Chapter 2: Predicate logic

CS1231 Discrete Structures

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> An important use of predicate calculus is in the formal specification of a piece of code. This is done by writing down conditions which must hold before and after the code is executed. [. . .] Propositional calculus, which gives us no access to program variables, is much more restrictive than we require. [. . .] Many of the propositions we would like to write down and make use of in programs depend on variables which are measurements of quantities likes time, weight or money.

Dowsing, Rayward-Smith, Walter (1986)

Predicate logic

Plan

- predicates — ______ likes ______.
- quantifiers — $\forall$, $\exists$
- negation — $\neg\forall$, $\neg\exists$
- nested quantification — $\forall\forall$, $\forall\exists$, $\exists\forall$, $\exists\exists$, $\ldots$

Variables

Definition 2.1.1

(1) A *variable* is a symbol that indicates a position in a sentence in which one can substitute (the name of) an object.

(2) A *valid* substitution for a variable replaces all free occurrences of that variable in the sentence by the same object.

(3) Saying a variable x *takes* an object a in a sentence means one substitutes the object a into the variable x in the sentence.

(4) Sometimes we may want to allow only certain objects to be substituted into a variable x. In this case, we call the set of all such objects the *domain* of x, and we may say that *x ranges over* these objects.

Free variables

Remark 2.1.2

- A phrase or a symbol may use, or more technically speaking, *bind* a variable occurring in the sentence.
- For example, the variable x in

  For every real number x, we must have $x^2 \geqslant 0$.

  is already used, or *bound*, by the phrase "for every": this sentence means

  No matter what real number one substitutes into the variable x, the sentence $x^2 \geqslant 0$ becomes true.

- A valid substitution should be applied *only* to the variable occurrences that are not already used or bound by anything in the sentence.
- Such occurrences are said to be *free*.

Diversion: sets

Remark 2.1.3

(1) A *set* is a (possibly empty, possibly infinite) collection of objects; these objects are called the *elements* of the set. We can write $z \in A$ for "z is an element of A".
Chapter 4 contains a more detailed treatment of sets.

(2) In Chapter 7, we will introduce the notion of the domain of a *function*. This is different from the domain of a variable.

(3) Some people insist that every variable has a domain. We do not.

Common sets (Table 2.1)

Note 2.1.4. Some define $0 \notin \mathbb{N}$, but we do *not*.

Symbol	Meaning	Elements	Non-elements
$\mathbb{N}$	the set of all natural numbers	$0, 1, 2, 3, 31$	$-1, \frac{1}{2}$
$\mathbb{Z}$	the set of all integers	$0, 1, -1, 2, -10$	$\frac{1}{2}, \sqrt{2}$
$\mathbb{Q}$	the set of all rational numbers	$-1, 10, \frac{1}{2}, -\frac{7}{5}$	$\sqrt{2}, \pi, \sqrt{-1}$
$\mathbb{R}$	the set of all real numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi$	$\sqrt{-1}, \sqrt{-10}$
$\mathbb{C}$	the set of all complex numbers	all of the above	
$\mathbb{Z}^+$	the set of all positive integers	$1, 2, 3, 31$	$0, -1, -12$
$\mathbb{Z}^-$	the set of all negative integers	$-1, -2, -3, -31$	$0, 1, 12$
$\mathbb{Z}_{\geqslant 0}$	the set of all non-negative integers	$0, 1, 2, 3, 31$	$-1, -12$

$\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geqslant m}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_{\geqslant m}$, etc. are defined similarly.

$\mathbb{Z}$ is for *Zahlen*.

$\mathbb{Q}$ is for quotients.

"Positive" means > 0.

"Negative" means < 0.

"Non-negative" means $\geqslant 0$.

Predicates

$$\underline{\qquad\qquad} \text{ likes } \underline{\qquad\qquad}.$$

Definition 2.1.5

Let P be a sentence and let $x_1, x_2, \ldots, x_n$ list all the variables that appear free in P.

(1) We may write P as $P(x_1, x_2, \ldots, x_n)$.

(2) If $z_1, z_2, \ldots, z_n$ are objects, then we denote by $P(z_1, z_2, \ldots, z_n)$ the sentence obtained from $P(x_1, x_2, \ldots, x_n)$ by substituting each z_i into x_i.

Definition 2.1.6

(1) A *predicate* is a sentence that becomes a proposition whenever one validly substitutes objects into all its variables.

(2) A sentence $P(x_1, x_2, \ldots, x_n)$ is a *predicate over sets $D_1, D_2, \ldots, D_n$* if $P(z_1, z_2, \ldots, z_n)$ is a proposition whenever $z_1, z_2, \ldots, z_n$ are respectively elements of $D_1, D_2, \ldots, D_n$; in the case when a variable x_i here has a domain, we additionally require this domain to contain every element of D_i.

(3) We may call a predicate over $D, D, \ldots, D$ simply a *predicate on D*.

Predicates: examples

Example 2.1.7

Let $P(x)$ be "$x^2 \geqslant x$", where x is a variable with domain $\mathbb{Q}$. Then

(1) $P(x)$ is a predicate over $\mathbb{Q}$;

(2) $P(1231)$ is "$1231^2 \geqslant 1231$", which is a true proposition; and

(3) $P(1/2)$ is "$(1/2)^2 \geqslant 1/2$", which is a false proposition because $(1/2)^2 = 1/4 < 1/2$.

Example 2.1.8

Let $Q(x, y)$ be "$x + y = 0$", where x and y are variables with domain $\mathbb{Z}$. Then

(1) $Q(x, y)$ is a predicate on $\mathbb{Z}$;

(2) $Q(0, 1)$ is "$0 + 1 = 0$", which is a false proposition; and

(3) $Q(2, -2)$ is "$2 + (-2) = 0$", which is a true proposition.

The universal quantifier

Let $P(x)$ be a sentence and D be a set.

$\forall$ll

Definition 2.2.1

(1) We denote by $\forall x\ P(x)$ the proposition "for all x, $P(x)$".

(2) The symbol $\forall$, read as "for all", is known as the *universal quantifier*.

(3) The proposition $\forall x\ P(x)$ is true if and only if $P(z)$ is true for all objects z.

(4) A *counterexample* to $\forall x\ P(x)$ is an object z for which $P(z)$ is not true.

(5) We denote by $\forall x \in D\ \ P(x)$ the sentence "for all x in D, $P(x)$", or symbolically $\forall x\ \big(x \in D \to P(x)\big)$.

Note 2.2.2

- The proposition $\forall x \in D\ P(x)$ is false if and only if it has a counterexample.
- In the case when $P(x)$ is a predicate over D, this in turn is equivalent to $P(z)$ being false for at least one object z in D.

The universal quantifier: examples

Example 2.2.3

(1) Let D be the set that contains precisely $1, 2, 3, 4, 5$. Then the proposition $\forall x \in D \;\; x^2 \geqslant x$ is true because

$$1^2 \geqslant 1 \quad \text{and} \quad 2^2 \geqslant 2 \quad \text{and} \quad 3^2 \geqslant 3 \quad \text{and} \quad 4^2 \geqslant 4 \quad \text{and} \quad 5^2 \geqslant 5.$$

(2) The number $1/2$ is a counterexample to $\forall x \in \mathbb{Q} \;\; x^2 \geqslant x$ because $1/2$ is an element of $\mathbb{Q}$ and $(1/2)^2 = 1/4 < 1/2$.

(3) So the proposition $\forall x \in \mathbb{Q} \;\; x^2 \geqslant x$ is false.

The existential quantifier

Let $P(x)$ be a sentence and D be a set.

$\exists$xist

Definition 2.2.4

(1) We denote by $\exists x\ P(x)$ the proposition "there exists x such that $P(x)$".

(2) The symbol $\exists$, read as "there exists", is known as the *existential quantifier*.

(3) The proposition $\exists x\ P(x)$ is true if and only if $P(z)$ is true for at least one object z.

(4) A *witness* to the proposition $\exists x\ P(x)$ is an object z for which $P(z)$ is true.

(5) We denote by $\exists x \in D\ \ P(x)$ the proposition "there exists x in D such that $P(x)$", or symbolically $\exists x\big(x \in D \wedge P(x)\big)$.

Note 2.2.5

- The proposition $\exists x \in D\ P(x)$ is true if and only if it has a witness.
- In the case when $P(x)$ is a predicate over D, the proposition $\exists x\ P(x)$ is false if and only if $P(z)$ is false for all objects z in D.

The existential quantifier: examples

Example 2.2.6

(1) 2 is a witness to $\exists x \in \mathbb{Q}\ \ x^2 \geqslant x$ because 2 is an element of $\mathbb{Q}$ and $2^2 = 4 \geqslant 2$.

(2) So the proposition $\exists x \in \mathbb{Q}\ \ x^2 \geqslant x$ is true.

(3) Let D be the set that contains precisely $1/2$, $1/3$, $1/4$, $1/5$. Then the proposition $\exists x \in D\ \ x^2 \geqslant x$ is false because

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2} \quad \text{and} \quad \left(\frac{1}{3}\right)^2 = \frac{1}{9} < \frac{1}{3} \quad \text{and}$$
$$\left(\frac{1}{4}\right)^2 = \frac{1}{16} < \frac{1}{4} \quad \text{and} \quad \left(\frac{1}{5}\right)^2 = \frac{1}{25} < \frac{1}{5}.$$

Nota bene

Let $P(x)$ be a predicate.

Convention 2.2.7

(1) In mathematics,

"there exists one x such that $P(x)$" or "there is one x such that $P(x)$"

means "there exists *at least* one x such that $P(x)$".

(2) More generally, if n is a non-negative integer, then

"there exist n x's such that $P(x)$" or "there are n x's such that $P(x)$"

means "there exist *at least* n x's such that $P(x)$".

(3) If the exact number is intended, then use the word "exactly", as in "there are exactly two x's such that $P(x)$".

Convention 2.2.8

(1) In informal contexts, some may write symbolically a quantifier, say $\forall x \in D$ or $\exists x$, *after* the expression it applies to.

(2) However, in this module, we do *not* do it: here a quantifier, when written symbolically, always comes *before* the expression it applies to.

(3) Here it applies *only* to the smallest predicate (over an appropriate set) that follows it.

Quantification in mathematics

Terminology 2.2.9

(1) In addition to "all', words that indicate universal quantification in mathematics include "every", "each", and "any".

(2) One may also express "for all x in D, $P(x)$" as

$$\text{“}P(x) \text{ whenever } x \in D\text{”} \quad \text{or} \quad \text{“If } x \in D, \text{ then } P(x)\text{”}.$$

(3) In addition to "exists", phrases that indicate existential quantification in mathematics include "some" and "there is".

Real example

Example 2.2.10

Let $\text{Even}(x)$ denote the predicate "x is even" over $\mathbb{Z}$. Express the following propositions symbolically using $\text{Even}(x)$.

(1) "The square of any even integer is even."

(2) "Any integer whose square is even must itself be even."

(3) "Some even integer n satisfies $n^2 = 2n$."

Solution

(1) $\forall n \in \mathbb{Z}\ \big(\text{Even}(n) \to \text{Even}(n^2)\big)$.

A *common mistake* is to answer $\forall n \in \mathbb{Z}\ \big(\text{Even}(n) \wedge \text{Even}(n^2)\big)$; this can be read as

"for every integer n, n is even and n^2 is even",

whose meaning is different from that of the given proposition.

(2) $\forall n \in \mathbb{Z}\ \big(\text{Even}(n^2) \to \text{Even}(n)\big)$.

(3) $\exists n \in \mathbb{Z}\ \big(\text{Even}(n) \wedge n^2 = 2n\big)$.

Domain of discourse

Remark 2.2.11

(1) In certain areas of mathematics, all variables have the same domain. This common domain is called the *domain of discourse*. For brevity, some authors may omit this in quantified expressions in the particular context.

(2) In this module, there is *no* domain of discourse, as we often need to consider variables with different domains. In particular, we will *not* abbreviate $\forall x \in D$ and $\exists x \in D$ as $\forall x$ and $\exists x$.

Exercise 2.2.12

Which of the following is/are true for every predicate $P(x)$ over $\mathbb{R}$? ✎ 2a

(1) If $\forall x \in \mathbb{Z}\ \ P(x)$ is true, then $\forall x \in \mathbb{R}\ \ P(x)$ is true.

(2) If $\forall x \in \mathbb{R}\ \ P(x)$ is true, then $\forall x \in \mathbb{Z}\ \ P(x)$ is true.

(3) If $\exists x \in \mathbb{Z}\ \ P(x)$ is true, then $\exists x \in \mathbb{R}\ \ P(x)$ is true.

(4) If $\exists x \in \mathbb{R}\ \ P(x)$ is true, then $\exists x \in \mathbb{Z}\ \ P(x)$ is true.

Negation of quantified sentences (1/2)

Theorem 2.3.1

The following are true for all predicates $P(x)$ over a set D.

(1,3) $\neg\forall x \in D\ P(x) \leftrightarrow \exists x \in D\ \neg P(x)$.

(2,4) $\neg\exists x \in D\ P(x) \leftrightarrow \forall x \in D\ \neg P(x)$.

Proof of (1)

Note that the following are true.

$$\begin{aligned}
\neg\forall x\ P(x) \text{ is true} &\leftrightarrow \forall x\ P(x) \text{ is false} \\
&\qquad\qquad \text{by the definition of } \neg. \\
\forall x\ P(x) \text{ is false} &\leftrightarrow P(z) \text{ is false for at least one object } z \\
&\qquad\qquad \text{by Note 2.2.2(1).} \\
P(z) \text{ is false for at least one object } z &\leftrightarrow \neg P(z) \text{ is true for at least one object } z \\
&\qquad\qquad \text{by the definition of } \neg. \\
\neg P(z) \text{ is true for at least one object } z &\leftrightarrow \exists x\ \neg P(x) \text{ is true} \\
&\qquad\qquad \text{by the definition of } \exists.
\end{aligned}$$

From these, we deduce that $\neg\forall x\ P(x)$ is true if and only if $\exists x\ \neg P(x)$ is true. $\square$

Negation of quantified sentences (2/2)

Theorem 2.3.1

The following are true for all predicates $P(x)$ over a set D.

(1,3) $\neg \forall x \in D\ P(x) \ \leftrightarrow\ \exists x \in D\ \neg P(x)$.

(2,4) $\neg \exists x \in D\ P(x) \ \leftrightarrow\ \forall x \in D\ \neg P(x)$.

Proof of (2)

Note that the following are true.

$$\begin{aligned}
\neg \exists x\ P(x) \text{ is true} \quad &\leftrightarrow \quad \exists x\ P(x) \text{ is false} && \text{by the definition of } \neg. \\
\exists x\ P(x) \text{ is false} \quad &\leftrightarrow \quad P(z) \text{ is false for all objects } z \\
& && \text{by Note 2.2.5(2).} \\
P(z) \text{ is false for all objects } z \quad &\leftrightarrow \quad \neg P(z) \text{ is true for all objects } z \\
& && \text{by the definition of } \neg. \\
\neg P(z) \text{ is true for all objects } z \quad &\leftrightarrow \quad \forall x\ \neg P(x) \text{ is true} && \text{by the definition of } \forall.
\end{aligned}$$

From these, we deduce that $\neg \forall x\ P(x)$ is true if and only if $\exists x\ \neg P(x)$ is true. □

Remark 2.3.2. We will introduce a more succinct way to write these proofs.

Real Example 2.3.4

Consider the following proposition:

"Not every integer is even."

We can express this symbolically as

$$\neg \forall n \in \mathbb{Z} \; \mathrm{Even}(n),$$

where $\mathrm{Even}(n)$ denotes the predicate "n is even" over $\mathbb{Z}$. In view of Theorem 2.3.1, this is equivalent to

$$\exists n \in \mathbb{Z} \; \neg \mathrm{Even}(n).$$

It follows that the given English proposition is equivalent to

"There is an integer that is not even."

Real Example 2.3.5

Consider the following proposition:

"No integer is both odd and even."

We can express this symbolically as

$$\neg\exists n \in \mathbb{Z}\ \big(\mathrm{Odd}(n) \wedge \mathrm{Even}(n)\big),$$

where $\mathrm{Even}(n)$ and $\mathrm{Odd}(n)$ denote respectively the predicates "n is even" and "n is odd" over $\mathbb{Z}$. In view of Theorem 2.3.1, this is equivalent to

$$\forall n \in \mathbb{Z}\ \neg\big(\mathrm{Odd}(n) \wedge \mathrm{Even}(n)\big).$$

By De Morgan's Laws, this is in turn equivalent to

$$\forall n \in \mathbb{Z}\ \big(\neg\mathrm{Odd}(n) \vee \neg\mathrm{Even}(n)\big).$$

It follows that the given English proposition is equivalent to

"For every integer, either it is not odd or it is not even."

Generalization

$\exists x\ \forall y\ Q(x, y)$ is the proposition $\exists x\ P(x)$, where $P(x)$ denotes the predicate $\forall y\ Q(x, y)$.

Definition 2.4.1 ($n = 1$)

Consider a sentence $Q(x, y)$ and a set E. Let z be an object. Assume additionally that z is in the domain of x if x has a domain.

(1) We denote by $\forall y\ Q(x, y)$ and $\exists y\ Q(x, y)$ the predicates "for all y, $Q(x, y)$" and "there exists y such that $Q(x, y)$" respectively. Both of these predicates may mention the variable x.

(2) Denote by $\forall y\ Q(z, y)$ and $\exists y\ Q(z, y)$ the propositions obtained respectively from the predicates $\forall y\ Q(x, y)$ and $\exists y\ Q(x, y)$ by substituting each z into x.

(3) The proposition $\forall y\ Q(z, y)$ is true if and only if $Q(z, w)$ is true for all objects w.

(5) The proposition $\exists y\ Q(z, y)$ is true if and only if $Q(z, w)$ is true for at least one object w.

(7) We denote by $\forall y \in E\ \ Q(x, y)$ the predicate "for all y in E, $Q(x, y)$", or symbolically $\forall y\ \big(y \in E \to Q(x, y)\big)$.

(8) We denote by $\exists y \in E\ \ Q(x, y)$ the predicate "there exists y in E such that $Q(x, y)$", or symbolically $\exists y\ \big(y \in E \wedge Q(x, y)\big)$.

Nesting distinct quantifiers

Example 2.4.3

(1) Consider the proposition "$\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "for every integer x, there is an integer y, such that $x + y = 0$".

  (b) This is *true* because, given any integer x, one can set $y = -x$ to make y an integer and $x + y = 0$.

(2) Consider the proposition "$\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "there is an integer x such that, for every integer y, $x + y = 0$".

  (b) Alternatively, one can express this as "there is an integer which, when added to any integer, gives a sum of 0".

  (c) This is *false* because, given any integer x, one can set

$$y = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0, \end{cases}$$

  so that y is an integer and $x + y \neq 0$.

The order of quantifiers matters!

Nesting like quantifiers

Example 2.4.3

(3) Consider the proposition "$\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "for every integer x, for every integer y, $x + y = 0$".
  (b) Alternatively, one can express this as "the sum of any two integers is 0".
  (c) This is *false* because 1 and 1 are integers and $1 + 1 = 2 \neq 0$.

(4) Consider the proposition "$\exists x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ \ x + y = 0$".

  (a) This reads "there exists an integer x, there exists an integer y, such that $x + y = 0$".
  (b) Alternatively, one can express this as "there are two integers which, when added together, gives 0".
  (c) This is *true* because 2 and -2 are integers and $2 + (-2) = 0$.

Real examples

Note 2.4.5

One can interpret the following sentences as any one of $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ \ x + y = 0$ and $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ \ x + y = 0$. As the two interpretations are not equivalent, *avoid* writing such ambiguous sentences in mathematics.

(1) "One can add any integer to some integer to get a sum of 0."

(2) "There is an integer x such that $x + y = 0$ for any integer y."

Example 2.4.6

One can express the proposition

"Every even integer is the sum of two odd integers."

from Example 1.1.2(3) symbolically as

$$\forall n \in \mathbb{Z}\ \ \big(\mathrm{Even}(n) \to \exists k \in \mathbb{Z}\ \exists \ell \in \mathbb{Z}\ \ (\mathrm{Odd}(k) \wedge \mathrm{Odd}(\ell) \wedge n = k + \ell)\big),$$

where $\mathrm{Even}(n)$ and $\mathrm{Odd}(n)$ are respectively the predicates "n is even" and "n is odd" over $\mathbb{Z}$.

Consecutive like quantifiers

Let $P(x_1, x_2, \ldots, x_n)$, $Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ be sentences. Let D, E be sets.

Notation 2.4.7

(1) We may abbreviate $\forall y_1 \in E\ \forall y_2 \in E\ \ldots\ \forall y_m \in E\ Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ as $\forall y_1, y_2, \ldots, y_m \in E\ Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

(2) We may abbreviate $\exists y_1 \in E\ \exists y_2 \in E\ \ldots\ \exists y_m \in E\ Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ as $\exists y_1, y_2, \ldots, y_m \in E\ Q(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

Note 2.4.8

(1) The proposition $\forall x_1, x_2, \ldots, x_n \in D\ \ P(x_1, x_2, \ldots, x_n)$ is true if and only if $P(z_1, z_2, \ldots, z_n)$ is true for all objects $z_1, z_2, \ldots, z_n$ in D.

(2) The proposition $\exists x_1, x_2, \ldots, x_n \in D\ \ P(x_1, x_2, \ldots, x_n)$ is true if and only if $P(z_1, z_2, \ldots, z_n)$ is true for some objects $z_1, z_2, \ldots, z_n$ in D.

The objects $z_1, z_2, \ldots, z_n$ above are *not* necessarily different.

- If the z_i's are really meant to be all different, then one can use the word "distinct" to indicate it, as in "for all distinct $z_1, z_2, \ldots, z_n$" and "there exist distinct $z_1, z_2, \ldots, z_n$".

Negating multiply quantified sentences

Theorem 2.4.9

The following are true for all predicates $Q(x, y)$.

(1) $\neg\forall x\, \forall y\; Q(x, y) \leftrightarrow \exists x\, \exists y\, \neg Q(x, y)$.

(2) $\neg\forall x\, \exists y\; Q(x, y) \leftrightarrow \exists x\, \forall y\, \neg Q(x, y)$.

(3) $\neg\exists x\, \exists y\; Q(x, y) \leftrightarrow \forall x\, \forall y\, \neg Q(x, y)$.

(4) $\neg\exists x\, \forall y\; Q(x, y) \leftrightarrow \forall x\, \exists y\, \neg Q(x, y)$.

Proof of (1)

We have the following equivalences by Theorem 2.3.1.

$$\begin{aligned} \neg\forall x\, \forall y\; Q(x, y) &\leftrightarrow \exists x\, \neg\forall y\; Q(x, y). \\ \exists x\, \neg\forall y\; Q(x, y) &\leftrightarrow \exists x\, \exists y\, \neg Q(x, y). \end{aligned}$$

So $\neg\forall x\, \forall y\; Q(x, y)$ is true if and only if $\exists x\, \exists y\, \neg Q(x, y)$ is true. $\square$

Proof of (2)

We have the following equivalences by Theorem 2.3.1.

$$\begin{aligned} \neg\forall x\, \exists y\; Q(x, y) &\leftrightarrow \exists x\, \neg\exists y\; Q(x, y). \\ \exists x\, \neg\exists y\; Q(x, y) &\leftrightarrow \exists x\, \forall y\, \neg Q(x, y). \end{aligned}$$

So $\neg\forall x\, \exists y\; Q(x, y)$ is true if and only if $\exists x\, \forall y\, \neg Q(x, y)$ is true. $\square$

Write proofs for (3) and (4).

✎ 2c

Nested quantifiers: exercise

Exercise 2.4.10

Let D be the set that contains precisely $-1, 0, 1$. Let E be the set that contains precisely $1, -1, 2, -2$. Which of the following propositions is/are true? ✐ 2d

(1) $\exists x \in D \ \forall y \in E \ \ xy = 0$.

(2) $\forall y \in E \ \exists x \in D \ \ xy = 0$.

(3) $\exists x \in D \ \forall y \in E \ \ xy < 0$.

(4) $\forall y \in E \ \exists x \in D \ \ xy < 0$.

(5) $\exists x_1, x_2 \in D \ \ x_1 + x_2 = 2$.

(6) $\forall y_1, y_2 \in E \ \ y_1 = y_2$.

Summary

Let $P(x)$ be a sentence and D be a set. Let p, q be propositions.

universal quantifier	for all	$\forall$
existential quantifier	there exists	$\exists$

$$\begin{array}{rcl} \forall x \in D \;\; P(x) & \text{means} & \forall x \; \big(x \in D \to P(x)\big). \\ \exists x \in D \;\; P(x) & \text{means} & \exists x \; \big(x \in D \wedge P(x)\big). \end{array}$$

A counterexample to $\forall x \in D \;\; P(x)$ is some z such that $P(z)$ is not true.
A witness to $\exists x \in D \;\; P(x)$ is some z such that $P(z)$ is true.

$$\begin{array}{rcll} \neg\forall x \in D \;\; P(x) & \leftrightarrow & \exists x \in D \; \neg P(x) & \text{by Theorem 2.3.1.} \\ \neg\exists x \in D \;\; P(x) & \leftrightarrow & \forall x \in D \; \neg P(x) & \text{by Theorem 2.3.1.} \\ \neg(p \wedge q) & \leftrightarrow & \neg p \vee \neg q & \text{by De Morgan's Laws.} \\ \neg(p \vee q) & \leftrightarrow & \neg p \wedge \neg q & \text{by De Morgan's Laws.} \\ \neg(p \to q) & \leftrightarrow & p \wedge \neg q & \text{by Example 1.4.23.} \\ \neg(\neg p) & \leftrightarrow & p & \text{by Double Negative Law.} \end{array}$$

- The order of distinct quantifiers matters.
- Like quantifiers can be grouped together.