Tutorial solutions for Chapter 8

Sometimes there are other correct answers.

8.1. (a) Suppose f and g are surjective. Let $z \in C$. Use the surjectivity of g to find $y \in B$ such that $z = g(y)$. Then use the surjectivity of f to find $x \in A$ such that $y = f(x)$. Now $z = g(y) = g(f(x)) = (g \circ f)(x)$ by Proposition 7.3.1, as required. $\square$

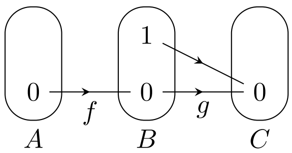
(b) Suppose f and g are injective. Let $x_1, x_2 \in A$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then $g(f(x_1)) = g(f(x_2))$ by Proposition 7.3.1. The injectivity of g then implies $f(x_1) = f(x_2)$. So the injectivity of f tells us $x_1 = x_2$, as required. $\square$

Additional comment. Let S be a relation from a set C to a set B and R be a relation from the set B to a set A. Our proof of Proposition 7.3.1 actually shows that if R and S both satisfy (F1) in the definition of functions, then $R \circ S$ also satisfies (F1). As the surjectivity of a function precisely says its inverse relation satisfies (F1), applying this to $R = f^{-1}$ and $S = g^{-1}$ gives (a) directly.

Similarly, our proof of Proposition 7.3.1 shows that if R and S both satisfy (F2) in the definition of functions, then $R \circ S$ also satisfies (F2). As the injectivity of a function precisely says its inverse relation satisfies (F2), applying this to $R = f^{-1}$ and $S = g^{-1}$ gives (b) directly.

8.2. We use the same example for all the three parts. Let $A = \{0\}$ and $B = \{0, 1\}$ and $C = \{0\}$. Define $f\colon A \to B$ and $g\colon B \to C$ by setting $f(0) = 0$ and $g(0) = 0$ and $g(1) = 0$.

Diagram.



Explanation. Then $g \circ f = \mathrm{id}_{\{0\}}$ and is thus bijective; see the solution to Exercise 8.3(1) below for a proof. The function f is not surjective because $1 \in B$ and $1 \neq f(x)$ for any $x \in A$. The function g is not injective because $0, 1 \in B$ satisfying $g(0) = 0 = g(1)$ and $0 \neq 1$. So neither f nor g is bijective.

8.3. (a) It suffices to show that the identity function id_A on A is a bijection $A \to A$. For surjectivity, given any $x \in A$, we have $\mathrm{id}_A(x) = x$. For injectivity, if $x_1, x_2 \in A$ such that $\mathrm{id}_A(x_1) = \mathrm{id}_A(x_2)$, then $x_1 = x_2$. $\square$

(b) If f is a bijection $A \to B$, then Proposition 7.4.3 tells us f^{-1} is a bijection $B \to A$. $\square$

(c) If f is a bijection $A \to B$ and g is a bijection $B \to C$, then $g \circ f$ is a bijection $A \to C$ by Exercise 8.1. $\square$

8.4. Pick $(u,v),(s,t)\in\mathbb{R}^2$. According to the information from Tutorial Exercise 6.3(b),

$$[(u,v)] = \{(x,y)\in\mathbb{R}^2 : y = 3x + (v-3u)\}, \text{ and}$$
$$[(s,t)] = \{(x,y)\in\mathbb{R}^2 : y = 3x + (t-3s)\}.$$

To show that $[(u,v)]$ and $[(s,t)]$ have the same cardinality, it suffices to find a bijection $[(u,v)]\to[(s,t)]$. There are many choices. Here we use the function $f\colon [(u,v)]\to[(s,t)]$ defined by setting, for all $(x,y)\in[(u,v)]$,

$$f(x,y) = \big(x, y + (t-3s) - (v-3u)\big).$$

(Well-defined) Let us first check that the definition of f given indeed assigns every element of $[(u,v)]$ exactly one element of $[(s,t)]$, as required by the definition of functions. Take any $(x_0,y_0)\in[(u,v)]$. Clearly, there is exactly one object that is equal to $\big(x_0, y_0 + (t-3s) - (v-3u)\big)$. So we only need to check that this object is indeed an element of the codomain $[(s,t)]$.

As $(x_0,y_0)\in[(u,v)]$, we know $y_0 = 3x_0 + (v-3u)$. So

$$y_0 + (t-3s) - (v-3u) = 3x_0 + (v-3u) + (t-3s) - (v-3u) = 3x_0 + (t-3s).$$

This shows $\big(x_0, y_0 + (t-3s) - (v-3u)\big)\in[(s,t)]$, as required.

(Surjective) Let $(z_0,w_0)\in[(s,t)]$. Then $w_0 = 3z_0 + (t-3s)$. So

$$w_0 + (v-3u) - (t-3s) = 3z_0 + (t-3s) + (v-3u) - (t-3s) = 3z_0 + (v-3u).$$

This shows $\big(z_0, w_0 + (v-3u) - (t-3s)\big)\in[(u,v)]$. Moreover,

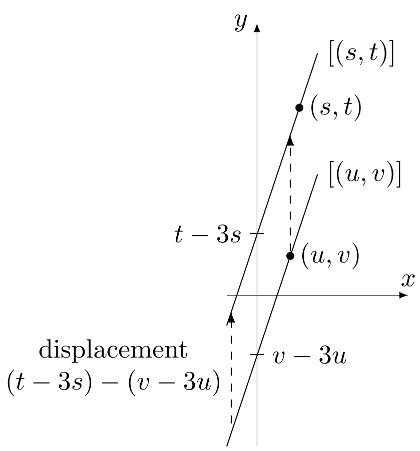
$$f\big(z_0, w_0 + (v-3u) - (t-3s)\big) = \big(z_0, w_0 + (v-3u) - (t-3s) + (t-3s) - (v-3u)\big) = (z_0,w_0).$$

(Injective) Let $(x_1,y_1),(x_2,y_2)\in[(u,v)]$ such that $f(x_1,y_1) = f(x_2,y_2)$. According to the definition of f, this means

$$\big(x_1, y_1 + (t-3s) - (v-3u)\big) = \big(x_2, y_2 + (t-3s) - (v-3u)\big).$$

So $x_1 = x_2$ and $y_1 + (t-3s) - (v-3u) = y_2 + (t-3s) - (v-3u)$. The latter implies $y_1 = y_2$. Thus together we have $(x_1,y_1) = (x_2,y_2)$. $\square$

Additional comment 1. It is probably easier to see how one may come up with our function f geometrically. When drawn on the plane, it moves the straight line $[(u,v)]$ vertically up/down to the straight line $[(s,t)]$. The exact displacement involved can be calculated from the points where these lines intersect the y-axis, for example.



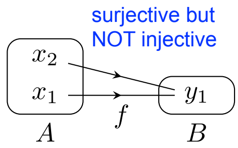
Additional comment 2 (repeated from the solutions to Exercise 8.2.5). Sometimes when one defines a function, it is not immediately clear that the definition given indeed defines a function, i.e., that it defines an object satisfying the definition of functions. In such cases, a proof should be provided. In many other cases, it is clear that the definition given really defines a function, and so no additional explanation is needed.

8.5. By symmetry, it suffices to show only one direction. Suppose A is finite. Use the definition of finiteness to find $n \in \mathbb{N}$ such that A has the same cardinality as $\{1, 2, \ldots, n\}$. Then the symmetry and the transitivity of same-cardinality from Proposition 8.2.4 tell us B has the same cardinality of $\{1, 2, \ldots, n\}$. So B is finite. $\square$

8.6. (a) In the induction step, the author claims that "no $i \in \{1, 2, \ldots, k\}$ makes $f(x_i) = y_\ell$". The author did not explain clearly why this is true.

Additional explanation. In fact, this claim may not be true. For example, consider the case when

$$\begin{aligned} k &= 1, & n &= k + 1 = 2, & m &= 1, \\ & & A &= \{x_1, x_2\}, & B &= \{y_1\}, \\ f(x_1) &= y_1, & f(x_2) &= y_1. \end{aligned}$$



As $f(x_{k+1}) = f(x_2) = y_1$, we have $\ell = 1$. If $i = 1$, then $i = 1 \in \{1\} = \{1, 2, \ldots, k\}$ and $f(x_i) = f(x_1) = y_1 = y_\ell$. So some $i \in \{1, 2, \ldots, k\}$ makes $f(x_i) = y_\ell$, contrary to what the author claims.

If we carry on following the author's argument with this f, then we would need to construct a function $\hat{f}\colon \{x_1\} \to \{\}$, which is not possible because $\{\}$ has no element, but such $\hat{f}$ must satisfy $\hat{f}(x_1) \in \{\}$.

(b) The attempt takes care of the case when no $i \in \{1, 2, \ldots, k\}$ makes $f(x_i) = y_\ell$. We can deal with the remaining case separately as follows.

Assume some $i \in \{1, 2, \ldots, k\}$ makes $f(x_i) = y_\ell$. Define $\hat{f}\colon \{x_1, x_2, \ldots, x_k\} \to \{y_1, y_2, \ldots, y_m\}$ by setting $\hat{f}(x_i) = f(x_i)$ for each $i \in \{1, 2, \ldots, k\}$. Then $\hat{f}$ is surjective because, for each y_h, the surjectivity of f gives some x_i such that $y_h = f(x_i)$, and we can require this $i \neq k+1$ by our assumption; so $y_h = f(x_i) = \hat{f}(x_i)$. As the x's are all different and the y's are all different, the induction hypothesis tells us $k \geqslant m$. So $k + 1 \geqslant m + 1 \geqslant m$.

Extra exercises

8.7. (a) Suppose $g \circ f$ is surjective. Take any $z \in C$. Use the surjectivity of $g \circ f$ to find $x \in A$ such that $(g \circ f)(x) = z$. Define $y = f(x)$. Then

$$\begin{aligned} g(y) &= g(f(x)) && \text{as } y = f(x); \\ &= (g \circ f)(x) && \text{by Proposition 7.3.1;} \\ &= z. \end{aligned}$$

$\square$

(b) Suppose $g \circ f$ is injective. Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} & g(f(x_1)) = g(f(x_2)). \\ \therefore\quad & (g \circ f)(x_1) = (g \circ f)(x_2) && \text{by Proposition 7.3.1.} \\ \therefore\quad & x_1 = x_2 && \text{as } g \circ f \text{ is injective.} \end{aligned}$$

$\square$

8.8. Suppose A has the same cardinality as both $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$. Use the definition of same cardinality to find bijections $f\colon A \to \{1, 2, \ldots, m\}$ and $g\colon A \to \{1, 2, \ldots, n\}$. Then f^{-1} is a bijection $\{1, 2, \ldots, m\} \to A$ by Proposition 7.4.3. In view of Exercise 8.1, these imply that $g \circ f^{-1}$ is a bijection $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$. Hence $m = n$ by Theorem 8.1.3. $\square$

8.9. (a) Let f be a surjection $A \to B$ that is not an injection. Use the failure of injectivity to find $i, j \in \{1, 2, \ldots, n\}$ such that $x_i \neq x_j$ but $f(x_i) = f(x_j)$. Now

$$x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$$

are $n-1$ objects with no repetition. Denote these by $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}$ respectively. Let $\hat{A} = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}\}$, so that $\hat{A} = A \setminus \{x_j\}$. Define a function $\hat{f} \colon \hat{A} \to B$ by setting $\hat{f}(x) = f(x)$ for all $x \in \hat{A}$.

We claim that $\hat{f}$ is surjective. To prove this, consider $y \in B$. Use the surjectivity of f to find $k \in \{1, 2, \ldots, n\}$ such that $y = f(x_k)$.

Case 1: Assume $k = j$. Then $x_i \in A \setminus \{x_j\} = \hat{A}$ because $x_i \neq x_j$. Also

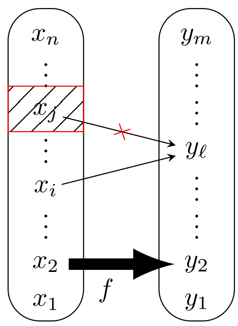
$$\begin{aligned} \hat{f}(x_i) &= f(x_j) && \text{by the definition of } \hat{f}; \\ &= f(x_j) && \text{by the choice of } i, j; \\ &= f(x_k) && \text{as } j = k \text{ by assumption;} \\ &= y && \text{by the choice of } k. \end{aligned}$$

Case 2: Assume $k \neq j$. Then $x_k \in A \setminus \{x_j\} = \hat{A}$ as the x's are different. So the definition of $\hat{f}$ and the choice of k tell us $\hat{f}(x_k) = f(x_k) = y$.

So $y = \hat{f}(x)$ for some $x \in \hat{A}$ in all cases. This completes the proof of the claim.

Now $\hat{f}$ is a surjection $\hat{A} \to B$, where $A = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}\}$ and $B = \{y_1, y_2, \ldots, y_m\}$. Since the $\hat{x}$'s are different and the y's are different, the Dual Pigeonhole Principle then implies $m \leqslant n - 1 < n$, as required. □

Diagram.



(b) Let f be an injection $A \to B$ that is not a surjection. Use the failure of surjectivity to find $\ell \in \{1, 2, \ldots, m\}$ such that $y_\ell \neq f(x)$ for any $x \in A$. Now

$$y_1, y_2, \ldots, y_{\ell-1}, y_{\ell+1}, \ldots, y_m$$

are $m-1$ objects with no repetition. Denote these by $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{m-1}$ respectively. Let $\hat{B} = \{\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{m-1}\}$, so that $\hat{B} = B \setminus \{y_\ell\}$. Define a function $\hat{f} \colon A \to \hat{B}$ by setting $\hat{f}(x) = f(x)$ for all $x \in A$. This function is well defined: for any $x \in A$, we know $f(x) \neq y_\ell$ by the choice of ℓ, and thus $f(x) \in B \setminus \{y_\ell\} = \hat{B}$. Note that $\hat{f}$ is injective, because if $i, j \in \{1, 2, \ldots, n\}$ such that $\hat{f}(x_i) = \hat{f}(x_j)$, then the definition of $\hat{f}$ tells us $f(x_i) = f(x_j)$, and so $x_i = x_j$ by the injectivity of f. Now $\hat{f}$ is an injection $A \to \hat{B}$, where $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{m-1}\}$. Since the x's are different and the y's are different, the Pigeonhole Principle then implies $n \leqslant m - 1 < m$, as required. □

Diagram.