CS1231 Chapter 2

Predicate logic

2.1 Predicates

Definition 2.1.1. (1) A *variable* is a symbol that indicates a position in a sentence in which one can substitute (the name of) an object.

(2) A *valid* substitution for a variable replaces all free occurrences of that variable in the sentence by the same object.

(3) Saying a variable x *takes* an object a in a sentence means one substitutes the object a into the variable x in the sentence.

(4) Sometimes we may want to allow only certain objects to be substituted into a variable x. In this case, we call the set of all such objects the *domain* of x, and we may say that x *ranges over* these objects.

Remark 2.1.2. A phrase or a symbol may use, or more technically speaking, *bind* a variable occurring in the sentence. For example, the variable x in

> For every real number x, we must have $x^2 \geqslant 0$.

is already used, or *bound*, by the phrase "for every": this sentence means

> No matter what real number one substitutes into the variable x, the sentence $x^2 \geqslant 0$ becomes true.

A valid substitution should be applied only to the variable occurrences that are not already used or bound by anything in the sentence. Such occurrences are said to be *free*.

Remark 2.1.3. (1) A *set* is a (possibly empty, possibly infinite) collection of objects; these objects are called the *elements* of the set. We can write $z \in A$ for "z is an element of A". Chapter 4 contains a more detailed treatment of sets.

(2) In Chapter 7, we will introduce the notion of the domain of a *function*. This is different from the domain of a variable.

(3) Some people insist that every variable has a domain. We do not.

Note 2.1.4. We define the natural numbers to include 0, but some authors do not.

Definition 2.1.5. Let P be a sentence and let $x_1, x_2, \ldots, x_n$ list all the variables that appear free in P.

(1) We may write P as $P(x_1, x_2, \ldots, x_n)$.

(2) If $z_1, z_2, \ldots, z_n$ are objects, then we denote by $P(z_1, z_2, \ldots, z_n)$ the sentence obtained from $P(x_1, x_2, \ldots, x_n)$ by substituting each z_i into x_i.

	Symbol	Meaning	Elements	Non-elements
Discrete	$\mathbb{N}$	the set of all natural numbers	$0, 1, 2, 3, 31$	$-1, \frac{1}{2}$
Discrete	$\mathbb{Z}$	the set of all integers	$0, 1, -1, 2, -10$	$\frac{1}{2}, \sqrt{2}$
Discrete	$\mathbb{Q}$	the set of all rational numbers	$-1, 10, \frac{1}{2}, -\frac{7}{5}$	$\sqrt{2}, \pi, \sqrt{-1}$
Not Discrete	$\mathbb{R}$	the set of all real numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi$	$\sqrt{-1}, \sqrt{-10}$
Not Discrete	$\mathbb{C}$	the set of all complex numbers	all of the above	
	$\mathbb{Z}^+$	the set of all positive integers	$1, 2, 3, 31$	$0, -1, -12$
	$\mathbb{Z}^-$	the set of all negative integers	$-1, -2, -3, -31$	$0, 1, 12$
	$\mathbb{Z}_{\geqslant 0}$	the set of all non-negative integers	$0, 1, 2, 3, 31$	$-1, -12$
	$\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geqslant m}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_{\geqslant m}$, etc. are defined similarly.			

Table 2.1: Common sets

Definition 2.1.6. (1) A *predicate* is a sentence that becomes a proposition whenever one validly substitutes objects into all its variables.

NOTE: these are not numbers, it is objects substituted into sentences

(2) A sentence $P(x_1, x_2, \ldots, x_n)$ is a *predicate over sets* $D_1, D_2, \ldots, D_n$ if $P(z_1, z_2, \ldots, z_n)$ is a proposition whenever $z_1, z_2, \ldots, z_n$ are respectively elements of $D_1, D_2, \ldots, D_n$; in the case when a variable x_i here has a domain, we additionally require this domain to contain every element of D_i.

(3) We may call a predicate over $D, D, \ldots, D$ simply a *predicate on D*.

Example 2.1.7. Let $P(x)$ be "$x^2 \geqslant x$", where x is a variable with domain $\mathbb{Q}$. Then

(1) $P(x)$ is a predicate over $\mathbb{Q}$;

(2) $P(1231)$ is "$1231^2 \geqslant 1231$", which is a true proposition; and

(3) $P(1/2)$ is "$(1/2)^2 \geqslant 1/2$", which is a false proposition because $(1/2)^2 = 1/4 < 1/2$.

Example 2.1.8. Let $Q(x, y)$ be "$x + y = 0$", where x and y are variables with domain $\mathbb{Z}$. Then

(1) $Q(x, y)$ is a predicate on $\mathbb{Z}$;

(2) $Q(0, 1)$ is "$0 + 1 = 0$", which is a false proposition; and

(3) $Q(2, -2)$ is "$2 + (-2) = 0$", which is a true proposition.

2.2 Quantifiers

Definition 2.2.1. Let $P(x)$ be a sentence.

(1) We denote by $\forall x\ P(x)$ the proposition "for all x, $P(x)$".

(2) The symbol $\forall$, read as "for all", is known as the *universal quantifier*.

(3) The proposition $\forall x\ P(x)$ is true if and only if $P(z)$ is true for all objects z.

(4) A *counterexample* to the proposition $\forall x\ P(x)$ is an object z for which $P(z)$ is not true.

(5) If D is a set, then we denote by $\forall x \in D\quad P(x)$ the sentence "for all x in D, $P(x)$", or symbolically $\forall x\ (x \in D \to P(x))$.

∀x∈D P(x) means
∀x (x∈D→ P(x))

Note 2.2.2. Let $P(x)$ be a sentence and D be a set.

(1) The proposition $\forall x\ P(x)$ is false if and only if it has a counterexample. In the case when $P(x)$ is a predicate, this in turn is equivalent to $P(z)$ being false for at least one object z.

(2) The proposition $\forall x \in D\ \ P(x)$ is false if and only if it has a counterexample. In the case when $P(x)$ is a predicate over D, this in turn is equivalent to $P(z)$ being false for at least one element z of D.

Example 2.2.3. (1) Let D be the set that contains precisely $1, 2, 3, 4, 5$. Then the proposition $\forall x \in D\ \ x^2 \geqslant x$ is true because

$$1^2 \geqslant 1 \quad \text{and} \quad 2^2 \geqslant 2 \quad \text{and} \quad 3^2 \geqslant 3 \quad \text{and} \quad 4^2 \geqslant 4 \quad \text{and} \quad 5^2 \geqslant 5.$$

(2) The number $1/2$ is a counterexample to $\forall x \in \mathbb{Q}\ \ x^2 \geqslant x$ because $1/2$ is an element of $\mathbb{Q}$ and $(1/2)^2 = 1/4 < 1/2$.

(3) So the proposition $\forall x \in \mathbb{Q}\ \ x^2 \geqslant x$ is false.

Definition 2.2.4. Let $P(x)$ be a sentence.

(1) We denote by $\exists x\ P(x)$ the proposition "there exists x such that $P(x)$".

(2) The symbol $\exists$, read as "there exists", is known as the *existential quantifier*.

(3) The proposition $\exists x\ P(x)$ is true if and only if $P(z)$ is true for at least one object z.

(4) A *witness* to the proposition $\exists x\ P(x)$ is an object z for which $P(z)$ is true.

(5) If D is a set, then we denote by $\exists x \in D\ \ P(x)$ the proposition "there exists x in D such that $P(x)$", or symbolically $\exists x\big(x \in D \wedge P(x)\big)$.

Note 2.2.5. Let $P(x)$ be a sentence and D be a set.

(1) The proposition $\exists x\ P(x)$ is true if and only if it has a witness.

(2) In the case when $P(x)$ is a predicate, the proposition $\exists x\ P(x)$ is false if and only if $P(z)$ is false for all objects z.

(3) The proposition $\exists x \in D\ \ P(x)$ is true if and only if it has a witness.

(4) In the case when $P(x)$ is a predicate over D, the proposition $\exists x \in D\ \ P(x)$ is false if and only if $P(z)$ is false for all elements z of D.

Example 2.2.6. (1) The number 2 is a witness to $\exists x \in \mathbb{Q}\ \ x^2 \geqslant x$ because 2 is an element of $\mathbb{Q}$ and $2^2 = 4 \geqslant 2$.

(2) So the proposition $\exists x \in \mathbb{Q}\ \ x^2 \geqslant x$ is true.

(3) Let D be the set that contains precisely $1/2,\ 1/3,\ 1/4,\ 1/5$. Then the proposition $\exists x \in D\ \ x^2 \geqslant x$ is false because

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2} \quad \text{and} \quad \left(\frac{1}{3}\right)^2 = \frac{1}{9} < \frac{1}{3} \quad \text{and} \quad \left(\frac{1}{4}\right)^2 = \frac{1}{16} < \frac{1}{4} \quad \text{and} \quad \left(\frac{1}{5}\right)^2 = \frac{1}{25} < \frac{1}{5}.$$

Convention 2.2.7. Let $P(x)$ be a predicate.

(1) In mathematics,

"there exists one x such that $P(x)$" or "there is one x such that $P(x)$"

means "there exists *at least* one x such that $P(x)$".

(2) More generally, if n is a non-negative integer, then

$$\text{“there exist } n\ x\text{'s such that } P(x)\text{”} \quad \text{or} \quad \text{“there are } n\ x\text{'s such that } P(x)\text{”}$$

means “there exist *at least* n x's such that $P(x)$”.

(3) If the exact number is intended, then use the word “exactly”, as in “there are exactly two x's such that $P(x)$”.

Convention 2.2.8. (1) In informal contexts, some may write symbolically a quantifier, say $\forall x \in D$ or $\exists x$, after the expression it applies to.

(2) However, in this module, we do *not* do it: here a quantifier, when written symbolically, always comes before the expression it applies to.

(3) More precisely, it applies *only* to the smallest predicate (over an appropriate set) that follows it.

Terminology 2.2.9. (1) In addition to “all', words that indicate universal quantification in mathematics include “every”, “each”, and “any”.

(2) One may also express “for all x in D, $P(x)$”, where $P(x)$ is a sentence and D is a set, as

$$\text{“}P(x) \text{ whenever } x \in D\text{”} \quad \text{or} \quad \text{“If } x \in D\text{, then } P(x)\text{”}.$$

(3) In addition to “exists”, phrases that indicate existential quantification in mathematics include “some” and “there is”.

Example 2.2.10. Let $\mathrm{Even}(x)$ denote the predicate “x is even” over $\mathbb{Z}$. (We will give a precise definition of this predicate in Chapter 3.) Express the following propositions symbolically using $\mathrm{Even}(x)$.

(1) “The square of any even integer is even.”

(2) “Any integer whose square is even must itself be even.”

(3) “Some even integer n satisfies $n^2 = 2n$.”

Solution. (1) $\forall n \in \mathbb{Z} \ \ \big(\mathrm{Even}(n) \rightarrow \mathrm{Even}(n^2)\big)$.

A *common mistake* is to answer $\forall n \in \mathbb{Z} \ \ \big(\mathrm{Even}(n) \wedge \mathrm{Even}(n^2)\big)$; this can be read as “for every integer n, n is even and n^2 is even”, whose meaning is different from that of the given proposition.

(2) $\forall n \in \mathbb{Z} \ \ \big(\mathrm{Even}(n^2) \rightarrow \mathrm{Even}(n)\big)$.

(3) $\exists n \in \mathbb{Z} \ \ \big(\mathrm{Even}(n) \wedge n^2 = 2n\big)$.

Remark 2.2.11. (1) In certain areas of mathematics, all variables have the same domain. This common domain is called the *domain of discourse*. For brevity, some authors may omit this in quantified expressions in the particular context.

(2) In this module, there is *no* domain of discourse, as we often need to consider variables with different domains. In particular, we will *not* abbreviate $\forall x \in D$ and $\exists x \in D$ as $\forall x$ and $\exists x$.

Exercise 2.2.12. Which of the following is/are true for every predicate $P(x)$ over $\mathbb{R}$? ✎ 2a

(1) If $\forall x \in \mathbb{Z} \ \ P(x)$ is true, then $\forall x \in \mathbb{R} \ \ P(x)$ is true.

(2) If $\forall x \in \mathbb{R} \ \ P(x)$ is true, then $\forall x \in \mathbb{Z} \ \ P(x)$ is true.

(3) If $\exists x \in \mathbb{Z} \ \ P(x)$ is true, then $\exists x \in \mathbb{R} \ \ P(x)$ is true.

(4) If $\exists x \in \mathbb{R} \ \ P(x)$ is true, then $\exists x \in \mathbb{Z} \ \ P(x)$ is true.

$\forall x \in D\ P(x) \to p(1) \wedge p(2) \wedge p(3)$
for all is like AND

$\exists x \in D\ P(x) \to p(1) \vee p(2) \vee p(3)$
for all is like OR

2.3 Negation

Theorem 2.3.1. The following are true for all predicates $P(x)$.

(1) $\neg\forall x\ P(x) \leftrightarrow \exists x\ \neg P(x)$.

Pushing in the negation

(2) $\neg\exists x\ P(x) \leftrightarrow \forall x\ \neg P(x)$.

The following are true for all predicates $P(x)$ over a set D.

(3) $\neg\forall x \in D\ \ P(x) \leftrightarrow \exists x \in D\ \ \neg P(x)$.

(4) $\neg\exists x \in D\ \ P(x) \leftrightarrow \forall x \in D\ \ \neg P(x)$.

Proof.

(1) Note that the following are true.

$$\begin{aligned}
\neg\forall x\ P(x) \text{ is true} \quad &\leftrightarrow \quad \forall x\ P(x) \text{ is false} \\
&\qquad\qquad \text{by the definition of } \neg. \\
\forall x\ P(x) \text{ is false} \quad &\leftrightarrow \quad P(z) \text{ is false for at least one object } z \\
&\qquad\qquad \text{by Note 2.2.2(1).} \\
P(z) \text{ is false for at least one object } z \quad &\leftrightarrow \quad \neg P(z) \text{ is true for at least one object } z \\
&\qquad\qquad \text{by the definition of } \neg. \\
\neg P(z) \text{ is true for at least one object } z \quad &\leftrightarrow \quad \exists x\ \neg P(x) \text{ is true} \\
&\qquad\qquad \text{by the definition of } \exists.
\end{aligned}$$

From these, we deduce that $\neg\forall x\ P(x)$ is true if and only if $\exists x\ \neg P(x)$ is true.

(2) The proof is similar: note that the following are true.

$$\begin{aligned}
\neg\exists x\ P(x) \text{ is true} \quad &\leftrightarrow \quad \exists x\ P(x) \text{ is false} \quad \text{by the definition of } \neg. \\
\exists x\ P(x) \text{ is false} \quad &\leftrightarrow \quad P(z) \text{ is false for all objects } z \\
&\qquad\qquad \text{by Note 2.2.5(2).} \\
P(z) \text{ is false for all objects } z \quad &\leftrightarrow \quad \neg P(z) \text{ is true for all objects } z \\
&\qquad\qquad \text{by the definition of } \neg. \\
\neg P(z) \text{ is true for all objects } z \quad &\leftrightarrow \quad \forall x\ \neg P(x) \text{ is true} \quad \text{by the definition of } \forall.
\end{aligned}$$

From these, we deduce that $\neg\exists x\ P(x)$ is true if and only if $\forall x\ \neg P(x)$ is true.

We can prove (3) and (4) in a similar way. ☐

Remark 2.3.2. We will introduce a more succinct way to write the proof of Theorem 2.3.1 in Chapter 3.

Exercise 2.3.3. Alternatively, parts (3) and (4) of Theorem 2.3.1 can be proved directly using parts (1) and (2) together with the symbolic definitions from Definition 2.2.1(5) and Definition 2.2.4(5). Verify this. ✎ 2b

Example 2.3.4. Consider the following proposition:

"Not every integer is even."

We can express this symbolically as

$$\neg\forall n \in \mathbb{Z}\ \ \mathrm{Even}(n),$$

where $\mathrm{Even}(n)$ denotes the predicate "n is even" over $\mathbb{Z}$. In view of Theorem 2.3.1, this is equivalent to

$$\exists n \in \mathbb{Z}\ \ \neg\mathrm{Even}(n).$$

It follows that the given English proposition is equivalent to

"There is an integer that is not even."

Example 2.3.5. Consider the following proposition:

"No integer is both odd and even."

We can express this symbolically as

$$\neg\exists n \in \mathbb{Z} \quad \big(\mathrm{Odd}(n) \wedge \mathrm{Even}(n)\big),$$

where $\mathrm{Even}(n)$ and $\mathrm{Odd}(n)$ denote respectively the predicates "n is even" and "n is odd" over $\mathbb{Z}$. In view of Theorem 2.3.1, this is equivalent to

$$\forall n \in \mathbb{Z} \quad \neg\big(\mathrm{Odd}(n) \wedge \mathrm{Even}(n)\big).$$

By De Morgan's Laws, this is in turn equivalent to

$$\forall n \in \mathbb{Z} \quad \big(\neg\mathrm{Odd}(n) \vee \neg\mathrm{Even}(n)\big).$$

It follows that the given English proposition is equivalent to

"For every integer, either it is not odd or it is not even."

2.4 Nested quantification

Definition 2.4.1 (generalizing Definitions 2.2.1 and 2.2.4)**.** Consider a sentence $Q(x_1, x_2, \ldots, x_n, y)$ and a set E. Let $z_1, z_2, \ldots, z_n$ be objects. For each i, assume additionally that z_i is in the domain of x_i if x_i has a domain.

(1) We denote by $\forall y\ Q(x_1, x_2, \ldots, x_n, y)$ and $\exists y\ Q(x_1, x_2, \ldots, x_n, y)$ the predicates "for all y, $Q(x_1, x_2, \ldots, x_n, y)$" and "there exists y such that $Q(x_1, x_2, \ldots, x_n, y)$" respectively. Both of these predicates may mention variables $x_1, x_2, \ldots, x_n$.

(2) Denote by $\forall y\ Q(z_1, z_2, \ldots, z_n, y)$ and $\exists y\ Q(z_1, z_2, \ldots, z_n, y)$ the propositions obtained respectively from the predicates $\forall y\ Q(x_1, x_2, \ldots, x_n, y)$ and $\exists y\ Q(x_1, x_2, \ldots, x_n, y)$ by substituting each z_i into x_i.

(3) The proposition $\forall y\ Q(z_1, z_2, \ldots, z_n, y)$ is true if and only if $Q(z_1, z_2, \ldots, z_n, w)$ is true for all objects w.

(4) A *counterexample* to the proposition $\forall y\ Q(z_1, z_2, \ldots, z_n, y)$ is an object w for which $Q(z_1, z_2, \ldots, z_n, w)$ is not true.

(5) The proposition $\exists y\ Q(z_1, z_2, \ldots, z_n, y)$ is true if and only if $Q(z_1, z_2, \ldots, z_n, w)$ is true for at least one object w.

(6) A *witness* to the proposition $\exists y\ Q(z_1, z_2, \ldots, z_n, y)$ is an object w for which $Q(z_1, z_2, \ldots, z_n, w)$ is true.

(7) We denote by $\forall y \in E \quad Q(x_1, x_2, \ldots, x_n, y)$ the predicate "for all y in E, $Q(x_1, x_2, \ldots, x_n, y)$", or symbolically $\forall y\ \big(y \in E \rightarrow Q(x_1, x_2, \ldots, x_n, y)\big)$.

(8) We denote by $\exists y \in E \quad Q(x_1, x_2, \ldots, x_n, y)$ the predicate "there exists y in E such that $Q(x_1, x_2, \ldots, x_n, y)$", or symbolically $\exists y\ \big(y \in E \wedge Q(x_1, x_2, \ldots, x_n, y)\big)$.

Example 2.4.2. Let $Q(x, y)$ be a predicate. Then $\exists x\ \forall y\ Q(x, y)$ is the proposition $\exists x\ P(x)$, where $P(x)$ denotes the predicate $\forall y\ Q(x, y)$.

Example 2.4.3. Recall the predicate $x + y = 0$ over $\mathbb{Z}$ from Example 2.1.8.

(1) Consider the proposition "$\forall x \in \mathbb{Z}\ \ \exists y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "for every integer x, there is an integer y, such that $x + y = 0$".

  (b) This is true because, given any integer x, one can set $y = -x$ to make y an integer and $x + y = 0$.

(2) Consider the proposition "$\exists x \in \mathbb{Z}\ \ \forall y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "there is an integer x such that, for every integer y, $x + y = 0$".

  (b) Alternatively, one can express this as "there is an integer which, when added to any integer, gives a sum of 0".

  (c) This is false because, given any integer x, one can set

$$y = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0, \end{cases}$$

  so that y is an integer and $x + y \neq 0$.

(3) Consider the proposition "$\forall x \in \mathbb{Z}\ \ \forall y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "for every integer x, for every integer y, $x + y = 0$".

  (b) Alternatively, one can express this as "the sum of any two integers is 0".

  (c) This is false because 1 and 1 are integers and $1 + 1 = 2 \neq 0$.

(4) Consider the proposition "$\exists x \in \mathbb{Z}\ \ \exists y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "there exists an integer x, there exists an integer y, such that $x+y = 0$".

  (b) Alternatively, one can express this as "there are two integers which, when added together, gives 0".

  (c) This is true because 2 and -2 are integers and $2 + (-2) = 0$.

Warning 2.4.4. As Example 2.4.3 demonstrates, there are predicates $Q(x, y)$ for which

$$\forall x\ \exists y\ Q(x, y) \quad \text{and} \quad \exists y\ \forall x\ Q(x, y)$$

are not equivalent. So the order of quantifiers matters.

Note 2.4.5. One can interpret the following sentences as any one of $\forall y \in \mathbb{Z}\ \ \exists x \in \mathbb{Z}\ \ x+y = 0$ and $\exists x \in \mathbb{Z}\ \ \forall y \in \mathbb{Z}\ \ x + y = 0$. As the two interpretations are not equivalent, *avoid* writing such ambiguous sentences in mathematics.

(1) "One can add any integer to some integer to get a sum of 0."

(2) "There is an integer x such that $x + y = 0$ for any integer y."

Example 2.4.6. One can express the proposition

"Every even integer is the sum of two odd integers."

from Example 1.1.2(3) symbolically as

$$\forall n \in \mathbb{Z}\ \ \big(\mathrm{Even}(n) \to \exists k \in \mathbb{Z}\ \ \exists \ell \in \mathbb{Z}\ \ (\mathrm{Odd}(k) \wedge \mathrm{Odd}(\ell) \wedge n = k + \ell)\big),$$

where $\mathrm{Even}(n)$ and $\mathrm{Odd}(n)$ are respectively the predicates "n is even" and "n is odd" over $\mathbb{Z}$.

Notation 2.4.7. Let $Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ be a sentence and E be a set.

(1) We may abbreviate

$$\forall y_1 \; \forall y_2 \; \ldots \; \forall y_m \; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$

as $\forall y_1, y_2, \ldots, y_m \; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

(2) We may abbreviate

$$\forall y_1 \in E \;\; \forall y_2 \in E \;\; \ldots \;\; \forall y_m \in E \;\; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$

as $\forall y_1, y_2, \ldots, y_m \in E \;\; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

(3) We may abbreviate

$$\exists y_1 \; \exists y_2 \; \ldots \; \exists y_m \; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$

as $\exists y_1, y_2, \ldots, y_m \; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

(4) We may abbreviate

$$\exists y_1 \in E \;\; \exists y_2 \in E \;\; \ldots \;\; \exists y_m \in E \;\; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$

as $\exists y_1, y_2, \ldots, y_m \in E \;\; Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

Note 2.4.8. Let $P(x_1, x_2, \ldots, x_n)$ be a sentence.

(1) The proposition $\forall x_1, x_2, \ldots, x_n \; P(x_1, x_2, \ldots, x_n)$ is true if and only if $P(z_1, z_2, \ldots, z_n)$ is true for all objects $z_1, z_2, \ldots, z_n$.

(2) The proposition $\exists x_1, x_2, \ldots, x_n \; P(x_1, x_2, \ldots, x_n)$ is true if and only if $P(z_1, z_2, \ldots, z_n)$ is true for some objects $z_1, z_2, \ldots, z_n$.

The objects $z_1, z_2, \ldots, z_n$ above are *not* necessarily different. The analogous statements for $\forall x_1, x_2, \ldots, x_n \in D \quad P(x_1, x_2, \ldots, x_n)$ and $\exists x_1, x_2, \ldots, x_n \in D \quad P(x_1, x_2, \ldots, x_n)$ are also true when D is a set. If the z_i's are really meant to be all different, then one can use the word "distinct" to indicate it, as in "for all distinct $z_1, z_2, \ldots, z_n$" and "there exist distinct $z_1, z_2, \ldots, z_n$".

Theorem 2.4.9. The following are true for all predicates $Q(x, y)$.

(1) $\neg\forall x \; \forall y \; Q(x, y) \leftrightarrow \exists x \; \exists y \; \neg Q(x, y)$.

(2) $\neg\forall x \; \exists y \; Q(x, y) \leftrightarrow \exists x \; \forall y \; \neg Q(x, y)$.

(3) $\neg\exists x \; \exists y \; Q(x, y) \leftrightarrow \forall x \; \forall y \; \neg Q(x, y)$.

(4) $\neg\exists x \; \forall y \; Q(x, y) \leftrightarrow \forall x \; \exists y \; \neg Q(x, y)$.

Proof. We consider here only (1) and (2); the proofs of parts (3) and (4) are left as exercises. ✎ 2c

(1) We have the following equivalences by Theorem 2.3.1.

$$\begin{aligned} \neg\forall x \; \forall y \; Q(x, y) &\leftrightarrow \exists x \; \neg\forall y \; Q(x, y). \\ \exists x \; \neg\forall y \; Q(x, y) &\leftrightarrow \exists x \; \exists y \; \neg Q(x, y). \end{aligned}$$

From these, we deduce that $\neg\forall x \; \forall y \; Q(x, y)$ is true if and only if $\exists x \; \exists y \; \neg Q(x, y)$ is true.

(2) We have the following equivalences by Theorem 2.3.1.

$$\begin{aligned} \neg\forall x\ \exists y\ Q(x,y) &\leftrightarrow \exists x\ \neg\exists y\ Q(x,y). \\ \exists x\ \neg\exists y\ Q(x,y) &\leftrightarrow \exists x\ \forall y\ \neg Q(x,y). \end{aligned}$$

From these, we deduce that $\neg\forall x\ \exists y\ Q(x,y)$ is true if and only if $\exists x\ \forall y\ \neg Q(x,y)$ is true. □

Exercise 2.4.10. Let D be the set that contains precisely $-1,0,1$. Let E be the set that contains precisely $1,-1,2,-2$. Which of the following propositions is/are true? ✎ 2d

(1) $\exists x \in D \quad \forall y \in E \quad xy = 0$. T : Pick x = 0

(2) $\forall y \in E \quad \exists x \in D \quad xy = 0$.

(3) $\exists x \in D \quad \forall y \in E \quad xy < 0$.

(4) $\forall y \in E \quad \exists x \in D \quad xy < 0$.

(5) $\exists x_1, x_2 \in D \quad x_1 + x_2 = 2$.

(6) $\forall y_1, y_2 \in E \quad y_1 = y_2$.

Tutorial exercises

2.1. Fix a relation R on a set A. (For this exercise, it does not matter what "relation" means.) Consider the following propositions from Chapter 6.

(a) $x\ R\ x$ for any element x of A. (reflexivity)

(b) For all elements x, y of A such that $x\ R\ y$, one must have $y\ R\ x$. (symmetry)

(c) No distinct elements x, y of A satisfy both $x\ R\ y$ and $y\ R\ x$. (antisymmetry)

Rewrite these propositions symbolically in terms of R and A.

2.2. Refer back to Question 2.1. Someone claims that the following is an equivalent formulation of antisymmetry.

If x, y are elements of A satisfying $x\ R\ y$ and $y\ R\ x$, then x and y must be equal.

Is this claim correct? Justify your answer.

2.3. Rewrite the following propositions symbolically.

(a) There is no largest natural number.

(b) Between any two distinct rational numbers, there exists another one.

2.4. Consider the following propositions, some of which come from Chapter 7.

How do know what is the negation?
Just add a ¬ () infront of the whole statement !!!!

(a) $\forall x \in \mathbb{R} \quad \exists y \in \mathbb{R} \quad y = x^2$.

(b) $\forall y \in \mathbb{R} \quad \exists x \in \mathbb{R} \quad y = x^2$.

(c) $\forall x_1, x_2, y_1, y_2 \in \mathbb{R} \quad (y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge x_1 \neq x_2 \rightarrow y_1 \neq y_2)$.

(d) $\forall x_1, x_2, y_1, y_2 \in \mathbb{R} \quad (y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge y_1 \neq y_2 \rightarrow x_1 \neq x_2)$.

(e) $\forall x \in \mathbb{R} \quad \big((\exists y \in \mathbb{R} \quad y = x^2) \rightarrow x \geqslant 0\big)$.

(f) $\forall y \in \mathbb{R} \quad \big((\exists x \in \mathbb{R} \quad y = x^2) \rightarrow y \geqslant 0\big)$.

Rewrite the negation of each of these into an equivalent proposition where no $\neg$ is followed by a proposition that mentions $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, or $\exists$. Do so with a minimal amount of rewrites. Which of these propositions are true? Which of these are false? Briefly explain your answers.

(Hint: to understand what these propositions and their negations mean, it may be helpful to read them out in words.)

Extra exercises

2.5. Consider the following proposition from Chapter 6.

> An integer d divides an integer n if and only if n is the product of d with some integer.

Rewrite this proposition symbolically in terms of the predicate $\text{Divides}(d, n)$, which stands for "d divides n".

2.6. Let D be the set that contains precisely $1, 3, 5, 7, 9, 11, 13$. Let E be the set that contains precisely $0, 2, 4, 6$. Consider the following propositions, some of which come from Chapter 7.

(a) $\forall x \in D \;\; \exists y \in E \;\; y < x$.

(b) $\exists y \in E \;\; \forall x \in D \;\; y < x$.

(c) $\forall x \in D \;\; \exists y \in E \;\; y + 1 = x$.

(d) $\forall x \in D \;\; \big(x < 6 \to \exists y \in E \;\; (y + 1 = x)\big)$.

Rewrite the negation of each of these into an equivalent proposition where no $\neg$ is followed by a proposition that mentions $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, $\forall$, or $\exists$. Do so with a minimal amount of rewrites. Which of these propositions are true? Which of these are false? Briefly explain your answers.