NATIONAL UNIVERSITY OF SINGAPORE

CS1231 - DISCRETE STRUCTURES

(Semester 1: AY2016/17)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This assessment paper contains **NINETEEN (19)** questions in **TWO (2)** parts and comprises **EIGHT (8)** printed pages, including this page.
2. Answer **ALL** questions.
3. This is an **OPEN BOOK** assessment.
4. You are allowed to use **NUS APPROVED CALCULATORS**.
5. You are to submit two documents: The **OCR form** and the **Answer Sheet**. You may keep this question paper.
6. **Shade** and **write** your **Student Number** completely and accurately on the OCR form.
7. You must use 2B pencil for the OCR form.
8. **Write** your **Student Number** on the Answer Sheet. **Do not write your name.**
9. You may use pen or pencil to write your answers, but please erase cleanly, and write legibly. Marks may be deducted for illegible handwriting.

Solution: SOLUTIONS

Part A: (30 marks) MCQ. Answer on the OCR form.

For each multiple choice question, choose the best answer and **shade** the corresponding choice on the OCR form. Remember to **shade** and **write** your **Student Number** (check that it is correct!) on the OCR form. Each multiple choice question is worth 2 marks. No mark is deducted for wrong answers. You should use a **2B pencil**.

Q1. Do you know the rule of inference used in the following argument?

If I can do it, then I really can do it! I can do it. • Therefore, I really can do it!

A. Yes, it is modus ponens.
B. Yes, it is modus tollens.
C. Yes, it is induction induction induction induction induction...
D. No, but I know it is contradiction.
E. No, I really don't know!

Q2. Let $P(x)$ be "x is an NUS student", $Q(x)$ be "x is a computer science major", $R(x, y)$ be "x takes module y", and $S(x)$ be "x is a CS module".

Which of the following means "Every NUS computer science major takes at least a CS module"?

A. $\forall x \forall y\, (P(x) \wedge Q(x) \wedge R(x, y) \wedge S(y))$
B. $\forall x \exists y\, (P(x) \wedge Q(x) \wedge R(x, y) \wedge S(y))$
C. $\exists x\, (\ (P(x) \wedge Q(x)) \rightarrow \exists y(R(x, y) \wedge S(y))\ )$
D. $\forall x\, (\ (P(x) \wedge Q(x)) \rightarrow \exists y(R(x, y) \wedge S(y))\ )$
E. None of the above.

Q3. Your good friend Dueet wants to write a new song and asks you to give him a list of 4 words, where each word is one of these 3 words: *pen*, *apple* and *pineapple*. Obviously repetition is allowed. He will then use the list you give him to write a lyric that contains the 4 words in your list in any order he wants. How many different ways can you select the 4 words for Dueet?

A. 4
B. 12
C. 15
D. 64
E. None of the above.

Q4. You are buying lunch for 20 guests. The menu offers 6 choices: chicken rice, nasi lemak, mee rebus, rendang daging, laksa and fish porridge. Two guests ask for chicken rice, three ask for nasi lemak, two ask for mee rebus and the rest have no preference. How many different selections can you make?

A. 1716
B. 8568
C. 11628
D. 15504
E. None of the above.

Q5. How many onto functions are there from a set with 4 elements to a set with 2 elements?

A. 12

B. 14

C. 16

D. 24

E. None of the above

Q6. Aiken is taking this exam paper and does not know the answers to three multiple-choice questions (MCQs). Each MCQ has five choices for the answer.

Aiken can eliminate two answer choices as incorrect for one of the three questions, and eliminate one answer choice as incorrect for the second question, but has no clue about the correct answer at all for the third question.

Assuming that Aiken's selection of a choice for one question does not affect her selection of a choice for another question, what is the probability that Aiken will answer at least one of the three questions correctly?

A. 2/5

B. 13/30

C. 3/5

D. 2/3

E. None of the above.

Q7. Which of the following statements is false?

A. Every complete graph K_n, where $n > 2$, contains a Hamiltonian cycle.

B. Every tree with n vertices, where $n > 2$, has an Euler circuit.

C. Every connected graph with n vertices, where $n > 2$, contains a spanning tree.

D. There are n^{n-2} spanning trees in a complete graph K_n, where $n > 2$.

E. A tree with n vertices, where $n > 2$, has at least two vertices with degree 1.

Q8. The n^{th} Fibonacci number is always less than or equal to the n^{th} Triangle number for integers n up to, and including,

A. 1.

B. 10.

C. 15.

D. 20.

E. None of the above.

Q9. Consider the equivalence relation $\equiv_{\pmod 5}$ defined on $\mathbb{Z}$. Integer x is such that $3x \in [17]$. Then x could be:

A. $\frac{17}{3}$.

B. -2.

C. 6.

D. 17.

E. None of the above.

The next six questions, Q10 to Q15, refer to the following definitions.

Seven binary relations are defined as follows:

$$\begin{aligned}
&\mathcal{R}_1 \subseteq \mathbb{R}\times\mathbb{R}, && \forall x,y\in\mathbb{R}\ (x\,\mathcal{R}_1\,y \ \leftrightarrow\ x=\sin y).\\
&\mathcal{R}_2 \subseteq A\times B, && \forall x\in A\ \forall y\in B\ (x\,\mathcal{R}_2\,y \ \leftrightarrow\ x=\sin y),\\
& && \text{where } A=\{-1,0,1\}, B=\left\{\frac{-\pi}{2},0,\frac{\pi}{2}\right\}.\\
&\mathcal{R}_3 \subseteq \mathbb{R}\times\mathbb{R}, && \forall x,y\in\mathbb{R}\ (x\,\mathcal{R}_3\,y \ \leftrightarrow\ |x|=y).\\
&\mathcal{R}_4 \subseteq \mathbb{R}\times\mathbb{R}, && \forall x,y\in\mathbb{R}\ (x\,\mathcal{R}_4\,y \ \leftrightarrow\ x=|y|).\\
&\mathcal{R}_5 \subseteq S\times S, && \forall (a,b),(c,d)\in S\ (\ (a,b)\,\mathcal{R}_5\,(c,d) \ \leftrightarrow\ ad=bc\ ),\\
& && \text{where } S=\mathbb{Z}\times(\mathbb{Z}-\{0\}).\\
&\mathcal{R}_6 \subseteq C\times C, && \forall x,y\in C\ (x\,\mathcal{R}_6\,y \ \leftrightarrow\ x\mid y),\\
& && \text{where } C=\{1,2,3,4,5,6\}.\\
&\mathcal{R}_7 \subseteq \mathbb{R}\times\mathbb{R}, && \forall x,y\in\mathbb{R}\ (x\,\mathcal{R}_7\,y \ \leftrightarrow\ xy\geq 1).
\end{aligned}$$

Note that $|x|$ is the absolute value of x, defined as:

$$\forall x\in\mathbb{R},\ |x| = \begin{cases} x, & \text{if } x\geq 0,\\ -x, & \text{otherwise.}\end{cases}$$

Solution: Comments:

a. $\mathcal{R}_1$ is not a function, but $\mathcal{R}_1^{-1}$ is.

b. $\mathcal{R}_2$ is a bijection.

c. $\mathcal{R}_3$ is the function $f(x)=|x|$. It is not 1-1 nor onto.

d. $\mathcal{R}_4$ is not a function, but $\mathcal{R}_4^{-1}=\mathcal{R}_3$ is.

e. $\mathcal{R}_5$ is an equivalence relation. It represents rationals as ordered pairs instead of usual ratios: $\frac{a}{b}=(a,b)$.

f. $\mathcal{R}_6$ is a partial order.

g. $\mathcal{R}_7$ is not reflexive, not transitive, but symmetric.

h. $\mathcal{R}_4\circ\mathcal{R}_3$ is an equivalence relation: $\forall x,y\in\mathbb{R}\ (x\,(\mathcal{R}_4\circ\mathcal{R}_3)\,y \ \leftrightarrow\ |x|=|y|)$.

Q10. Which of the above relations are functions?

A. $\mathcal{R}_1$ and $\mathcal{R}_4$.

B. $\mathcal{R}_2$ and $\mathcal{R}_3^{-1}$.

C. $\mathcal{R}_1^{-1},\mathcal{R}_2$ and $\mathcal{R}_4^{-1}$.

D. $\mathcal{R}_5,\mathcal{R}_6$ and $\mathcal{R}_7$.

E. $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$ and $\mathcal{R}_4$.

Q11. Which of the above relations are onto functions?

A. $\mathcal{R}_2$ **and** $\mathcal{R}_2^{-1}$**.**

B. $\mathcal{R}_3$ and $\mathcal{R}_4^{-1}$.

C. $\mathcal{R}_3^{-1}$ and $\mathcal{R}_4$.

D. $\mathcal{R}_5$ and $\mathcal{R}_7$.

E. $\mathcal{R}_1$ and $\mathcal{R}_6^{-1}$.

Q12. Which relation(s) is/are symmetric?

A. $\mathcal{R}_6$.

B. $\mathcal{R}_3$ and $\mathcal{R}_4^{-1}$.

C. $\mathcal{R}_6$ and $\mathcal{R}_7$.

D. $\mathcal{R}_5, \mathcal{R}_7$ **and** $\mathcal{R}_4 \circ \mathcal{R}_3$**.**

E. None of the above.

Q13. Which relation(s) is/are transitive?

A. $\mathcal{R}_6$.

B. $\mathcal{R}_2 \circ \mathcal{R}_2$.

C. $\mathcal{R}_3 \circ \mathcal{R}_4$ and $\mathcal{R}_7$.

D. $\mathcal{R}_5, \mathcal{R}_6$ **and** $\mathcal{R}_4 \circ \mathcal{R}_3$**.**

E. All of the above.

Q14. Which relation has a maximal element of 1?

A. $\mathcal{R}_2$.

B. $\mathcal{R}_6$.

C. $\mathcal{R}_7$.

D. $\mathcal{R}_2^{-1}$.

E. $\mathcal{R}_6^{-1}$**.**

Q15. Find all $y \in \mathbb{R}$ such that $2\,(\mathcal{R}_7 \circ \mathcal{R}_7)\,y$.

A. $y > 0$**.**

B. $y > \frac{1}{2}$.

C. $0 < y \leq 2$.

D. $0 < y \leq \frac{1}{2}$.

E. $y > \frac{1}{2}$ or $y < -\frac{1}{2}$.

Part B: (50 marks) Structured questions. Write your answer in the Answer Sheet. Marks may be deducted for illegible handwriting and unnecessary statements in proofs.

Q16. (13 marks) **Probability and Counting.**

(a) (3 marks) For a biased 6-sided die, the probability of the smaller numbers (1, 2, 3) turning up is three times the probability of the larger numbers (4, 5, 6) turning up.

i. (1 mark) If we roll this die, what is the expected value that would turn up? Write your computation on a single line.

> **Solution:** $3/12 \times (1 + 2 + 3) + 1/12 \times (4 + 5 + 6) = 2.75$

ii. (2 marks) If we roll two such dice, what is the probability of getting a sum of 6? Leave your answer as a fraction in its lowest terms.

> **Solution:** $3/12 \times 1/12$ (rolling 1,5) + $3/12 \times 1/12$ (rolling 2,4) + $3/12 \times 3/12$ (rolling 3,3) + $1/12 \times 3/12$ (rolling 4,2) + $1/12 \times 3/12$ (rolling 5,1) = $7/48$

(b) (4 marks) Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.
(*Hint:* Apply the Pigeonhole Principle.)

> **Solution:**
> 1. Divide the unit square into 25 equal smaller squares of side 1/5 each. Then at least one of these small squares would contain at least three points. (Otherwise, if every square contains two points or less, the total number of points is no more than $2 \times 25 = 50$, which contradicts our assumption that there are 51 points.)
> 2. Now, the circle circumvented around the square with the three points inside also contains these three points and has radius $\sqrt{(1/10)^2 + (1/10)^2)} = \sqrt{2/100} = \sqrt{1/50} < \sqrt{1/49} = 1/7$

(c) (6 marks) In the classical *Tuesday Birthday Problem (TBP)*, a man says:
"I have two children. One is a boy born on a Tuesday. What is the probability that I have two boys?"
What is the answer? It is known that the answer is neither 1/2 nor 1/3.
(*Hint:* Use Bayes' theorem, or alternatively, explain with a diagram using these three categories: (1) a boy born on a Tuesday, (2) a boy born on one of the other six days of the week, and (3) a girl.)

> **Solution:**
> 1. For each child, there are 14 possible outcomes (2 genders $\times$ 7 days of the week).
> 2. There are $14 \times 14 = 196$ possible combinations for 2 children.

	Boy born on Tuesday	Boy born on other days	Girl	
Boy born on Tuesday	1	6	7	14
Boy born on other days	6	36	42	84
Girl	7	42	49	98
	14	84	98	196

3. The first row and first column represent the 27 possible outcomes of at least one son born on a Tuesday.
4. Out of these 27 outcomes, 13 are possible outcomes of both being boys.
5. Hence the probability that the man has two boys is 13/27.

Alternatively,

1. By Bayes' Theorem, the probability of two boys given that one boy was born on a Tuesday is: $P(BB|B_T) = P(B_T|BB) \times P(BB)/P(B_T)$
2. $P(B_T|BB) = 1 - (6/7)^2 = 13/49$ (One minus the probability that neither boy is born on Tuesday.)
3. $P(BB) = 1/4$ (BB out of 4 possibilities: BB, BG, GB and GG.)
4. For $P(B_T)$, there are 4 cases to consider, each with 1/4 chance: BB, BG, GB, and GG.
   - 4.1. Case BB: Probability is $1/7 + 1/7 - (1/7)^2 = 13/49$. (Principle of inclusion-exclusion)
   - 4.2. Case BG: Probability is 1/7.
   - 4.3. Case GB: Probability is 1/7.
   - 4.4. Case GG: Probability is 0.
   - 4.5. Hence, $P(B_T) = 1/4 \times (13/49 + 1/7 + 1/7 + 0) = 1/4 \times 27/49$.
5. Therefore, $P(BB|B_T) = (13/49 \times 1/4)/(1/4 \times 27/49) = 13/27$.

Q17. (12 marks) **Graphs.**

(a) (3 marks) How many simple graphs on 3 vertices are there? In general, how many simple graphs are there on n ($n > 1$) vertices?

Solution:

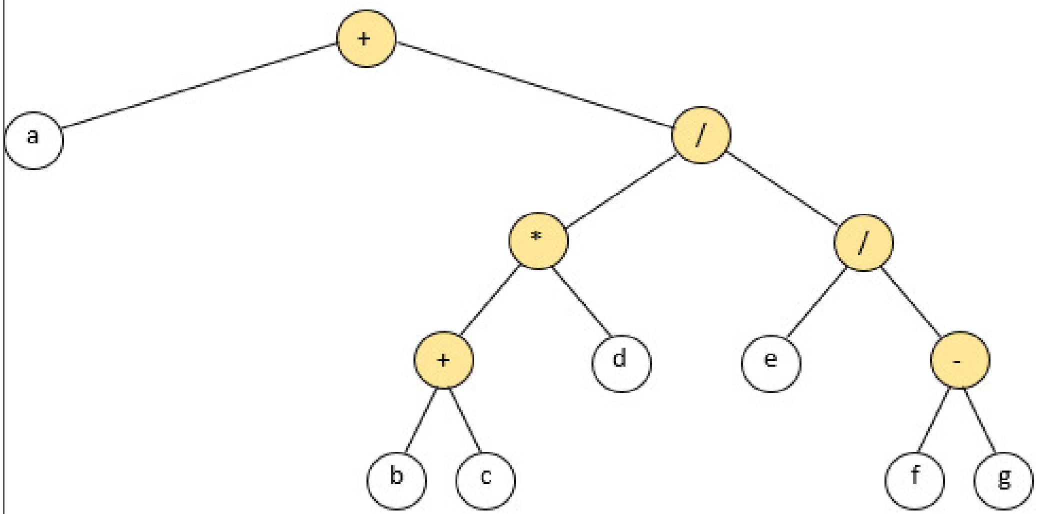
There are 2^3 or 8 simple graphs on 3 vertices.

There are $2^{\binom{n}{2}}$ simple graphs on n vertices. (There are $\binom{n}{2}$ pairs of distinct vertices in a graph of n vertices. Each of these pairs is one possible edge. A set with $\binom{n}{2}$ members has $2^{\binom{n}{2}}$ subsets.)

(b) (5 marks) Complete the following parts.

i. (2 marks) Draw a binary tree for the following algebraic expression:
$a + (((b + c) * d)/(e/(f - g)))$

Solution:



ii. (3 marks) Figure 1 shows a graph where the vertices are Pokestops. Using either Kruskal's or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex, vertex A.

You are to trace the MST edge by edge on the answer sheet, in the order of insertion by following your chosen algorithm. (In other words, draw a solid line representing the first edge of the MST in the first diagram, two edges in the second diagram, and so on, until you get all six edges in the last diagram.)

You must indicate which algorithm you are using, failing which no mark will be awarded even if your MST is correct.

Solution:

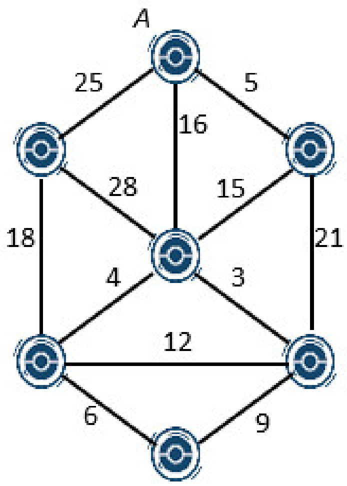
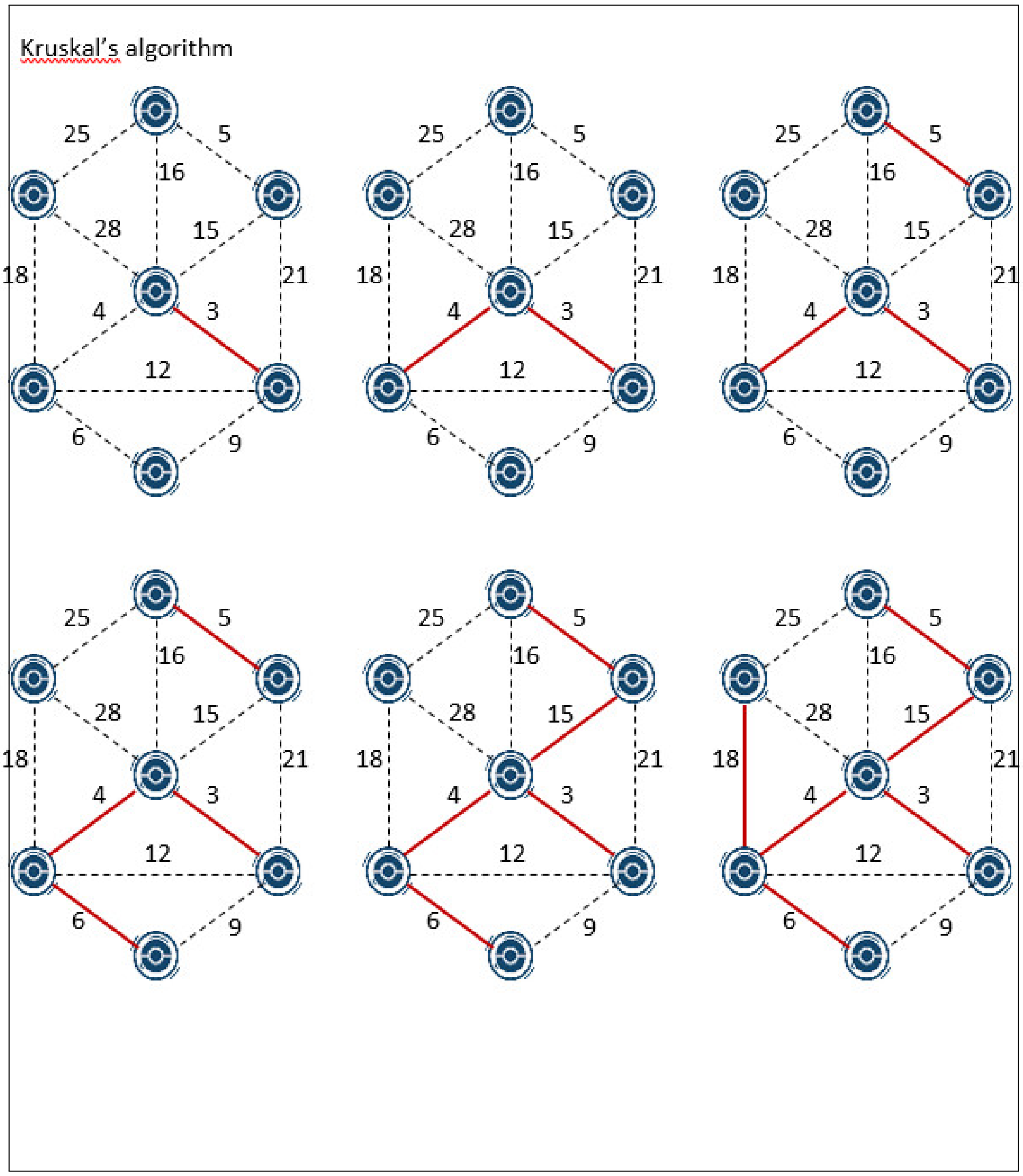
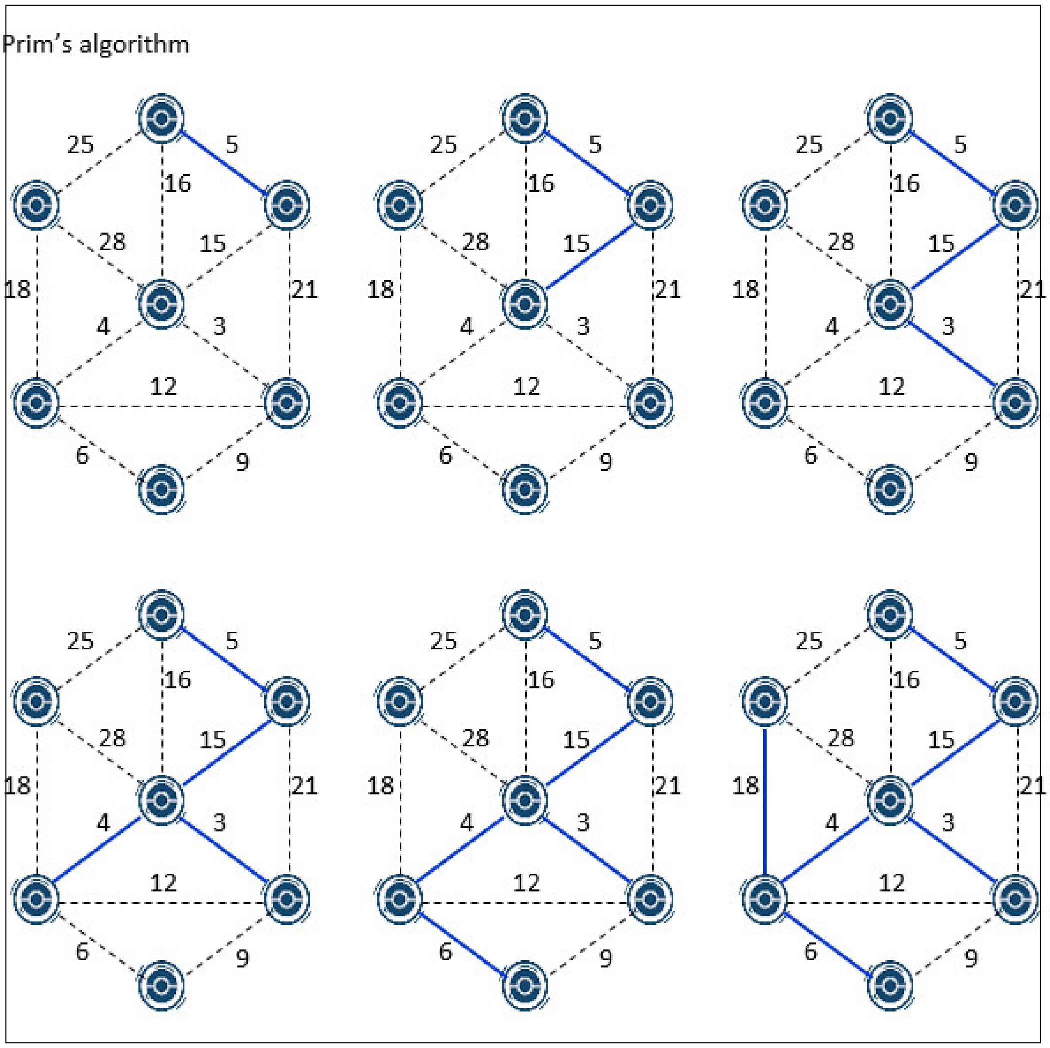


Figure 1: Pokestop graph.





(c) (4 marks) Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected.

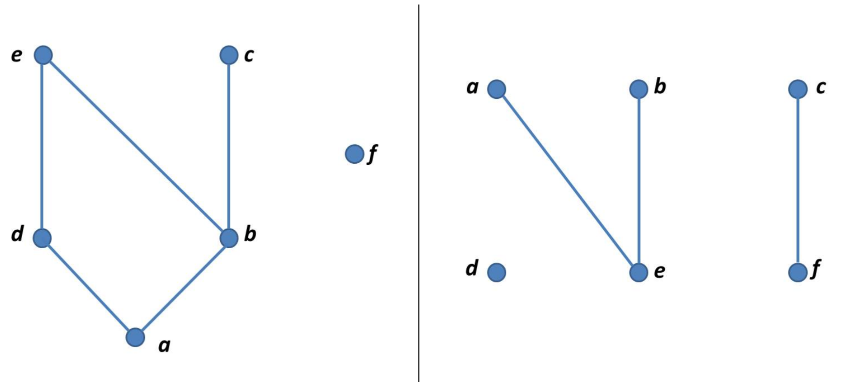
Solution:

1. Prove by contradiction. Assuming that the graph is not connected.
2. Let u and v be the vertices in two separate connected components.
3. The number of vertices in the union of their neighbourhood, including u and v, is at least $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 \geq n + 1$.
4. This exceeds the number of vertices in the graph, hence contradiction.

Q18. (10 marks) Figure 2(a) shows the Hasse diagram of a partial order $\preceq_1$ defined on $A = \{a, b, c, d, e, f\}$.

(a) (3 marks) List all the ordered pairs in $\preceq_1$.

Solution: $\preceq_1 =$



(a) Hasse diagram for $\preceq_1$ (b) Hasse diagram for $\preceq_2$.

Figure 2: Hasse diagrams.

$\{ (a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,e),(c,c),(d,d),(d,e),(e,e),(f,f) \}$.

(b) (3 marks) Find all the minimal, minimum, maximal, and maximum elements in $(A, \preceq_1)$.

Solution: Minimal $= a, f$, maximal $= c, e, f$, no maximum, no minimum.

(c) (1 mark) Another partial order $\preceq_2$ is defined on A, whose Hasse diagram is shown in Figure 2(b). Is $\preceq_1 \cup \preceq_2$ a partial order? Explain briefly.

Solution: No, it is not a partial order. Anti-symmetry is broken by the ordered pairs (b,e) and (e,b), and by (a,e) and (e,a).

(d) (3 marks) Given any two anti-symmetric binary relations $\mathcal{R}, \mathcal{S}$ defined on a set B, is $\mathcal{R} \cup \mathcal{S}$ anti-symmetric? Prove or give a counterexample.

Solution: $\mathcal{R} \cup \mathcal{S}$ is not anti-symmetric. Counterexample: the two partial orders shown above.

Q19. (15 marks) A sequence a_n is defined by the third-order recurrence relation:

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}, \text{ for } n \in \mathbb{Z}_{\geq 3}.$$

with initial values: $a_0 = a_1 = 1$ and $a_2 = 3$.

(a) (2 marks) Explicitly calculate a_3, a_4, a_5 and a_6.

Solution:

$$a_3 = 11, \; a_4 = 35, \; a_5 = 99, \; a_6 = 259.$$

(b) (4 marks) Using Mathematical Induction, prove that:

$$\sum_{r=0}^{n-1} r2^r \; = \; 2^n(n-2) + 2, \text{ for all } n \in \mathbb{Z}^+.$$

Solution:

Proof. (ny Regular Induction)

1. Let $P(n) = (\sum_{r=0}^{n-1} r2^r = 2^n(n-2)+2),\ \forall n \in \mathbb{Z}^+$.
2. Base case: $n = 1$
   - 2.1. Left hand side $= \sum_{r=0}^{0} r2^r = 0 \times 2^0 = 0$.
   - 2.2. Right hand side $= 2^1(1-2)+2 = 0 =$ left hand side.
   - 2.3. Hence $P(1)$ is true.
3. Inductive step: For any $k \in \mathbb{Z}^+$:
   - 3.1. Assume $P(k)$ is true, ie. $\sum_{r=0}^{k-1} r2^r = 2^k(k-2)+2$.
     - 3.1.1. Consider the $k+1$ case:
     - 3.1.2. Left hand side $= \sum_{r=0}^{k} r2^r = k2^k + \sum_{r=0}^{k-1} r2^r$, by basic algebra.
     - 3.1.3. $= k2^k + 2^k(k-2)+2$, by applying the Inductive Hypothesis on the second term.
     - 3.1.4. $= 2^k(2k-2)+2 = 2^{k+1}(k+1-2)+2 =$ right hand side.
     - 3.1.5. Hence $P(k+1)$ is true.
4. Hence the statement is true by Mathematical Induction. ■

(c) (9 marks) Derive an explicit closed-form formula for a_n, for all $n \in \mathbb{N}$.

Hint: Define a new sequence b_n in terms of a_n. Derive a second-order linear homogeneous recurrence relation with constant coefficients for b_n, solve it, then solve for a_n.

Solution: Re-write the recurrence relation for a_n as:

$$a_n - a_{n-1} = 4a_{n-1} - 4a_{n-2} - 4a_{n-2} + 4a_{n-3}$$

Now define $b_n = a_n - a_{n-1}$:

$$\Rightarrow \quad b_n = 4b_{n-1} - 4b_{n-2}, \text{ by substitution}$$
$$\text{Initial values: } b_1 = a_1 - a_0 = 0, \text{ and } b_2 = a_2 - a_1 = 2.$$

This may be solved using Theorem 5.8.5 (Epp). The characteristic equation is:

$$t^2 - 4t + 4 = 0$$
$$\Rightarrow \quad (t-2)^2 = 0$$
$$\text{The single repeated root is thus: } t = 2$$

By Theorem 5.8.5 (Epp), the solution is:

$$b_n = C2^n + Dn2^n, \text{ for constants } C, D$$
$$\text{Now, } b_1 = 0 = 2C + 2D$$
$$\text{And } b_2 = 2 = C \cdot 2^2 + D2 \cdot 2^2$$
$$\Rightarrow \quad 2C + 4D = 1$$

Solving these simultaneous equations yields:

$$C = -\frac{1}{2}, \text{ and } D = \frac{1}{2}.$$
$$\text{And thus } b_n = (n-1)2^{n-1}, \ \forall n \in \mathbb{Z}^+.$$

Now solve for a_n:

$$\begin{aligned}
b_n = a_n - a_{n-1} &= (n-1)2^{n-1} \\
\text{Thus: } a_{n-1} - a_{n-2} &= (n-2)2^{n-2} \\
\text{Thus: } a_{n-2} - a_{n-3} &= (n-3)2^{n-3} \\
\vdots &= \vdots \\
a_2 - a_1 &= 1 \cdot 2^1 \\
a_1 - a_0 &= 0 \cdot 2^0
\end{aligned}$$

Adding all these equations:

$$a_n - 1 = \sum_{r=0}^{n-1} r2^r$$
$$\text{So } a_n - 1 = 2^n(n-2) + 2, \text{ by the result of the previous part}$$
$$\text{Thus, } a_n = 2^n(n-2) + 3, \ \forall n \in \mathbb{N}.$$

END OF PAPER