Chapter 11: Graphs

CS1231 Discrete Structures

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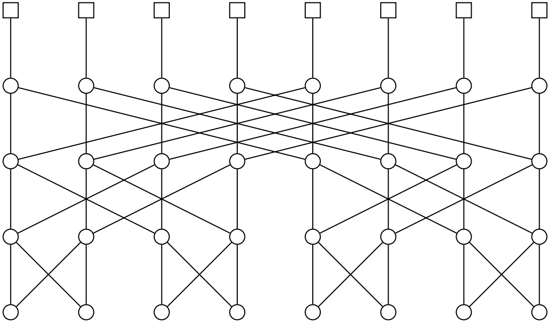
2022/23 Semester 2



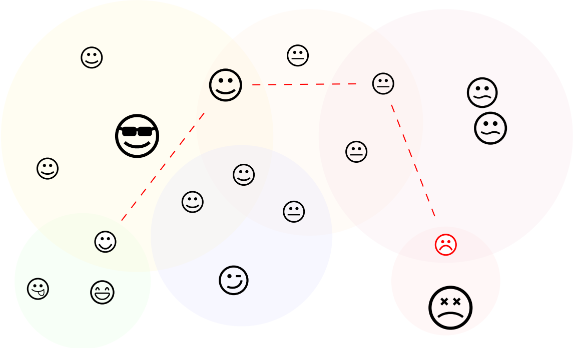
Butterfly network for 8 processors

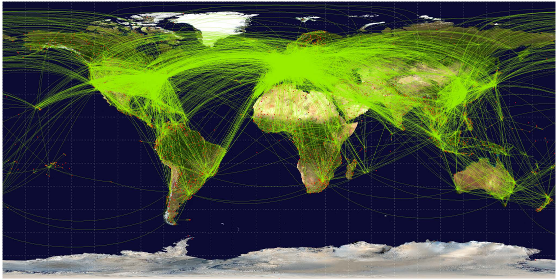
Why graphs?

- They provide a more combinatorial view (as opposed to an algebraic view) of relations.
- They are useful in representing all kinds of situations where linkages are involved.
- So theorems about graphs are widely applicable.

Plan

- paths
- cycles
- connectedness





Undirected finite graphs

Let G be an undirected graph, where $G = (V, E)$.

Warning 11.1.1

There are several commonly used, conflicting sets of terminologies for graphs. Always check the definitions being used when looking into the literature.

Definition 11.1.2

(1) Denote by $\mathrm{V}(G)$ and $\mathrm{E}(G)$ the set of all vertices and the set of all edges in G respectively, i.e.,
$$\mathrm{V}(G) = V \quad \text{and} \quad \mathrm{E}(G) = E.$$

(2) When there is no risk of ambiguity, we may write an edge $\{x, y\}$ as xy.

(3) The graph G is *finite* if $\mathrm{V}(G)$ is finite, else it is *infinite*.

Definition 11.1.3

(1) We say that an undirected graph H is a *subgraph* of G, or G *contains* H (as a subgraph), if $\mathrm{V}(H) \subseteq \mathrm{V}(G)$ and $\mathrm{E}(H) \subseteq \mathrm{E}(G)$.

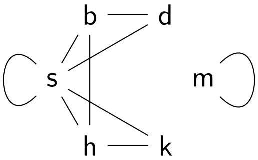
(2) A *proper subgraph* of G is a subgraph H of G such that $H \neq G$.

Subgraphs: examples

Example 11.1.4

Consider the graph G, where

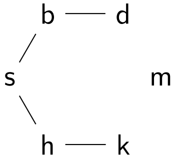
$$V(G) = \{b, d, h, k, m, s\},$$
$$E(G) = \{bd, bh, bs, ds, hk, hs, ks, mm, ss\}.$$



- The graph G_1, where

$$V(G_1) = \{b, d, h, k, m, s\},$$
$$E(G_1) = \{bd, bs, hk, hs\},$$

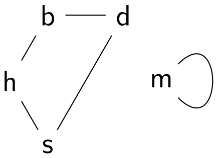
  is a subgraph of G.



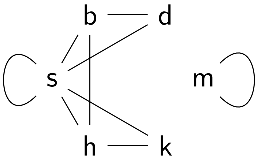
- The graph G_2, where

$$V(G_2) = \{b, d, h, m, s\},$$
$$E(G_2) = \{bd, bh, ds, hs, mm\},$$

  is a subgraph of G.



Subgraphs: non-examples



Example 11.1.4

Consider the graph G, where

$$V(G) = \{b, d, h, k, m, s\},$$
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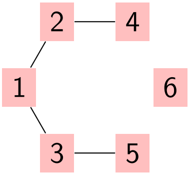
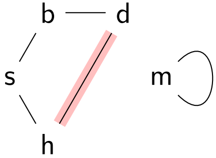


- The graph G_3, where
$$V(G_3) = \{1, 2, 3, 4, 5, 6\},$$
$$E(G_3) = \{12, 13, 24, 35\},$$
is *not* a subgraph of G because $V(G_3) \not\subseteq V(G)$.



- The graph G_4, where
$$V(G_4) = \{b, d, h, m, s\},$$
$$E(G_4) = \{bd, bs, dh, hs, mm\},$$
is *not* a subgraph of G because $E(G_4) \not\subseteq E(G)$.

Paths

Definition 11.1.5

Let G be an undirected graph, and u, v be vertices in G. A *path* between u and v in G is a subgraph of G of the form

$$\big(\{x_0, x_1, \dots, x_\ell\}, \{x_0x_1, x_1x_2, \dots, x_{\ell-1}x_\ell\}\big),$$

where the x's are all different and $\ell \in \mathbb{N}$, satisfying $u = x_0$ and $v = x_\ell$.

- Here ℓ is called the *length* of the path.
- When there is no risk of ambiguity, we may denote the subgraph above by $x_0x_1 \dots x_\ell$.

Picture

$$x_0 \;\text{———}\; x_1 \;\text{———}\; x_2 \;\text{———}\; \cdots \;\text{———}\; x_{\ell-1} \;\text{———}\; x_\ell$$

Remark 11.1.6

(1) Informally speaking, a path links two vertices in a graph via a sequence of edges, each joined to the next, that has no repeated vertex.

(2) Some consider paths of infinite length. We do not.

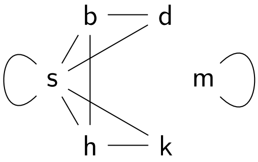
Paths: examples

Example 11.1.7

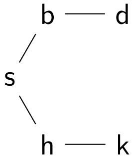
Consider the graph G, where

$$\mathrm{V}(G) = \{\mathrm{b}, \mathrm{d}, \mathrm{h}, \mathrm{k}, \mathrm{m}, \mathrm{s}\},$$

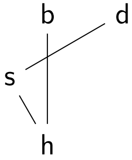
$$\mathrm{E}(G) = \{\mathrm{bd}, \mathrm{bh}, \mathrm{bs}, \mathrm{ds}, \mathrm{hk}, \mathrm{hs}, \mathrm{ks}, \mathrm{mm}, \mathrm{ss}\}.$$



- The graph P, where $P = $ dbshk, is a path of length 4 between d and k in G.



- The graph Q, where $Q = $ dshb, is a path of length 3 between b and d in G.



Remark 11.1.8

(1) The subgraph $(\{\mathrm{s}\}, \{\})$, which we may write as s, is a path of length 0 in G. It is essentially the vertex s with no edge, not even a loop.

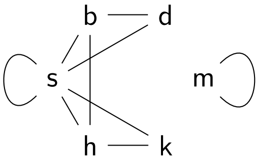
(2) The subgraph $(\{\mathrm{s}, \mathrm{h}\}, \{\mathrm{sh}\})$, which we may write as sh, is a path of length 1 in G. It is essentially the edge sh.

Paths: non-examples

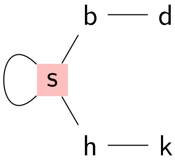
Example 11.1.7

Consider the graph G, where

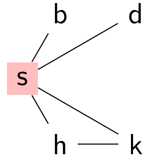
$$V(G) = \{b, d, h, k, m, s\},$$
$$E(G) = \{bd, bh, bs, ds, hk, hs, ks, mm, ss\}.$$



- The graph H_1, where $H_1 = \text{dbsshk}$, is not a path in G because s is in three edges in H_1.



- The graph H_2, where $H_2 = \text{bshksd}$, is not a path in G because s is in four edges in H_2.

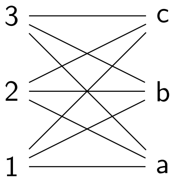


(Note that each vertex in a path is in at most two edges in the path.)

Paths: exercise

Exercise 11.1.9

How many paths are there between 1 and 3 in the undirected graph G with the drawing below? ✐ 11a



Combining paths

Lemma 11.1.10

Let G be an undirected graph and u, v, w be vertices in G. Suppose there are a path P between u and v in G, and a path Q between v and w in G. Then there is a path between u and w in G.

Proof

Let $P = x_0x_1 \ldots x_k$ and $Q = y_0y_1 \ldots y_\ell$, so that $k, \ell \in \mathbb{N}$ and

$$x_0 = u, \quad x_k = v = y_0, \quad w = y_\ell.$$

As $x_k = y_0 \in \mathrm{V}(Q)$, we know some $t \in \{0, 1, \ldots, k\}$ satisfies $x_t \in \mathrm{V}(Q)$. Let t be the smallest element of $\{0, 1, \ldots, k\}$ such that $x_t \in \mathrm{V}(Q)$. By the smallestness of t, none of $x_0, x_1, \ldots, x_{t-1}$ is in $\mathrm{V}(Q)$. So if $s \in \{0, 1, \ldots, \ell\}$ such that $x_t = y_s$, then

$$x_0x_1 \ldots x_t y_{s+1} y_{s+2} \ldots y_\ell$$

is a path between u and w in G. □

Remark 11.1.11

The k above is an element of $\{t \in \{0, 1, \ldots, k\} : x_t \in \mathrm{V}(Q)\}$. This set is thus a nonempty subset of $\mathbb{N}$, and so must have a smallest element by the Well-Ordering Principle.

Cycles

Definition 11.2.1(1)

A *cycle* in an undirected graph G is a subgraph of G of the form

$$\big(\{x_1, x_2, \ldots, x_\ell\}, \big\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{\ell-1}, x_\ell\}, \{x_\ell, x_1\}\big\}\big),$$

where the x's are all different and $\ell \in \mathbb{N}_{\geqslant 3}$. Here ℓ is called the *length* of the cycle. When there is no risk of ambiguity, we may denote the subgraph above by $x_1 x_2 \ldots x_\ell x_1$.

Picture

$$\begin{array}{ccccccc}
x_1 & \text{———} & x_2 & \text{———} & x_3 & \text{———} & x_4 \\
| & & & & & & | \\
x_\ell & \text{———} & x_{\ell-1} & \text{———} & \cdots & \text{———} & x_5
\end{array}$$

Remark 11.2.2

(1) Informally speaking, a cycle in a graph is a sequence of at least three edges, each joined to the next, and the last joined to the first, that has no repeated vertex.

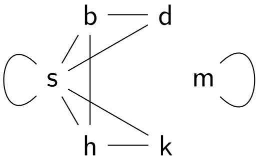
(2) By definition, a cycle has at least three vertices (and thus at least three edges). Therefore, in no sense can a loop be a cycle.

Cycles: examples

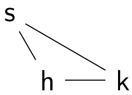
Example 11.2.3

Consider the graph G, where

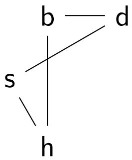
$$\mathrm{V}(G) = \{\mathrm{b}, \mathrm{d}, \mathrm{h}, \mathrm{k}, \mathrm{m}, \mathrm{s}\},$$
$$\mathrm{E}(G) = \{\mathrm{bd}, \mathrm{bh}, \mathrm{bs}, \mathrm{ds}, \mathrm{hk}, \mathrm{hs}, \mathrm{ks}, \mathrm{mm}, \mathrm{ss}\}.$$



- The graph C, where $C = \mathrm{shk}$, is a cycle of length 3 in G.



- The graph D, where $D = \mathrm{sdbh}$, is a cycle of length 4 in G.

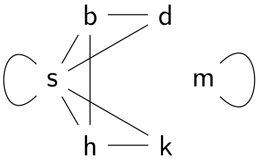


Cycles: non-examples

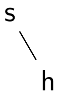
Example 11.2.3

Consider the graph G, where

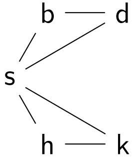
$$\begin{aligned} \mathrm{V}(G) &= \{\mathrm{b}, \mathrm{d}, \mathrm{h}, \mathrm{k}, \mathrm{m}, \mathrm{s}\}, \\ \mathrm{E}(G) &= \{\mathrm{bd}, \mathrm{bh}, \mathrm{bs}, \mathrm{ds}, \mathrm{hk}, \mathrm{hs}, \mathrm{ks}, \mathrm{mm}, \mathrm{ss}\}. \end{aligned}$$



- The graph H_3, where shk, is not a cycle in G because it has only two vertices.



- The graph H_4, where $H_4 =$ bshksdb, is not a cycle in G because s is in four different edges in H_4.



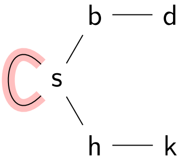
(Note that every cycle by definition has at least three vertices, and each vertex in a cycle is in exactly two edges in the cycle.)

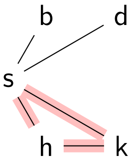
Cyclic graphs

Definition 11.2.1(2)

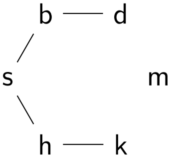
An undirected graph is *cyclic* if it has a loop or a cycle; else it is *acyclic*.

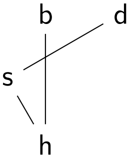
Drawings of cyclic graphs:





Drawings of acyclic graphs:





At least two paths (1/2)

Theorem 11.2.5

An undirected graph G with no loop is cyclic if and only if
it has two vertices between which there are two distinct paths.

Proof of $\Rightarrow$

Assume G is cyclic. According to the definition of cyclic graphs, as G has no loop, it must have a cycle, say,

$$x_1 x_2 \dots x_\ell x_1.$$

From this, we find two paths between x_1 and x_ℓ:

$$x_1 x_\ell \quad \text{and} \quad x_1 x_2 \dots x_\ell x_1.$$

These two paths are distinct because the first one has two vertices and the second one has at least three vertices as $\ell \geqslant 3$. $\square$

At least two paths (2/2)

Theorem 11.2.5

An undirected graph G with no loop is cyclic if and only if it has two vertices between which there are two distinct paths.

Proof sketch of $\Leftarrow$

Let $u, v \in \mathrm{V}(G)$ with two distinct paths between them, say,

$$P = x_0 x_1 \dots x_k \quad \text{and} \quad Q = y_0 y_1 \dots y_\ell,$$

where $x_0 = u = y_0$ and $x_k = v = y_\ell$, and $k \leqslant \ell$. As $P \neq Q$, we know $x_i \neq y_i$ for some $i \in \{0, 1, \dots, k\}$. Let r be the smallest element of $\{0, 1, \dots, k\}$ such that $x_r \neq y_r$. Here $r \neq 0$ because $x_0 = y_0$. So the smallestness of r tells us $x_{r-1} = y_{r-1}$. Let s be the smallest element of $\{r, r+1, \dots, k\}$ which makes $x_s \in \{y_r, y_{r+1}, \dots, y_\ell\}$. If t is the element of $\{r, r+1, \dots, \ell\}$ which makes $x_s = y_t$, then

$$x_{r-1} x_r \dots x_s y_{t-1} y_{t-2} \dots y_r y_{r-1}$$

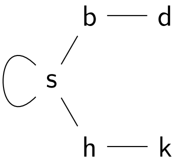
is a cycle in G. $\square$

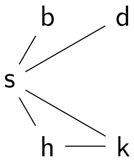
Connected graphs

Definition 11.3.1

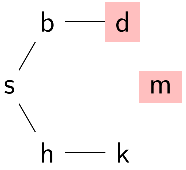
An undirected graph is *connected* if there is a path between any two vertices.

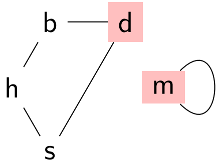
Drawings of connected graphs:





Drawings of unconnected graphs:



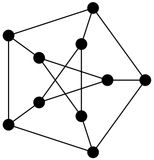


Connected graphs: quick check

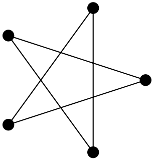
Exercise 11.3.3

Which of the following are drawings of connected graphs? 11b

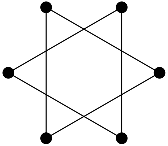
(a)



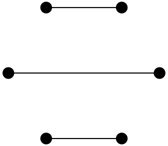
(b)



(c)



(d)

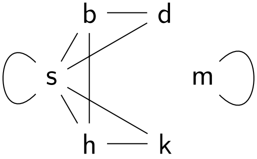


Connected components

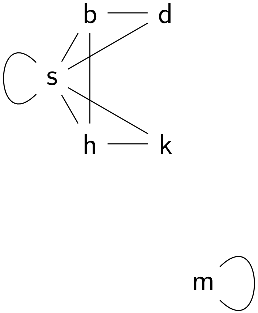
Definition 11.3.4

Let G be an undirected graph. A *connected component* of G is a maximal connected subgraph of G, i.e., it is a connected subgraph H of G such that no connected subgraph of G contains H as a proper subgraph.

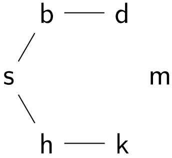
graph:



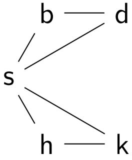
connected components:



not connected components:



not connected



not maximal

Every vertex is in a connected component

Proposition 11.3.6

Every vertex v in an undirected graph G is in some connected component of G.

Proof sketch

Define the subgraph H of G by setting

$$\mathsf{V}(H) = \{x \in \mathsf{V}(G) : \text{there is a path between } v \text{ and } x \text{ in } G\} \quad \text{and}$$
$$\mathsf{E}(H) = \{xy \in \mathsf{E}(G) : x, y \in \mathsf{V}(H)\}.$$

We know $v \in \mathsf{V}(H)$ because $(\{v\}, \{\})$ is a path between v and v. To finish the proof, we show H is a connected component of G. Note H is connected by Lemma 11.1.10.
Let H^+ be a connected subgraph of G which contains H as a subgraph. We want to show that $H^+ = H$. The definition of subgraphs tells us already $\mathsf{V}(H) \subseteq \mathsf{V}(H^+)$ and $\mathsf{E}(H) \subseteq \mathsf{E}(H^+)$. So it remains to show $\mathsf{V}(H^+) \subseteq \mathsf{V}(H)$ and $\mathsf{E}(H^+) \subseteq \mathsf{E}(H)$.
Take any $x \in \mathsf{V}(H^+)$. As $v \in \mathsf{V}(H) \subseteq \mathsf{V}(H^+)$ and H^+ is connected, there is a path between v and x in H^+, hence in G. So $x \in \mathsf{V}(H)$ by the definition of $\mathsf{V}(H)$.
Take any $xy \in \mathsf{E}(H^+)$. Then $x, y \in \mathsf{V}(H^+) \subseteq \mathsf{V}(H)$ by the previous paragraph. As $xy \in \mathsf{E}(H^+) \subseteq \mathsf{E}(G)$, the definition of $\mathsf{E}(H)$ tells us $xy \in \mathsf{E}(H)$. $\square$

Connectedness (1/2)

Theorem 11.3.7

Let u, v be vertices in an undirected graph G. Then there is a path between u and v in G if and only if there is a connected component of G that has both u and v in it.

Proof of $\Leftarrow$

Assume there is a connected component, say H, of G that has both u and v in it. As H is connected, there is a path between u and v in H, hence in G. $\square$

Connectedness (2/2)

Proof of $\Rightarrow$ for Theorem 11.3.7

Assume there is a path between u and v in G, say,

$$P = x_0 x_1 \dots x_\ell,$$

where $u = x_0$ and $v = x_\ell$. Use Proposition 11.3.6 to find a connected component H_u of G that has u in it. Define $H = \big(\mathrm{V}(H_u) \cup \mathrm{V}(P), \mathrm{E}(H_u) \cup \mathrm{E}(P)\big)$. We claim that H is a connected subgraph of G. From this, we will deduce $H = H_u$ using the maximality of the connected component H_u; thus $v \in \mathrm{V}(P) \subseteq \mathrm{V}(H) = \mathrm{V}(H_u)$. Take $a, b \in \mathrm{V}(H)$.

Case 1: suppose $a, b \in \mathrm{V}(H_u)$. As H_u is connected, there is a path between a and b in H_u, hence in H.

Case 2: suppose $a, b \in \mathrm{V}(P)$. Say $a = x_r$ and $b = x_s$, where $r \leqslant s$. Then a path between a and b in H is $x_r x_{r+1} \dots x_s$.

Case 3: suppose one of a, b is in H_u and another is in P. Say $a \in \mathrm{V}(H_u)$ and $b \in \mathrm{V}(P)$. On the one hand, as H_u is connected, and $a, u \in \mathrm{V}(H_u)$, one can find a path, say Q_1, between a and u in H_u, hence in H. On the other hand, if $b = x_t$ where $t \in \{0, 1, \dots, \ell\}$, then $Q_2 = x_0 x_1 \dots x_t$ a path between u and b in H. Combining Q_1 and Q_2 using Lemma 11.1.10, we get a path between a and b in H. $\square$

Summary

Remark 11.1.6(1)

Informally speaking, a path links two vertices in a graph via a sequence of edges, each joined to the next, that has no repeated vertex.

Remark 11.2.2(1)

Informally speaking, a cycle in a graph is a sequence of at least three edges, each joined to the next, and the last joined to the first, that has no repeated vertex.

Definition 11.2.1(2)

An undirected graph is *cyclic* if it has a loop or a cycle; else it is *acyclic*.

Theorem 11.2.5

An undirected graph G with no loop is cyclic if and only if it has two vertices between which there are two distinct paths.

Theorem 11.3.7

Let u, v be vertices in an undirected graph G. Then there is a path between u and v in G if and only if there is a connected component of G that has both u and v in it.