NATIONAL UNIVERSITY OF SINGAPORE

CS1231 — DISCRETE STRUCTURES
(Semester 2: AY2021/22)

Time allowed: 2 hours (exclusive of the time for scanning and submission)

INSTRUCTIONS TO STUDENTS

1. Do **NOT** scroll past this cover page and do **NOT** start writing until your invigilator tells you to do so.
2. Write your Student Number only. Do not write your name.
3. This exam paper contains **NINE** questions in **TWO** sections. It comprises **THREE** pages excluding this cover page.
4. Answer **ALL** questions.
5. Write your answers on paper. Specify the question numbers clearly. Do not copy the questions.
6. This is an **OPEN BOOK** exam. You may refer to any materials on physical paper or stored locally on your exam device.
7. Do **NOT** look up the Internet. Do **NOT** communicate with anyone except your invigilator.
8. The use of handheld calculators is **NOT** allowed. The use of any electronic device, except the proctoring device, the exam device, and a scanning device, is **NOT** allowed.
9. After the end of the exam, you will be given **10 MINUTES** to scan and submit your work.
10. When instructed by the invigilator, scan or take pictures of your work. Put all the images into a single `pdf` file in the right order for submission.
11. Name your submission as ⟨*your Student Number*⟩`.pdf`, for example, `A123456R.pdf`.
12. Submit your work on LumiNUS > Files > Exam > Submission > ⟨*your exam group*⟩.

Short questions **(total 10 marks from 5 questions)**

There is **no need** to explain your answers for questions in this section.

1. Consider the following propositions.

   (I) The set $\{(x, x^2) : x \in \mathbb{Z}\}$ can be viewed as a function $\mathbb{Z} \to \mathbb{Q}$.

   (II) The set $\{(x^2, x) : x \in \mathbb{Z}\}$ can be viewed as a function $\mathbb{Q} \to \mathbb{Z}$.

   Which of the following is true?

   A. (I) and (II) are both true.
   B. (I) is true, but (II) is false.
   C. (II) is true, but (I) is false.
   D. (I) and (II) are both false. [2 marks]

2. Consider the following propositions.

   (I) Any function $A \to A$, where A is a set, is a bijection.

   (II) If f is a bijection $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, then for all $X \in \mathcal{P}(\mathbb{N})$, the image $f(X)$ has the same cardinality as X.

   Which of the following is true?

   A. (I) and (II) are both true.
   B. (I) is true, but (II) is false.
   C. (II) is true, but (I) is false.
   D. (I) and (II) are both false. [2 marks]

3. Consider the following propositions.

   (I) There are two infinite sets that do not have the same cardinality.

   (II) The union of two uncountable sets is uncountable.

   Which of the following is true?

   A. (I) and (II) are both true.
   B. (I) is true, but (II) is false.
   C. (II) is true, but (I) is false.
   D. (I) and (II) are both false. [2 marks]

4. How many permutations of the string OCCURRENCE are there?

   A. $10!$.
   B. $P(10, 6) \times 2 \times 2 \times 3$.
   C. $\dfrac{10!}{2 \times 2 \times 3}$.
   D. $\dfrac{10!}{2!\,2!\,3!}$.
   E. None of the above. [2 marks]

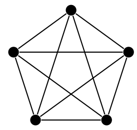
5. We have 100 undirected graphs. We know 72 of them are connected and 64 are acyclic. Assuming all our graphs are either connected or acyclic, how many of them are trees?

   A. 8.
   B. 36.
   C. 92.
   D. 136.
   E. None of the above. [2 marks]

Structured questions (total 30 marks from 4 questions)

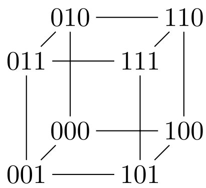
6. For this question, you may express your answers as products of expressions of the form n, n^r, $P(n, r)$, $n!$ or $\binom{n}{r}$ where $n, r \in \mathbb{N}$.

   (i) Calculate the number of cycles of length three in the graph drawn below. Briefly explain your answer.



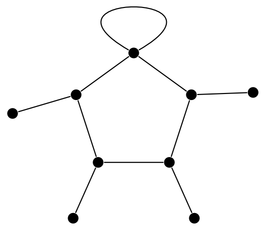
[2 marks]

   (ii) Calculate the number of paths between the vertices 000 and 111 in the graph drawn below. Briefly explain your answer.



[3 marks]

   (iii) Calculate the number of graphs, whose vertices are precisely $1, 2, \ldots, 9$, that are isomorphic to the graph drawn below. Briefly explain your answer.



[3 marks]

7. For a finite undirected graph G with at least one vertex, denote by $\delta(G)$ the smallest element of
$$\{d \in \mathbb{N} : \text{some vertex is in exactly } d \text{ edges in } G\}.$$

   (a) Prove by induction that, for all $n \in \mathbb{N}$ and all undirected graphs G with at least one vertex and with no loop, if $\delta(G) \geqslant n$, then G has a path of length n. [4 marks]

   (b) Someone attempts to prove the claim that any finite undirected graph G with at least one vertex has a path of maximum length as follows.

   > Consider $S = \{\ell \in \mathbb{N} : \text{there is no path of length at least } \ell \text{ in } G\}$. We know $|\mathrm{V}(G)| \in S$ because paths of length at least $|\mathrm{V}(G)|$ have at least $|\mathrm{V}(G)| + 1$ vertices, but G has only $|\mathrm{V}(G)|$ vertices. Thus $S \neq \varnothing$. Apply the ____________________ to find a smallest element, say k, of S. Since $(\{v\}, \{\})$ is a path of length 0 in G for any vertex v in G, we know $0 \notin S$ and so $k \geqslant 1$. Therefore, by the smallestness of k, we have a path P of length $k - 1$ in G. As $k \in S$, no path in G is strictly longer than P.

   What can one fill into the blank space to make this into a proof? [1 mark]

   (c) Use the claim established in (b) to prove that, for any finite undirected graph G with at least one vertex and with no loop, if $\delta(G) \geqslant 2$, then G has a cycle of length at least $\delta(G) + 1$. [4 marks]

   (d) Explain where you used the condition that $\delta(G) \geqslant 2$ in your proof for (c). If you already have such an explanation in your proof, then simply point out where it is. [1 mark]

8. Let G be an undirected graph with no loop and v be a vertex in G such that, for every vertex w in G, there is a unique path between v and w in G. Prove that G is a tree. [5 marks]

9. Fix a finite alphabet Γ. Recall that ε denotes the empty string and Γ^* denotes the set of all strings over Γ. For a string s over Γ, let us denote by $\operatorname{len}(s)$ the length of s, i.e., the number of symbols s has (including repetition).

   Suppose we have a subset $V \subseteq \Gamma^*$ that contains the empty string ε and satisfies the following condition:

   for all $k \in \mathbb{N}$ and $a_1 a_2 \ldots a_k a_{k+1} \in \Gamma^*$, if $a_1 a_2 \ldots a_k a_{k+1} \in V$, then $a_1 a_2 \ldots a_k \in V$.

   Define a graph G by setting $\mathrm{V}(G) = V$ and

   $$\mathrm{E}(G) = \big\{\{a_1 a_2 \ldots a_k, a_1 a_2 \ldots a_k a_{k+1}\} : a_1 a_2 \ldots a_k a_{k+1} \in V \text{ and } k \in \mathbb{N}\big\}.$$

   (a) Prove using the Well-Ordering Principle that, if

   $$\big(\{s_0, s_1, \ldots, s_\ell\}, \big\{\{s_0, s_1\}, \{s_1, s_2\}, \ldots, \{s_{\ell-1}, s_\ell\}\big\}\big)$$

   is a path in G where s_0 is the empty string ε and the s's are all different, then $\operatorname{len}(s_0) < \operatorname{len}(s_1) < \cdots < \operatorname{len}(s_\ell)$. [3 marks]

   (b) Use the propositions shown in (a) and in Question 8 to prove that G is a tree. [4 marks]

END OF PAPER