NATIONAL UNIVERSITY OF SINGAPORE

CS1231 DISCRETE STRUCTURES

(Semester 1: 2022/2023)
[In the notation and the terminology for 2022/23 Semester 2]

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Write your Student Number only. Do not write your name.
2. This assessment paper contains **FOUR** questions and comprises **NINE** printed pages.
3. Answer **ALL** questions. The marks for each question are indicated in brackets.
4. Write your solutions in the spaces provided.
5. This is an **OPEN** book examination.

STUDENT NUMBER: ________________________________

EXAMINER'S USE ONLY		
Question	Marks	Score
Q1	20	
Q2	5	
Q3	19	
Q4	6	
Total	50	

1. Consider the relation f from $\mathbb{Z}$ to $\mathbb{Z}$ defined by

$$f = \{(m + 12, 31m) : m \in \mathbb{Z}\}.$$

(i) Prove that f is a function $\mathbb{Z} \to \mathbb{Z}$. [4 marks]

Solution: (F1) Let $r \in \mathbb{Z}$. Define $m = r - 12$. Then $m \in \mathbb{Z}$ and $m + 12 = (r - 12) + 12 = r$. Thus $(r, 31m) \in f$.

(F2) Let $r, s_1, s_2 \in \mathbb{Z}$ such that $(r, s_1), (r, s_2) \in f$. Use the definition of f to find $m_1, m_2 \in \mathbb{Z}$ such that $(r, s_1) = (m_1+12, 31m_1)$ and $(r, s_2) = (m_2+12, 31m_2)$. Then $m_1+12 = r = m_2+12$. This implies $m_1 = m_2$. So $s_1 = 31m_1 = 31m_2 = s_2$.

(ii) What is the codomain of f? [1 mark]

Solution: The codomain of f is $\mathbb{Z}$.

(iii) What is the range of f? [1 mark]

Solution: The range of f is $\{31m : m \in \mathbb{Z}\}$.

(iv) Is f surjective? Justify your answer. [3 marks]

Solution: No, because the range of f is not equal to the codomain of f. As $1 \in \mathbb{Z}$, it suffices to argue that $f(r) \neq 1$ for any $r \in \mathbb{Z}$.
Suppose we have $r \in \mathbb{Z}$ such that $f(r) = 1$. Use the definition of f to find $m \in \mathbb{Z}$ such that $(r, 1) = (m + 12, 31m)$. Then $1 = 31m$ and thus $m = 1/31 \notin \mathbb{Z}$. This contradicts the fact that $m \in \mathbb{Z}$. So no $r \in \mathbb{Z}$ can make $f(r) = 1$.

1. (continued from the previous page)

(v) Is f injective? Justify your answer. [3 marks]

Solution: Yes, as shown below.
Let $r_1, r_2 \in \mathbb{Z}$ such that $f(r_1) = f(r_2)$. Say $f(r_1) = s = f(r_2)$. Use the definition of f to find $m_1, m_2 \in \mathbb{Z}$ such that $(r_1, s) = (m_1 + 12, 31m_1)$ and $(r_2, s) = (m_2 + 12, 31m_2)$. Then $31m_1 = s = 31m_2$. This implies $m_1 = m_2$. So $r_1 = m_1 + 12 = m_2 + 12 = r_2$.

(vi) Define $f^1 = f$ and $f^{n+1} = f^n \circ f$ for each $n \in \mathbb{Z}^+$. Prove that f^n is injective for any $n \in \mathbb{Z}^+$. [3 marks]

Solution: We proceed by induction on n.

(Base step) Note that $f^1 = f$ by definition. So f^1 is injective because we know from (v) that f is injective.

(Induction step) Let $k \in \mathbb{Z}^+$ such that f^k is injective. By the Induction Hypothesis and the assumption, we know f^k and f are both injective. So Tutorial Exercise 8.1(b) tells us $f^k \circ f$ is injective. As $f^{k+1} = f^k \circ f$ by definition, we conclude that f^{k+1} is injective.

This completes the induction.

1. (continued from the previous page)

(vii) Does the relation f, considered as a set, have the same cardinality as $\mathbb{Z} \times \mathbb{Z}$? Justify your answer. [5 marks]

Solution: Yes, as shown below.
Define $g\colon \mathbb{Z} \to f$ by setting $g(m) = (m+12, 31m)$ for all $m \in \mathbb{Z}$. It is surjective because, given any $(m + 12, 31m) \in f$, we have $m \in \mathbb{Z}$ and $g(m) = (m + 12, 31m)$. It is injective because, if $m_1, m_2 \in \mathbb{Z}$ such that $f(m_1) = f(m_2)$, then $(m_1 + 12, 31m_1) = (m_2 + 12, 31m_2)$, and thus $m_1 + 12 = m_2 + 12$ and $31m_1 = 31m_2$, implying $m_1 = m_2$. Thus g is a bijection $\mathbb{Z} \to f$.
The previous paragraph shows f has the same cardinality as $\mathbb{Z}$. Note that $\mathbb{Z}$ is countable by Proposition 9.1.4. As $\mathbb{N} \subseteq \mathbb{Z}$, we also know that $\mathbb{Z}$ is infinite by Exercise 8.2.7 and Proposition 9.2.6(1). So f is countable and infinite by Lemma 8.2.8 and Lemma 9.2.1. This means f has the same cardinality as $\mathbb{N}$.
Now $\mathbb{Z} \times \mathbb{Z}$ is countable by Proposition 9.1.4 and Extra Exercise 9.9, as $\mathbb{Z}$ is infinite. As f is infinite and $f \subseteq \mathbb{Z} \times \mathbb{Z}$, Proposition 9.2.6(1) tells us $\mathbb{Z} \times \mathbb{Z}$ is infinite. Thus $\mathbb{Z} \times \mathbb{Z}$ has the same cardinality as $\mathbb{N}$.
Now both f and $\mathbb{Z} \times \mathbb{Z}$ have the same cardinality as $\mathbb{N}$. So they must have the same cardinality by Proposition 8.2.4.

2. Let A be an uncountable subset of $\mathbb{R}$.

(i) Define $A^- = A \cap \mathbb{R}^-$ and $A_{\geqslant 0} = A \cap \mathbb{R}_{\geqslant 0}$. Explain why A^- and $A_{\geqslant 0}$ cannot both be countable. [1 mark]

Solution: Note that $A = A^- \cup A_{\geqslant 0}$. So if A^- and $A_{\geqslant 0}$ are both countable. then $A^- \cup A_{\geqslant 0}$ is also countable by Tutorial Exercise 9.2, contradicting the hypothesis that A is uncountable.

(ii) Define $B = \{x^2 : x \in A\}$. Using (i), or otherwise, prove that B is uncountable. [4 marks]

Solution: From (i), we know that either A^- or $A_{\geqslant 0}$ is uncountable.

Case 1: suppose A^- is uncountable. Define $f\colon A^- \to B$ by setting $f(x) = x^2$ for each $x \in A^-$. Then f is injective because, if $x_1, x_2 \in A^-$ such that $f(x_1) = f(x_2)$, then

$$
\begin{aligned}
& {x_1}^2 = {x_2}^2 && \text{by the definition of } f; \\
\therefore\quad & x_1 = x_2 && \text{as } x_1 < 0 \text{ and } x_2 < 0.
\end{aligned}
$$

Since A^- is uncountable, we deduce from Corollary 9.2.7(2) that B is uncountable.

Case 2: suppose $A_{\geqslant 0}$ is uncountable. Define $g\colon A_{\geqslant 0} \to B$ by setting $f(x) = x^2$ for each $x \in A_{\geqslant 0}$. Then g is injective because, if $x_1, x_2 \in A^-$ such that $g(x_1) = g(x_2)$, then

$$
\begin{aligned}
& {x_1}^2 = {x_2}^2 && \text{by the definition of } g; \\
\therefore\quad & x_1 = x_2 && \text{as } x_1 \geqslant 0 \text{ and } x_2 \geqslant 0.
\end{aligned}
$$

Since $A_{\geqslant 0}$ is uncountable, we deduce from Corollary 9.2.7(2) that B is uncountable.

3. Consider an undirected graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{12, 13, 14, 23, 34, 45\}$.

(i) Draw G. [1 mark]

Solution:

$$\begin{tikzcd} 2 \arrow[r, dash] \arrow[d, dash] & 3 \arrow[d, dash] & \\ 1 \arrow[ur, dash] \arrow[r, dash] & 4 \arrow[r, dash] & 5 \end{tikzcd}$$

(ii) How many connected components are there in G? [1 mark]

Solution: 1

(iii) How many cycles are there in G? [1 mark]

Solution: 3

(iv) How many paths are there between the vertices 1 and 5 in G? [1 mark]

Solution: 3

(v) Draw a subgraph (V', E') of G that is not a tree but satisfies $|E'| = |V'| - 1$. [1 mark]

Solution:

$$\begin{tikzcd} 2 \arrow[r, dash] \arrow[d, dash] & 3 & \\ 1 \arrow[ur, dash] & 4 \arrow[r, dash] & 5 \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} 2 & 3 \arrow[d, dash] \\ 1 \arrow[ur, dash] \arrow[r, dash] & 4 \end{tikzcd} \quad \text{or} \quad \text{4 other possibilities.}$$

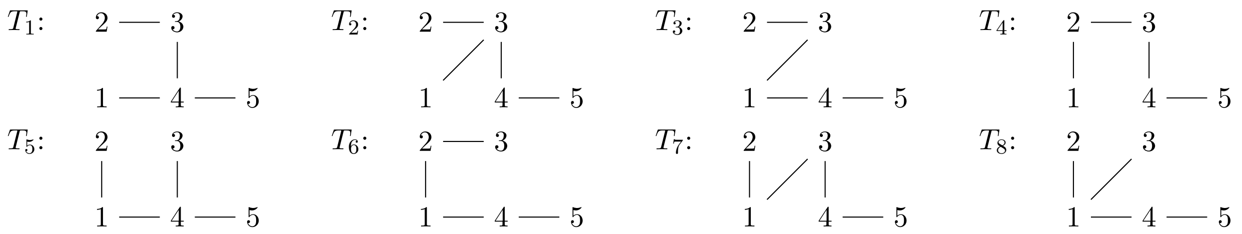
(vi) Identify an edge xy such that $(V, E \setminus \{xy\})$ is not connected. [1 mark]

Solution: 45

3. (continued from the previous page)

(vii) Draw all spanning trees of G. [3 marks]

Solution:



(viii) Which of the spanning trees from (vii) are isomorphic to each other? [3 marks]

Solution: T_1, T_2, T_3, T_8 are isomorphic. T_3, T_4, T_6, T_7 are isomorphic.

(ix) Pick two non-isomorphic trees T and T' from (viii). Explain why T and T' are not isomorphic. [2 marks]

Solution: Consider $T = T_1$ and $T' = T_3$. These are not isomorphic because

- T has a vertex, namely 4, that is in 3 edges, but
- T' does not have any vertex that is in 3 edges.

3. (continued from the previous page)

(x) Determine the number of graphs with vertex set $\{1,2,3,4,5\}$ that are isomorphic to G. Briefly explain your answer. [2 marks]

Solution: We count the number of ways to assign vertices from $\{1,2,3,4,5\}$ to a,b,c,d,e in the following graph without repetition:

$$\begin{array}{ccccc} b & — & c & & \\ | & / & | & & \\ a & — & d & — & e \end{array}$$

There are $5!$ ways to do so by Corollary 10.3.7. Each such assignment gives rise to a graph with vertex set $\{1,2,3,4,5\}$ that is isomorphic to G. However, each such graph comes from exactly two assignments:

$$\begin{array}{ccccc} b & — & c & & \\ | & / & | & & \\ a & — & d & — & e \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} b & — & a & & \\ | & / & | & & \\ c & — & d & — & e \end{array}$$

give rise to the same graph. Thus the total number of such graphs is $5!/2 = 60$.

Alternative solution: First, we assign vertices from $\{1,2,3,4,5\}$ to a,b,c in the following graph without repetition:

$$\begin{array}{ccccc} c & — & \bullet & & \\ | & / & | & & \\ \bullet & — & b & — & a \end{array}$$

There are $5 \times 4 \times 3 = 60$ ways to do this by the Multiplication Rule. All ways to assign vertices in $\{1,2,3,4,5\} \setminus \{a,b,c\}$ to the two $\bullet$'s result in the same graph, whose vertex set is $\{1,2,3,4,5\}$ and which is isomorphic to G. Since different assignments give rise to different graphs, there are exactly 60 such graphs.

(xi) Suppose we add a vertex 6 to V and an edge 26 to E. Determine the number of graphs with vertex set $\{1,2,3,4,5,6\}$ that are isomorphic to $(V \cup \{6\}, E \cup \{26\})$. Briefly explain your answer. [3 marks]

Solution: Here is a drawing of the new graph:

$$\begin{array}{ccccccc} 6 & — & 2 & — & 3 & & \\ & & | & / & | & & \\ & & 1 & — & 4 & — & 5 \end{array}$$

Call this G_6. We count the number of ways to assign vertices from $\{1,2,3,4,5,6\}$ to a,b,c,d,e,f in the following graph without repetition:

$$\begin{array}{ccccccc} f & — & b & — & c & & \\ & & | & / & | & & \\ & & a & — & d & — & e \end{array}$$

There are $6!$ ways to do so by Corollary 10.3.7. Each such assignment gives rise to a graph with vertex set $\{1,2,3,4,5,6\}$ that is isomorphic to G_6. However, each such graph comes from exactly four assignments:

$$\begin{array}{ccccccc} f & — & b & — & c & & \\ & & | & / & | & & \\ & & a & — & d & — & e \end{array}, \quad \begin{array}{ccccccc} e & — & d & — & c & & \\ & & | & / & | & & \\ & & a & — & b & — & f \end{array}, \quad \begin{array}{ccccccc} f & — & b & — & a & & \\ & & | & / & | & & \\ & & c & — & d & — & e \end{array}, \quad \begin{array}{ccccccc} e & — & d & — & a & & \\ & & | & / & | & & \\ & & c & — & b & — & f \end{array}$$

give rise to the same graph. Thus the total number of such graphs is $6!/4 = 180$.

Alternative solution: The number of ways to choose a subset $\{a,b\} \subseteq \{1,2,3,4,5,6\}$, then a subset $\{c,d\} \subseteq \{1,2,3,4,5,6\} \setminus \{a,b\}$, then two different vertices e,f from $\{1,2,3,4,5,6\} \setminus \{a,b,c,d\}$ is $\binom{6}{2} \times \binom{4}{2} \times 2 = 180$ by Theorem 10.3.15 and the Multiplication Rule. Each such sequence of choices gives rise to a unique graph

$$\begin{array}{ccccccc} b & — & e & — & c & & \\ & & | & / & | & & \\ & & d & — & f & — & a \end{array}$$

with vertex set $\{1,2,3,4,5,6\}$ that is isomorphic to G_6, and each such graph can be obtained in this way. So the number of such graphs is also 180.

4. Let H be a subgraph of a finite connected undirected graph G. Prove that H is a subgraph of a spanning tree of G if and only if H is acyclic. [6 marks]

Solution: ($\Rightarrow$) Suppose H is a subgraph of a spanning tree T of G. As T is a tree, we know it is acyclic, i.e., it has no loop and no cycle. This implies all subgraphs of T have no loop and no cycle. In particular, this is true for the subgraph H of T. Thus H is acyclic.

($\Leftarrow$) Suppose H is acyclic. Run the following procedure.

1. Set $H_0 = H$ and initialize $m = 0$.
2. While H_m is unconnected do:
   // Being unconnected, the graph H_m must have at least two connected components. As G
   // is connected, any two of these connected components can be connected by a path in G.
   2.1. Use the Well-Ordering Principle to find a path P_m in G of smallest length which connects two different connected components A_m, B_m of H_m, i.e., the path P_m is between a vertex a_m in A_m and a vertex b_m in B_m.
   2.2. Set $H_{m+1} = \big(\mathrm{V}(H_m) \cup \mathrm{V}(P), \mathrm{E}(H_m) \cup \mathrm{E}(P)\big)$.
   // The graph H_{m+1} has at least one fewer connected components than H_m because a_i
   // and b_2 are not in the same connected component in H_m, but they are in H_{m+1} by
   // Theorem 11.3.7.
   // This while-loop must stop because H is finite, and thus have only finitely many connected
   // components to connect.
3. Set $L_0 = H_m$ and initialize $n = 0$.
4. While $\mathrm{V}(G) \setminus \mathrm{V}(L_n) \neq \varnothing$ do:
   4.1. Use the connectedness of G to find an edge $x_n y_n \in \mathrm{E}(G)$ where $x_n \notin \mathrm{V}(L_n)$ and $y_n \in \mathrm{V}(L_n)$.
   4.2. Set $L_{n+1} = \big(\mathrm{V}(L_n) \cup \{x_n\}, \mathrm{E}(L_n) \cup \{x_n y_n\}\big)$.
   4.3. Increment n to $n + 1$.
   // This while-loop must stop because G is finite.

We verify that each H_i is acyclic. The graph H_0 is acyclic because H is acyclic. Consider some $i \in \{0, 1, \ldots, m-1\}$ such that H_i is acyclic. Say $P_i = u_0 u_1 \ldots u_k$, where $u_0 = a_i$ and $u_k = b_i$. Since neither H_i nor P has any loop, the union H_{i+1} also has no loop. Suppose, towards a contradiction, that H_{i+1} has a cycle, say C. As H_i is acyclic, this cycle C must involve some edge in P. However, by the smallestness of the length of P, we know $u_1, u_2, \ldots, u_{k-1} \notin \mathrm{V}(H_i)$. Thus one by one we see that C must have in it all the edges in P. If $C = u_0 u_1 \ldots u_k v_1 v_2 \ldots v_\ell u_0$, then $u_k v_1 v_2 \ldots v_\ell u_0$ is a path between a_i and b_i in H_i, contradicting the fact that a_i and b_i are in different connected components of H_i via Theorem 11.3.7.

We verify that each L_j is a tree. Recall $L_0 = H_m$. We know H_m is acyclic from the previous paragraph. We know H_m is connected because the stopping condition is reached in block 2. So L_0 is a tree. Consider some $j \in \{0, 1, \ldots, n-1\}$ such that L_j is a tree. The graph L_{j+1} cannot have any loop because L_j has no loop and $x_j \neq y_j$. Recall $x_j \notin \mathrm{V}(K_j)$. So the new edge $x_j y_j$ is the only edge in L_{j+1} that contains x_j. Since every vertex is in two edges in a cycle, no cycle in L_{j+1} can contain x_j. However, no cycle in L_{j+1} can omit x_j because L_j has no cycle, and x_j is the only difference between L_j and L_{j+1}. Hence L_{j+1} can have no cycle.

Since the stopping condition is reached in block 4, we know $\mathrm{V}(L_n) = \mathrm{V}(G)$. So L_n is a spanning tree of G. Here H is a subgraph of L_n because L_n is obtained from H by successively adding vertices and edges.

— END OF PAPER —