Tutorial solutions for Chapter 2

Sometimes there are other correct answers.

2.1. (a) $\forall x \in A \;\; x \mathrel{R} x$.

(b) $\forall x, y \in A \;\; (x \mathrel{R} y \to y \mathrel{R} x)$.
The answer $\forall x, y \in A \;\; (x \mathrel{R} y \land y \mathrel{R} x)$ is *not* correct; see Example 2.2.10(1).

(c) $\neg \exists x, y \in A \;\; (x \neq y \land x \mathrel{R} y \land y \mathrel{R} x)$.
Another acceptable answer is $\forall x, y \in A \;\; \neg(x \neq y \land x \mathrel{R} y \land y \mathrel{R} x)$, but this resembles the given sentence less.

2.2. One can rewrite this English sentence symbolically as

$$\forall x, y \in A \;\; (x \mathrel{R} y \land y \mathrel{R} x \to x = y).$$

This is equivalent to

$$\forall x, y \in A \;\; \big(\neg(x \mathrel{R} y \land y \mathrel{R} x) \lor x = y\big)$$

by the logical identity on implication. In view of De Morgan's Laws, this in turn is equivalent to

$$\forall x, y \in A \;\; \neg\big((x \mathrel{R} y \land y \mathrel{R} x) \land x \neq y\big)$$

and thus, via Theorem 2.3.1, to

$$\neg \exists x, y \in A \;\; (x \neq y \land x \mathrel{R} y \land y \mathrel{R} x),$$

which is what we gave for Exercise 2.1(c).

2.3. (a) $\neg \exists x \in \mathbb{N} \;\; \forall y \in \mathbb{N} \;\; (x \geqslant y)$.
Another acceptable answer is $\forall x \in \mathbb{N} \;\; \exists y \in \mathbb{N} \;\; (y > x)$, but this resembles the given sentence less.

(b) $\forall x, y \in \mathbb{Q} \;\; \big(x \neq y \to \exists z \in \mathbb{Q} \;\; \big((x < z \land z < y) \lor (y < z \land z < x)\big)\big)$.
Another acceptable answer is

$$\forall x, y \in \mathbb{Q} \;\; \exists z \in \mathbb{Q} \;\; (x \neq y \to (x < z \land z < y) \lor (y < z \land z < x)),$$

but this resembles the given sentence less. Yet less preferable but still acceptable answers include

$$\forall x, y \in \mathbb{Q} \;\; \big(x < y \to \exists z \in \mathbb{Q} \;\; (x < z \land z < y)\big) \quad \text{and}$$
$$\forall x, y \in \mathbb{Q} \;\; \exists z \in \mathbb{Q} \;\; \big(x < y \to x < z \land z < y\big).$$

One may write $x < z < y$ for $x < z \land z < y$.

2.4. (a)
- By Theorem 2.4.9, the negation is equivalent to

$$\exists x \in \mathbb{R} \;\; \forall y \in \mathbb{R} \;\; \neg(y = x^2),$$

or simply $\exists x \in \mathbb{R} \;\; \forall y \in \mathbb{R} \;\; y \neq x^2$.

- The original proposition says any real number has a real square.
- This is **true**.

(b)
- By Theorem 2.4.9, the negation is equivalent to
$$\exists y \in \mathbb{R} \ \ \forall x \in \mathbb{R} \ \ \neg(y = x^2),$$
or simply $\exists y \in \mathbb{R} \ \ \forall x \in \mathbb{R} \ \ y \neq x^2$.
- This negation says some real number is not the square of any real number.
- The negated proposition is true: for instance, the real number -1 is not the square of any real number.
- The original proposition is **false**, because its negation is true.

(c)
- By Theorem 2.4.9, the negation is equivalent to
$$\exists x_1, x_2, y_1, y_2 \in \mathbb{R} \ \ \neg(y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge x_1 \neq x_2 \to y_1 \neq y_2).$$
In view of Example 1.4.23, this is in turn equivalent to
$$\exists x_1, x_2, y_1, y_2 \in \mathbb{R} \ \ (y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge x_1 \neq x_2 \wedge \neg(y_1 \neq y_2)),$$
or simply $\exists x_1, x_2, y_1, y_2 \in \mathbb{R} \ \ (y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge x_1 \neq x_2 \wedge y_1 = y_2)$.
- The negated proposition says there are distinct real numbers who squares are equal.
- This is true: for instance, the real numbers 1 and -1 are not equal, but their squares are equal.
- The original proposition is **false**, because its negation is true.

(d)
- By Theorem 2.4.9, the negation is equivalent to
$$\exists x_1, x_2, y_1, y_2 \in \mathbb{R} \ \ \neg(y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge y_1 \neq y_2 \to x_1 \neq x_2).$$
In view of Example 1.4.23, this is in turn equivalent to
$$\exists x_1, x_2, y_1, y_2 \in \mathbb{R} \ \ (y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge y_1 \neq y_2 \wedge \neg(x_1 \neq x_2))$$
or simply $\exists x_1, x_2, y_1, y_2 \in \mathbb{R} \ \ (y_1 = {x_1}^2 \wedge y_2 = {x_2}^2 \wedge y_1 \neq y_2 \wedge x_1 = x_2)$.
- The original proposition says that if the squares of two real numbers are different, then these two real numbers must be different too.
- To put this contrapositively, this says if two real numbers are equal, then their squares are also equal.
- This is true.
- So by Theorem 1.4.12(1), the given proposition is **true** as well.

(e)
- By Theorem 2.4.9, the negation is equivalent to
$$\exists x \in \mathbb{R} \ \ \neg\big((\exists y \in \mathbb{R} \ \ y = x^2) \to x \geqslant 0\big).$$
In view of Example 1.4.23, this is in turn equivalent to
$$\exists x \in \mathbb{R} \ \ \big((\exists y \in \mathbb{R} \ \ y = x^2) \wedge \neg(x \geqslant 0)\big),$$
or simply $\exists x \in \mathbb{R} \ \ \big((\exists y \in \mathbb{R} \ \ y = x^2) \wedge x < 0\big)$.
- The negated proposition says there is a real number that has a real square but is negative.
- This is true: for instance, the real number -1 has a real square 1 but it is negative.
- The original proposition is **false**, because its negation is true.

(f)
- By Theorem 2.3.1, the negation is equivalent to

$$\exists y \in \mathbb{R} \ \ \neg\big((\exists x \in \mathbb{R} \ \ y = x^2) \to y \geqslant 0\big).$$

  In view of Example 1.4.23, this is in turn equivalent to

$$\exists y \in \mathbb{R} \ \ \big((\exists x \in \mathbb{R} \ \ y = x^2) \wedge \neg(y \geqslant 0)\big),$$

  or simply $\exists y \in \mathbb{R} \ \ \big((\exists x \in \mathbb{R} \ \ y = x^2) \wedge y < 0\big)$.
- The original proposition says any real number that is the square of some real number must be non-negative.
- This is **true**.

Additional information. Consider the proposition

$$\forall x \in \mathbb{R} \ \ \exists y \in \mathbb{R} \ \ (y = x^2 \to x \geqslant 0).$$

It says that for every real number x, there is a real number y such that if $y = x^2$, then $x \geqslant 0$. This is **true**: no matter which real number x is, one can set y to be the real number -1 such that $y = x^2$ is false and thus the conditional proposition $y = x^2 \to x \geqslant 0$ is vacuously true. In particular, this proposition is *not* equivalent to that in (e).

Extra exercises

2.5. $\forall d, n \in \mathbb{Z} \ \ \big(\mathrm{Divides}(d, n) \leftrightarrow \exists k \in \mathbb{Z} \ \ (n = dk)\big)$.

The proposition $\forall d, n \in \mathbb{Z} \ \ \exists k \in \mathbb{Z} \ \ (\mathrm{Divides}(d, n) \leftrightarrow n = dk)$ is *not* a correct answer. To see this, let us replace $\mathrm{Divides}(d, n)$ by "$d + n \neq d + n$".

- The first proposition becomes false because 2 and 6 are integers where $2+6 \neq 2+6$ is false, but for some integer k, namely $k = 3$, we have $6 = 2 \times k$.
- The second proposition remains true because, given any integers d and n, one can choose k to be the integer $n+1$ such that $d+n \neq d+n$ and $n = dk$ are both false.

Hence the two propositions do not "mean" the same.

2.6. (a)
- By Theorem 2.4.9, the negation is equivalent to

$$\exists x \in D \ \ \forall y \in E \ \ \neg(y < x)$$

  or simply $\exists x \in D \ \ \forall y \in E \ \ y \geqslant x$.
- The original proposition says every element x of D is strictly bigger than some element y of E.
- This is **true**: no matter what x is, we can choose $y = 0$ to guarantee $y < x$.

(b)
- By Theorem 2.4.9, the negation is equivalent to

$$\forall y \in E \ \ \exists x \in D \ \ \neg(y < x)$$

  or simply $\forall y \in E \ \ \exists x \in D \ \ y \geqslant x$.
- The original proposition says some element y of E is strictly less than all the elements x of D.
- This is **true**: take $y = 0$ so that, no matter what x is, we must have $y < x$.

(c)
- By Theorem 2.4.9, the negation is equivalent to

$$\exists x \in D \ \ \forall y \in E \ \ \neg(y + 1 = x)$$

  or simply $\exists x \in D \ \ \forall y \in E \ \ y + 1 \neq x$.

- This negation says some element x of D is not equal to any element y of E plus 1.
- This is true: the element 13 of D is not equal to any element of E plus 1.
- Thus the original proposition is **false**, because its negation is true.

(d)
- By Theorem 2.3.1, the negation is equivalent to

$$\exists x \in D \ \ \neg\big(x < 6 \to \exists y \in E \ \ (y + 1 = x)\big).$$

  In view of Example 1.4.23, this is equivalent to

$$\exists x \in D \ \ \big(x < 6 \wedge \neg\exists y \in E \ \ (y + 1 = x)\big).$$

  This is in turn equivalent to

$$\exists x \in D \ \ \big(x < 6 \wedge \forall y \in E \ \ \neg(y + 1 = x)\big)$$

  by Theorem 2.3.1, or simply $\exists x \in D \ \ \big(x < 6 \wedge \forall y \in E \ \ (y + 1 \neq x)\big)$.
- The original proposition says every element x of D that is strictly less than 6 is equal to some element y of E plus 1.
- This is **true**: the only elements of D that are strictly less than 6 are 1, 3, and 5; if x is equal of 1, 3, or 5, then we can take y to be 0, 2, or 4 respectively such that $y + 1 = x$.