Tutorial solutions for Chapter 10

Sometimes there are other correct answers.

10.1. Let A, B, C, D be finite sets. Then

$$\begin{aligned}|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|.\end{aligned}$$

Proof. By the Inclusion–Exclusion Rule for two sets,

$$|A \cup B \cup C \cup D| = |A| + |B \cup C \cup D| - |A \cap (B \cup C \cup D)|.$$

We know how to handle the term $|B \cup C \cup D|$ using the Inclusion–Exclusion Rule for three sets. For the last term,

$$\begin{aligned}& |A \cap (B \cup C \cup D)| \\ & = |(A \cap B) \cup (A \cap C) \cup (A \cap D)| \\ & = |A \cap B| + |A \cap C| + |A \cap D| \\ & \quad - |(A \cap B) \cap (A \cap C)| - |(A \cap C) \cap (A \cap D)| - |(A \cap D) \cap (A \cap B)| \\ & \quad + |(A \cap B) \cap (A \cap C) \cap (A \cap D)| \\ & = |A \cap B| + |A \cap C| + |A \cap D| - |A \cap B \cap C| - |A \cap C \cap D| - |A \cap B \cap D| \\ & \quad + |A \cap B \cap C \cap D|.\end{aligned}$$

Putting all these together gives

$$\begin{aligned}|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| - |B \cap C| - |C \cap D| - |D \cap B| \\ & + |B \cap C \cap D| - |A \cap B| - |A \cap C| - |A \cap D| \\ & + |A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap D| - |A \cap B \cap C \cap D|. \qquad \square\end{aligned}$$

10.2. Suppose $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

(a) By definition, the relations from A to B are precisely the subsets of $A \times B$. So the number of relations from A to B is

$$\begin{aligned}|\mathcal{P}(A \times B)| &= 2^{|A \times B|} && \text{by Theorem 10.2.7;} \\ &= 2^{|A| \times |B|} && \text{by Corollary 10.2.4(1);} \\ &= 2^{mn}.\end{aligned}$$

(b) We know from Proposition 7.2.7 that a function $f\colon A \to B$ is completely determined by $f(a_1), f(a_2), \ldots, f(a_m)$. As $|B| = n$, there are n choices for each of these. So by the General Multiplication Rule, the number of functions $A \to B$ is

$$\underbrace{n \times n \times \cdots \times n}_{m\text{-many } n\text{'s}} = n^m.$$

Additional comment. The argument above suggests that functions $A \to B$ may be viewed as tuples in B^m.

(c)* We know from the Dual Pigeonhole Principle that, if $m < n$, then there is no surjection $A \to B$. So suppose $m \geqslant n$. Let F denote the set of all functions $A \to B$. For each $b_j \in B$, define

$$F_j = \{f \in F : b_j \notin \operatorname{range}(f)\}.$$

As $|B| = n = 3$, the set of all surjections $A \to B$ is $F \setminus (F_1 \cup F_2 \cup F_3)$. By the Difference Rule, the size of this set is $|F| - |F_1 \cup F_2 \cup F_3|$. From (b), we know $|F| = n^m = 3^m$. Let us calculate $|F_1 \cup F_2 \cup F_3|$ using the Inclusion–Exclusion Principle for three sets: as in (b),

$$|F_1| = |F_2| = |F_3| = \underbrace{2 \times 2 \times \cdots \times 2}_{m\text{-many 2's}} = 2^m;$$

$$|F_1 \cap F_2| = |F_2 \cap F_3| = |F_3 \cap F_1| = \underbrace{1 \times 1 \times \cdots \times 1}_{m\text{-many 1's}} = 1;$$

$$\begin{aligned} & |F_1 \cap F_2 \cap F_3| = 0 \quad && \text{as any } f \in F_1 \cap F_2 \cap F_3 \text{ must have an empty range,} \\ & && \text{which is not possible as } A \neq \varnothing; \\ \therefore\quad & |F_1 \cup F_2 \cup F_3| = {} && |F_1| + |F_2| + |F_3| - |F_1 \cap F_2| - |F_2 \cap F_3| - |F_3 \cap F_1| \\ & && + |F_1 \cap F_2 \cap F_3| \\ & && = 2^m + 2^m + 2^m - 1 - 1 - 1 + 0 = 2^m 3 - 3. \end{aligned}$$

It follows that the number of surjections $A \to B$ is $3^m - 2^m 3 + 3$.

(d) We know from the Pigeonhole Principle that, if $m > n$, then there is no injection $A \to B$. So suppose $m \leqslant n$. Similar to the situation in (b), we choose $f(a_1), f(a_2), \ldots, f(a_m)$ to determine the function $f\colon A \to B$. Now, we additionally need to ensure that f is injective. This means $f(a_1), f(a_2), \ldots, f(a_m)$ are all different. So there are

$$\begin{array}{lll} n & \text{choices for} & f(a_1), \\ n-1 & \text{choices for} & f(a_2), \\ \vdots & & \vdots \\ n-(m-1) & \text{choices for} & f(a_m). \end{array}$$

Therefore, by the General Multiplication Rule, the number of injections $A \to B$ is

$$n \times (n-1) \times (n-(m-1)) = \frac{n!}{(n-m)!} = P(n, m).$$

Additional comment. The argument above suggests that injections $A \to B$ may be viewed as m-permutations of B.

(e) We know from the two Pigeonhole Principles that, if $m \neq n$, then there is no bijection $A \to B$. So let us concentrate on the case when $m = n$. Now we know from (d) that there are $n \times (n-1) \times \cdots \times (n-(n-1)) = n!$ injections $A \to B$ As A and B are finite and $|A| = m = n = |B|$, all these injections are surjective by Extra Exercise 8.9(b) So there are exactly $n!$ bijections $A \to B$.

Additional comment. The argument above suggests that bijections $A \to B$ may be viewed as permutations of B.

10.3. (a) Applying Theorem 10.3.15,

$$\begin{aligned}
\binom{n}{r} + \binom{n}{r-1} &= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-(r-1))!} \\
&= \frac{(n-r+1)(n!)}{r!\,(n-r+1)((n-r)!)} + \frac{r(n!)}{r((r-1)!)\,(n-r+1)!} \\
&= (n-r+1+r)\,\frac{n!}{r!\,(n-r+1)!} \\
&= \frac{(n+1)(n!)}{r!\,(n-r+1)!} = \frac{(n+1)!}{r!\,((n+1)-r)!} = \binom{n+1}{r}. \qquad \square
\end{aligned}$$

(b) Let $A = \{1, 2, \ldots, n+1\}$. By definition, there are $\binom{n+1}{r}$ subsets of A of size r. For any such subset X, either $n+1 \notin X$ or $n+1 \in X$ but not both.

- The subsets $X \subseteq A$ of size r where $n+1 \notin X$ are precisely the subsets of $A \setminus \{n+1\} = \{1, 2, \ldots, n\}$. So there are precisely $\binom{n}{r}$ of these.
- The subsets $X \subseteq A$ of size r where $n+1 \in X$ are precisely those $X_0 \cup \{n+1\}$ where X_0 is a subset of $A \setminus \{n+1\} = \{1, 2, \ldots, n\}$ of size $r-1$. So there are precisely $\binom{n}{r-1}$ of these.

Hence $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ by the Addition Rule. $\square$

Additional comment. This counting argument is based on the following:

$$\begin{aligned}
&\{X \subseteq A : |X| = r\} \\
&= \{X \subseteq A : |X| = r \text{ and } n+1 \notin X\} \cup \{X \subseteq A : |X| = r \text{ and } n+1 \in X\} \\
&= \{X \subseteq A \setminus \{n+1\} : |X| = r\} \\
&\quad \cup \{X_0 \cup \{n+1\} : X_0 \subseteq A \setminus \{n+1\} \text{ and } |X| = r-1\}.
\end{aligned}$$

Pascal's Triangle.

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & & & \searrow+\swarrow & & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & & & \searrow+\swarrow & & \searrow+\swarrow & & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & & & \searrow+\swarrow & & \searrow+\swarrow & & \searrow+\swarrow & & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
 & & \searrow+\swarrow & & \searrow+\swarrow & & \searrow+\swarrow & & \searrow+\swarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

10.4. (a) Fix $r \in \mathbb{N}$. We proceed by induction on n.

(Base step) If $r = 0$, then $\binom{0}{r} = \binom{0}{0} = 1 = \binom{1}{1} = \binom{0+1}{r+1}$. If $r > 0$, then $\binom{0}{r} = 0 = \binom{1}{r+1} = \binom{0+1}{r+1}$. So $\binom{0}{r} = \binom{0+1}{r+1}$ in all cases.

(Induction step) Let $k \in \mathbb{N}$ such that $\binom{0}{r} + \binom{1}{r} + \cdots + \binom{k}{r} = \binom{k+1}{r+1}$. Then

$$\begin{aligned}
&\binom{0}{r} + \binom{1}{r} + \cdots + \binom{k}{r} + \binom{k+1}{r} \\
&= \binom{k+1}{r+1} + \binom{k+1}{r} && \text{by the induction hypothesis;} \\
&= \binom{k+2}{r+1} && \text{by Exercise 10.3;} \\
&= \binom{(k+1)+1}{r+1}.
\end{aligned}$$

This completes the induction. $\square$

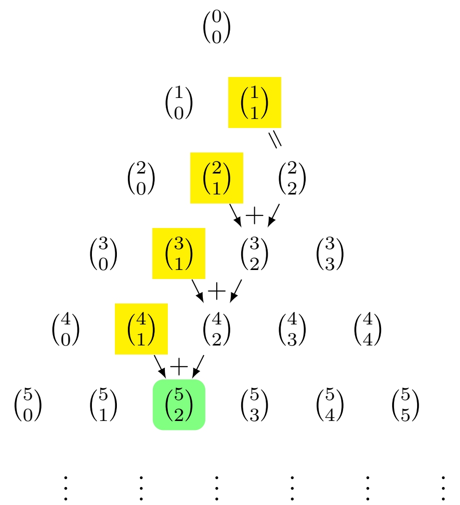
(b) Let $A = \{0, 1, 2, \ldots, n\}$. Note that $|A| = n + 1$. By definition, there are $\binom{n+1}{r+1}$ subsets of A of size $r + 1$. The largest element of any such subset is exactly one of $0, 1, 2, \ldots, n$. For each $m \in \{0, 1, \ldots, n\}$, the subsets of $X \subseteq A$ of size $r + 1$ of which the largest element is m are precisely those $X_0 \cup \{m\}$ where X_0 is a subset of $\{0, 1, \ldots, m - 1\}$ of size r; so there are exactly $\binom{m}{r}$ of these. Hence, by the Addition Rule,

$$\binom{n+1}{r+1} = \binom{0}{r} + \binom{1}{r} + \cdots + \binom{n}{r}. \qquad \square$$

Additional comment. This counting argument is based on the following:

$$\begin{aligned}
&\{X \subseteq \{0, 1, \ldots, n\} : |X| = r + 1\} \\
&= \{X \subseteq \{0, 1, \ldots, n\} : |X| = r + 1 \text{ and } \max X = 0\} \\
&\quad \cup \{X \subseteq \{0, 1, \ldots, n\} : |X| = r + 1 \text{ and } \max X = 1\} \\
&\quad \cup \{X \subseteq \{0, 1, \ldots, n\} : |X| = r + 1 \text{ and } \max X = 2\} \\
&\quad \cup \cdots \cup \{X \subseteq \{0, 1, \ldots, n\} : |X| = r + 1 \text{ and } \max X = n\} \\
&= \{X_0 \cup \{0\} : X_0 \subseteq \{\} \text{ and } |X| = r\} \\
&\quad \cup \{X_0 \cup \{1\} : X_0 \subseteq \{0\} \text{ and } |X| = r\} \\
&\quad \cup \{X_0 \cup \{2\} : X_0 \subseteq \{0, 1\} \text{ and } |X| = r\} \\
&\quad \cup \cdots \cup \{X_0 \cup \{n\} : X_0 \subseteq \{0, 1, \ldots, n - 1\} \text{ and } |X| = r\}.
\end{aligned}$$

Figure.



Extra exercises

10.5. (a) By definition, a relation on A is a relation from A to A. So there are $2^{|A| \times |A|} = 2^{m^2}$ of them by Exercise 10.2(a).

(b) Consider the following questions about a relation R on A, where xy ranges over 2-permutations of A.

"Is $x \mathrel{R} y$?"

Each way of answering these questions gives rise to a unique reflexive relation R on A because any such relation R must have $x \mathrel{R} x$ for any $x \in A$ by reflexivity. Moreover, all reflexive relations on A can be obtained in this way, and different ways to answer these questions give rise to different reflexive relations on A. So the number of reflexive relations on A is equal to the number of ways to answer these questions. Since $|A| = m$ and the number of 2-permutations of A is $P(|A|, 2)$, there are exactly $P(m, 2)$ such questions. There are exactly two ways to answer each of these questions. So, by the Multiplication Rule, the number of ways to answer all these questions, and hence the number of reflexive relations on A, is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{P(m,2)} = 2^{P(m,2)}.$$

(c) Consider the following questions about a relation R on A.

(i) "Is $x \mathrel{R} x$?" where $x \in A$.

(ii) "Is $x \mathrel{R} y \wedge y \mathrel{R} x$ or $x \not\mathrel{R} y \wedge y \not\mathrel{R} x$?" where $\{x, y\} \subseteq A$ of size 2.

In view of Tutorial Exercise 3.3, each way of answering these questions gives rise to a (unique) symmetric relation R on A, and every symmetric relation on A can be obtained in this way. Moreover, different ways to answer these questions give rise to different symmetric relations on A. Therefore, the number of symmetric relations on A is equal to the number of ways to answer these questions.

Since $|A| = m$, there are exactly m questions of type (i). Since there are $\binom{|A|}{2}$ subsets of A of size 2 by definition, there are exactly $\binom{m}{2}$ questions of type (ii). There are exactly two ways to answer each of these questions. So, by the Multiplication Rule, the number of ways to answer all these questions, and hence the number of symmetric relations on A, is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{m\text{-many 2's}} \times \underbrace{2 \times 2 \times \cdots \times 2}_{\binom{m}{2}\text{-many 2's}} = 2^{m+\binom{m}{2}}.$$

(d) Consider the following questions about a relation R on A.

(i) "Is $x \mathrel{R} x$?" where $x \in A$.

(ii) "Is $x \mathrel{R} y \wedge y \not\mathrel{R} x$ or $x \not\mathrel{R} y \wedge y \mathrel{R} x$ or $x \not\mathrel{R} y \wedge y \not\mathrel{R} x$?" where $\{x, y\} \subseteq A$ of size 2.

As in (c), the number of antisymmetric relations on A is equal to the number of ways to answer these questions. There are exactly two ways to answer each question of type (i), and there are exactly three ways to answer each question of type (ii). So, by the Multiplication Rule, the number of ways to answer all these questions, and hence the number of antisymmetric relations on A, is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{m\text{-many 2's}} \times \underbrace{3 \times 3 \times \cdots \times 3}_{\binom{m}{2}\text{-many 2's}} = 2^m 3^{\binom{m}{2}}.$$

Additional comment. In (b)–(d), we essentially defined bijections from the set of all ways to answer all the stated questions to the set of all relations to be counted.