NATIONAL UNIVERSITY OF SINGAPORE

CS1231 DISCRETE STRUCTURES

(Semester 1: 2021/2022)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Write your Student Number only. Do not write your name.
2. This assessment paper contains **FIVE** questions and comprises **EIGHT** printed pages.
3. Answer **ALL** questions. The marks for each question are indicated in brackets.
4. Write your solutions in the spaces provided.
5. This is an **OPEN** book examination.

STUDENT NUMBER: ______________________________

EXAMINER'S USE ONLY		
Question	Marks	Score
Q1	5	
Q2	4	
Q3	8	
Q4	10	
Q5	13	
Total	40	

1. Let A be a countable set with at least 2 different elements. Define $B = \{X \subseteq A : X \text{ is finite and } |X| = 2\}$.

   (a) Define a set $B_2 \supseteq B$ and a surjection $h\colon \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to B_2$. [2 marks]

   (b) Prove that the h you defined in (a) is indeed surjective. [1 mark]

   (c) Use this h to show that B is countable. [2 marks]

   (Hint: Tutorial 7 Question 9 tells us that a nonempty set S is countable if and only if there is a surjection $\mathbb{Z}^+ \to S$.)

2. Let C be an uncountable set. Define $D = \{Y \subseteq C : Y \text{ is finite and } |Y| = 2\}$. We use Corollary 9.3.1 in the notes to prove the uncountability of D as follows.

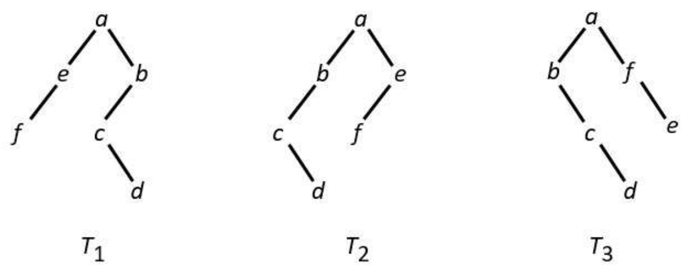
   (a) Define an uncountable set C_1 and an injection $f\colon C_1 \to D$. [2 marks]

   (b) Explain why the C_1 you defined in (a) is indeed uncountable. [1 mark]

   (c) Prove that the f you defined in (a) is indeed injective. [1 mark]

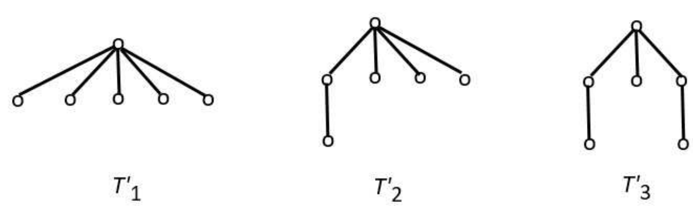
   (Hint: Tutorial 8 Question 5(b) tells us that if S is an uncountable set and S_0 is a countable set, then $S \setminus S_0$ is uncountable.)

3. Consider the rooted trees T_1, T_2, T_3 below (a is the root):



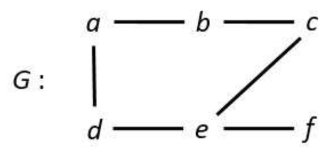
(a) Explain why (i) $T_1 = T_2$ and (ii) $T_2 \neq T_3$. [2 marks]

(b) In (a), we say T_2 and T_3 are two different ways of using $\{a, b, c, d, e, f\}$ to *label* the vertices of the same tree (whereas T_1 and T_2 are the same tree labeled the same way).
Consider the rooted trees T_1', T_2' and T_3' below.



For each tree, calculate the number of different ways of using $\{a, b, c, d, e, f\}$ to label the tree. [6 marks]

4. Consider the following undirected graph G:

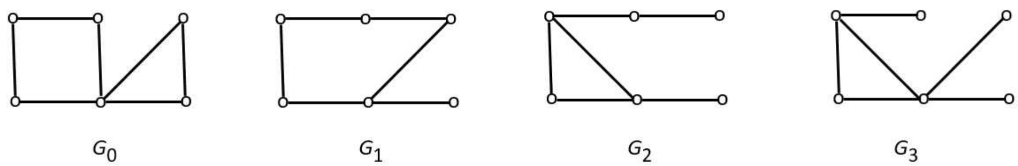


(a) Draw all spanning trees for G. [2 marks]

(b) For your answer to (a), identify 4 different spanning trees S_1, S_2, S_3, S_4 such that S_1 and S_2 are isomorphic ($S_1 \simeq S_2$), and S_3 and S_4 are isomorphic ($S_3 \simeq S_4$). [4 marks]

(c) For your choice of S_1, S_2, S_3 and S_4 in (b), define a permutation π of $\{a, b, c, d, e, f\}$ that shows $S_1 \simeq S_2$, and a permutation π' that shows $S_3 \simeq S_4$. [4 marks]

5. For an undirected graph without loops, a vertex ℓ is called a *leaf* if and only if there is exactly one edge containing ℓ. In the following examples, G_0 has 0 leaves, G_1 has 1 leaf, G_2 has 2 leaves and G_3 has 3 leaves:



(a) Let H_0, H_1, H_2 and H_3 be undirected and loopless connected graphs with 5 vertices each; H_0 has 0 leaves, H_1 has 1 leaf, H_2 has 2 leaves and H_3 has 3 leaves. Draw one example each for H_0, H_1, H_2 and H_3. [4 marks]

(b) Which of your examples in (a) are cyclic, and which are acyclic? [2 marks]

(c) Let (V, E) be an (unrooted) tree, and $\{b, c\} \in E$. Prove that $(V, E \setminus \{\{b, c\}\})$ is a forest with two trees. [3 marks]

(d) Using (c) and induction, or otherwise, prove that a nontrivial (unrooted) tree must have at least two leaves. [4 marks]

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