Tutorial solutions for Chapter 9

Sometimes there are other correct answers.

9.1. Define $A_0 = A \setminus B$ and $B_0 = B$. Then $A_0 \cap B_0 = \varnothing$ and $A_0 \cup B_0 = A \cup B$ by Extra Exercise 4.9. So it suffices to show that $A_0 \cup B_0$ is finite. Observe that, as $A_0 = \{x \in A : x \notin B\} \subseteq A$ and A is finite, we know A_0 is also finite by Proposition 9.2.6(1).

Use the definition of finite sets to find bijections

$$f\colon \{1, 2, \ldots, m\} \to A_0 \quad \text{and} \quad g\colon \{1, 2, \ldots, n\} \to B_0$$

where $m, n \in \mathbb{N}$. Define the function $h\colon \{1, 2, \ldots, m+n\} \to A_0 \cup B_0$ by setting, for each $x \in \{1, 2, \ldots, m+n\}$,

$$h(x) = \begin{cases} f(x), & \text{if } x \leqslant m; \\ g(x-m), & \text{if } x > m. \end{cases}$$

If $x \in \{1, 2, \ldots, m+n\}$ such that $x \leqslant m$, then x is in the domain of f, and so we can indeed apply f to x. If $x \in \{1, 2, \ldots, m+n\}$ such that $x > m$, then $0 = m - m < x - m \leqslant (m+n) - m = n$, and so we can indeed apply g to $x - m$. For every $x \in \{1, 2, \ldots, m+n\}$, if $x \leqslant m$, then $h(x) = f(x) \in A_0$; conversely, if $x > m$, then $h(x) = g(x-m) \in B_0$, and so $h(x) \notin A_0$ as $A_0 \cap B_0 = \varnothing$. Hence

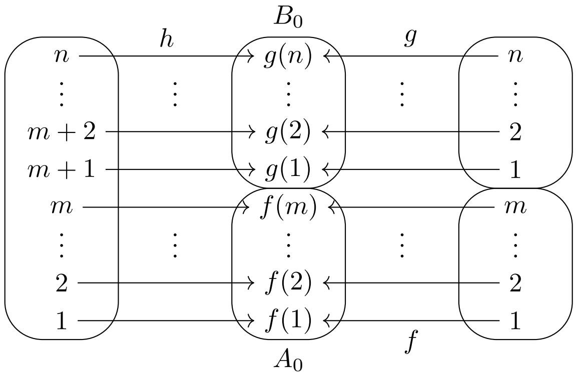
$$\forall x \in \{1, 2, \ldots, m+n\} \qquad \big(x \leqslant m \quad \Leftrightarrow \quad h(x) \in A_0\big). \tag{$*$}$$

Now we verify that h is bijective; this will finish the proof.

(Surjectivity) Let $y \in A_0 \cup B_0$. Then $y \in A_0$ or $y \in B_0$ by the definition of $A_0 \cup B_0$. If $y \in A_0$, then the surjectivity of f gives $v \in \{1, 2, \ldots, m\}$ such that $y = f(v) = h(v)$. If $y \in B_0$, then the surjectivity of g gives $w \in \{1, 2, \ldots, n\}$ such that $y = g(w) = g\big((w+m) - m\big) = h(w+m)$ as $w + m > 0 + m = m$. So we have $x \in \{1, 2, \ldots, m+n\}$ which makes $y = f(x)$ in all cases.

(Injectivity) Let $x_1, x_2 \in \{1, 2, \ldots, m+n\}$ such that $h(x_1) = h(x_2)$. In view of ($*$), this implies either $x_1 \leqslant m$ and $x_2 \leqslant m$, or $x_1 > m$ and $x_2 > m$. If $x_1 \leqslant m$ and $x_2 \leqslant m$, then $f(x_1) = h(x_1) = h(x_2) = f(x_2)$, and thus $x_1 = x_2$ by the injectivity of f. If $x_1 > m$ and $x_2 > m$, then $g(x_1 - m) = h(x_1) = h(x_2) = g(x_2 - m)$, and thus $x_1 - m = x_2 - m$ by the injectivity of g, from which we can deduce $x_1 = x_2$. So $x_1 = x_2$ in all cases. □

Additional comment. The following diagram gives some idea of why we chose to define h in this way:



One reason why small sets should be closed under (binary) union. We may view two sets as equivalent if and only if their (symmetric) difference is small. In particular, if S is a set and A, B are small sets, then S should be equivalent to $S \setminus A$, and $S \setminus A$ should be equivalent to $(S \setminus A) \setminus B = S \setminus (A \cup B)$. Any notion of equivalence should be (reflexive, symmetric, and) transitive. So if S is a set and A, B are small sets, then S should be equivalent to $S \setminus (A \cup B)$ by transitivity; this indicates that $A \cup B$ should also be small.

9.2. Define $A_0 = A \setminus B$ and $B_0 = B$. Then $A_0 \cap B_0 = \varnothing$ and $A_0 \cup B_0 = A \cup B$ by Extra Exercise 4.9. So it suffices to show that $A_0 \cup B_0$ is countable. Observe that, as $A_0 = \{x \in A : x \notin B\} \subseteq A$ and A is countable, we know A_0 is also countable by Proposition 9.2.6(2).

If A_0 and B_0 are both finite, then Exercise 9.1 tells us $A_0 \cup B_0$ is finite and thus countable. So suppose at least one of A_0 and B_0 is infinite.

Case 1: suppose exactly one of A_0 and B_0 is infinite. Say A_0 is finite and B_0 is infinite; the case when A_0 is infinite and B_0 is finite can be handled similarly. As B_0 is infinite and countable, it has the same cardinality as $\mathbb{N}$. Use this fact and the definition of finite sets to find bijections

$$f\colon \{1, 2, \ldots, m\} \to A_0 \quad \text{and} \quad g\colon \mathbb{N} \to B_0$$

where $m \in \mathbb{N}$. Define the function $h_1\colon \mathbb{N} \to A_0 \cup B_0$ by setting, for each $x \in \mathbb{N}$,

$$h_1(x) = \begin{cases} f(x+1), & \text{if } x \leqslant m-1; \\ g(x-m), & \text{if } x \geqslant m. \end{cases}$$

If $x \in \mathbb{N}$ such that $x \leqslant m-1$, then $1 = 0+1 \leqslant x+1 \leqslant (m-1)+1 = m$, and so we can indeed apply f to x. If $x \in \mathbb{N}$ such that $x \geqslant m$, then $x - m \geqslant m - m = 0$, and so we can indeed apply g to $x - m$. For every $x \in \mathbb{N}$, if $x \leqslant m-1$, then $h_1(x) = f(x+1) \in A_0$; conversely, if $x \geqslant m$, then $h_1(x) = g(x-m) \in B_0$, and so $h_1(x) \notin A_0$ as $A_0 \cap B_0 = \varnothing$. Hence

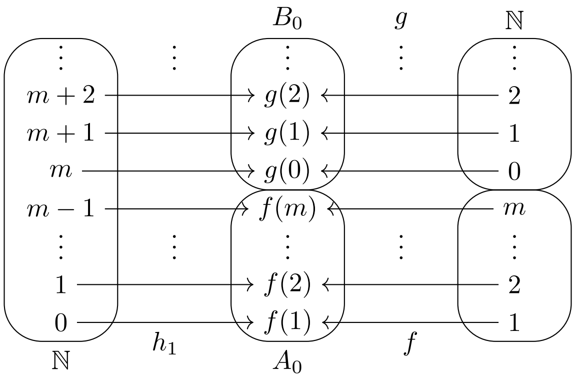
$$\forall x \in \mathbb{N} \quad \big(x \leqslant m \quad \Leftrightarrow \quad h_1(x) \in A_0\big). \tag{$\dagger$}$$

Now we verify that h_1 is bijective; this will give the countability of $A_0 \cup B_0$.

(Surjectivity) Let $y \in A_0 \cup B_0$. Then $y \in A_0$ or $y \in B_0$ by the definition of $A_0 \cup B_0$. If $y \in A_0$, then the surjectivity of f gives $v \in \{1, 2, \ldots, m\}$ such that $y = f(v) = h_1(v-1)$. If $y \in B_0$, then the surjectivity of g gives $w \in \mathbb{N}$ such that $y = g(w) = g\big((w+m) - m\big) = h_1(w+m)$ as $w + m \geqslant 0 + m = m$. So we have $x \in \mathbb{N}$ which makes $y = h_1(x)$ in all cases.

(Injectivity) Let $x_1, x_2 \in \mathbb{N}$ such that $h_1(x_1) = h_1(x_2)$. In view of (†), this implies either $x_1 \leqslant m-1$ and $x_2 \leqslant m-1$, or $x_1 \geqslant m$ and $x_2 \geqslant m$. If $x_1 \leqslant m-1$ and $x_2 \leqslant m-1$, then $f(x_1+1) = h_1(x_1) = h_1(x_2) = f(x_2+1)$, and thus $x_1+1 = x_2+1$ by the injectivity of f, from which we can deduce $x_1 = x_2$. If $x_1 \geqslant m$ and $x_2 \geqslant m$, then $g(x_1 - m) = h_1(x_1) = h_1(x_2) = g(x_2 - m)$, and thus $x_1 - m = x_2 - m$ by the injectivity of g, from which we can deduce $x_1 = x_2$. So $x_1 = x_2$ in all cases.

Figure.



Case 2: suppose both A_0 and B_0 are infinite. As A_0 and B_0 are both infinite and countable, they have the same cardinality as $\mathbb{N}$. Use this fact to find bijections

$$f : \mathbb{N} \to A_0 \quad \text{and} \quad g : \mathbb{N} \to B_0.$$

Define the function $h_2 : \mathbb{Z} \to A_0 \cup B_0$ by setting, for each $x \in \mathbb{Z}$,

$$h_2(x) = \begin{cases} f(x), & \text{if } x \geqslant 0; \\ g(-x-1), & \text{if } x < 0. \end{cases}$$

If $x \in \mathbb{Z}$ such that $x \geqslant 0$, then x is in the domain of f, and so we can indeed apply f to x. If $x \in \mathbb{Z}$ such that $x < 0$, then $-x-1 > -0-1 = -1$, and thus $x \geqslant 0$; so we can indeed apply g to $-x-1$.

For every $x \in \mathbb{Z}$, if $x \geqslant 0$, then $h_2(x) = f(x) \in A_0$; conversely, if $x < 0$, then $h_2(x) = g(-x-1) \in B_0$, and so $h_2(x) \notin A_0$ as $A_0 \cap B_0 = \varnothing$. Hence

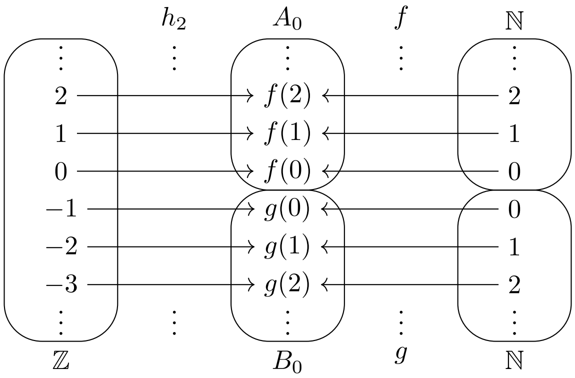
$$\forall x \in \mathbb{N} \qquad \big(x \geqslant 0 \quad \Leftrightarrow \quad h_2(x) \in A_0\big). \tag{‡}$$

We claim that h_2 is a bijection. This will tell us $\mathbb{Z}$ has the same cardinality as $A_0 \cup B_0$, and thus $A_0 \cup B_0$ is countable by Proposition 9.1.4 and Lemma 9.2.1.

(Surjectivity) Let $y \in A_0 \cup B_0$. Then $y \in A_0$ or $y \in B_0$ by the definition of $A_0 \cup B_0$. If $y \in A_0$, then the surjectivity of f gives $v \in \mathbb{N}$ such that $y = f(v) = h_2(v)$. If $y \in B_0$, then the surjectivity of g gives $w \in \mathbb{N}$ such that $y = g(w) = g\big(-(-w-1)-1\big) = h_2(-w-1)$ as $-w-1 \leqslant -0-1 = -1$. So we have $x \in \mathbb{Z}$ which makes $y = h_2(x)$ in all cases.

(Injectivity) Let $x_1, x_2 \in \mathbb{Z}$ such that $h_2(x_1) = h_2(x_2)$. In view of (‡), this implies either $x_1 \geqslant 0$ and $x_2 \geqslant 0$, or $x_1 < 0$ and $x_2 < 0$. If $x_1 \geqslant 0$ and $x_2 \geqslant 0$, then $f(x_1) = h_2(x_1) = h_2(x_2) = f(x_2)$, and thus $x_1 = x_2$ by the injectivity of f. If $x_1 < 0$ and $x_2 < 0$, then $g(-x_1-1) = h_2(x_1) = h_2(x_2) = g(-x_2-1)$, and thus $-x_1-1 = -x_2-1$ by the injectivity of g, from which we can deduce $x_1 = x_2$. So $x_1 = x_2$ in all cases. □

Figure.



Alternative solution when A_0 and B_0 are both infinite. As A_0 and B_0 are both infinite and countable, they have the same cardinality as $\mathbb{N}$. Use this fact to find bijections

$$f\colon \mathbb{N} \to A_0 \quad \text{and} \quad g\colon \mathbb{N} \to B_0.$$

Define the function $h_3\colon \mathbb{N} \to A_0 \cup B_0$ by setting, for each $x \in \mathbb{N}$,

$$h(x) = \begin{cases} f(\frac{x}{2}), & \text{if } x \text{ is even;} \\ g(\frac{x-1}{2}), & \text{if } x \text{ is odd.} \end{cases}$$

If $x \in \mathbb{N}$ such that x is even, say $x = 2u$ where $u \in \mathbb{Z}$, then $0 = \frac{0}{2} \leqslant \frac{x}{2} = \frac{2u}{2} = u$; this implies $\frac{x}{2} = u \in \mathbb{N}$ and thus we can indeed apply f to $\frac{x}{2}$. If $x \in \mathbb{N}$ such that x is odd, say $x = 2u+1$ where $u \in \mathbb{Z}$, then $-\frac{1}{2} = \frac{0-1}{2} \leqslant \frac{x-1}{2} = \frac{(2u+1)+1}{2} = u+1$, and so $u+1 \geqslant 0$ as $u+1 \in \mathbb{Z}$; this implies $\frac{x-1}{2} = u+1 \in \mathbb{N}$ and thus we can indeed apply g to $\frac{x-1}{2}$.

For every $x \in \mathbb{Z}$, if x is even, then $h_3(x) = f(\frac{x}{2}) \in A_0$; conversely, if x is odd, then $h_3(x) = g(\frac{x-1}{2}) \in B_0$, and so $h_3(x) \notin A_0$ as $A_0 \cap B_0 = \varnothing$. Hence

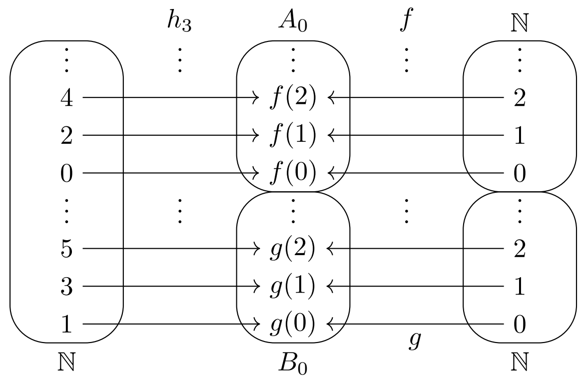
$$\forall x \in \mathbb{N} \qquad \bigl(x \text{ is even} \quad \Leftrightarrow \quad h_3(x) \in A_0\bigr). \tag{§}$$

Now we verify that h_3 is bijective; this will give the countability of $A_0 \cup B_0$.

(Surjectivity) Let $y \in A_0 \cup B_0$. Then $y \in A_0$ or $y \in B_0$ by the definition of $A_0 \cup B_0$. If $y \in A_0$, then the surjectivity of f gives $v \in \mathbb{N}$ such that $y = f(v) = f(\frac{2v}{2}) = h_2(2v)$ as $2v$ is even. If $y \in B_0$, then the surjectivity of g gives $w \in \mathbb{N}$ such that $y = g(w) = g(\frac{(2w+1)-1}{2}) = h_1(2w+1)$ as $2w+1$ is odd. So we have $x \in \mathbb{N}$ which makes $y = h_3(x)$ in all cases.

(Injectivity) Let $x_1, x_2 \in \mathbb{N}$ such that $h_3(x_1) = h_3(x_2)$. In view of (§), this implies either x_1, x_2 are both even, or x_1, x_2 are both odd. If x_1, x_2 are both even, then $f(\frac{x_1}{2}) = h_3(x_1) = h_3(x_2) = f(\frac{x_2}{2})$, and thus $\frac{x_1}{2} = \frac{x_2}{2}$ by the injectivity of f, from which we can deduce $x_1 = x_2$. If x_1, x_2 are both odd, then $g(\frac{x_1-1}{2}) = h_3(x_1) = h_3(x_2) = g(\frac{x_2-1}{2})$, and thus $\frac{x_1-1}{2} = \frac{x_2-1}{2}$ by the injectivity of g, from which we can deduce $x_1 = x_2$. So $x_1 = x_2$ in all cases. $\square$

Figure.



9.3. **(Base step)** $\bigcup_{i=0}^{0} A_i = A_0$, which is countable. So the proposition is true for $n = 0$.

(Induction step) Let $k \in \mathbb{N}$ such that the proposition is true for $n = k$. We will prove the proposition for $n = k + 1$.

$$\begin{aligned} \bigcup_{i=0}^{k+1} A_i &= A_0 \cup A_1 \cup \cdots \cup A_k \cup A_{k+1} && \text{by the definition of } \bigcup_{i=0}^{k+1} A_i; \\ &= \Big(\bigcup_{i=0}^{k} A_i\Big) \cup A_{k+1} && \text{by the definition of } \bigcup_{i=0}^{k} A_i. \end{aligned}$$

Here $\bigcup_{i=0}^{k} A_i$ is countable by the induction hypothesis, and A_{k+1} is countable by assumption. So Exercise 9.2 tells us $\big(\bigcup_{i=0}^{k} A_i\big) \cup A_{k+1} = \bigcup_{i=0}^{k+1} A_i$ is also countable.

This completes the induction. □

9.4. (a) Suppose $A_0, A_1, A_2, \ldots$ are countable sets. Exercise 9.3 only tells us the following sets are countable:

$$\bigcup_{i=0}^{0} A_i, \quad \bigcup_{i=0}^{1} A_i, \quad \bigcup_{i=0}^{2} A_i, \quad \bigcup_{i=0}^{3} A_i, \quad \bigcup_{i=0}^{4} A_i, \quad \bigcup_{i=0}^{5} A_i, \quad \bigcup_{i=0}^{6} A_i, \quad \bigcup_{i=0}^{7} A_i, \quad \ldots$$

There is no specific reason why any of these should be equal to $\bigcup_{i=0}^{\infty} A_i$. Even given the fact that the union of all the sets displayed above is equal to $\bigcup_{i=0}^{\infty} A_i$, it is still not clear why the countability of these individual sets imply the countability of the union; in fact, this is essentially what one is trying to prove here.

Extra argument. In view of Exercise 9.1, we may perform an induction like what we did in Exercise 9.3 to prove that if $A_0, A_1, A_2, \ldots$ are finite sets, then $\bigcup_{i=0}^{n} A_i$ is finite for every $n \in \mathbb{N}$. As deductions are supposed to depend on the forms of the propositions involved alone according to Terminology 3.1.10, if what is claimed were indeed correct, then the same argument would allow one to deduce that $\bigcup_{i=0}^{\infty} A_i$ is finite whenever $A_0, A_1, A_2, \ldots$ are finite. However, this conclusion is not true: let if we let $A_i = \{i\}$ for each $i \in \mathbb{N}$, then each A_i is finite, but $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$, which is infinite.

Moral. A predicate that is satisfied by arbitrarily large finite sets may not be satisfied by infinite sets. There is a gap between "arbitrarily large finite" and "infinite".

(b) For each $i \in \mathbb{N}$, use the countability of A_i to find a bijection $f_i \colon \mathbb{N} \to A_i$ or a bijection $f_i \colon \{1, 2, \ldots, \ell\} \to A_i$ where $\ell \in \mathbb{N}$. Define $g \colon \bigcup_{i=0}^{\infty} A_i \to \mathbb{N} \times \mathbb{N}$ by setting, for each $x \in \bigcup_{i=0}^{\infty} A_i$,

$g(x) = (i_x, j_x)$ where i_x is the smallest element of $\{i \in \mathbb{N} : x \in A_i\}$ and $j_x \in \mathbb{N}$ such that $f_{i_x}(j_x) = x$.

(Well-defined) Let us verify that the definition of g above indeed assigns to every element of $\bigcup_{i=0}^{\infty} A_i$ exactly one element of $\mathbb{N} \times \mathbb{N}$. Take $x \in \bigcup_{i=0}^{\infty} A_i$. Then the definition of $\bigcup_{i=0}^{\infty} A_i$ tells us $x \in A_i$ for some $i \in \mathbb{N}$. So $\{i \in \mathbb{N} : x \in A_i\}$ is nonempty, and thus must have a smallest element, say i_x, by the Well-Ordering Principle. Such an i_x must be unique because, if i is another smallest element of this set, then $i_x \leqslant i$ and $i \leqslant i_x$ by the smallest-ness of i_x and i respectively, and thus $i_x = i$. As $x \in A_{i_x}$ and f_{i_x} is a surjection $\mathbb{N} \to A_{i_x}$, we get $j_x \in \mathbb{N}$ such that $f_{i_x}(j_x) = x$. This j_x is also unique because f_{i_x} is injective.

(Injectivity) Let $x, y \in \bigcup_{i=0}^{\infty} A_i$ such that $g(x) = g(y)$. Say $g(x) = (i, j) = g(y)$. Then the definition of g tells us that $x = f_i(j) = y$.

Now g is an injection $\bigcup_{i=0}^{\infty} A_i \to \mathbb{N} \times \mathbb{N}$. Recall from Theorem 9.1.5 that $\mathbb{N} \times \mathbb{N}$ is countable. So $\bigcup_{i=0}^{\infty} A_i$ must also be countable by Corollary 9.2.7(2). □

Extra explanation. Here is a procedure that gives the function g defined above.

1. Input $x \in \bigcup_{i=0}^{\infty} A_i$.
2. Search for the smallest $i \in \mathbb{N}$ such that $x \in A_i$.
3. Search for the smallest $j \in \mathbb{N}$ such that $f_i(j) = x$.
4. Output (i, j).

Moral. Even wrong proofs may have true conclusions.

9.5. (a) **True**, because this is the contrapositive of Proposition 9.2.6(2).

(b) **True**, as shown below.

We prove this by contraposition. Let A, B be sets such that $A \setminus B$ is countable. We will show that either A is countable or B is uncountable. If B is uncountable, then there is nothing to prove. So suppose B is countable. As $A \setminus B$ and B are both countable, we know $(A \setminus B) \cup B$ is also countable by Exercise 9.2. Note that Extra Exercise 4.9 tells us $(A \setminus B) \cup B = A \cup B$. Thus $A \cup B$ is countable. As $A \subseteq A \cup B$ by Tutorial Exercise 4.6, Proposition 9.2.6(2) implies A is countable too. □

Intuition. As mentioned in Exercise 9.2, one may view countable sets as small sets, and uncountable set as large sets.

(a) This proposition is intuitive because, if a set B has a large subset A, then B should itself be large.

(b) This proposition is intuitive because, if we take away a small set B from a large set A, then the result $A \setminus B$ should remain large.

9.6. Let A be an infinite set. Apply Proposition 9.2.4 to find a countable infinite subset $A_0 \subseteq A$. Then $\mathcal{P}(A_0)$ is uncountable by Corollary 9.3.2. We know $\mathcal{P}(A_0) \subseteq \mathcal{P}(A)$ because, if $S \in \mathcal{P}(A_0)$, then the definition of $\mathcal{P}(A_0)$ tells us $S \subseteq A_0 \subseteq A$, and thus $S \in \mathcal{P}(A)$ by the transitivity of $\subseteq$ from Example 6.1.4. Therefore, Proposition 9.2.6(2) implies that $\mathcal{P}(A)$ is also uncountable. □

Intuition. On the one hand, as verified above, the power set of a larger set is larger. On the other hand, we know that the countable infinity is the smallest infinity. Since the power set of a countable infinite set is already large, the power set of any infinite set should also be large.

Extra exercises

9.7. Define $A^* = A \cup \mathbb{N}$, so that $A \subseteq A^*$ by Tutorial Exercise 4.6. From Example 9.1.3(1), we know $\mathbb{N}$ is countable. As A is countable, this implies A^* is countable too by Exercise 9.2.

Consider the function $f_A\colon \mathbb{N} \to A^*$ satisfying $f_A(x) = x$ for all $x \in A$. This function is injective because if $x_1, x_2 \in \mathbb{N}$ such that $f_A(x_1) = f_A(x_2)$, then $x_1 = x_2$ by the definition of f_A. Now $\mathbb{N}$ is infinite by Exercise 8.2.7, and f_A is an injection $\mathbb{N} \to A^*$. So Corollary 9.2.7(1) tells us A^* must also be infinite. $\square$

Additional comment. We could not have used this exercise to avoid the case-splitting for Exercise 9.2 because Exercise 9.2 is used in the proof here.

9.8. ($\Rightarrow$) Suppose A is countable. Apply Exercise 9.7 to find a countable infinite set A^* such that $A \subseteq A^*$. According to the definition of countability, this set A^* has the same cardinality as $\mathbb{N}$. Use the definition of same-cardinality to find a bijection $g\colon A^* \to \mathbb{N}$. Define $f\colon A \to \mathbb{N}$ by setting $f(x) = g(x)$ for all $x \in A$. Then f is an injection because if $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$, then

$$\begin{array}{lcl} & g(x_1) = g(x_2) & \text{by the definition of } f; \\ \therefore & x_1 = x_2 & \text{as } g \text{ is injective.} \end{array}$$

($\Leftarrow$) Let f be an injection $A \to \mathbb{N}$. From Example 9.1.3(1), we know $\mathbb{N}$ is countable. So A cannot be uncountable in view of Corollary 9.2.7(2). This means A must be countable. $\square$

9.9. Apply Exercise 9.7 to find countable infinite sets A^*, B^* such that $A \subseteq A^*$ and $B \subseteq B^*$. According to the definition of countability, both A^* and B^* have the same cardinality as $\mathbb{N}$. Use the definition of same-cardinality to find bijections $g_A\colon A^* \to \mathbb{N}$ and $g_B\colon B^* \to \mathbb{N}$.

Consider the function $f\colon A \times B \to \mathbb{N} \times \mathbb{N}$ satisfying $f(x, y) = \big(g_A(x), g_B(y)\big)$ for all $(x, y) \in A \times B$. This function is injective because if $(x_1, y_1), (x_2, y_2) \in A \times B$ such that $f(x_1, y_1) = f(x_2, y_2)$, then

$$\begin{array}{lcl} & \big(g_A(x_1), g_B(y_1)\big) = \big(g_A(x_2), g_B(y_2)\big) & \text{by the definition of } f; \\ \therefore & g_A(x_1) = g_A(x_2) \quad \text{and} \quad g_B(y_1) = g_B(y_2) & \text{by the definition of ordered pairs;} \\ \therefore & x_1 = x_2 \quad \text{and} \quad y_1 = y_2 & \text{as } g_A, g_B \text{ are injective;} \\ \therefore & (x_1, y_1) = (x_2, y_2) & \text{by the definition of ordered pairs.} \end{array}$$

Now we have an injection $f\colon A \times B \to \mathbb{N} \times \mathbb{N}$, of which the codomain $\mathbb{N} \times \mathbb{N}$ is countable by Theorem 9.1.5. So, in view of Corollary 9.2.7(2), the domain $A \times B$ must also be countable. $\square$