CS1231 Chapter 1

Propositional logic

1.1 Propositions

Definition 1.1.1. (1) A *proposition* (or a *statement*) is a sentence that is either true or false but not both.

(2) The *truth value* of a proposition is true if the proposition is true; it is false otherwise.

(3) Often we abbreviate the truth values true and false by T and F respectively.

Example 1.1.2. Each of the following is a proposition.

(1) $1 + 2 = 3$.

(2) $2 + 3 < 4$.

(3) Every even integer is the sum of two odd integers.

(4) Every even integer that is greater than four is the sum of two prime numbers.

The truth values of (1), (2), and (3) are T, F, and T respectively. The truth value of (4) is unknown.

Remark 1.1.3. One does not need to know the truth value of a sentence to know that the sentence has a unique truth value.

Remark 1.1.4. We consider *only* sentences whose truth or falsity depends neither on time nor on anyone's knowledge.

Example 1.1.5. None of the following is a proposition when x and y are variables.

(1) $1 + 2$

(2) Good morning!

(3) Please explain this in more detail.

(4) Why is this true?

(5) $x + y = 0$.

When one substitutes numbers into the variables, the sentence in (5) would become a proposition. We will discuss this more in Chapter 2.

Theorem 1.1.6 (Liar Paradox, extra material)**.** The sentence below is not a proposition:

$$\text{The sentence } (*) \text{ is not true.} \tag{$*$}$$

Proof. We prove this by contradiction. Suppose $(*)$ is a proposition. The meaning of $(*)$ tells us

(a) if $(*)$ is true, then $(*)$ is not true; and

(b) if $(*)$ is false, then $(*)$ is true.

Split into two cases.

- Case 1: assume $(*)$ is true. Then $(*)$ is not true by (a). This contradicts our assumption that $(*)$ is true.
- Case 2: assume $(*)$ is false. Then $(*)$ is true by (b). This contradicts our assumption that $(*)$ is false, as $(*)$ cannot be both true and false as a proposition.

In either case, we get a contradiction. So the proof is finished. $\square$

1.2 Boolean connectives

Definition 1.2.1. Let p be a proposition.

(1) We denote by $\neg p$ (or $\sim p$) the proposition "it is not the case that p". Often we read $\neg p$ as "not p", and call it the *negation* of p.

(2) The proposition $\neg p$ is true if p is false; it is false otherwise.

Definition 1.2.2. Let p, q be propositions.

(1) We denote by $p \wedge q$ the proposition "p and q". We call $p \wedge q$ the *conjunction* of p and q.

(2) The conjunction $p \wedge q$ is true if p and q are both true; it is false otherwise.

Definition 1.2.3. Let p, q be propositions.

(1) We denote by $p \vee q$ the proposition "p or q". We call $p \vee q$ the *disjunction* of p and q.

(2) The disjunction $p \vee q$ is false if p and q are both false; it is true otherwise.

Remark 1.2.4. Let p, q be propositions. Note that the truth values of the negated proposition $\neg p$, the conjunction $p \wedge q$, and the disjunction $p \vee q$ depend *only* on the truth values of p and q, not on their other aspects. So one can summarize the behaviours of $\neg$, $\wedge$ and $\vee$ as in Table 1.1, where each possible combination of the truth values of p and q appears as a row. These tables are called *truth tables.*

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.1: Truth tables for $\neg$, $\wedge$ and $\vee$

Warning 1.2.5. In everyday English, if two propositions p and q are both true, then "p or q" is false. However, in common mathematical parlance, if two propositions p and q are both true, then "p or q" is true. Moreover, including the word "either" does not change this. If we want to express the everyday English sense of "or" in mathematics, then we may write "p or q but not both", as we did in Definition 1.1.1.

p	q	r	
T	T	T	...
T	T	F	...
T	F	T	...
T	F	F	...
F	T	T	...
F	T	F	...
F	F	T	...
F	F	F	...

Table 1.2: Listing out all possibilities in a truth table

Example 1.2.6. Consider a fixed (not a variable) real number x. Let p and q be the propositions "$x > 1$" and "$x^2 > 1$" respectively.

(1) $p \vee q$ is "$x > 1$ or $x^2 > 1$", or equivalently "$x^2 > 1$".

(2) $\neg(p \vee q)$ is "it is not the case that $x > 1$ or $x^2 > 1$", or equivalently "$x^2 \leqslant 1$".

(a) Suppose $x = 12.31$. Then p is true and q is true. So $p \vee q$ is true and $\neg(p \vee q)$ is false.

(b) Suppose $x = 0$. Then p is false and q is false. So $p \vee q$ is false and $\neg(p \vee q)$ is true.

(c) Suppose $x = -2$. Then p is false and q is true. So $p \vee q$ is true and $\neg(p \vee q)$ is false.

1.3 Conditional propositions

Note 1.3.1. Let p, q be propositions. In mathematics, we interpret the proposition "if p then q" as a *guarantee*: it guarantees that whenever the proposition p is true, the proposition q must also be true. This guarantee is false (or, more grammatically, this guarantee fails) only when p is true but q is false; otherwise it is true. In this interpretation, the truth value of "if p then q" depends *only* on the truth values of p and q; there is no requirement on whether p and q have related subject matters, unlike in everyday English.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.3: Truth tables for $\to$ and $\leftrightarrow$

Definition 1.3.2. Let p, q be propositions.

(1) We denote by $p \to q$ the proposition "if p then q". Often we read this as "p implies q", and call it a *conditional proposition* or an *implication*.

(2) The conditional proposition $p \to q$ is false if p is true and q is false; it is true otherwise.

(3) In the conditional proposition $p \to q$, we call p the *hypothesis* (or the *antecedent*) and q the *conclusion* (or the *consequent*).

Terminology 1.3.3. In mathematics, the following are alternative ways to express "if p then q" when p, q are propositions.

- q if p.
- q whenever p.
- p only if q.
- p is sufficient for q.
- p is a sufficient condition for q.
- q is necessary for p.
- q is a necessary condition for p.

Example 1.3.4. Consider a fixed (not a variable) real number x. Let p and q be the propositions "$x > 1$" and "$x^2 > 1$" respectively.

(1) $p \to q$ is "if $x > 1$, then $x^2 > 1$".

(2) $q \to p$ is "if $x^2 > 1$, then $x > 1$".

(a) Suppose $x = 3$. Then p is true and q is true. So $p \to q$ is true and $q \to p$ is true.

(b) Suppose $x = 0$. Then p is false and q is true. So $p \to q$ is true and $q \to p$ is true.

(c) Suppose $x = -2$. Then p is false and q is true. So $p \to q$ is true but $q \to p$ is false.

Terminology 1.3.5. A *vacuously true* conditional proposition is one in which the hypothesis is false.

Example 1.3.6. In Example 1.3.4, the implication $p \to q$ is vacuously true when $x = 0$ and when $x = -2$

Exercise 1.3.7. Why are all vacuously true conditional propositions true? ✎ 1a

Vacuously true conditional propositions are true as hypothesis cannot be satisfied (false), it does not matter whether the conclusion is true or not; the implication will be true.

Definition 1.3.8. Let p, q be propositions.

(1) The *converse* of $p \to q$ is $q \to p$.

(2) The *inverse* of $p \to q$ is $(\neg p) \to (\neg q)$.

(3) The *contrapositive* of $p \to q$ is $(\neg q) \to (\neg p)$.

Note 1.3.9. Example 1.3.4 shows that a conditional proposition and its converse may have a different truth values. The same goes for the inverse, but not for the contrapositive, as the following truth table shows.

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$(\neg p) \to (\neg q)$	$(\neg q) \to (\neg p)$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Exercise 1.3.10. Let a, b, c, d be propositions. What are the converse, the inverse, and the contrapositive of $(a \wedge b) \to (c \vee d)$? ✎ 1b

Definition 1.3.11. Let p, q be propositions.

(1) We denote by $p \leftrightarrow q$ the proposition "p if and only if q". Sometimes we read this as "p is equivalent to q" and call it a *biconditional proposition* or an *equivalence*. Some abbreviate "if and only if" to "iff".

(2) The biconditional proposition $p \leftrightarrow q$ is true if p and q have the same truth value; it is false otherwise.

Terminology 1.3.12. In mathematics, the following are alternative ways to express "p if and only if q" when p, q are propositions.

- p is necessary and sufficient for q.
- p is a necessary and sufficient condition for q.
- p exactly if q.
- p precisely if q.
- p exactly when q.
- p precisely when q.

Example 1.3.13. Consider a fixed (not a variable) real number x. As in Example 1.3.4, let p and q be the propositions "$x > 1$" and "$x^2 > 1$" respectively. Then $p \leftrightarrow q$ is

$$\text{“}x > 1 \text{ if and only if } x^2 > 1\text{”}.$$

- Case 1 :
  - Proposition is True
    - If P is not not false (i.e false), then P is both true and false (contradicting that P is a proposition)
    - Therefore, P is not false (P is True)

(a) Suppose $x = 3$. Then p is true and q is true. So $p \leftrightarrow q$ is true.

(b) Suppose $x = -2$. Then p is false and q is true. So $p \leftrightarrow q$ is false.

Exercise 1.3.14. Explain why a proposition is true if and only if it is not false. 1c

Remark 1.3.15. The phrase "if and only if" is rarely used in everyday English: frequently, people state only one of "if" and "only if" and leave the other part implicit. For example, in everyday English,

You will get ice cream if you behave.

sometimes actually means

You will get ice cream if and only if you behave.

Here the "only if" part is left implicit. In mathematical contexts, we leave no part implicit: "if" means "if", and "only if" means "only if", *except* when following Convention 1.4.8.

1.4 Equivalence

Definition 1.4.1. (1) A *propositional variable* is a variable for substituting in an arbitrary proposition.

(2) A *compound expression* is an expression constructed (grammatically) from propositional variables using $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

Example 1.4.2. Let p, q be propositional variables. Then $\neg(p \vee q)$ and $(\neg p) \vee q$ are compound expressions.

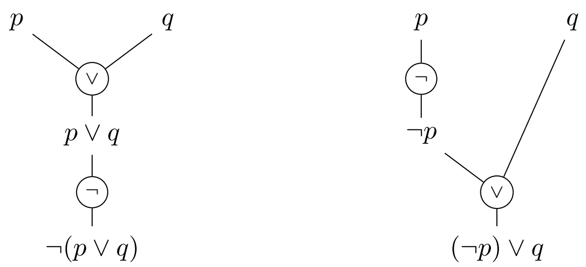

Figure 1.4: Building up compound expressions from propositional variables

Convention 1.4.3. (1) When there is a choice, one always performs first the propositional connective nearer to the top in the following figure.

$$\begin{array}{cc} \multicolumn{2}{c}{\neg} \\ \wedge & \vee \\ \to & \leftrightarrow \end{array}$$

(2) We do not set up a rule to decide whether to perform $\wedge$ or $\vee$ first when there is a choice, *except* when following Convention 1.4.22. Do *not* leave it to the reader to choose which of these to perform first. The same goes for $\to$ and $\leftrightarrow$.

(3) Use parentheses $(\dots)$ to prevent ambiguities.

Example 1.4.4. Let p, q, r be propositional variables.

(1) One can write $(\neg p) \vee q$ from Example 1.4.2 alternatively as $\neg p \vee q$.

(2) One can write $(\neg q) \to (\neg p)$ from Definition 1.3.8(3) alternatively as $\neg q \to \neg p$.

(3) Do *not* write $p \wedge q \vee r$ because our convention does not specify whether one should perform $\wedge$ or $\vee$ first.

Terminology 1.4.5. When one substitutes propositions into all the propositional variables occurring in a compound expression, one obtains a proposition. If the proposition obtained is true, then we say that the compound expression *evaluates to* T under this substitution; else we say that it *evaluates to* F.

Exercise 1.4.6. Let p and q be propositional variables. When one substitutes a true proposition into p and a false proposition into q, what does the compound expression $\neg p \to \neg q$ evaluates to? ✎ 1d

- p = true , q = false
- ¬p = false, ¬q = true
- ¬p → ¬q = true

Definition 1.4.7. Two compound expressions P, Q are *equivalent* if (and only if) they evaluate to the same truth value under any substitution of propositions into the propositional variables. In this case, we write $P \equiv Q$.

Convention 1.4.8. In mathematical definitions, people often use "if" between the term being defined and the phrase being used to define the term. This is the *only* situation in mathematics when "if" should be understood as a (special) "if and only if".

Remark 1.4.9. Definition 1.3.11 and Definition 1.4.7 give two different notions of equivalences. We will look into the relationship between two in Tutorial Question 1.7.

Exercise 1.4.10. Convince yourself that the following are true for all compound expressions P, Q, and R. ✎ 1e

(1) $P \equiv P$. P and P evaluate to the same truth value under any substitution of propositions into the propositional variables.

(2) If $P \equiv Q$, then $Q \equiv P$. If P and Q evaluate to the same truth value under any substitution of propositions into the propositional variables, then so do Q and P .

(3) If $P \equiv Q$ and $Q \equiv R$, then $P \equiv R$. If P and Q evaluate to the same truth value under any substitution of propositions into the propositional variables, and Q and R evaluate to the same truth value under any substitution of propositions into the propositional variables, then so do P and R .

We will demonstrate in Chapter 6 that these are *precisely* the properties that a reasonable notion of equivalence should satisfy.

Technique 1.4.11. One way to check whether two compound expressions are equivalent is to draw a truth table for the two expressions: if the columns for the two expressions are exactly the same, then the expressions are equivalent, else they are not.

Justification. Suppose the columns for the two compound expressions P and Q are exactly the same. As all possible combinations of truth values of propositions (to be substituted into P and Q) are listed in the truth table, this indicates that P and Q evaluate to the same truth value under any substitution of propositions into the propositional variables. Thus P and Q are equivalent. Note that we use here implicitly the fact that what a compound expression evaluates to depends only on the truth values of the propositions being substituted in, not on their other aspects, as alluded to in Remark 1.2.4.

Suppose the columns for P and Q are not exactly the same. Say, they differ in row i. This row i corresponds to a way to substitute true and false propositions into the propositional variables occurring in P and Q. As the columns for P and Q are different in row i, these two expressions evaluate to different truth values under this substitution. This shows that P and Q are not equivalent. $\square$

Theorem 1.4.12. Let p, q be propositional variables. Consider the conditional proposition $p \to q$.

(1) The conditional proposition $p \to q$ is equivalent to its contrapositive $\neg q \to \neg p$.

(2) The converse $q \to p$ is equivalent to the inverse $\neg p \to \neg q$.

(3) The conditional proposition $p \to q$ is not equivalent to its converse $p \to q$.

Proof. (1) We see that the truth table in Note 1.3.9 the column for $p \to q$ is exactly the same as that for $\neg q \to \neg p$. So these two compound expressions are equivalent.

(2) Similarly, the compound expressions $q \to p$ and $\neg p \to \neg q$ are equivalent because their columns in the truth table in Note 1.3.9 are exactly the same.

(3) In the truth table in Note 1.3.9, the columns for $p \to q$ and $q \to p$ differ in the row where p is true and q is false. So these compound expressions are not equivalent. $\square$

Remark 1.4.13. For Theorem 1.4.12(3), the columns involved actually differ also in the row where p is false and q is true, but one differing row is enough to show the non-equivalence.

Theorem 1.4.14. Let p, q be propositional variables. Then $p \to q$ is equivalent to $\neg p \vee q$.

Proof. Here is the truth table for $p \to q$ and $\neg p \vee q$.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The columns for $p \to q$ and $\neg p \vee q$ are exactly the same. So the two compound expressions are equivalent. $\square$

Technique 1.4.15. To prove that two compound expressions are not equivalent, it suffices to find *one* way to substitute true and false propositions into the propositional variables to make the expressions evaluate to different truth values.

Example 1.4.16. Let p, q be propositional variables. Then $\neg(p \vee q)$ and $\neg p \vee q$ are not equivalent.

Proof. Let us substitute true propositions into both p and q. Under this substitution, the expression $p \vee q$ evaluates to T, and thus $\neg(p \vee q)$ evaluates to F. However, under the same substitution, the expression $\neg p \vee q$ evaluates to T. So these two compound expressions are not equivalent. $\square$

Exercise 1.4.17. Let p, q, r be propositional variables. Is $(p \wedge q) \vee r$ equivalent to $p \wedge (q \vee r)$? Give a proof for your answer. 1f

Let p be F, q be T and r be T.
(p ∧ q) ∨ r is T while
p ∧ (q ∨ r) is F

Definition 1.4.18. (1) A *tautology* is a compound expression that evaluates to true no matter what propositions are substituted into all its propositional variables.

(2) A *contradiction* is a compound expression that evaluates to false no matter what propositions are substituted into all its propositional variables.

Example 1.4.19. Let p be a propositional variable. Here is the truth table for $p \vee \neg p$ and $p \wedge \neg p$.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Since all the entries in the column for $p \vee \neg p$ are T, we see that $p \vee \neg p$ is a tautology. Since all the entries in the column for $p \wedge \neg p$ are F, we see that $p \wedge \neg p$ is a contradiction.

Theorem 1.4.20 (logical identities)**.** Let t be a tautology and c be a contradiction. For all propositional variables p, q, r, the following equivalences hold.

Commutativity	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associativity	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributivity	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Idempotence	$p \vee p \equiv p$	$p \wedge p \equiv p$
Absorption	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Identities	$p \vee c \equiv p$	$p \wedge t \equiv p$
Annihilators	$p \vee t \equiv t$	$p \wedge c \equiv c$
Negation	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
Double Negative Law	$\neg\neg p \equiv p$	
Top and bottom	$\neg c \equiv t$	$\neg t \equiv c$
Implication	$p \to q \equiv \neg p \vee q$	

Proof. We saw the Negation and the Implication parts already in Theorem 1.4.14 and Example 1.4.19 respectively. We will prove the Double Negative Law and De Morgan's Laws below. Distributivity, Idempotence, Absorption, and Top and bottom will appear in the tutorial. The rest (i.e., Commutativity, Identities, and Annihilators) are left as exercises. ✎ 1g

For the Double Negative Law and the De Morgan Law on the left, we use the following truth tables.

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

We see that the columns for $\neg\neg p$ and p are exactly the same in the left truth table. So these expressions are equivalent. Similarly, the columns for $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are exactly the same in the right truth table. So these expressions are also equivalent.

For the De Morgan Law on the right, we can also use a truth table, but let us proceed as follows to demonstrate another approach:

$$\begin{aligned}
\neg(p \wedge q) &\equiv \neg(\neg\neg p \wedge \neg\neg q) && \text{by the Double Negative Law;} \\
&\equiv \neg\neg(\neg p \vee \neg q) && \text{by the left De Morgan Law;} \\
&\equiv \neg p \vee \neg q && \text{by the Double Negative Law.}
\end{aligned}$$ □

Remark 1.4.21. Our proof of the right De Morgan Law above used the fact that if one substitutes equivalent compound expressions into the same propositional variable in equivalent compound expressions, then one obtains again equivalent compound expressions.

Convention 1.4.22. Let p, q, r be propositional variables. There are two reasonable interpretations of $p \vee q \vee r$: it can stand for $(p \vee q) \vee r$ or $p \vee (q \vee r)$. These two interpretations are equivalent by the associativity of $\vee$ from Theorem 1.4.20. It does not matter which interpretation we take because we will treat equivalent compound expressions as "the same". So we may write $p \vee q \vee r$ without parentheses. Similarly, in view of the commutativity of $\vee$, we will not need to distinguish between $p \vee q$ and $q \vee p$. These generalize to any number of $\vee$'s. The same goes for $\wedge$, but not a mixture of the two; cf. Convention 1.4.3.

Example 1.4.23. Let p, q be propositional variables. Then $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

Proof.

$$\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by the logical identity on the implication;} \\
&\equiv \neg\neg p \wedge \neg q && \text{by De Morgan's Laws;} \\
&\equiv p \wedge \neg q && \text{by the Double Negative Law.}
\end{aligned}$$ □

Note 1.4.24. Different (looking) compound expressions may be equivalent. Therefore, showing that a compound expression P is equivalent (via the logical identities, say) to one that is different from another compound expression Q alone is not sufficient to imply that P and Q are not equivalent.

Example 1.4.25. Let A, B be sets. (For this example, it does not matter what "set" means.) Consider the following propositions from Chapter 8.

(1) If A is a subset of B, then the countability of B implies the countability of A.

(2) If A is not countable and A is a subset of B, then B is not countable.

One can rewrite these two propositions symbolically as

$$p \rightarrow (r \rightarrow q) \quad \text{and} \quad (\neg q \wedge p) \rightarrow \neg r$$

respectively, where

- p denotes the proposition "A is a subset of B";
- q denotes the proposition "A is countable"; and
- r denotes the proposition "B is countable".

Treating p, q, r as propositional variables, we see that

$$\begin{aligned}
p \to (r \to q) &\equiv \neg p \vee (\neg r \vee q) && \text{by the logical identity on the implication;}\\
&\equiv (q \vee \neg p) \vee \neg r && \text{by the commutativity and the associativity of } \vee;\\
&\equiv (\neg\neg q \vee \neg p) \vee \neg r && \text{by the Double Negative Law;}\\
&\equiv \neg(\neg q \wedge p) \vee \neg r && \text{by De Morgan's Laws;}\\
&\equiv (\neg q \wedge p) \to \neg r && \text{by the logical identity on the implication.}
\end{aligned}$$

So propositions (1) and (2) have the same truth value.

Example 1.4.26. Let p, q be propositional variables. Then $\big((p \to q) \wedge p\big) \to q$ is a tautology.

Proof.

$$\begin{aligned}
\big((p \to q) \wedge p\big) \to q &\equiv \neg\big((p \to q) \wedge p\big) \vee q && \text{by the logical identity on the implication.}\\
&\equiv \neg(p \to q) \vee \neg p \vee q && \text{by De Morgan's Laws;}\\
&\equiv \neg(p \to q) \vee (p \to q) && \text{by the logical identity on the implication.}
\end{aligned}$$

So $\big((p \to q) \wedge p\big) \to q$ is a tautology by Example 1.4.19. □

Remark 1.4.27. In Example 1.4.26, our proof used the fact that a compound expression is a tautology if it is equivalent to one. In fact, it is also true that a compound expression is a contradiction if it is equivalent to one.

Tutorial exercises

An asterisk (*) indicates a more challenging question.

1.1. Let p, q be false propositions and r be a true proposition. What are the truth values of the following propositions?

(a) $p \wedge q \to r$.

(b) $p \leftrightarrow q \vee \neg r$.

(c) $p \wedge \big(q \vee (\neg r \to p)\big)$.

(d) $\big(q \vee r \vee (\neg p \leftrightarrow q)\big) \wedge (q \vee r)$.

1.2. Fix a graph G. (For this exercise, it does not matter what "graph" means.) Consider the following propositions from Chapter 12.

(a) The graph G is a tree if and only if it is connected and acyclic.

(b) For G to be a tree, it is necessary that G is acyclic.

(c) Being a tree is a sufficient condition for the graph G to be connected.

Rewrite these propositions symbolically in terms of c, a, t, where

- c denotes the proposition "G is connected";
- a denotes the proposition "G is acyclic"; and
- t denotes the proposition "G is a tree".

1.3. Fix a tautology t and a contradiction c. Let p, q, r be propositional variables.

(a) Use truth tables to prove the following logical identities.

(i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

(ii) $p \wedge p \equiv p$.

(iii) $p \wedge (p \vee q) \equiv p$.

(iv) $\neg t \equiv c$.

(b) Use part (a) and the logical identities proved (in the exercises) in the notes to prove the following logical identities.

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(ii) $p \vee p \equiv p$.

(iii) $p \vee (p \wedge q) \equiv p$.

(iv) $\neg c \equiv t$.

1.4. In this exercise, we show that "if and only if" really means "if" and "only if".

Let p, q be propositional variables. Prove that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

1.5. Let p, q, r be propositional variables. Prove that $\big((p \to q) \wedge (q \to r)\big) \to (p \to r)$ is a tautology.

This tautology is sometimes called the *transitivity of* $\to$. It is used all the time, albeit implicitly, in proofs.

1.6. Let p, q be propositional variables. Is $(p \to q) \to (q \to p)$ is tautology? Give a proof for your answer.

1.7.* We saw two different notions of equivalences: the symbol $\leftrightarrow$ refers to one between propositions, while the symbol $\equiv$ refers to one between compound expressions. Although these are different notions, they are closely related to each other, as we will show in this exercise.

Let P and Q be compound expressions. Prove that $P \leftrightarrow Q$ is a tautology if and only if $P \equiv Q$.

(Hint: Study carefully the definitions of $\leftrightarrow$, tautology, and $\equiv$.)

Extra exercises

1.8. Let p, q, r be propositional variables. Consider the compound expressions

$$p \wedge q \to r \quad \text{and} \quad (p \to r) \wedge (q \to r).$$

Are they equivalent? Give a proof for your answer.

1.9. Let p, q, r be propositional variables, and c be a contradiction. Prove that the following compound expressions are tautologies:

(a) $(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$;

(b) $(\neg p \to c) \to p$. Use proof by contradiction

As we will see in Chapter 3, proofs that split into cases and proofs by contradictions are based on these tautologies respectively.

1.10. We omitted the proof of Remark 1.4.21. Here let us get a taste of what the proof is like by looking at a particular example.

Let P, Q be compound expressions. Prove that, if $P \equiv Q$, then $\neg P \equiv \neg Q$.