CS1231 Chapter 5

Relations

5.1 Basics

Definition 5.1.1. An *ordered pair* is an expression of the form

$$(x, y).$$

Let (x_1, y_1) and (x_2, y_2) be ordered pairs. Then

$$(x_1, y_1) = (x_2, y_2) \quad \Leftrightarrow \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Example 5.1.2. (1) $(1, 2) \neq (2, 1)$, although $\{1, 2\} = \{2, 1\}$.

(2) $(3, 0.5) = (\sqrt{9}, \frac{1}{2})$.

Definition 5.1.3. Let A, B be sets. The *Cartesian product* of A and B, denoted $A \times B$, is defined to be

$$\text{A × B} = \{(x, y) : x \in A \text{ and } y \in B\}.$$

Read $A \times B$ as "A cross B".

The Cartesian product is a set formed from two or more given sets and contains all ordered pairs of elements

Example 5.1.4. $\{\text{a}, \text{b}\} \times \{1, 2, 3\} = \{(\text{a}, 1), (\text{a}, 2), (\text{a}, 3), (\text{b}, 1), (\text{b}, 2), (\text{b}, 3)\}$.

Definition 5.1.5. Let A, B be sets.

(1) A *relation* from A to B is a subset of $A \times B$.

(2) Let R be a relation from A to B and $(x, y) \in A \times B$. Then we may write

$$x \mathrel{R} y \text{ for } (x, y) \in R \quad \text{and} \quad x \not\mathrel{R} y \text{ for } (x, y) \notin R.$$

We read "$x \mathrel{R} y$" as "x is *R-related* to y" or simply "x is *related* to y".

Example 5.1.6. Let $\Gamma = \{\text{A}, \text{B}, \ldots, \text{Z}, 0, 1, 2, \ldots, 9\}$ and $\Phi = \{\text{A}, \text{B}, \ldots, \text{Z}, \text{a}, \text{b}, \ldots, \text{z}\}$. As in Figure 5.1, define

$$SN = \{(\text{001R}, \text{Gates}), (\text{012B}, \text{Brin}), (\text{062E}, \text{Bezos}), (\text{126N}, \text{Ma}), (\text{254E}, \text{Zuckerberg})\}.$$

Then SN is a relation from Γ^* to Φ^*.

Example 5.1.7. Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4\}$. Define the relation R from A to B by setting

$$x \mathrel{R} y \quad \Leftrightarrow \quad x < y.$$

Then $0 \mathrel{R} 1$ and $0 \mathrel{R} 2$, but $2 \not\mathrel{R} 1$. Thus

$$R = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

identity	
Student ID	name
001R	Gates
012B	Brin
062E	Bezos
126N	Ma
254E	Zuckerberg

$$SN = \{ (\text{001R, Gates}), (\text{012B, Brin}), (\text{062E, Bezos}), (\text{126N, Ma}), (\text{254E, Zuckerberg}) \}$$

is enrolled in	
Student ID	module
126N	CS3234
254E	CS3234
001R	MA2001
012B	MA2001
062E	MA2001
126N	MA2001
012B	MU2109
001R	PC2130
062E	PL3103
254E	PL3103

$$SM = \{ (\text{126N, CS3234}), (\text{254E, CS3234}), (\text{001R, MA2001}), (\text{012B, MA2001}), (\text{062E, MA2001}), (\text{126N, MA2001}), (\text{012B, MU2109}), (\text{001R, PC2130}), (\text{062E, PL3103}), (\text{254E, PL3103}) \}$$

progress		
Student ID	faculty	year
062E	Arts	1
254E	Arts	2
012B	Science	2
001R	Science	1
126N	Science	3

$$SFY = \{ (\text{062E, Arts, 1}), (\text{254E, Arts, 2}), (\text{012B, Science, 2}), (\text{001R, Science, 1}), (\text{126N, Science, 3}) \}$$

teaching			
module	department	faculty	instructor
CS3234	CS	Computing	Turing
MA2001	Mathematics	Science	Gauss
MU2109	Music	Arts	Mozart
PC2130	Physics	Science	Newton
PL3103	Psychology	Arts	Freud

$$MDFI = \{ (\text{CS3234, CS, Computing, Turing}), (\text{MA2001, Mathematics, Science, Gauss}), (\text{MU2109, Music, Arts, Mozart}), (\text{PC2130, Physics, Science, Newton}), (\text{PL3103, Psychology, Arts, Freud}) \}$$

The set $\{SM, SN, SFY, MDFI\}$ represents the relational database.

Figure 5.1: A fictitious miniature university database and its set-theoretic representation

Definition 5.1.8. One can draw a diagram representing a relation R from a set A to a set B as follows.

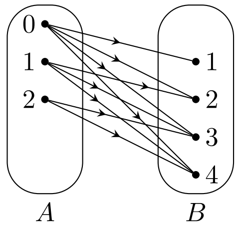
(1) On the left, draw all the elements $x \in A$ that satisfy $x \mathrel{R} y$ for some $y \in B$.

(2) On the right, draw all the elements $y \in B$ that satisfy $x \mathrel{R} y$ for some $x \in A$.

(3) For all $x \in A$ and all $y \in B$ that are drawn, draw an arrow from x to y if and only if $x \mathrel{R} y$.

Such a diagram is called an *arrow diagram* for R.

Example 5.1.9. The following is an arrow diagram for the relation R in Example 5.1.7.



(Base step) P (2) is true by the definition of ordered pairs.
(Induction step) Let k ∈ Z ⩾ 2 such that P(k) is true,
(x1,x2,...,xk) = (y1,y2,...,yk) ⇔ x1 = y1 and x2 ... and xk = yk

Definition 5.1.10. Let $n \in \mathbb{Z}_{\geqslant 2}$.

(1) An expression of the form $(x_1, x_2, \ldots, x_n)$ is called an *ordered n-tuple*.

(2) Ordered n-tuples are defined recursively by setting, for all objects $x_1, x_2, \ldots, x_{n+1}$,

$$(x_1, x_2, \ldots, x_{n+1}) = ((x_1, x_2, \ldots, x_n), x_{n+1}).$$

Exercise 5.1.11. Prove by induction on n that, for all $n \in \mathbb{Z}_{\geqslant 2}$ and all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$, ✎ 5a

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } \ldots \text{ and } x_n = y_n.$$

Example 5.1.12. (1) $(1, 2, 5) \neq (2, 1, 5)$, although $\{1, 2, 5\} = \{2, 1, 5\}$.

(2) $(3, (-2)^2, 0.5, 0) = (\sqrt{9}, 4, \frac{1}{2}, 0)$

Definition 5.1.13. Let $n \in \mathbb{Z}_{\geqslant 2}$ and $A_1, A_2, \ldots, A_n$ be sets. The *Cartesian product of $A_1, A_2, \ldots, A_n$*, denoted $A_1 \times A_2 \times \cdots \times A_n$, is defined to be

$$\{(x_1, x_2, \ldots, x_n) : x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \ldots \text{ and } x_n \in A_n\}.$$

If A is a set, then $A^n = \underbrace{A \times A \times \cdots \times A}_{n\text{-many } A\text{'s}}$.

Example 5.1.14. $\{0, 1\} \times \{0, 1\} \times \{\mathrm{a}, \mathrm{b}\} = \{(0, 0, \mathrm{a}), (0, 0, \mathrm{b}), (0, 1, \mathrm{a}), (0, 1, \mathrm{b}), (1, 0, \mathrm{a}), (1, 0, \mathrm{b}), (1, 1, \mathrm{a}), (1, 1, \mathrm{b})\}$.

Definition 5.1.15. Let $n \in \mathbb{Z}_{\geqslant 2}$ and $A_1, A_2, \ldots, A_n$ be sets. A *n-ary relation over* $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

Example 5.1.16. Following Example 5.1.6, let $\Gamma = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, 0, 1, 2, \ldots, 9\}$ and $\Phi = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, \mathrm{a}, \mathrm{b}, \ldots, \mathrm{z}\}$. As in Figure 5.1, define

$$\begin{aligned} \mathit{MDFI} = \{&(\mathrm{CS3234}, \mathrm{CS}, \mathrm{Computing}, \mathrm{Turing}), (\mathrm{MA2001}, \mathrm{Mathematics}, \mathrm{Science}, \mathrm{Gauss}), \\ &(\mathrm{MU2109}, \mathrm{Music}, \mathrm{Arts}, \mathrm{Mozart}), (\mathrm{PC2130}, \mathrm{Physics}, \mathrm{Science}, \mathrm{Newton}), \\ &(\mathrm{PL3103}, \mathrm{Psychology}, \mathrm{Arts}, \mathrm{Freud})\}. \end{aligned}$$

$\Gamma*$ $\Phi*$ $\Phi*$ $\Phi*$

Then *MDFI* is a 4-ary relation over $\Gamma^*, \Phi^*, \Phi^*, \Phi^*$.

follows the order in the brackets

5.2 Operations on relations

Note: S ∘ R means undergo
R relation -> S relation

Figure 5.2: Relation composition and inversion

Definition 5.2.1. Let R be a relation from A to B, and S be a relation from B to C. Then $S \circ R$ is the relation from A to C defined by

$$S \circ R = \{(x, z) \in A \times C : (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y \in B\}.$$

We read $S \circ R$ as "S composed with R" or "S circle R".

Note 5.2.2. We compose two binary relations together only when there is a common middle set.

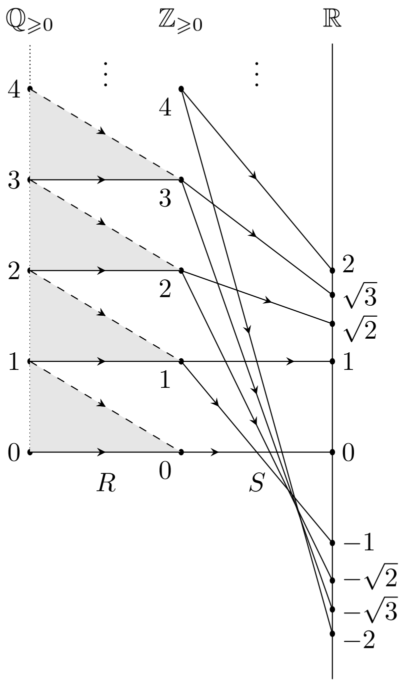
Definition 5.2.3. The *floor* of a real number x, denoted $\lfloor x \rfloor$, is the greatest integer that is less than or equal to x.

Example 5.2.4. Define a relation R from $\mathbb{Q}_{\geqslant 0}$ to $\mathbb{Z}_{\geqslant 0}$ and a relation S from $\mathbb{Z}_{\geqslant 0}$ to $\mathbb{R}$ by:

$$R = \{(x, y) \in \mathbb{Q}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} : \lfloor x \rfloor = y\}, \quad \text{and}$$
$$S = \{(y, z) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{R} : y = z^2\}.$$

- $(4.8, 2) \in S \circ R$ because $4 \in \mathbb{Z}_{\geqslant 0}$ such that $(4.8, 4) \in R$ and $(4, 2) \in S$.
- $(5/2, -\sqrt{2}) \in S \circ R$ because $2 \in \mathbb{Z}_{\geqslant 0}$ such that $(5/2, 2) \in R$ and $(2, -\sqrt{2}) \in S$.

In general, we have $S \circ R = \{(x, z) \in \mathbb{Q}_{\geqslant 0} \times \mathbb{R} : \lfloor x \rfloor = z^2\}$.



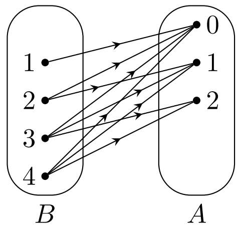
Definition 5.2.5. Let R be a relation from A to B. Then the *inverse of R* is the relation R^{-1} from B to A defined by

$$R^{-1} = \{(y,x) \in B \times A : (x,y) \in R\}.$$

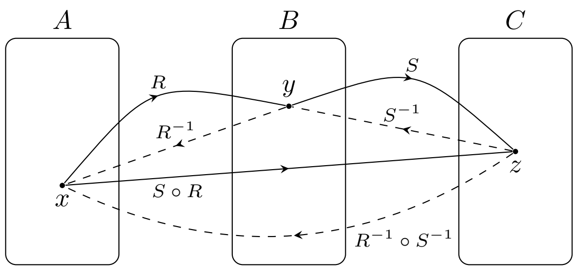
Example 5.2.6. As in Example 5.1.7, let R be the relation from A to B where

$$\begin{aligned} &A = \{0,1,2\}, \qquad B = \{1,2,3,4\}, \\ &R = \{(0,1),(0,2),(0,3),(0,4),(1,2),(1,3),(1,4),(2,3),(2,4)\}. \end{aligned}$$

Then $R^{-1} = \{(1,0),(2,0),(3,0),(4,0),(2,1),(3,1),(4,1),(3,2),(4,2)\}$.



Proposition 5.2.7. Let R be a relation from A to B, and S be a relation from B to C. Then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.



Proof. Since $S \circ R$ is a relation from A to C, we know $(S \circ R)^{-1}$ is a relation from C to A. Since S^{-1} is a relation from C to B, and R^{-1} is a relation from B to A, we know $R^{-1} \circ S^{-1}$ is a relation from C to A as well. Now for all $(z,x) \in C \times A$,

$$\begin{aligned} (z,x) \in (S \circ R)^{-1} \quad &\Leftrightarrow \quad (x,z) \in S \circ R && \text{by the definition of inverses;} \\ \Leftrightarrow \quad &(x,y) \in R \text{ and } (y,z) \in S \ \text{ for some } y \in B && \text{by the definition of composition;} \\ \Leftrightarrow \quad &(y,x) \in R^{-1} \text{ and } (z,y) \in S^{-1} \ \text{ for some } y \in B && \text{by the definition of inverses;} \\ \Leftrightarrow \quad &(z,x) \in R^{-1} \circ S^{-1} && \text{by the definition of composition.} \end{aligned}$$

So $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. ☐

Exercise 5.2.8. Let $A = \{0,1,2\}$. Define two relations R, S from A to A by: 5b

A^2 = A × A

$$R = \{(x,y) \in A^2 : x < y\} \quad \text{and} \quad S = \{(0,1),(1,2),(2,0)\}.$$

Is $R \circ S = S \circ R$? Prove that your answer is correct.

A x A = {(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)}.

No. For instance, we have (2,2) ∈ R ∘ S because (2,0) ∈ S and (0,2) ∈ R, but (2,2) ∉ S ∘ R because no y ∈ A makes (2,y) ∈ R and (y,2) ∈ S.

5.3 Graphs

A binary relation is a set of ordered pairs of elements from two sets.

Definition 5.3.1. A *(binary) relation on a set A* is a relation from A to A.

Remark 5.3.2. It follows from Definition 5.1.5 and Definition 5.3.1 that the relations on a set A are precisely the subsets of $A \times A$.

Definition 5.3.3. One can draw a diagram representing a relation R on a set A as follows.

(1) Draw all the elements of A.

(2) For all $x, y \in A$, draw an arrow from x to y if and only if $x \mathrel{R} y$.

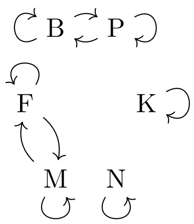
Such a diagram is called an *arrow diagram* for R.

Example 5.3.4. Let

elements / nodes
$$V = \{\mathrm{B}, \mathrm{P}, \mathrm{F}, \mathrm{M}, \mathrm{K}, \mathrm{N}\}, \quad \text{and}$$
edge
$$D = \{(\mathrm{B}, \mathrm{P}), (\mathrm{P}, \mathrm{B}), (\mathrm{F}, \mathrm{M}), (\mathrm{M}, \mathrm{F}), (\mathrm{B}, \mathrm{B}), (\mathrm{P}, \mathrm{P}), (\mathrm{F}, \mathrm{F}), (\mathrm{M}, \mathrm{M}), (\mathrm{K}, \mathrm{K}), (\mathrm{N}, \mathrm{N})\}.$$

b → p

Then the following is an arrow diagram for D as a relation on V.



of vertices

Definition 5.3.5. A *directed graph* is an ordered pair (V, D) where V is a set and D is a binary relation on V. In the case when (V, D) is a directed graph,

(1) the *vertices* or the *nodes* are the elements of V;

(2) the *edges* are the elements of D;

(3) an edge *from x to y* is the element $(x, y) \in D$;

(4) a *loop* is an edge from a vertex to itself;

(5) a *drawing* of (V, D) is an arrow diagram for D as a relation on A in the sense of Definition 5.3.3.

Example 5.3.6. Let V and D be as defined in Example 5.3.4. Then (V, D) is a directed graph. The diagram in Example 5.3.4 is a drawing of (V, D).

Definition 5.3.7. An *undirected graph* is an ordered pair (V, E) where V is a set and E is a set all of whose elements are of the form $\{x, y\}$ with $x, y \in V$. In the case when (V, E) is an undirected graph,

(1) the *vertices* or the *nodes* are the elements of V;

(2) the *edges* are the elements of E;

(3) an edge *between x and y* is the element $\{x, y\} \in E$;

(4) a *loop* is an edge between a vertex and itself.

Example 5.3.8. Following Example 5.3.4, define

$$
\begin{aligned}
V &= \{\mathrm{B}, \mathrm{P}, \mathrm{F}, \mathrm{M}, \mathrm{K}, \mathrm{N}\}, \quad \text{and} \\
E &= \{\{\mathrm{B}, \mathrm{P}\}, \{\mathrm{F}, \mathrm{M}\}, \{\mathrm{B}, \mathrm{B}\}, \{\mathrm{P}, \mathrm{P}\}, \{\mathrm{F}, \mathrm{F}\}, \{\mathrm{M}, \mathrm{M}\}, \{\mathrm{K}, \mathrm{K}\}, \{\mathrm{N}, \mathrm{N}\}\}.
\end{aligned}
$$

{B,P} = {P,B}
{B,B} = {B}

Then (V, E) is an undirected graph.

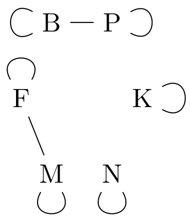
Definition 5.3.9 (drawings of undirected graphs)**.** One can draw a diagram representing an undirected graph (V, E) as follows:

(1) Draw all the elements of V.

(2) For all $x, y \in V$, draw a line between x and y if and only if $\{x, y\} \in E$.

Such a diagram is called a *drawing* of (V, E).

Example 5.3.10. Here is a drawing of the undirected graph from Example 5.3.8.



Tutorial exercises

An asterisk (*) indicates a more challenging question.

5.1. Let $M = \{\mathrm{MA1100}, \mathrm{CS1231}\}$, $G = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, and $S = \{+, -\}$. Write the following sets in roster notation:

(a) $M \times G$; (b) $M \times G \times S$; (c) $\mathcal{P}(\mathcal{P}(\varnothing)) \times S$.

5.2. Consider the database relations in Figure 5.1.

(a) Draw arrow diagrams for SN, SM and $SM \circ (SN)^{-1}$.

(b) What does $x \; SM \circ (SN)^{-1} \; y$ say about x and y?

5.3. Let $A = \{1, 2, 3, 4\}$, let $B = \{-1, 0, 1\}$, and let $C = \{2, 3, 5, 7\}$. Consider the relation R from A to B and the relation S from B to C where

$$
\begin{aligned}
R &= \{(a, b) \in A \times B : ab \text{ is even}\}, \quad \text{and} \\
S &= \{(b, c) \in B \times C : b + 2c \text{ is odd}\}.
\end{aligned}
$$

Draw arrow diagrams for R, S, R^{-1}, S^{-1}, $S \circ R$, and $R^{-1} \circ S^{-1}$.

5.4. In this exercise, we look at a commonly studied relation in number theory called *congruence modulo* 2. Intuitively, two integers are congruent modulo 2 if and only if they have the same parity, i.e., they are both odd or both even. Here we adopt a slightly different, yet equivalent, formulation of the definition.

Define a relation R on $\mathbb{Z}$ by setting, for all $a, b \in \mathbb{Z}$,

$$
a \; R \; b \quad \Leftrightarrow \quad a - b \text{ is even}.
$$

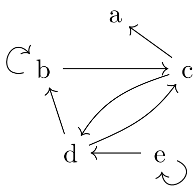
(a) Prove that $R^{-1} = R$.

(b) Prove that $R \circ R = R$.

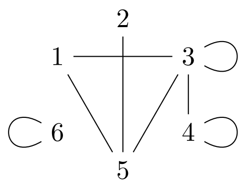
5.5. The aim of this exercise is to show the associativity of relation composition.

Let R be a relation from A to B, let S be a relation from B to C, and let T be a relation from C to D. Prove that $T \circ (S \circ R) = (T \circ S) \circ R$.

5.6. The following are drawings of a directed graph (V, D) and an undirected graph (W, E).



directed graph
must use
ordered pair!
(x,y)



undirected graphs can
just use curly brackets
{x,y}

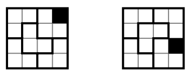
Write down V, D, W, E in roster notation.

5.7.* (Induction corner) To help you gradually get used to induction, we will have one tutorial exercise on induction in each of the chapters to come. This exercise need not be related to the topic of the chapter.

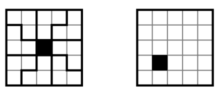
An *L-tromino* is the following L-shape formed by three squares of the checkerboard:

We are interested in whether it is possible to cover an $\ell \times \ell$ checkerboard with one square removed for various choices of the integer ℓ.

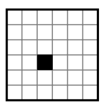
- No matter which square is removed from a 4×4 checkerboard, one can always cover the remaining squares using L-trominos. Here are two examples.



- For a 5×5 checkerboard, it depends on which square we remove. For the left one below, we can cover the remaining squares by L-trominos, but we cannot do it for the right one, as one can verify by exhaustion.



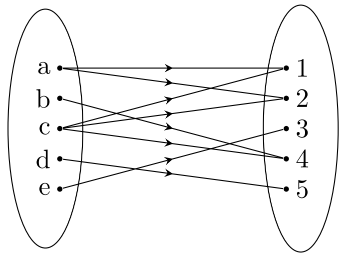
- No matter which square is removed from a 6×6 checkerboard, one can never cover the remaining squares using L-trominos because the number of remaining squares is $6 \times 6 - 1 = 35$, which is not a multiple of 3.



Prove by induction that, for all $n \in \mathbb{Z}^+$, if one square is removed from a $2^n \times 2^n$ checkerboard, then the remaining squares can be covered by L-trominos.

Extra exercises

5.8. The following is an arrow diagram for a relation R from a set A to a set B.



(a) Write down R, R^{-1}, $R^{-1} \circ R$, and $R \circ R^{-1}$ in roster notation.

(b) Draw the directed graphs $(A, R^{-1} \circ R)$ and $(B, R \circ R^{-1})$.

5.9. Let R be a relation from A to B. Prove that $(R^{-1})^{-1} = R$.