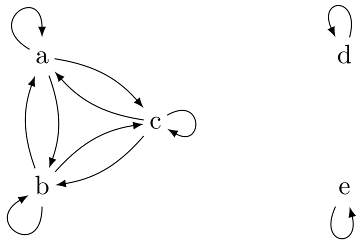
Tutorial solutions for Chapter 6

Sometimes there are other correct answers.

6.1. (a)



(b) Yes, this R is **an equivalence relation**: as one can verify exhaustively, it is reflexive, symmetric, and transitive.

The equivalence classes are $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ and $\{\mathrm{d}\}$ and $\{\mathrm{e}\}$.

(c) No, this R is **not a partial order**. It is not antisymmetric because, for example, we have $(\mathrm{a}, \mathrm{b}) \in R$ and $(\mathrm{b}, \mathrm{a}) \in R$, but $\mathrm{a} \neq \mathrm{b}$. As R is not antisymmetric, it cannot be a partial order.

No, this R is **not a total order**, because it is not even a partial order.

Moral. It is often helpful to draw a diagram.

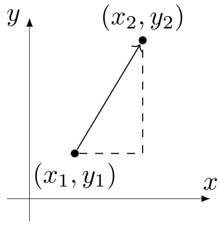
6.2. (a) R is **reflexive** because $xx = x^2 \geqslant 0$ for all $x \in \mathbb{Q}$.

R is **symmetric** because $xy \geqslant 0$ implies $yx \geqslant 0$ for all $x, y \in \mathbb{Q}$.

R is **not antisymmetric** because $1 \times 2 \geqslant 0$ and $2 \times 1 \geqslant 0$ but $1 \neq 2$, for instance.

R is **not transitive** because $1 \times 0 \geqslant 0$ and $0 \times (-1) \geqslant 0$ but $1 \times (-1) = -1 < 0$, for instance.

(b)



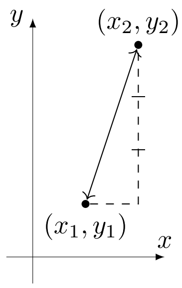
S is **reflexive** because $x \leqslant x$ and $y \leqslant y$ for all $(x, y) \in \mathbb{R}^2$.

S is **not symmetric** because $(1,1)\ S\ (2,2)$ but $(2,2)\ \not S\ (1,1)$, for instance.

S is **antisymmetric**. To see this, let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that $(x_1, y_1)\ S\ (x_2, y_2)$ and $(x_2, y_2)\ S\ (x_1, y_1)$. The former implies $x_1 \leqslant x_2$ and $y_2 \leqslant y_2$, while the latter implies $x_2 \leqslant x_1$ and $y_2 \leqslant y_1$. Thus $x_1 = x_2$ and $y_1 = y_2$. Hence $(x_1, y_1) = (x_2, y_2)$.

S is **transitive**. To see this, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ such that $(x_1, y_1)\ S\ (x_2, y_2)$ and $(x_2, y_2)\ S\ (x_3, y_3)$. These imply $x_1 \leqslant x_2 \leqslant x_3$ and $y_1 \leqslant y_2 \leqslant y_3$. Thus $x_1 \leqslant x_3$ and $y_1 \leqslant y_3$. Hence $(x_1, y_1)\ S\ (x_3, y_3)$.

6.3.



(a) (Reflexivity) Let $(x, y) \in \mathbb{R}^2$. Then $3(x-x) = 3\times 0 = 0 = y-y$. So $(x, y)\ R\ (x, y)$.

(Symmetry) Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that $(x_1, y_1)\ R\ (x_2, y_2)$. In view of the definition of R, this means $3(x_1 - x_2) = y_1 - y_2$. Thus $3(x_2 - x_1) = y_2 - y_1$. So $(x_2, y_2)\ R\ (x_1, y_1)$ by the definition of R.

(Transitivity) Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ such that $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$. In view of the definition of R, this means $3(x_1 - x_2) = y_1 - y_2$ and $3(x_2 - x_3) = y_2 - y_3$. Thus

$$3(x_1 - x_3) = 3(x_1 - x_2) + 3(x_2 - x_3) = (y_1 - y_2) + (y_2 - y_3) = y_1 - y_3.$$

So $(x_1, y_1)\ R\ (x_3, y_3)$ by the definition of R.

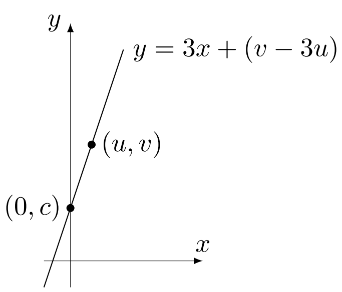
As R is reflexive, symmetric and transitive, it is an equivalence relation.

(b) $[(u, v)]$ is the straight line with slope 3 passing through the point (u, v) in the plane $\mathbb{R}^2$.

Explanations.

$$\begin{aligned}
[(u, v)] &= \{(x, y) \in \mathbb{R}^2 : (u, v)\ R\ (x, y)\} && \text{by the definition of } [(u, v)]; \\
&= \{(x, y) \in \mathbb{R}^2 : 3(u - x) = v - y\} && \text{by the definition of } R; \\
&= \{(x, y) \in \mathbb{R}^2 : y = 3x + (v - 3u)\}.
\end{aligned}$$

From this, we see that $[(u, v)]$ is a straight line with slope 3. We know $(u, v) \in [(u, v)]$ by Lemma 6.3.5(1).



(c) Let $(u, v) \in \mathbb{R}^2$. Define $c = v - 3u$. Then $v - c = v - (v - 3u) = 3u = 3(u - 0)$. So the definition of R tells us $(u, v)\ R\ (0, c)$. Hence $(0, c) \in [(u, v)]$ by the definition of $[(u, v)]$. $\square$

How one may find this c**.** Unravelling the definitions, we see that we want $c \in \mathbb{R}$ such that $(u, v)\ R\ (0, c)$. This means $3(u - 0) = v - c$. Solving gives $c = v - 3u$. What we showed here is that the only $c \in \mathbb{R}$ that can possibly make $(0, c) \in [(u, v)]$ is $v - 3u$. As verified above, this choice of c actually works.

Additional comment. We do not need to explain how we found our c in a proof that such a c exists.

6.4. ($\Rightarrow$) Suppose $x \sim y$. We want to show $[x] = [y]$.

($[x] \subseteq [y]$) Take $z \in [x]$. This means $x \sim z$ by the definition of $[x]$. As $x \sim y$, the symmetry of $\sim$ tells us $y \sim x$. In view of the transitivity of $\sim$, having both $y \sim x$ and $x \sim z$ implies $y \sim z$. So $z \in [y]$ by the definition of $[y]$.

($[y] \subseteq [x]$) This is essentially the same as the proof of the converse, except that we interchange x and y. Take $z \in [y]$. This means $y \sim z$ by the definition of $[y]$. In view of the transitivity of $\sim$, having both $x \sim y$ and $y \sim z$ implies $x \sim z$. So $z \in [x]$ by the definition of $[x]$.

($\Leftarrow$) Suppose $[x] = [y]$. Note $y \sim y$ by the reflexivity of $\sim$. So the definition of $[y]$ tells us $y \in [y] = [x]$. So $x \sim y$ by the definition of $[x]$. ☐

Alternative proof using Lemma 6.3.5 and Lemma 6.3.6. ($\Rightarrow$) Suppose $x \sim y$. Then $y \in [x]$ by the definition of $[x]$. We also know $y \in [y]$ by Lemma 6.3.5(1). These tell us $y \in [x] \cap [y]$ and thus $[x] \cap [y] \neq \varnothing$. Hence $[x] = [y]$ by Lemma 6.3.6.

($\Leftarrow$) Suppose $[x] = [y]$. Then Lemma 6.3.5(1) implies $y \in [y] = [x]$. So $x \sim y$ by the definition of $[x]$. ☐

6.5. (a) The symmetry of a relation R on A states that $\forall x, y \in A \;\; (x \mathrel{R} y \Rightarrow y \mathrel{R} x)$. The attempt claims that this implies $\forall x, y \in A \;\; (x \mathrel{R} y \wedge y \mathrel{R} x)$. This is not justified (and actually not true).

(b) One counterexample is the relation $R = \{(0,0)\}$ on the set $A = \{0,1\}$.

Explanation. The given proposition can be written semi-symbolically as

$$\forall \text{set } A \; \forall \text{relation } R \text{ on } A \; \big(\mathrm{Sym}(A,R) \wedge \mathrm{Tran}(A,R) \rightarrow \mathrm{Refl}(A,R)\big),$$

where $\mathrm{Sym}(A,R)$ stands for "R is a symmetric relation on A", etc. So a counterexample consists of a set A and a relation R on A which makes the conditional proposition

$$\mathrm{Sym}(A,R) \wedge \mathrm{Tran}(A,R) \rightarrow \mathrm{Refl}(A,R)$$

false. According to Example 1.4.23, the negation of this conditional proposition is

$$\mathrm{Sym}(A,R) \wedge \mathrm{Tran}(A,R) \wedge \neg\mathrm{Refl}(A,R).$$

The relation given above satisfies this: it is symmetric and transitive, as one can verify exhaustively, but it is not reflexive because $1 \in A$ but $(1,1) \notin R$.

Moral. The compound expressions $p \rightarrow q$ and $p \wedge q$, where p, q are propositional variables, are *not* equivalent, as one can verify using a truth table.

6.6. (a) This proposition is **false**. One counterexample is the relation $R = \{(0,0),(1,1)\}$ on the set $A = \{0,1\}$. It is both symmetric and antisymmetric, as one can verify exhaustively.

(b) This proposition is also **false**. One counterexample is the divisibility relation on $\mathbb{Z}$ from Example 6.1.8 and Example 6.4.5.

Moral. Antisymmetry is not the negation of symmetry. In fact, it is neither a necessary nor a sufficient condition for the negation of symmetry.

6.7. Let $P(n)$ be the predicate

there exist no $x_1, x_2, \ldots, x_n \in A$ satisfying $x_1 \neq x_2$ and

$$x_1 \mathrel{R} x_2 \quad \text{and} \quad x_2 \mathrel{R} x_3 \quad \text{and} \quad \ldots \quad \text{and} \quad x_{n-1} \mathrel{R} x_n \quad \text{and} \quad x_n \mathrel{R} x_1$$

over $\mathbb{Z}_{\geqslant 2}$.

(Base step) The antisymmetry of R tells us there exist no $x_1, x_2 \in A$ satisfying $x_1 \neq x_2$ and $x_1 \, R \, x_2$ and $x_2 \, R \, x_1$. So $P(2)$ is true.

(Induction step) Let $k \in \mathbb{Z}_{\geqslant 2}$ such that $P(k)$ is true, i.e., there exist no $x_1, x_2, \ldots, x_k \in A$ satisfying $x_1 \neq x_2$ and

$$x_1 \, R \, x_2 \quad \text{and} \quad x_2 \, R \, x_3 \quad \text{and} \quad \ldots \quad \text{and} \quad x_{k-1} \, R \, x_k \quad \text{and} \quad x_k \, R \, x_1.$$

Suppose $P(k+1)$ is false. This gives $x_1, x_2, \ldots, x_k, x_{k+1} \in A$ satisfying $x_1 \neq x_2$ and

$$x_1 \, R \, x_2 \quad \text{and} \quad x_2 \, R \, x_3 \quad \text{and} \quad \ldots \quad \text{and} \quad x_{k-1} \, R \, x_k \quad \text{and} \quad x_k \, R \, x_{k+1} \quad \text{and} \quad x_{k+1} \, R \, x_1.$$

In view of the transitivity of R, the last two conditions imply $x_k \; R \; x_1$. This, together with the rest of the conditions above, contradict the induction hypothesis. So $P(k+1)$ is true.

Hence $\forall n \in \mathbb{Z}_{\geqslant 2} \;\; P(n)$ is true by MI. $\square$

Extra exercises

6.8. (a) R is **not reflexive** because $0 \times 0 = 0 \leqslant 0$.

R is **symmetric** because $xy > 0$ implies $yx >$ for all $x, y \in \mathbb{Q}$.

R is **not antisymmetric** because $1 \times 2 > 0$ and $2 \times 1 >$ but $1 \neq 2$, for instance.

R is **transitive**. To see this, suppose $x, y, z \in \mathbb{Q}$ such that $xy > 0$ And $yz > 0$. As $xy > 0$, either x, y are both positive, or x, y are both negative.

- Suppose $x > 0$ and $y > 0$. Then $z > 0$ too as $yz > 0$. It follows that $xz > 0$.
- Suppose $x < 0$ and $y < 0$. Then $z < 0$ too as $yz > 0$. It follows that $xz > 0$.

(b) S is **not reflexive** because $x \neq x + 1$ for all $x \in \mathbb{Z}$.

S is **not symmetric** because $y = x + 1$ implies $x = y - 1 \neq y + 1$ for all $x, y \in \mathbb{Z}$.

S is **antisymmetric** vacuously because if $y = x + 1$ and $x = y + 1$, then $x = y + 1 = (x + 1) + 1 = x + 2 \neq x$, which is a contradiction.

S is **not transitive** because if $y = x + 1$ and $z = y + 1$, then $z = (x + 1) + 1 = x + 2 \neq x + 1$.

6.9.
$$\begin{array}{lll} & R = R^{-1} & \\ \Leftrightarrow & \forall (x, y) \in A^2 \;\; \big((x, y) \in R \Leftrightarrow (x, y) \in R^{-1}\big) & \text{as } R, R^{-1} \subseteq A^2; \\ \Leftrightarrow & \forall (x, y) \in A^2 \;\; \big((x, y) \in R \Leftrightarrow (y, x) \in R\big) & \text{by the definition of } R^{-1}; \\ \Leftrightarrow & \forall (x, y) \in A^2 \;\; \big((x, y) \in R \Rightarrow (y, x) \in R\big) & \text{by Exercise 3.3;} \\ \Leftrightarrow & R \text{ is symmetric} & \text{by the definition of symmetry.} \end{array}$$

6.10. (a) (Reflexivity) Let $a \in \mathbb{Z}$. Note that $a - a = 0 = 2 \times 0$, where $0 \in \mathbb{Z}$. So $a - a$ is even, and thus $a \, R \, a$ by the definition of R.

(Symmetry) Let $a, b \in \mathbb{Z}$ such that $a \, R \, b$. Then $a - b$ is even by the definition of R. Use the definition of even integers to find $x \in \mathbb{Z}$ such that $a - b = 2x$. Then $b - a = 2(-x)$, where $-x \in \mathbb{Z}$. So $b - a$ is even, and thus $b \, R \, a$ by the definition of R.

(Alternative proof of symmetry) Let $a, b \in \mathbb{Z}$ such that $a \, R \, b$. Then $a - b$ is even by the definition of R. So Exercise 3.2.6(1) implies $b - a = -(a - b)$ is also even. Thus $b \, R \, a$ by the definition of R.

(Yet another proof of symmetry) We know $R^{-1} = R$ from Tutorial Exercise 5.4(a) So R is symmetric by Tutorial Exercise 6.9.

(Transitivity) Let $a, b, c \in \mathbb{Z}$ such that $a \mathrel{R} b$ and $b \mathrel{R} c$. Then $a - b$ and $b - c$ are both even by the definition of R. Use the definition of even integers to find $x, y \in \mathbb{Z}$ such that $a-b = 2x$ and $b-c = 2y$. Then $a-c = (a-b)+(b-c) = 2x+2y = 2(x+y)$, where $x + y \in \mathbb{Z}$. So $a - c$ is even, and thus $a \mathrel{R} c$ by the definition of R.

(Alternative proof of transitivity) We know $R \circ R = R \subseteq R$ from Tutorial Exercise 5.4(b) and Remark 4.2.4(3). So R is transitive by Exercise 6.1.10.

As R is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$

(b) $[0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\} = [2] = [-2] = [4] = [-4] = \cdots$

$[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\} = [-1] = [3] = [-3] = [5] = \cdots$