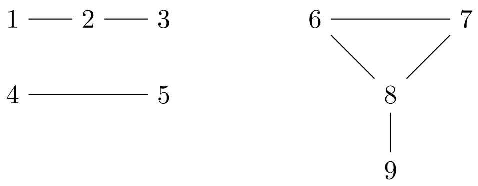
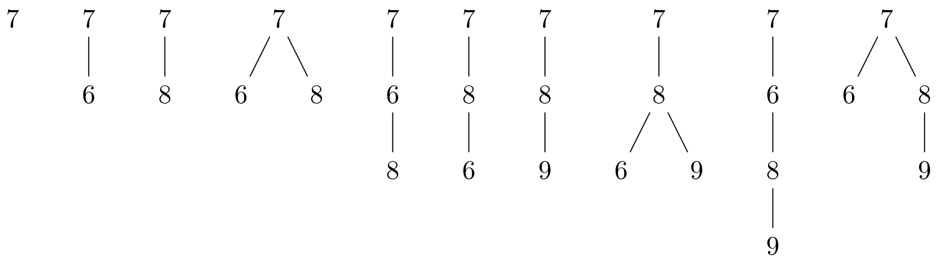
Tutorial solutions for Chapter 12

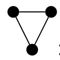
Sometimes there are other correct answers.

12.1. (a)



(b)



(c)
- There are $\binom{9}{3}$ ways to choose three vertices for •—•—•; amongst these three vertices, there are $\binom{3}{1}$ ways to choose the middle vertex.
- There are $\binom{9-3}{2} = \binom{6}{2}$ ways left to choose two vertices for •—•.
- There are $\binom{9-3-2}{3} = \binom{4}{3}$ ways left to choose three vertices for ; amongst these three vertices, there are $\binom{3}{1}$ ways to choose where to attach the final vertex.

By the Multiplication Rule, the number of such graphs is

$$\binom{9}{3}\binom{3}{1}\binom{6}{2}\binom{4}{3}\binom{3}{1} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 3 \times \frac{6 \times 5}{2 \times 1} \times \frac{4 \times 3 \times 2}{3 \times 2 \times 1} \times 3 = 45360.$$

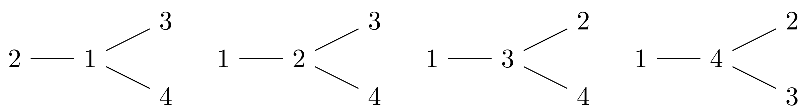
12.2. (a) For $n = 1, 2, 3, 4$, the numbers are $1, 1, 3, 16$ respectively, as listed below.

($n = 1$) 1

($n = 2$) 1 — 2

($n = 3$) 1 — 2 — 3   1 — 3 — 2   2 — 1 — 3

($n = 4$)
1 — 2 — 3 — 4   1 — 2 — 4 — 3   1 — 3 — 2 — 4
1 — 3 — 4 — 2   1 — 4 — 2 — 3   1 — 4 — 3 — 2
2 — 1 — 3 — 4   2 — 1 — 4 — 3   2 — 3 — 1 — 4
2 — 4 — 1 — 3   3 — 1 — 2 — 4   3 — 2 — 1 — 4



Additional comment. We can check whether any tree is missing or double-counted via some extra counting. For example, consider trees of the shape

whose vertices are precisely $1, 2, 3, 4$. There are $4!$ ways to assign $1, 2, 3, 4$ to the four vertices here, say, from left to right. Each such assignment gives the same graph as exactly one other assignment. More specifically, if a, b, c, d are distinct elements of $\{1, 2, 3, 4\}$, then

$$a \text{ --- } b \text{ --- } c \text{ --- } d \qquad \text{and} \qquad d \text{ --- } c \text{ --- } b \text{ --- } a$$

are drawings of the same graph

$$(\{1, 2, 3, 4\}, \{ab, bc, cd\}),$$

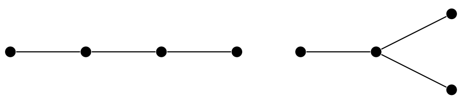
and no other assignment gives a drawing of this graph. This tells us that the number of such vertex assignments is exactly twice the number of such graphs. Thus there are exactly $4!/2 = 12$ such graphs; this number matches with what we had in the solutions.

(b) For $n = 1, 2, 3, 4$, the numbers are $1, 1, 1, 2$ respectively.

$(n = 1)$

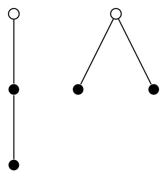
$(n = 2)$

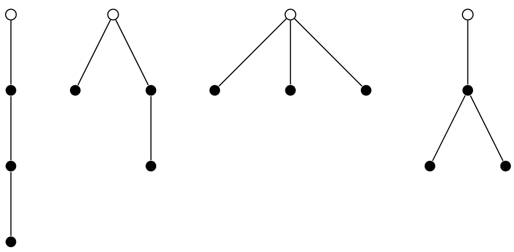
$(n = 3)$

$(n = 4)$ 

Additional comment. These are the "shapes" of the trees we got in (a).

(c) For $n = 1, 2, 3, 4$, the numbers are $1, 1, 2, 4$ respectively.

$(n = 1)$ $(n = 2)$  $(n = 3)$ 

$(n = 4)$ 

Additional comment. These are the different ways to choose roots for the trees we got in (b).

Moral. It is best if we can count without counting. Nevertheless, it is sometimes easier to count by counting.

12.3. $(\Rightarrow)$ Suppose G is a tree. Let $u, v \in \mathrm{V}(G)$ such that $uv \notin \mathrm{E}(G)$. If $u = v$, then uv is a loop and thus $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$ is cyclic. So suppose $u \neq v$. Use connectedness to find a path P between u and v in G. Note that P must have length at least 2 because $uv \notin \mathrm{E}(G)$. So $\big(\mathrm{V}(P), \mathrm{E}(P) \cup \{uv\}\big)$ is a cycle in $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$, making $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$ cyclic.

($\Leftarrow$) We prove by contraposition. Suppose G is not a tree. As G is acyclic, this tells us G is unconnected. Find $u, v \in \mathrm{V}(G)$ between which there is no path in G. Note that $u \neq v$ because $(\{u\}, \{\})$ is a path between u and u in G. Also $uv \notin \mathrm{E}(G)$.

Suppose, towards a contradiction, we have a cycle in $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$, say,

$$C = x_1 x_2 \ldots x_\ell x_1.$$

Note that $uv \in \mathrm{E}(C)$ because G is acyclic. Renaming the x's if needed, we may assume $u = x_1$ and $v = x_\ell$. Then, as $\ell \geqslant 3$, we have a path $x_1 x_2 \ldots x_\ell$ between u and v in G, which contradicts our choice of u, v. $\square$

Alternative proof for $\Leftarrow$. Assume that adding any new edge makes G cyclic. We show G is a tree, i.e., it is both acyclic and connected. We know that G is acyclic by hypothesis. For connectedness, pick any $u, v \in \mathrm{V}(G)$.

Case 1: suppose $uv \in \mathrm{E}(G)$. There are two subcases.

Case 1.1: suppose $u = v$. Then $(\{u\}, \{\})$ is a path between u and v in G.

Case 1.2: suppose $u \neq v$. Then uv is a path between u and v in G.

Case 2: suppose $uv \notin \mathrm{E}(G)$. Consider the graph

$$\hat{G} = (\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\}).$$

By assumption, we know $\hat{G}$ is cyclic. So the definition of cyclic graphs tells us that $\hat{G}$ either has a loop or has a cycle.

Case 2.1: suppose $\hat{G}$ has a loop. As G is acyclic, it has no loop. Now $\hat{G}$ has a loop, and the only difference between G and $\hat{G}$ is the additional edge uv in $\hat{G}$. So it must be the case that uv is a loop, i.e., that $u = v$. In this case, we have the path $(\{u\}, \{\})$ between u and v in G.

Case 2.2: suppose $\hat{G}$ has a cycle. As G is a acyclic, it has no cycle. Now $\hat{G}$ has a cycle, and the only difference between G and $\hat{G}$ is the additional edge uv in $\hat{G}$. So it must be the case that some/all cycles in $\hat{G}$ have uv in it. Consider a cycle $x_1 x_2 \ldots x_\ell x_1$ in $\hat{G}$. Renaming the x's if needed, we may assume $u = x_1$ and $v = x_\ell$. Then, as $\ell \geqslant 3$, we have the path $x_1 x_2 \ldots x_\ell$ between u and v in G.

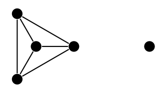
So there is a path between u and v in G in all cases. $\square$

12.4. As G is finite, it has only finitely many connected components. Let $H_1, H_2, \ldots, H_k$ list all the connected components of G without repetition. As G is unconnected, we know $k > 1$. By the definition of connected components, each H_i is connected. In addition, each H_i is acyclic because G is acyclic. So each H_i is a tree. It follows that

$$\begin{aligned}
&|\mathrm{E}(G)| \\
&= |\mathrm{E}(H_1) \cup \mathrm{E}(H_2) \cup \cdots \cup \mathrm{E}(H_k)| && \text{by the additional comment in Tutorial Exercise 11.4(b);} \\
&= |\mathrm{E}(H_1)| + |\mathrm{E}(H_2)| + \cdots + |\mathrm{E}(H_k)| && \text{by the Addition Rule;} \\
&= |\mathrm{V}(H_1)| - 1 + |\mathrm{V}(H_2)| - 1 + \cdots + |\mathrm{V}(H_k)| - 1 && \text{by Theorem 12.1.6;} \\
&= |\mathrm{V}(H_1)| + |\mathrm{V}(H_2)| + \cdots + |\mathrm{V}(H_k)| - k \\
&= |\mathrm{V}(H_1) \cup \mathrm{V}(H_2) \cup \ldots \mathrm{V}(H_k)| - k && \text{by the Addition Rule;} \\
&= |\mathrm{V}(G)| - k && \text{by Tutorial Exercise 11.4(a);} \\
&< |\mathrm{V}(G)| - 1 && \text{as } k > 1.
\end{aligned}$$

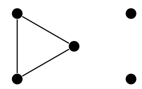
$\square$

12.5. No. One counterexample is the graph G drawn below.



This graph is not connected, but $|\mathrm{E}(G)| = 6 \geqslant 4 = 5 - 1 = |\mathrm{V}(G)| - 1$.

12.6. No. One counterexample is the graph G drawn below.



This graph is cyclic, but $|\mathrm{E}(G)| = 3 \leqslant 4 = 5 - 1 = |\mathrm{V}(G)| - 1$.

12.7. We proceed by strong induction on the number of vertices in T.

(Base step) Let T be a rooted tree with exactly one vertex. Then this vertex is terminal, and there is no internal vertex. So

$$\text{number of internal vertices} = 0 = 1 - 1 = \text{number of terminal vertices} - 1.$$

(Induction step) Let $k \in \mathbb{Z}^+$ such that the theorem is true for all rooted trees with at most k vertices in which every internal vertex has exactly two children. Consider a rooted tree T with exactly $k+1$ vertices in which every internal vertex has exactly two children. Let r be the root of T. Then r must be internal because otherwise T has exactly one vertex, and $1 < 1+1 \leqslant k+1$. So r has exactly two children by assumption, say u and v.

Observe that urv is a path between u and v in T. By Proposition 12.1.3, this is the only path between u and v in T. Therefore, in the graph

$$H = (\mathrm{V}(T) \setminus \{r\}, \mathrm{E}(T) \setminus \{ru, rv\}),$$

there is no path between u and v, and thus the vertices u, v are in different connected components by Theorem 11.3.7. Let H_u be a connected component of H containing u, and let H_v be a connected component of H containing v. Being connected components, we know H_u and H_v are each connected in particular. Also H_u and H_v are acyclic because they are subgraphs of the acyclic graph T. So both H_u and H_v are trees. We will treat u and v as the roots of H_u and H_v respectively. Denote by t_u and t_v the number of terminal vertices in H_u and H_v respectively. As neither H_u nor H_v has the vertex r in it, both H_u and H_v have at most k vertices.

There is a path P between the root r and any other vertex x in T by the connectedness of T. As u and v are the only children of the root r in T, any such path P has either u or v in it. Deleting the vertex r and the edge ru or rv depending on whether u or v is in P, we obtain a path between either u or v and the same vertex x in H. In view of Proposition 12.1.3, one can obtain any path between u or v and a vertex in H in this way.

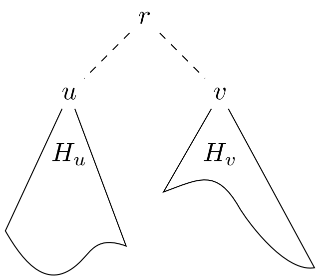
One consequence of the previous paragraph is that, in view of Theorem 11.3.7 and Tutorial Exercise 11.4(a), any vertex $x \in \mathrm{V}(T) \setminus \{r\}$ is either in H_u or in H_v. Another consequence is that, for all vertices x, y in H_u, we know x is a parent of y in T if and only if x is a parent of y in H_u; similarly for H_v. Thus, since r is not the child of any vertex in T, every internal vertex has exactly two children both in H_u and in H_v. Applying the induction hypothesis to the rooted trees H_u and H_v, we deduce that H_u has $t_u - 1$ internal vertices and H_v has $t_v - 1$ internal vertices.

On the one hand, all the terminal vertices in H_u and in H_v are terminal vertices in T. Moreover, the rooted tree T has no other terminal vertex because r is not a terminal vertex in T as discussed above. So T has exactly $t_u + t_v$ terminal vertices. On the other hand, all the internal vertices in H_u and in H_v are internal vertices in T. In addition to these, the rooted tree T has exactly one more internal vertex r. So T has exactly $(t_u - 1) + (t_v - 1) + 1 = t_u + t_v - 1$ internal vertices. So

$$(\text{number of internal vertices in } T) = (\text{number of terminal vertices in } T) - 1.$$

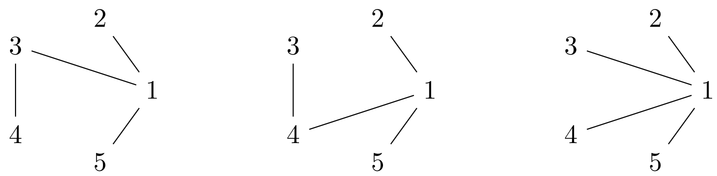
This completes the induction. $\square$

Diagram.



Extra exercises

12.8. (a)



(b) Note that G is itself a spanning tree of G. So G has at least one spanning tree. To show that G has at most one spanning tree, it suffices to prove that any spanning tree of G must be equal to G.

Let T be any spanning tree of G. The definition of spanning trees already tells us $\mathrm{V}(G) = \mathrm{V}(T)$. We know $\mathrm{E}(T) \subseteq \mathrm{E}(G)$ as T is a subgraph of G.

Finally, we prove $\mathrm{E}(G) \subseteq \mathrm{E}(T)$ by contradiction. Suppose we have an edge $e \in \mathrm{E}(G) \setminus \mathrm{E}(T)$. Then $(\mathrm{V}(T), \mathrm{E}(T) \cup \{e\})$ is cyclic by Tutorial Exercise 12.3 because T is a tree. However, this is a subgraph of an acyclic graph G. So we have the required contradiction. $\square$

(c)* Assume G has exactly one spanning tree, say T. It suffices to show that $G = T$. The definition of spanning trees already tells us $\mathrm{V}(G) = \mathrm{V}(T)$. We know $\mathrm{E}(T) \subseteq \mathrm{E}(G)$ as T is a subgraph of G. We prove $\mathrm{E}(G) \subseteq \mathrm{E}(T)$ by contradiction.

Suppose we have an edge $uv \in \mathrm{E}(G) \setminus \mathrm{E}(T)$. Thus $H = (\mathrm{V}(T), \mathrm{E}(T) \cup \{uv\})$ is cyclic by Tutorial Exercise 12.3. Since H is a subgraph of a graph G without any loop, it must contain a cycle, say $C = x_1 x_2 \ldots x_k x_1$. Now T, being a tree, has no cycle, and the only difference between H and T is the additional edge uv. So $uv \in \mathrm{E}(C)$. Renaming the x's if needed, we may assume $u = x_1$ and $v = x_2$. Define

$$\hat{T} = (\mathrm{V}(G), (\mathrm{E}(T) \cup \{uv\}) \setminus \{x_1 x_k\}).$$

Note that $uv = x_1 x_2 \neq x_1 x_k$ because they appear in different positions in the cycle C. So $\hat{T} \neq T$. Since G has exactly one spanning tree, to reach a contradiction, we verify that $\hat{T}$ is a spanning tree of G. By definition, we know $\mathrm{V}(\hat{T}) = \mathrm{V}(G)$.

(Connectedness) Take any $a, b \in \mathrm{V}(\hat{T})$. Note that $\mathrm{V}(\hat{T}) = \mathrm{V}(G) = \mathrm{V}(T)$. Use the connectedness of T to find a path $P = y_0 y_1 \ldots y_\ell$ in T where $y_0 = a$ and $y_\ell = b$.

Case 1: suppose $x_1 x_k \notin \mathrm{E}(P)$. Then P is a path between a and b in $\hat{T}$.

Case 2: suppose $x_1x_k \in \mathrm{E}(P)$. Swapping a and b if needed, say $x_1 = y_r$ and $x_k = y_{r+1}$. Now in $\hat{T}$,

- between a and y_r there is a path $y_0y_1 \ldots y_r$;
- between y_r and y_{r+1} there is a path $x_1x_2 \ldots x_k$;
- between y_{r+1} and b there is a path $y_{r+1}y_{r+2} \ldots y_\ell$.

So two applications of Lemma 11.1.10 give us a path between a and b in $\hat{T}$.

(Acyclicity) Note that $\hat{T}$, being a subgraph of a graph G with no loop, cannot have any loop. Suppose $\hat{T}$ has a cycle, say $D = z_1z_2 \ldots z_mz_1$. Now T does not contain a cycle, and the only edge that is in $\hat{T}$ but not in T is uv. So $uv \in \mathrm{E}(D)$. Renaming the z's if needed, we may assume $u = z_1$ and $v = z_m$. Note that

$$x_2x_3 \ldots x_kx_1 \quad \text{and} \quad z_1z_2 \ldots z_m$$

are both paths between u and v in T. These two paths are distinct because the edge x_1x_k is in the first path but not the second one. So T is cyclic by Theorem 11.2.5. This contradicts the assumption that T is a tree. □

Additional comment. Notice that the proof of connectedness above resembles a part of the proof of Theorem 12.1.4. One can extract from this a fact that is used in both proofs: if C is a cycle in a connected undirected graph G, then removing one $e \in \mathrm{E}(C)$ from G does not disconnect G.

12.9. There is no path of length $|\mathrm{V}(T)|$ in T because this would require $|\mathrm{V}(T)| + 1$ vertices in T, which T does not have. So there is $\ell \in \mathbb{N}$ such that T contains no path of length ℓ. Apply the Well-Ordering Principle to find the smallest such $\ell \in \mathbb{N}$. In view of the smallestness of ℓ, the tree T has a path of length $\ell - 1$, say $x_0x_1 \ldots x_{\ell-1}$.

We claim that x_0x_1 is the only edge containing x_0 in T. Suppose not. Say $x_0y \in \mathrm{E}(T)$, where $y \neq x_1$. If $y = x_i$, where $y \in \{2, 3, \ldots, \ell - 1\}$, then $x_0x_1 \ldots x_ix_0$ is a cycle in T, contradicting the acyclicity of T. So $y \neq x_i$ for any $y \in \{1, 2, \ldots, \ell - 1\}$. This makes $yx_0x_1 \ldots x_{\ell-1}$ a path of length ℓ in T, contradicting the choice of ℓ. This contradiction proves the claim.

In a similar way, one can prove that $x_{\ell-2}x_{\ell-1}$ is the only edge containing $x_{\ell-1}$ in T.

Note that T is a tree and is thus connected. So, since T has at least two vertices, it has a path of length at least two, and hence it has at least one edge. This gives a path of length one in T. So $\ell \geqslant 2$. As $\ell - 1 \geqslant 2 - 1 = 1$, we know $x_0 \neq x_{\ell-1}$. □