Chapter 8: Cardinality

CS1231 Discrete Structures

Wong Tin Lok

National University of Singapore

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> According to [Brouwer's] view and reading of history, classical logic was abstracted from the mathematics of finite sets and their subsets. [. . .] Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of set-theory, for which it is justly punished by the antinomies.
>
> Weyl (1946)

Plan

- Pigeonhole Principles
- when a set has the same number of elements as another set
- when a set has finitely many elements
- cardinalities of finite sets

Late at night, Tin Lok walks in to the Hilbert Hotel to see whether there is a vacant room for him. Unfortunately, the hotel is already full. Nevertheless, the clerk is able to make a special arrangement for him. The clerk says, “Let me ask the guest in Room 1 to move to Room 2, the guest in Room 2 to move to Room 3, etc. Then you can check in to Room 1.” Fortunately, the Hilbert Hotel has infinitely many rooms!

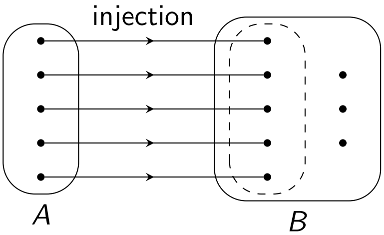
Injections and # elements

Theorem 8.1.1 (Pigeonhole Principle)

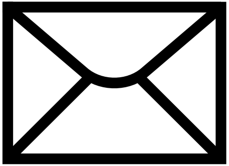
Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. If there is an injection $A \to B$, then $n \leqslant m$.

$x \in A$

Contrapositive. Let $n, m \in \mathbb{N}$ with $n > m$. If n letters are put into m pigeonholes, then there must be (at least) one pigeonhole with (at least) two letters.



Injectivity means that no two arrows point to the same dot.



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Proving the PHP

Theorem 8.1.1 (Pigeonhole Principle)

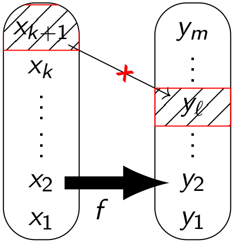
Let $A = \{x_1, x_2, \dots, x_n\}$ and $B = \{y_1, y_2, \dots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. If there is an injection $A \to B$, then $n \leqslant m$.

Proof by induction on n (sketch)

Base step: Prove the theorem for $n = 0$. If $n = 0$, then $m \geqslant 0 = n$ for all $m \in \mathbb{N}$.

Induction step: Let $k \in \mathbb{N}$ such that the theorem is true for $n = k$; we want to prove the theorem for $n = k + 1$.

Let $A = \{x_1, x_2, \dots, x_{k+1}\}$ and $B = \{y_1, y_2, \dots, y_m\}$, where $m \in \mathbb{N}$, such that the x's are different, and the y's are different. Suppose we have an injection $f : A \to B$. Suppose $f(x_{k+1}) = y_\ell$. Delete from f the arrow from x_{k+1} to y_ℓ, together with its endpoints. As one can verify, what remains is an injection $\{x_1, x_2, \dots, x_k\} \to \{\hat{y}_1, \hat{y}_2, \dots, \hat{y}_{m-1}\}$. So the induction hypothesis tells us $k \leqslant m - 1$. Hence $k + 1 \leqslant m$. $\square$



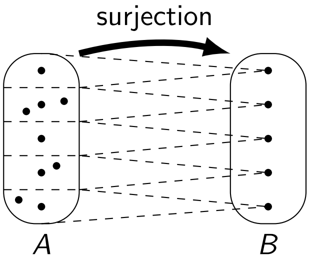
Surjections and # elements

Theorem 8.1.2 (Dual Pigeonhole Principle)

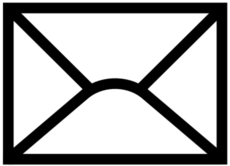
Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. If there is a surjection $A \to B$, then $n \geqslant m$.

x ∈ A y ∈ B

Contrapositive. Let $n, m \in \mathbb{N}$ with $n < m$. If n letters are put into m pigeonholes, then there must be (at least) one pigeonhole with no letter.



Surjectivity means that any dot on the right has an arrow pointing to it.



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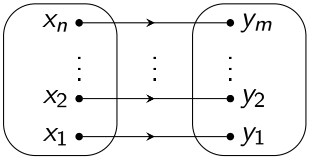
Bijections and # elements



Theorem 8.1.3

Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different. Then $n = m$ if and only if there is a bijection $A \to B$.

Proof

The $\Leftarrow$ part follows from the two Pigeonhole Principles. For the $\Rightarrow$ part, note that if $n = m$, then the function $f : A \to B$ satisfying $f(x_i) = y_i$ for all i is a bijection. $\square$

Comparing # elements using injections and surjections

Exercise 8.1.4

Prove the converse to the Pigeonhole Principle. Prove also the converse to the Dual Pigeonhole Principle when $B \neq \varnothing$. ✐ 8a

Same cardinality

Let A, B be sets.

Definition 8.2.1 (Cantor)

A is said to have the *same cardinality* as B if there is a bijection $A \to B$.

Note 8.2.2. We defined what it means for a set to have the same cardinality as another set without defining what the cardinality of a set is.

Example 8.2.3

(1) Let $n \in \mathbb{N}$. Then $\{0, 1, \ldots, n-1\}$ has the same cardinality as $\{1, 2, \ldots, n\}$ because the function $f\colon \{0, 1, \ldots, n-1\} \to \{1, 2, \ldots, n\}$ satisfying $f(x) = x + 1$ for all $x \in \{0, 1, \ldots, n-1\}$ is a bijection.

(2) $\mathbb{N}$ has the same cardinality as $\mathbb{N} \setminus \{0\}$ because the function $g\colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$ satisfying $g(x) = x + 1$ for all $x \in \mathbb{N}$ is a bijection.

(3) $\mathbb{N}$ has the same cardinality as $\mathbb{N} \setminus \{1, 3, 5, \ldots\}$ because the function $h\colon \mathbb{N} \to \mathbb{N} \setminus \{1, 3, 5, \ldots\}$ satisfying $h(x) = 2x$ for all $x \in \mathbb{N}$ is a bijection.

$$\begin{tikzcd} 0 \arrow[dr, "g"'] & 1 \arrow[dr] & 2 \arrow[dr] & 3 \arrow[dr] & 4 \arrow[dr] & \cdots \\ & 1 & 2 & 3 & 4 & 5 & \cdots \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[d, "h"'] & 1 \arrow[dr] & 2 \arrow[drr] & \cdots \\ 0 & & 2 & & 4 & \cdots \end{tikzcd}$$

Sanity checks

Proposition 8.2.4

The same-cardinality "relation" on "the set of all sets" is reflexive, symmetric, and transitive. ✎ Tutorial Exercise 8.3

Exercise 8.2.5

Prove that, if f is an injection $A \to B$, then A has the same cardinality as $\operatorname{range}(f)$. ✎ 8b

Definition 8.2.6

A set A is *finite* if it has the same cardinality as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. In this case, we call n the *cardinality* or the *size* of A, and we denote it by $|A|$. A set is *infinite* if it is not finite.

Exercise 8.2.7

Prove that no function $\mathbb{N} \to \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, can be injective. Deduce that $\mathbb{N}$ is infinite. ✎ 8c

Lemma 8.2.8

✎ Tutorial Exercise 8.5

Let A and B be sets of the same cardinality. Then A is finite if and only if B is finite.

Quick check

Question 8.2.9

Which of the following is/are true for all sets A, B? ✎ 8d

(1) If there is a bijection $A \to B$, then A has the same cardinality as B.

(2) If there is a surjection $A \to B$ that is not an injection, then A does not have the same cardinality as B.

(3) If there is an injection $A \to B$ that is not a surjection, then A does not have the same cardinality as B.

(4) If there is a function $A \to B$ that is neither a surjection nor an injection, then A does not have the same cardinality as B.

Summary

Let A, B be sets.

Theorems 8.1.1, 8.1.2, and 8.1.3

Suppose $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$, the x's are different, and the y's are different.

- (Pigeonhole Principle) If there is an injection $A \to B$, then $n \leqslant m$.
  (The converse is also true.)
- (Dual Pigeonhole Principle) If there is a surjection $A \to B$, then $n \geqslant m$.
  (The converse is also true if $B \neq \varnothing$.)
- $n = m$ if and only if there is a bijection $A \to B$.

Definition 8.2.1 (Cantor)

The set A is said to have the *same cardinality* as B if there is a bijection $A \to B$.

Definition 8.2.6

The set A is *finite* if it has the same cardinality as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. In this case, we call n the *cardinality* or the *size* of A, and we denote it by $|A|$. A set is *infinite* if it is not finite.