Chapter 7: Functions

CS1231 Discrete Structures

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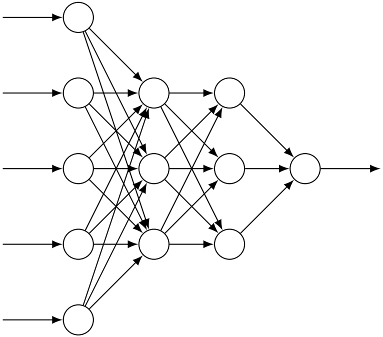
> Our early experience of *functions* is in many ways similar to that of (real) numbers: that is, we first get to know and to represent particular examples such as n^2 and 2^n. But the transition from such particular examples to a suitably general function-concept is far from automatic. Indeed, [. . .] it does not seem to me unreasonable to suggest that *the difficulty inherent in making this transition* from specific examples of what we would now call "functions" to an adequately general function-concept *was one of the main obstacles in the way of explaining precisely why, when and how the methods of the calculus could be trusted.*
>
> Gardiner (1982)

Plan

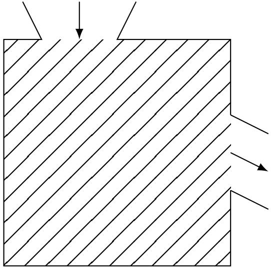
- functions as relations (and thus sets)
- images — an operational view
- composition
- inverse and bijectivity



Use sets to represent functions

What are functions?

- An operation which, when given any permitted input, returns one and only one output.
- We are *not* interested in what the operation does to the inputs to produce the outputs.



Representation

- We represent such an operation by the set of all input–output ordered pairs.
- This is a relation from the set of all permitted inputs (called the *domain* of the function) to a set that contains all the possible outputs (called the *codomain* of the function).

input	output
a	$f(a)$
b	$f(b)$
c	$f(c)$
$\vdots$	$\vdots$

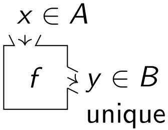
Definition of functions

Definition 7.1.1

Let A, B be sets. A *function* or a *map* from A to B is a relation f from A to B such that any element of A is f-related to a unique element of B, i.e.,

(F1) $\forall x \in A \ \exists y \in B \ (x, y) \in f$; and

(F2) $\forall x \in A \ \forall y_1, y_2 \in B \ \big((x, y_1) \in f \wedge (x, y_2) \in f \ \Rightarrow \ y_1 = y_2\big)$.



We write $f\colon A \to B$ for "f is a function from A to B".
Here A is called the *domain* of f, and B is called the *codomain* of f.

Remark 7.1.2+

The two conditions above can be expressed in words respectively as

(F1) every element of A is f-related to at least one element of B; and

(F2) every element of A is f-related to at most one element of B.

The negations of (F1) and (F2) can be expressed respectively as

($\neg$F1) $\exists x \in A \ \forall y \in B \ (x, y) \notin f$; and

($\neg$F2) $\exists x \in A \ \exists y_1, y_2 \in B \ \big((x, y_1) \in f \wedge (x, y_2) \in f \wedge y_1 \neq y_2\big)$.

Finite examples

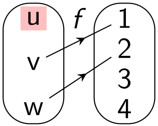
Example 7.1.3

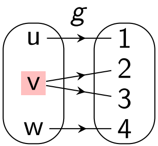
Let $A = \{\mathrm{u}, \mathrm{v}, \mathrm{w}\}$ and $B = \{1, 2, 3, 4\}$.

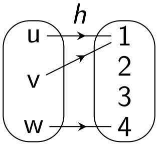
(1) $f = \{(\mathrm{v}, 1), (\mathrm{w}, 2)\}$ is *not* a function $A \to B$
because $\mathrm{u} \in A$ such that no $y \in B$ makes $(\mathrm{u}, y) \in f$, violating (F1).

(2) $g = \{(\mathrm{u}, 1), (\mathrm{v}, 2), (\mathrm{v}, 3), (\mathrm{w}, 4)\}$ is *not* a function $A \to B$
because $\mathrm{v} \in A$ and $2, 3 \in B$ such that $(\mathrm{v}, 2), (\mathrm{v}, 3) \in g$ but $2 \neq 3$, violating (F2).

(3) $h = \{(\mathrm{u}, 1), (\mathrm{v}, 1), (\mathrm{w}, 4)\}$ *is* a function $A \to B$
because both (F1) and (F2) are satisfied.







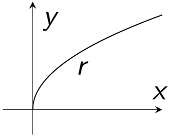
Infinite examples

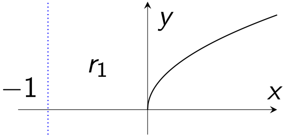
Example 7.1.4

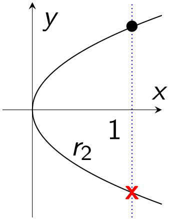
(1) $r = \{(x, y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} : x = y^2\}$ *is* a function $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$
because for every $x \in \mathbb{R}_{\geqslant 0}$, there is a unique $y \in \mathbb{R}_{\geqslant 0}$ such that $(x, y) \in r$, namely $y = \sqrt{x}$.

(2) $r_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_{\geqslant 0} : x = y^2\}$ is *not* a function $\mathbb{R} \to \mathbb{R}_{\geqslant 0}$
because $-1 \in \mathbb{R}$ that is not equal to y^2 for any $y \in \mathbb{R}_{\geqslant 0}$, violating (F1).

(3) $r_2 = \{(x, y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R} : x = y^2\}$ is *not* a function $\mathbb{R}_{\geqslant 0} \to \mathbb{R}$
because $1 \in \mathbb{R}_{\geqslant 0}$ and $-1, 1 \in \mathbb{R}$ such that $1 = (-1)^2$ and $1 = 1^2$ but $-1 \neq 1$, violating (F2).







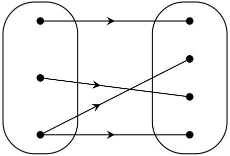
Quick check

Question 7.1.5

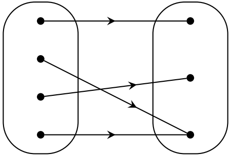
Which of the following arrow diagrams represent a function from the LHS set to the RHS set?

7a

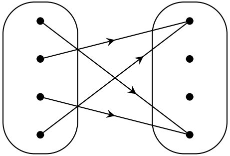
(a)



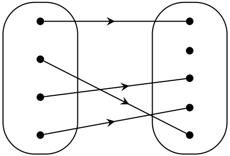
(b)



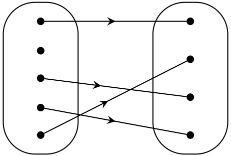
(c)



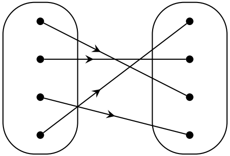
(d)



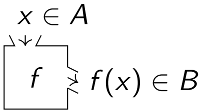
(e)



(f)



Images



Definition 7.2.1

(1) If $x \in A$, then $f(x)$ denotes the unique element $y \in B$ such that $(x, y) \in f$. We call $f(x)$ the *image* of x under f.

(2) The *range* of f, denoted $\operatorname{range}(f)$, is defined by $\operatorname{range}(f) = \{f(x) : x \in A\}$.

Remark 7.2.2

It follows from the definition of images that if $f : A \to B$ and $x \in A$, then for all $y \in B$,

$$(x, y) \in f \quad \Leftrightarrow \quad y = f(x).$$

Remark 7.2.3

(1) The range of a function is the set that contains all the outputs of the function *and nothing else*, while the codomain is the set associated to the function as part of its specification that contains all the outputs *but maybe also other objects*.

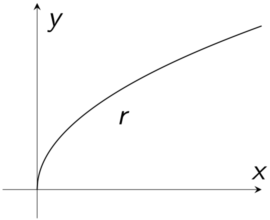
(2) For any function, the range is a subset of the codomain.

Square root

Example 7.2.4

Consider the function $r = \{(x, y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} : x = y^2\}$ from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}_{\geqslant 0}$.

- We know $\forall x, y \in \mathbb{R}_{\geqslant 0} \ \big(y = r(x) \Leftrightarrow x = y^2\big)$.
- Note that $\mathrm{range}(r) \supseteq \mathbb{R}_{\geqslant 0}$, because for every $y \in \mathbb{R}_{\geqslant 0}$, there is $x \in \mathbb{R}_{\geqslant 0}$ such that $y = r(x)$, namely $x = y^2$.
- So $\mathrm{range}(r) = \mathbb{R}_{\geqslant 0}$ by Remark 7.2.3(2).



Boolean functions

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 7.2.5

A *Boolean function* is a function $\{\mathrm{T}, \mathrm{F}\}^n \to \{\mathrm{T}, \mathrm{F}\}$ where $n \in \mathbb{Z}^+$.

Example 7.2.6

We can represent the Boolean expression $p \vee q$, where p, q be statement variables, using the Boolean function $d \colon \{\mathrm{T}, \mathrm{F}\}^2 \to \{\mathrm{T}, \mathrm{F}\}$ where, for all $p_0, q_0 \in \{\mathrm{T}, \mathrm{F}\}$,

> $d(p_0, q_0)$ is the truth value that $p \vee q$ evaluates to when one substitutes propositions of truth values p_0 and q_0 into the statement variables p and q respectively.

For instance, we have $d(\mathrm{T}, \mathrm{T}) = \mathrm{T}$ and $d(\mathrm{F}, \mathrm{F}) = \mathrm{F}$. Hence $\mathrm{range}(d) = \{\mathrm{T}, \mathrm{F}\}$ by Remark 7.2.3(2).

The images uniquely determine a function

Proposition 7.2.7

Let $f, g\colon A \to B$. Then $f = g$ if and only if $f(x) = g(x)$ for all $x \in A$.

Proof

$(\Rightarrow)$ Assume $f = g$. Let $x \in A$. Then

$$
\begin{array}{llr}
 & (x, f(x)) \in f & \text{by the } \Leftarrow \text{ part of Remark 7.2.2.} \\
\therefore & (x, f(x)) \in g & \text{as } f = g. \\
\therefore & f(x) = g(x) & \text{by the } \Rightarrow \text{ part of Remark 7.2.2.}
\end{array}
$$

$(\Leftarrow)$ Assume $f(x) = g(x)$ for all $x \in A$. For each $x \in A$ and each $y \in B$,

$$
\begin{array}{rclr}
(x, y) \in f & \Leftrightarrow & y = f(x) & \text{by Remark 7.2.2;} \\
 & \Leftrightarrow & y = g(x) & \text{by our assumption;} \\
 & \Leftrightarrow & (x, y) \in g & \text{by Remark 7.2.2.}
\end{array}
$$

So $f = g$. $\square$

Defining a function in terms of its images

Example 7.2.8

The following are true in view of Proposition 7.2.7.

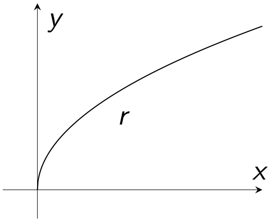
- There is *exactly one* function $r\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ satisfying

$$\forall x, y \in \mathbb{R}_{\geqslant 0}\ \left(y = r(x) \Leftrightarrow x = y^2\right).$$

- There is *exactly one* function $d\colon \{\mathrm{T}, \mathrm{F}\}^2 \to \{\mathrm{T}, \mathrm{F}\}$ such that, for all $p_0, q_0 \in \{\mathrm{T}, \mathrm{F}\}$,

  $d(p_0, q_0)$ is the truth value that $p \vee q$ evaluates to when one substitutes propositions of truth values p_0 and q_0 into the statement variables p and q respectively.

So these descriptions in terms of $r(x)$ and $d(p_0, q_0)$ can serve as definitions of r and d.



p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

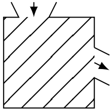
Functions as blackbox operations

Example 7.2.9

Let $f\colon \{0,2\} \to \mathbb{Z}$ and $g\colon \{0,2\} \to \mathbb{Z}$ defined by setting, for all $x \in \{0,2\}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2.$$

Then $f = g$ by Proposition 7.2.7, because $f(x) = g(x)$ for every $x \in \{0,2\}$.



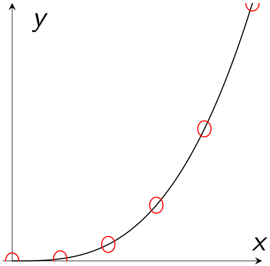
x	y
0	0
2	4

Example 7.2.10

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Q} \to \mathbb{Q}$ defined by

$$\forall x \in \mathbb{Z}\ (f(x) = x^3) \quad \text{and} \quad \forall x \in \mathbb{Q}\ (g(x) = x^3).$$

Then $f \neq g$ because $(1/2, 1/8)$ is an element of g but not of f.

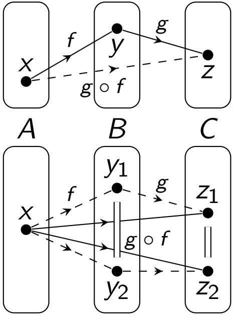


Function composition

Proposition 7.3.1

Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f$ is a function $A \to C$. For every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$



Proof

(F1) Let $x \in A$. Use (F1) for f to find $y \in B$ such that $(x, y) \in f$. Use (F1) for g to find $z \in C$ such that $(y, z) \in g$. Then $(x, z) \in g \circ f$ by the definition of $g \circ f$.

(F2) Let $x \in A$ and $z_1, z_2 \in C$ such that $(x, z_1), (x, z_2) \in g \circ f$. Use the definition of $g \circ f$ to find $y_1, y_2 \in B$ such that $(x, y_1), (x, y_2) \in f$ and $(y_1, z_1), (y_2, z_2) \in g$. Then (F2) for f implies $y_1 = y_2$. So $z_1 = z_2$ as g satisfies (F2).

(image) Now, for every $x \in A$,

$$\begin{aligned}
& (\underset{\text{input}}{x}, \underset{\text{output}}{f(x)}) \in \underset{\text{function}}{f} \quad \text{and} \quad (f(x), g(f(x))) \in g && \text{by the} \Leftarrow \text{part of Remark 7.2.2;}\\
\therefore\quad & (x, g(f(x))) \in g \circ f && \text{by the definition of } g \circ f;\\
\therefore\quad & g(f(x)) = (g \circ f)(x) && \text{by the} \Rightarrow \text{part of Remark 7.2.2.}
\end{aligned}$$

□

Noncommutativity of function composition

Example 7.3.2

Let $f, g \colon \mathbb{Z} \to \mathbb{Z}$ such that for every $x \in \mathbb{Z}$,

$$f(x) = 3x \quad \text{and} \quad g(x) = x + 1.$$

By Proposition 7.3.1, for every $x \in \mathbb{Z}$,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = g(3x) = 3x + 1 \quad \text{and} \\ (f \circ g)(x) &= f(g(x)) = f(x + 1) = 3(x + 1). \end{aligned}$$

Note $(g \circ f)(0) = 1 \neq 3 = (f \circ g)(0)$.
So $g \circ f \neq f \circ g$ by Proposition 7.2.7.

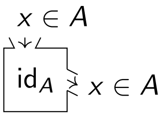
$$\begin{tikzcd}
 & x \arrow[rr, mapsto, "f"] & & 3x \arrow[dd, mapsto, "g"] \\
x \arrow[dd, mapsto, "g"'] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} \arrow[d, "g"] & \\
 & \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} & 3x + 1 \\
x + 1 \arrow[rr, mapsto, "f"'] & & 3(x + 1) &
\end{tikzcd}$$

Identity functions

Definition 7.3.3

Let A be a set. Then the *identity function* on A, denoted id_A, is the function $A \to A$ which satisfies, for all $x \in A$,

$$\mathrm{id}_A(x) = x.$$



Example 7.3.4

Let $f \colon A \to B$. Then $f \circ \mathrm{id}_A = f$ by Proposition 7.2.7, as Proposition 7.3.1 implies

- $f \circ \mathrm{id}_A$ is a function $A \to B$; and
- $(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x)$ for all $x \in A$.

Exercise 7.3.5

Prove that $\mathrm{id}_B \circ f = f$ for all functions $f \colon A \to B$. 7b

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[dr, dashed, "f"'] & A \arrow[d, "f"] \arrow[dr, dashed, "f"] & \\
& B \arrow[r, "\mathrm{id}_B"'] & B
\end{tikzcd}$$

Idempotent functions

Question 7.3.6

Which of the following define a function $f\colon \mathbb{Z} \to \mathbb{Z}$ that satisfies $f \circ f = f$? 7c

(1) $f(x) = 1231$ for all $x \in \mathbb{Z}$.

(2) $f(x) = x$ for all $x \in \mathbb{Z}$.

(3) $f(x) = -x$ for all $x \in \mathbb{Z}$.

(4) $f(x) = 3x + 1$ for all $x \in \mathbb{Z}$.

(5) $f(x) = x^2$ for all $x \in \mathbb{Z}$.

$$\begin{tikzcd}
& \mathbb{Z} \arrow[dr, "f"] & \\
\mathbb{Z} \arrow[ur, "f"] \arrow[rr, "f"'] & & \mathbb{Z}
\end{tikzcd}$$

Surjectivity, injectivity, and bijectivity

Let $f\colon A \to B$.

Definition 7.4.1

(1) f is *surjective* or *onto* if $(\mathrm{F}^{-1}1)$ $\forall y \in B\ \exists x \in A\ \ y = f(x)$.

(2) f is *injective* or *one-to-one* if $(\mathrm{F}^{-1}2)$ $\forall x_1, x_2 \in A\ \ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.

(3) f is *bijective* if it is both surjective and injective.

surjection = surjective function.
injection = injective function.
bijection = bijective function.

True by design

- f satisfies $(\mathrm{F}^{-1}1)$ $\Leftrightarrow$ f^{-1} satisfies (F1).
- f satisfies $(\mathrm{F}^{-1}2)$ $\Leftrightarrow$ f^{-1} satisfies (F2).
- f satisfies $(\mathrm{F}^{-1}1) \wedge (\mathrm{F}^{-1}2)$ $\Leftrightarrow$ f^{-1} satisfies $(\mathrm{F}1) \wedge (\mathrm{F}2)$.

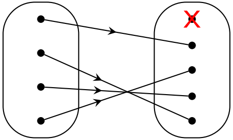
Change f to f^{-1}, noting that $(f^{-1})^{-1} = f$

- f^{-1} satisfies $(\mathrm{F}^{-1}1)$ $\Leftrightarrow$ f satisfies (F1).
- f^{-1} satisfies $(\mathrm{F}^{-1}2)$ $\Leftrightarrow$ f satisfies (F2).
- f^{-1} satisfies $(\mathrm{F}^{-1}1) \wedge (\mathrm{F}^{-1}2)$ $\Leftrightarrow$ f satisfies $(\mathrm{F}1) \wedge (\mathrm{F}2)$.

Proposition 7.4.3

If f is a bijection $A \to B$, then f^{-1} is a bijection $B \to A$.

Surjectivity



Example 7.4.4

The function $f\colon \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is surjective.

Proof

Take any $y \in \mathbb{Q}$. Let $x = (y-1)/3$. Then $x \in \mathbb{Q}$ and $f(x) = 3x + 1 = y$. $\square$

Remark 7.4.5

negation

A function $f\colon A \to B$ is *not* surjective if and only if $\exists y \in B\ \forall x \in A\ (y \neq f(x))$.

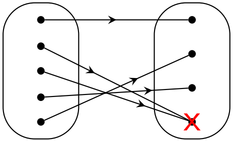
Example 7.4.6

Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not surjective.

Proof

Note $g(x) = x^2 \geqslant 0 > -1$ for all $x \in \mathbb{Z}$. So $g(x) \neq -1$ for all $x \in \mathbb{Z}$, although $-1 \in \mathbb{Z}$. $\square$

Injectivity



Example 7.4.7

The function $f\colon \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is injective.

Proof

Let $x_1, x_2 \in \mathbb{Q}$ such that $f(x_1) = f(x_2)$. Then $3x_1 + 1 = 3x_2 + 1$. So $x_1 = x_2$. $\square$

Remark 7.4.8

A function $f\colon A \to B$ is *not* injective if and only if $\overset{\text{negation}}{\exists x_1, x_2 \in A\ (f(x_1) = f(x_2) \wedge x_1 \neq x_2)}$.

Example 7.4.9

Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not injective.

Proof

Note $g(1) = 1^2 = 1 = (-1)^2 = g(-1)$, although $1 \neq -1$. $\square$

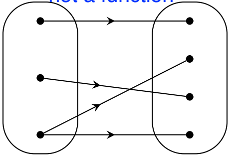
Quick check

Note : surjection = surjective function
must fufill both SURJECTIVE & FUNCTION

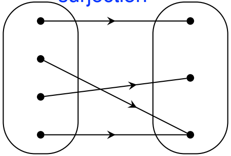
Question 7.4.10

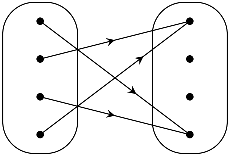
Amongst the arrow diagrams below, which ones represent injections, which ones represent surjections, and which ones represent bijections? 7d

not a function

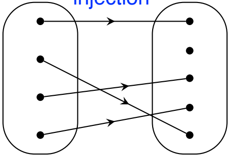
(a) 

surjection

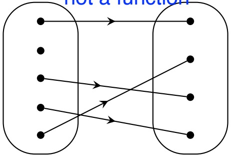
(b) 

(c) 

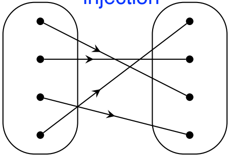
injection

(d) 

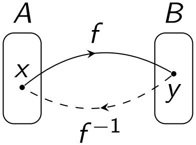
not a function

(e) 

surjection
injection

(f) 

Operational inverse



Proposition 7.4.11

Let $f\colon A \to B$ and $g\colon B \to A$. Then

$$g = f^{-1} \quad\Leftrightarrow\quad \forall x \in A\ \forall y \in B\ \big(g(y) = x \Leftrightarrow y = f(x)\big).$$

Proof

$$\begin{aligned}
g = f^{-1} \quad &\Leftrightarrow\quad \forall y \in B\ \forall x \in A\ \big((y,x) \in g \Leftrightarrow (y,x) \in f^{-1}\big) && \text{as } g, f^{-1} \subseteq B \times A;\\
&\Leftrightarrow\quad \forall x \in A\ \forall y \in B\ \big((y,x) \in g \Leftrightarrow (x,y) \in f\big) && \text{by the definition of } f^{-1};\\
&\Leftrightarrow\quad \forall x \in A\ \forall y \in B\ \big(g(y) = x \Leftrightarrow y = f(x)\big) && \text{by Remark 7.2.2.} \qquad \square
\end{aligned}$$

Finding the inverse of a function

Example 7.4.12

Define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Note that for all $x, y \in \mathbb{Q}$,

$$y = 3x + 1 \quad \Leftrightarrow \quad \underline{x = (y-1)/3}.$$

Let $g\colon \mathbb{Q} \to \mathbb{Q}$ such that $g(y) = (y-1)/3$ for all $y \in \mathbb{Q}$. The equivalence above implies

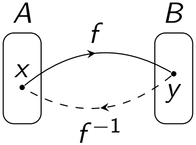
$$\forall x, y \in \mathbb{Q} \ \big(y = f(x) \Leftrightarrow \underline{x = g(y)}\big).$$

So Proposition 7.4.11 tells us $g = f^{-1}$.

Note 7.4.13

Unlike in Example 7.4.12, in general we are *not* guaranteed a description of the inverse of a bijection f that is significantly different from the trivial description that it is the inverse of f.

Algebraic inverse



Proposition 7.4.14

If f is a bijection $A \to B$, then $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

Proof of $f^{-1} \circ f = \mathrm{id}_A$

We know f^{-1} is a function by Proposition 7.4.3, because f is bijection. Let $x \in A$. Define $y = f(x)$. Then

$$\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}(f(x)) && \text{by Proposition 7.3.1;} \\
&= f^{-1}(y) && \text{by the definition of } y; \\
&= x && \text{by Proposition 7.4.11, as } y = f(x); \\
&= \mathrm{id}_A(x) && \text{by the definition of } \mathrm{id}_A.
\end{aligned}$$

So $f^{-1} \circ f = \mathrm{id}_A$ by Proposition 7.2.7. □

Exercise

The proof of $f \circ f^{-1} = \mathrm{id}_B$ is similar, and is left as an exercise. 7e

Summary

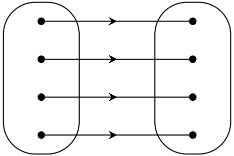
Definition 7.1.1. Let A, B be sets. A *function* from A to B is a relation f from A to B such that any element of A is f-related to a unique element of B, i.e.,

(F1) $\forall x \in A\ \exists y \in B\ \ (x, y) \in f$; and

(F2) $\forall x \in A\ \forall y_1, y_2 \in B\ \ \big((x, y_1) \in f \wedge (x, y_2) \in f\ \Rightarrow\ y_1 = y_2\big)$.

Remark 7.2.2. If $f\colon A \to B$, then $\forall x \in A\ \forall y \in B\ \ \big((x, y) \in f \ \Leftrightarrow\ y = f(x)\big)$.

Proposition 7.3.1. Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f\colon A \to C$.
For every $x \in A$,
$$(g \circ f)(x) = g(f(x)).$$



Definition 7.4.1. Let $f\colon A \to B$.

(1) f is *surjective* if $\forall y \in B\ \exists x \in A\ \ y = f(x)$.

(2) f is *injective* if $\forall x_1, x_2 \in A\ \ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.

(3) f is *bijective* if it is both surjective and injective.

Proposition 7.4.3. If f is a bijection $A \to B$, then f^{-1} is a bijection $B \to A$.

Proposition 7.4.11. Let $f\colon A \to B$ and $g\colon B \to A$. Then
$$g = f^{-1} \quad \Leftrightarrow \quad \forall x \in A\ \forall y \in B\ \ \big(g(y) = x \Leftrightarrow y = f(x)\big).$$

Proposition 7.4.14. If f is a bijection $A \to B$, then $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.