions are *equivalent* if and only if they evaluate to the same truth value under any substitution of propositions into the propositional variables.
- Truth tables can tell us whether two compound expressions are equivalent or not.
- One can use the logical identities to derive new equivalences, tautologies, etc.

Symbols?

- Symbols can save space.
- A proper use of symbols eliminates any ambiguity in the logical structure of a proposition.
- A proper use of symbols makes the logical structure of a proposition clearer.
  - Example: $\neg(p \vee q)$ vs $(\neg p) \vee q$.
- Having more symbols alone does *not* make a piece of mathematics better.
- Having too many symbols may make your work harder to understand.
- It is considered bad style to mix words and symbols in formal writing.