CS1231S
AY20/21 sem 1
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01. PROOFS

sets of numbers

$\mathbb{N}$: natural numbers ($\mathbb{Z}_{\geq 0}$)
$\mathbb{Z}$: integers
$\mathbb{Q}$: rational numbers
$\mathbb{R}$: real numbers
$\mathbb{C}$: complex numbers

basic properties of integers

closure (under addition and multiplication)
$x + y \in \mathbb{Z} \wedge xy \in \mathbb{Z}$
commutativity
$a + b = b + a \wedge ab = ba$
associativity
$a + b + c = a + (b + c) = (a + b) + c$
$abc = a(bc) = (ab)c$
distributivity
$a(b + c) = ab + ac$
trichotomy
$(a < b) \vee (a > b) \vee (a = b)$
transitive law
$(a < b) \wedge (b < c) \implies (a < c)$

definitions

even/odd
n is even $\leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k$
n is odd $\leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k + 1$
prime/composite
n is prime $\leftrightarrow n > 1$ and $\forall r, s \in \mathbb{Z}^+, n = rs \rightarrow (r = n) \vee (r = s)$
n is composite $\leftrightarrow n > 1$ and $\exists r, s \in \mathbb{Z}^+ s.t. n = rs$ and $1 < r < n$ and $1 < s < n$
divisibility (d divides n)
$d \mid n \leftrightarrow \exists k \in \mathbb{Z} \mid n = kd$
rationality
r is rational $\leftrightarrow \exists a, b \in \mathbb{Z} \mid r = \frac{a}{b}$ and $b \neq 0$
floor/ceiling
$\lfloor x \rfloor$: largest integer y such that $y \leq x$
$\lceil x \rceil$: smallest integer y such that $y \geq x$

rules of inference

generalisation	elimination
$p, \therefore p \vee q$	$p \vee q;\ \sim q,\ \therefore p$
specialisation	transitivity
$p \wedge q,\ \therefore p$	$p \rightarrow q;\ q \rightarrow r;\ \therefore p \rightarrow r$

04. METHODS OF PROOF

Proof by Exhaustion/Cases

1. list out possible cases
   1.1. Case 1: n is odd OR If $n = 9$, ...
   1.2. Case 2: n is even OR If $n = 16$, ...
2. therefore ...

Proof by Contradiction

1. Suppose that ...
   1.1. ¡proof¿
   1.2. ...but this contradicts ...
2. Therefore the assumption that ... is false. Hence ....

Proof by Contraposition

1. Contrapositive statement: $\sim q \rightarrow \sim p$
2. let $\sim q$
   2.1. ¡proof¿
   2.2. hence $\sim p$
3. $\therefore p \rightarrow q$

Proof by Construction

1. Let $x = 3, y = 4, z = 5$.
2. Then $x, y, z \in \mathbb{Z}_{\geq 1}$ and $x^2 + y^2 = 3^2 + 4^2 = 9 + 16 = 25 = 5^2$.
3. Thus $\exists x, y, z \in \mathbb{Z}_{\geq 1}$ such that $x^2 + y^2 = z^2$.

Proof by Induction

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition "..."
2. (base step) $P(1)$ is true because ¡manual method¿
3. (induction step)
   3.1. let $k \in \mathbb{Z}_{\geq 1}$ s.t. $P(k)$ is true
   3.2. Then ...
   3.3. proof that $P(k + 1)$ is true - e.g. $P(k + 1) = P(k) + term_{k+1}$
   3.4. So $P(k + 1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geq 1} P(n)$ is true by MI.

Proofs for Sets

Equality of Sets (A=B)

1. $(\Rightarrow)$
   1.1. Take any $z \in A$.
   1.2. ...
   1.3. $\therefore z \in B$.
2. $(\Leftarrow)$
   2.1. Take any $z \in B$.
   2.2. ...
   2.3. $\therefore z \in A$.

Element Method

1. $A \cap (B \backslash C) = \{x : x \in A \wedge x \in (B \backslash C)\}$ (by def. of $\cap$)
2. $= \{x : x \in A \wedge (x \in B \wedge x \notin C)\}$ (by def. of $\backslash$)
3. ...
4. $= (A \cap B) \backslash C$ (by def. of $\backslash$)

Other Proofs

iff ($A \leftrightarrow B$)

1. $(\Rightarrow)$ Suppose A.
   1.1. ... ¡proof¿ ...
   1.2. Hence $A \rightarrow B$
2. $(\Leftarrow)$ Suppose B.
   2.1. ... ¡proof¿ ...
   2.2. Hence $B \rightarrow A$

02. COMPOUND STATEMENTS

operations

1 $\sim$: negation (not)
2 $\wedge$: conjunction (and)
2 $\vee$: disjunction (or) - coequal to $\wedge$
3 $\rightarrow$: if-then

logical equivalence

- identical truth values in truth table
- definitions
- to show non-equivalence:
  - truth table method (only needs 1 row)
  - counter-example method

conditional statements

hypothesis $\rightarrow$ conclusion
$antecedent \rightarrow consequent$

- **vacuously true** : hypothesis is false
- **implication law** : $p \rightarrow q \equiv \sim p \vee q$
- common if/then statements:
  - if p then q: $p \rightarrow q$
  - p if q: $q \rightarrow p$
  - p only if q: $p \rightarrow q$
  - p iff q: $p \leftrightarrow q$
- **contrapositive** : $\sim q \rightarrow \sim p$
- **inverse** : $\sim p \rightarrow \sim q$
- **converse** : $q \rightarrow p$

converse $\equiv$ inverse
statement $\equiv$ contrapositive

- r is a **necessary** condition for s: $\sim r \rightarrow \sim s$ and $s \rightarrow r$
- r is a **sufficient** condition for s: $r \rightarrow s$
- **necessary** & **sufficient** : $\leftrightarrow$

valid arguments

- determining validity: construct truth table
  - valid $\leftrightarrow$ conclusion is true when premises are true
- **syllogism** : (argument form) 2 premises, 1 conclusion
- **modus ponens** : $p \rightarrow q;\ p;\ \therefore q$
- **modus tollens** : $p \rightarrow q;\ \sim q;\ \therefore \sim p$
- **sound argument** : is valid & all premises are true

fallacies

converse error	inverse error
$p \rightarrow q$	$p \rightarrow q$
q	$\sim p$
$\therefore p$	$\therefore \sim q$

03. QUANTIFIED STATEMENTS

- **truth set** of $P(x) = \{x \in D \mid P(x)\}$
- $P(x) \Rightarrow Q(x) : \forall x (P(x) \rightarrow Q(x))$
- $P(x) \Leftrightarrow Q(x) : \forall x (P(x) \leftrightarrow Q(x))$

relation between $\forall, \exists, \wedge, \vee$

- $\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$
- $\exists x \in D \mid Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$

05. SETS

notation

- set roster notation [1]: $\{x_1, x_2, \ldots, x_n\}$
- set roster notation [2]: $\{x_1, x_2, x_3, \ldots\}$
- set-builder notation: $\{x \in \mathbb{U} : P(x)\}$

definitions

- **equal sets** : $A = B \leftrightarrow \forall x (x \in A \leftrightarrow x \in B)$
  - $A = B \leftrightarrow (A \subseteq B) \wedge (A \supseteq B)$
- **empty set**, $\emptyset$: $\emptyset \subseteq$ all sets
- **subset** : $A \subseteq B \leftrightarrow \forall x (x \in A \rightarrow x \in B)$
- **proper subset** : $A \subsetneq B \leftrightarrow (A \subseteq B) \wedge (A \neq B)$
- **power set** of A : $\mathcal{P}(A) = \{X \mid X \subseteq A\}$
  - $|\mathcal{P}(A)| = 2^{|A|}$, given that A is a finite set
- **cardinality** of a set, $|\ A\ |$: number of distinct elements
- **singleton** : sets of size 1
- **disjoint** : $A \cap B = \emptyset$

methods of proof for sets

- direct proof
- element method
- truth table

boolean operations

- **union:** $A \cup B = \{x : x \in A \vee x \in B\}$
- **intersection:** $A \cap B = \{x : x \in A \wedge x \in B\}$
- **complement** (of B in A): $A \backslash B = \{x : x \in A \wedge x \notin B\}$
- **complement** (of B): $\bar{B}$ or $B^c = U \backslash B$
  - set difference law: $A \backslash B = A \cap \bar{B}$

ordered pairs and cartesian products

- **ordered pair** : (x, y)
  - $(x, y) = (x', y') \leftrightarrow x = x'$ and $y = y'$
- **Cartesian product** : $A \times B = \{(x, y) : x \in A$ and $y \in B\}$
  - $|A \times B| = |A| \times |B|$
- **ordered tuples** : expression of the form $(x_1, x_2, \ldots, x_n)$

06. FUNCTIONS

definitions

- **function/map** from A to B : assignment of each element of A to exactly one element of B.
  - $f : A \rightarrow B$: "f is a function from A to B"
  - $f : x \rightarrow y$: "f maps x to y"
  - **domain** of f = A
  - **codomain** of f = B
  - **range/image** of f = $\{f(x) : x \in A\}$ $= \{y \in B \mid y = f(x)$ for some $x \in A\}$
- **identity function** on A, $\text{id}_A : A \rightarrow A$
  - $\text{id}_A : x \rightarrow x$
  - range = domain = codomain = A
  - (E6.1.24) $f \circ \text{id}_A = f$ and $\text{id}_A \circ f = f$
- **well-defined function** : every element in the domain is assigned to exactly one element in the codomain

equality of functions

- same codomain and domain
- for all $x \in$ codomain, same output

function composition

- $(g \circ f)(x) = g(f(x))$
- for $(g \circ f)$ to be well defined, codomain of f must be equal to the domain of g
- × commutative
- ✓ **associative** - (T6.1.26) $f \circ (g \circ h) = (f \circ g) \circ h$

image & pre-image

for $f : A \to B$

- if $X \subseteq A$, **image** of X, $f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\}$
- if $Y \subseteq B$, **pre-image** of Y, $f^{-1}(Y) = \{x \in A : y = f(x) \text{ for some } y \in Y\}$

injection & surjection

- **surjective** (onto) : codomain = range
  - $\forall y \in B, \exists x \in A\ (y = f(x))$
  - surjective test: $\forall Y \subseteq B, Y \subseteq f(f^{-1}(Y))$
- **injective** : one-to-one
  - $\forall x, x' \in A (f(x) = f(x') \Rightarrow x = x')$
  - injective test: $\forall X \subseteq A, X \subseteq f^{-1}(f(X))$
- **bijective** : both surjective & injective
  - bijective $\Leftrightarrow$ has an inverse (T6.2.28)

inverse

- $\forall x \in A, \forall y \in B (f(x) = y \Leftrightarrow g(y) = x)$
- **uniqueness** of inverses (P2.6.16)
  - if g, g' are inverses of $f : A \to B$, then $g = g'$

07. INDUCTION

mathematical induction

to prove that $\forall n \in \mathbb{Z}_{\geq m}(P(n))$ is true,

- base step: show that $P(m)$ is true
- induction step: show that $\forall k \in \mathbb{Z}_{\geq m}(P(k) \Rightarrow P(k+1))$ is true.
  - induction hypothesis: assumption that $P(k)$ is true

strong MI

to prove that $\forall n \in \mathbb{Z}_{\geq 0}(P(n))$ is true,

- base step: show that $P(0), P(1)$ are true
- induction step: show that $\forall k \in \mathbb{Z}_{\geq 0}(P(0) \cdots \wedge P(k+1) \Rightarrow P(k+2))$ is true.

justification:

- $P(0) \wedge P(1)$ by base case
- $P(0) \wedge P(1) \to P(2)$ by induction with $k = 0$
- $P(0) \wedge P(1) \wedge P(2) \to P(3)$ by induction with $k = 1$
- $\ldots$
- we deduce that $P(0), P(1), \ldots$ are all true by a series of **modus ponens**

well-ordering principle

- every nonempty subset of $\mathbb{Z}_{\geq 0}$ has a smallest element.
- application: recursion has a base case

RECURSION

a sequence is **recursively defined** if the definition of a_n involves $a_0, a_1, \ldots, a_{n-1}$ for all but finitely many $n \in \mathbb{Z}_{\geq 0}$.

recursive definitions

e.g. recursive definition for $\mathbb{Z}$

1. (**base clause**) $0 \in \mathbb{Z}_{\geq 0}$
2. (**recursion clause**) If $x \in \mathbb{Z}_{\geq 0}$, then $x + 1 \in \mathbb{Z}_{\geq 0}$
3. (**minimality clause**) Membership for $\mathbb{Z}_{\geq 0}$ can be demonstrated by (finitely many) successive applications of the clauses above

recursion vs induction

- **recursion** - to define the set
- **induction** - to show things about the set

well-formed formulas (WFF)

in propositional logic

define the set of WFF(Σ) as follows

1. (base clause) every element ρ of Σ is in WFF(Σ)
2. (recursion clause) if x, y are in WFF(Σ), then $\sim x$ and $(x \wedge y)$ and $(x \vee y)$ are in WFF(Σ)
3. (minimality clause) Membership for WFF(Σ) can be demonstrated by (finitely many) successive applications of the clauses above

08. NUMBER THEORY

divisibility

transitivity of divisibility
If $a \mid b$ and $b \mid c$, then $a \mid c$.

closure lemma (non-standard name)
Let $a, b, d, m, n \in \mathbb{Z}$. If $d \mid m$ and $d \mid n$, then $d \mid am + bn$.

division theorem
$\forall n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, $\exists! q, r \in \mathbb{Z}$ s.t.
$n = dq + r$ and $0 \leq r < d$
$q = n\ div\ d = \lfloor n/d \rfloor$
$r = n \bmod d = n - dq$

base-b representation

of positive integer n is $(a_\ell a_{\ell-1} \ldots a_0)_b$
where $\ell \in \mathbb{Z}_{\geq 0}$ and $a_0, a_1, \ldots, a_\ell \in \{0, 1, \ldots, b-1\}$
s.t. $n = a_\ell b^\ell + a_{\ell-1} b^{\ell-1} + \cdots + a_0 b^0$ and $a_\ell \neq 0$

greatest common divisor

- if $m \neq 0$ and $n \neq 0$, then $\gcd(m, n)$ exists and is positive.
  - gcd: *Euclidean Algorithm*
  - integer linear combination: *Extended Euclidean Algorithm*

Bezout's Lemma:
For all $m, n \in \mathbb{Z}$ with $n \neq 0$, there exist $s, t \in \mathbb{Z}$ such that $\gcd(m, n) = ms + nt$.
Euclid's Lemma:
Let $m, n \in Z^+$. If p is prime and $p \mid mn$, then $p \mid m$ or $p \mid n$.

- (E8.4.3) $m \bmod n = 0 \Leftrightarrow \gcd(m, n) = n$
- (L8.4.11) $\forall x, y, r \in \mathbb{Z}$, $x \bmod y = r \Rightarrow \gcd(x, y) = \gcd(y, r)$

prime factorization thoerem

- **(aka Fundamental Theorem of Arithmetic)**: Every integer $n \geq 2$ has a unique prime factorization in which the prime factors are arranged in nondecreasing order.

modular arithmetic

$n \bmod d$ is always non-negative.

Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$.
congruence
$a \equiv b \pmod n \Leftrightarrow a \bmod n = b \bmod n$
Then $\exists k \in \mathbb{Z}(a = nk + b$ and $n \mid (a - b))$
reflexivity
$a \equiv a \pmod n$
symmetry
$a \equiv b \pmod n \to b \equiv a \pmod n$
transitivity
$a \equiv b \pmod n \wedge b \equiv c \pmod n \to a \equiv c \pmod n$

addition & multiplication

If $a \equiv b(\bmod n)$ and $c \equiv d(\bmod n)$,

- (P8.6.6) $a + c \equiv (b + d)(\bmod n)$
- (P8.6.13) $ac \equiv bd(\bmod n)$

additive inverse

b is an *additive inverse* of $a \bmod n \Leftrightarrow a + b \equiv 0 \pmod n$.
b is an *additive inverse* of $a \bmod n \Leftrightarrow b \equiv -a \pmod n$.

multiplicative inverse

b is a multiplicative inverse of $a \bmod n \Leftrightarrow ab \equiv 1 \pmod n$.

- If b, b' are multiplicative inverses of a, then $b \equiv b' \pmod n$.
- exists $\Leftrightarrow \gcd(a, n) = 1$.
  - a, n are coprime
- to find multiplicative inverse: **Euclidean Algorithm**

09. EQUIVALENCE RELATIONS

relations

Let R be a relation from A to B and $(x, y) \in A \times B$. Then:
xRy for $(x, y) \in R$ and $x\cancel{R}y$ for $(x, y) \notin R$

- a relation from A to B is a subset of $A \times B$.
- a **(binary) relation** on set A is a relation from A to A.
  - subset of A^2
- **inverse relation**: $xR^{-1}y \Leftrightarrow yRx$

reflexivity, symmetry, transitivity

Let A be a set and R be a relation on A.

reflexive
$\forall x \in A\ (xRx)$
symmetric
$\forall x, y \in A\ (xRy \Rightarrow yRx)$
transitive
$\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$

- **equivalence relation**: a relation that is reflexive, symmetric and transitive
- **equivalence class**: the set of all things equivalent to x

equivalence classes

Let A be a set and R be an equivalence relation on A.

- $[x]_R$: **equivalence class** of x with respect to R
  $\forall x \in A, [x]_R = \{y \in A : xRy\}$
- A/R : The set of all equivalent classes
  $A/R = \{[x]_R : x \in A\}$

$xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap [y] \neq \emptyset$

partitions

- a **partition** of a set A is a set $\mathscr{C}$ of *non-empty subsets* of A such that
  $(\geq 1)\ \ \forall x \in A,\ \exists S \in \mathscr{C}(x \in S)$
  $(\leq 1)\ \ \forall x \in A,\ \forall S, S' \in \mathscr{C}(x \in S \wedge x \in S' \Rightarrow S = S')$
- **components** : elements of a partition
- every partition comes from an equivalence relation

partial orders

Let A be a set and R be a relation on A.

- R is **antisymmetric** if $\forall x, y \in A\ (xRy \wedge yRx \to x = y)$
  - includes vacuously true cases (e.g. $xRy \Leftrightarrow x < y$)
- x and y are **comparable** if $\forall x, y \in A\ (xRy \vee yRx)$
- R is a **(non-strict) partial order** if R is reflexive, antisymmetric and transitive.
  - $\preccurlyeq$ - partial order
  - $x \prec y \Leftrightarrow x \preccurlyeq y \wedge x \neq y$ (NOT a partial order)
  - Hasse diagram
- R is a **(non-strict) total order** if R is a partial order and x and y are comparable

min and max

Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

- c is a **minimal element** if $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$
  - nothing is strictly below it
- c is a **maximal element** if $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x)$
  - nothing is strictly above it
- c is the **smallest element** or **minimum element** if $\forall x \in a\ (c \preccurlyeq x)$.
- c is the **largest element** or **maximum element** if $\forall x \in a\ (x \preccurlyeq c)$.

linearization

Let A be a set and $\preccurlyeq$ be a partial order on A.
Then there exists a total order $\preccurlyeq^*$ on A such that
$\forall x, y \in A\ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y)$

10A. COUNTING

permutations

$P(n, r) = \frac{n!}{(n-r)!}$ (also ${}_nP_r, P^n_r$)

- **multiplication/product rule:** An operation of k steps can be performed in $n_1 \times n_2 \times \cdots \times n_k$ ways.
- **addition/sum rule:** Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then
  $|A| = |A_1| + |A_2| + \cdots + |A_k|$
- **difference rule:** if A is a finite set and $B \subseteq A$, then
  $|A \backslash B| = |A| = |B|$
- **complement:** $P(\bar{A}) = 1 - P(A)$
- **inclusion/exclusion rule:** $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

permutations with indistinguishable objects

For n objects with n_k of type k indistinguishable from each other, the total number of distinguishable permutations
$= \frac{n!}{n_1!n_2!\dots n_k!}$

pigeonhole principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < \frac{n}{m}$, then there is some $y \in Y$ such that y is the image of at least $k+1$ distinct elements of X.

- A function from a finite set to a smaller finite set cannot be injective.
- **presentation:**
  - There are m ¡object M¿ (pigeons) and n ¡object N¿
  - Thus, by Pigeonhole Principle, ...

combinations

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (also $C(n,r)$, ${}_nC_r$, $C_{n,r}$, ${}^nC_r$)
r-combinations from n elements with **repetition**
$= \binom{r+n-1}{r}$

pascal's formula

Suppose $n, r \in \mathbb{Z}^+$ with $r \leq n$. Then
$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

binomial theorem

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
binomial coefficient: $\binom{n}{k}$

10B. PROBABILITY

probability

Let S be a sample space. For all events A and B in S, a *probability function* P satisfies the following axioms:

1. $0 \leq P(A) \leq 1$
2. $P(\emptyset) = 0$ and $P(S) = 1$
3. $(A \cap B = \emptyset) \Rightarrow [P(A \cup B) = P(A) + P(B)]$
4. $P(\bar{A}) = 1 - P(A)$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

expected value

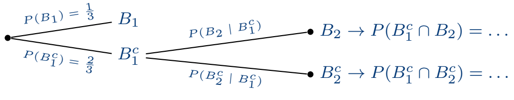
For possible outcomes $a_1, a_2, \dots, a_n$ which occur with probabilities $p_1, p_2, \dots, p_n$, the **expected value** is
$\sum_{k=1}^{n} = a_k p_k$

- **linearity** of expectation
  - $E[X+Y] = e[X] + E[Y]$
  - $E\left[\sum_{i=1}^{n} c_i \cdot X_i\right] = \sum_{i=1}^{n} (c_i \cdot E[X_i])$

conditional probability

The conditional probability of A given B,
$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

probability tree:



Bayes' theorem

Suppose a sample space S is a union of mutually disjoint events $B_1, B_2, \dots, B_n$ and A is an event in S.
For $k \in \mathbb{Z}$ and $1 \leq k \leq n$,

$$P(B_k \mid A) = \frac{P(A|B_k) \cdot P(B_k)}{\sum_{i=1}^{n} \left(P(A|B_i) \cdot P(B_i)\right)}$$

application of Bayes' theorem

$$P(B_1 \mid A) = \frac{P(A|B_1) \cdot P(B_1)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2)}$$

Let A be the event that the person test positive for a disease.
B_1: the person actually has the disease.
B_2: the person does not have the disease.

true positives: $P(B_1 \mid A)$	false negatives: $P(\bar{A} \mid B_1)$
false positives: $P(A \mid B_2)$	true negatives: $P(\bar{A} \mid B_2)$

independent events

A and B are **independent** *iff*
$P(A \cap B) = P(A) \cdot P(B)$
A, B and C are **pairwise independent** *iff*
1. $P(A \cap B) = P(A) \cdot P(B)$
2. $P(B \cap C) = P(B) \cdot P(C)$
3. $P(A \cap C) = P(A) \cdot P(C)$

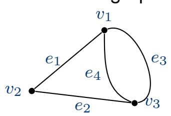
A, B and C are **mutually independent** *iff*
1. A, B and C are pairwise independent
2. $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

11. GRAPHS

- mathematical structures used to model pairwise relations between objects

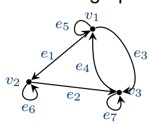
types of graphs

undirected graph



$e_1 = \{v_1, v_2\}$

directed graph



$e_1 = (v_1, v_2)$

undirected graph

- denoted by $G = (V, E)$, comprising
  - nonempty set of *vertices/nodes*, $V = \{v_1, v_2, \dots, v_n\}$
  - a set of *edges*, $E = \{e_1, e_2, \cdots, e_k\}$
- $e = \{v, w\}$ for an undirected edge E incident on vertices v and w

directed graph

- denoted by $G = (V, E)$, comprising
  - nonempty set V of *vertices*
  - a set E of *directed edges* (ordered pair of vertices)
- $e = (v, w)$ for an directed edge E from vertex v to vertex w

simple graph

- **undirected graph** with no loops or parallel edges

complete graph

- a complete graph on n vertices, $n > 0$, denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices

bipartite graph

- a simple graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V
- **complete bipartite graph:** $K_{m,n}$
  - bipartite graph on two disjoint sets U and V such that every vertex in U connects to every vertex in V
  - denoted $K_{m,n}$ where $|U| = m, |V| = n$

subgraph of a graph

H is a subgraph of $G \Leftrightarrow$
- every vertex in H is also a vertex in G
- every edge in H is also an edge in G
- every edge in H has the same endpoints as it has in G

degree

- **degree** of v, $deg(v)$ = number of edges incident on v
- **total degree** of G = sum of the degrees of all vertices of G
  total degree of G = $2\times$ (number of edges of G)
- (C10.1.2) the total degree of a graph is even
- (P10.1.3) in any graph there are an even number of vertices of odd degree

trails, paths and circuits

Let G be a graph; let v and w be vertices of G.
- **walk** (from v to w): a finite alternating sequence of adjacent vertices and edges of G.
  - e.g. $v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n$
  - **length** of walk: the number of edges, n
- a **trivial walk** from v to v consists of the single vertex v
- **trail** (from v to w): a walk from v to w that does not contain a repeated edge
- **path** (from v to w): a trail that does not contain a repeated vertex
- **closed walk**: walk that starts and ends at the same vertex
- **circuit/cycle**: an undirected graph $G(V, E)$ where
  - $V = \{x_1, x_2, \dots, x_n\}$
  - $E = \{\{x_1,x_2\},\{x_2,x_3\},\dots,\{x_{n-1},x_n\},\{x_n,x_1\}\}$
  - $n \in \mathbb{Z}_{\geq 3}$
  - aka a closed walk that does not contain a repeated edge
- **simple circuit/cycle**: does not have any other repeated vertex except the first and last
- (an undirected graph is) **cyclic** if it contains a loop/cycle

connectedness

- vertices v and w are connected $\Leftrightarrow \exists$ a walk from v to w
- graph G is connected $\Leftrightarrow \forall$ vertices $v, w \in V, \exists$ a walk from v to w

connected component

- a connected subgraph of the largest possible size
- graph H is a connected component of graph $G \Leftrightarrow$
  1. H is a subgraph of G
  2. H is connected
  3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H

Euler circuit

- **Euler circuit**: a circuit that contains every vertex and traverses every edge of G exactly once
- **Eulerian graph**: graph that contains an Euler circuit

T10.2.3
Euler circuit $\Leftrightarrow$ connected and every vertex has positive even degree
T10.2.4
Eulerian graph $\Leftrightarrow$ every vertex has positive even degree

- **Euler trail** (from v to w): a sequence of adjacent edges and vertices that starts at v, ends at w, and passes through every vertex of G at least once, and traverses every edge of G exactly once.

C10.2.5
$\exists$ Euler trail $\Leftrightarrow$ G is connected; v, w have odd degree; all other vertices of G have positive even degree

Hamiltonian circuit

- **Hamiltonian circuit** (for G): a *simple circuit* that includes every vertex of G.
  - does not need to include all the edges of G (unlike Euler circuit)
- **Hamilton(ian) graph**: contains a Hamiltonian circuit
- If G is a Hamiltonian circuit, then G has subgraph H where:
  1. H contains every vertex of G
  2. H is connected
  3. H has the same number of edges as vertices
  4. every vertex of H has degree 2

matrix representations of graphs

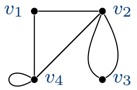
- **equal matrices** $\Leftrightarrow$ A and B are the same size and $a_{ij} = b_{ij}$ for all $i = 1, 2, \dots, m$ and $i = 1, 2, \dots, n$
- **square matrix**: equal number of rows and columns
- **main diagonal**: all entries $a_{11}, a_{22}, \dots, a_{nn}$
- **symmetric matrix** $\Leftrightarrow \forall i, j \in \mathbb{Z}^+_{\leq n} (a_{ij} = a_{ji})$

adjacency matrix

The adjacency matrix of a **directed graph** G is the $n \times n$ matrix $A = (a_{ij})$ over the set of non-negative integers such that
a_{ij} = number of **arrows** from v_i to v_j $\forall i, j = 1, 2, \dots, n$

$$A = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of an **undirected graph** G is the $n \times n$ matrix $A = (a_{ij})$ over the set of non-negative integers such that
a_{ij} = number of **edges** from v_i to v_j $\forall i, j = 1, 2, \dots, n$



$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 2 & 1 \\ v_3 & 0 & 2 & 0 & 0 \\ v_4 & 1 & 1 & 0 & 1 \end{array}$$

identity matrix

The $n \times n$ identity matrix,

$$I_n = (\delta_{ij}) = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \quad \text{for all } i, j = 1, 2, \ldots, n$$

matrix multiplication

scalar product

$$[a_{i1}\ a_{i2}\ \ldots\ a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

matrix product

Let $A = (a_{ij})$ be an $m \times k$ matrix and $B = (b_{ij})$ be a $k \times n$ matrix with real entries.

$$AB = (c_{ij}) = \sum_{r=1}^{k} a_{ir} b_{rj}$$

$\times$ commutative $\quad\checkmark$ associative

nth power of a matrix

For any $n \times n$ matrix **A**, the powers of A are defined as follows:

$A^0 = I$ where I is the $n \times n$ identity matrix

$A^n = AA^{n-1} \quad \forall n \in \mathbb{Z}_{\geq 1}$

counting walks of length N

number of walks of length n from v_i to v_j
$=$ the ij-th entry of A^n

isomorphism

- graph isomorphism ($\cong$) is an equivalence relation.

Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be two graphs.
$G \cong G' \Leftrightarrow$ there exist bijections $g : V_G \to V'_G$ and $h : E_G \to E'_G$ that preserve the edge-edgepoint functions of G and G' in the sense that $\forall v \in V_G$ and $e \in E_G$, v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

planar graph

- a graph that can be drawn on a two-dimensional plane without edges crossing.
  - divides a plane into *regions/faces* (includes 'outside' the graph)

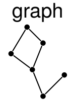
Euler's formula:
For a connected planar simple graph $G = (V, E)$ with $e = |E|$ and $v = |V|$ and f faces,
$f = e - v + 2$

Kuratowski's Theorem
A finite graph is planar $\Leftrightarrow$ does not contain a subgraph that is a subdivision of the complete graph K_5 or the complete bipartite graph K_3

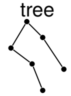
trees

- **tree** $\Leftrightarrow$ graph that is circuit-free and connected
  - **(L10.5.4)** If G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.
- **trivial tree**: graph that comprises a single vertex
- **forest** $\Leftrightarrow$ graph is circuit-free and not connected
  - a group of trees
- **terminal vertex**: a vertex of degree 1
- **internal vertex**: a vertex of degree greater than 1

graph



tree



forest

rooted trees

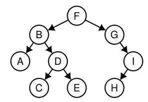
- **rooted tree**: a tree in which there is one vertex that is distinguished from the others and is called the root.
- **level** (of a vertex): the number of edges along the unique path between it and the root
- **height** (of a rooted tree): the maximum level of any vertex of the tree
- children, parent, siblings, ancestor, decendant

binary tree

- **binary tree**: a rooted tree in which every parent has at most 2 children
  - at most one left child and at most one right child
- **full binary tree**: a binary tree in which every parent has exactly 2 children
- (left/right) **subtree**: Given any parent v in a binary tree T, the binary tree whose root is the (left/right) child of v, whose vertices consist of the left child of v and all its descendants, and whose edges consist of all those edges of T that connect the vertices of the left subtree.

T10.6.1: Full Binary Tree Theorem
If T is a full binary tree with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices.

binary tree traversal



F-B-G-A-D-I-C-E-H

Breadth-First Search (BFS)
- starts at the root
- visits its adjacent vertices
- visits the next level

Depth-First Search (DFS)
- **pre-order**
  - current vertex $\to$ left subtree $\to$ right subtree
- **in-order**
  - left subtree $\to$ current vertex $\to$ right subtree
- **post-order**
  - left subtree $\to$ right subtree $\to$ current vertex

spanning trees

- **spanning tree** (for a graph G): a subgraph of G that contains every vertex of G and is a tree.
  - $w(e)$ - weight of edge e
  - $w(G)$ - total weight of G
- **weighted graph**: each edge has an associated positive real number weight
  - **total weight**: sum of the weights of all edges
- **minimum spanning tree**: least possible total weight compared to all other spanning trees

Kruskal's algorithm

For a connected weighted graph G with n vertices:

1. initialise T to have all the vertices of G and no edges.
2. let E be the set of all edges in G; let $m = 0$
3. while $(m < n - 1)$
   - 3.1. find and remove the edge e in E of least weight
   - 3.2. if adding e to the edge set of T does not produce a circuit:
     - i. add e to the edge set of T
     - ii. set $m = m + 1$

Prim's algorithm

For a connected weighted graph G with n vertices:

1. pick any vertex v of G and let T be the graph with this vertex only
2. let V be the set of all vertices of G except v
3. for ($i = 0$ to $n - 1$)
   - 3.1. find the edge e in G with the least weight of all the edges connected to T. let w be the endpoint of e.
   - 3.2. add e and w to the edge and vertex sets of T
   - 3.3. delete w from v

LOGICAL EQUIVALENCES

commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
identity laws	$p \wedge true \equiv p$	$p \vee false \equiv p$
idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
annihilators laws	$p \vee true \equiv true$	$p \wedge false \equiv false$
negation laws	$p \vee \sim p \equiv true$	$p \wedge \sim p \equiv false$
double negation law	$\sim(\sim p) \equiv p$	—
absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
De Morgan's Laws	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$
Implication law	$p \to q \equiv \sim p \vee q$	-

SET IDENTITIES

commutative laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
associative laws	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
identity laws	$A \cap U = A$	$A \cup \emptyset = A$
idempotent laws	$A \cap A = A$	$A \cup A = A$
universal bound laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
complement laws	$A \cap \overline{A} = \emptyset$	$A \cup \overline{A} = U$
double **complement** law	$\overline{(\overline{A})} = A$	—
absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$

proven:

number theory

- E1.1 - the product of 2 consecutive odd numbers is always odd.
- E1.5 - the difference between 2 consecutive squares is always odd
- E1.4 - the sum of any 2 even integers is even
- T4.6.1 - there is no greatest integer
- T8.2.8 - there are infinitely many prime numbers
- T4.3.1 - for all positive integers a and b, if $a|b$, then $a \leq b$.
- P4.6.4 - for all integers n, if n^2 is even then n is even
- T4.2.1 - all integers are rational numbers
- T4.2.2 - the sum of any 2 rational numbers is rational
- E1.7 - there exist irrational numbers p and q such that p^q is rational
- T4.7.1 - $\sqrt{2}$ is irrational.
- T4.3.2 - the only divisors of 1 are 1 and -1.

divisibility

- L8.1.5 - Let $d, n \in \mathbb{Z}$ with $d \neq 0$. Then $d \mid n \Leftrightarrow n/d \in \mathbb{Z}$
- L8.1.9 - Let $d, n \in \mathbb{Z}$. If $d \mid n$, then $-d \mid n$ and $d \mid -n$ and $-d \mid -n$
- L8.1.10 - Let $d, n \in \mathbb{Z}$. If $d \mid n$ and $d \neq 0$, then $|d| \leq |n|$
- L8.2.5 - **Prime Divisor Lemma** (non-standard name):
  - Let $n \in \mathbb{Z}_{\geq 2}$. Then n has a prime divisor.
- P8.2.6 - **sizes of prime divisors**:
  - Let n be a composite positive integer. Then n has a prime divisor $p \leq \sqrt{n}$.

base-b representation

- T8.3.13 - $\forall n \in \mathbb{Z}^+, \exists! \ell \in \mathbb{Z}_{\geq 0}$ and $a_0, a_1, \ldots, a_\ell \in \{0, 1, \ldots, b-1\}$ such that ¡the definition of base-b representation¿ holds.

logic

- T3.2.1 - negation of a universal statement:
  - $\sim(\forall x \in D, P(x)) \equiv \exists x \in D \mid \sim P(x)$
- T3.2.2 - negation of an existential statement:
  - $\sim(\exists x \in D \mid P(x)) \equiv \forall x \in D, \sim P(x)$

sets

- T4.1.18 - there exists a unique set with no element. It is denoted by $\emptyset$.
- E4.3.7 - for all A, B: $(A \cap B) \cup (A \backslash B) = A$
- E4.3.9(1) - $A \cap B \subseteq A$
- E4.3.9(2) - $A \subseteq A \cup B$
- T4.6 - $A \subseteq B \leftrightarrow \cup A \cup B = B$
- T5.3.11(1) - let A, B be disjoint finite sets. Then $|A \cup B| = |A| + |B|$
- T5.3.11(2) - let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets. Then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$
- T5.3.12 - **Inclusion-Exclusion Principle**:
  - for all finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$

induction

- L7.3.19 - If $x \in \text{WFF}^+(\Sigma)$, then assigning false to all elements of Σ makes x evaluate to false.
- T7.3.20 - $\sim(\forall x \in \text{WFF}(\Sigma), \exists y \in \text{WFF}^+(\Sigma) \;\; y \equiv x) \equiv$ $\exists x \in \text{WFF}(\Sigma) \;\; \forall y \in \text{WFF}^+(\Sigma) \;\; y \not\equiv x$ aka $\sim$ (not) must be included in the definition of WFF.

relations

- E9.2.11 - The equality relation R on a set A has equivalence classes of the form $[x] = \{y \in A : x = y\} = \{x\}$ where $x \in A$
- T9.3.4 - Let R be an equivalence relation on a set A. Then A/R is a partition of A.
- T9.3.5 - If $\mathscr{C}$ is a partition of A, then there is an equivalence relation of R on A such that $A/R = \mathscr{C}$.
- L9.5.5 - Consider a partial order $\preccurlyeq$ on set A.
  - A smallest element is minimal.
  - There is at most one smallest element.

graphs

- L10.2.1 - Let G be a graph.
  - L10.2.1a - If G is connected, then any two distinct vertices of G can be connected by a path
  - L10.2.1b - If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.
  - L10.2.1c - If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.
- L10.5.1 - Any non-trivial tree has at least one vertex of degree 1.
- T10.5.2 - Any tree with n vertices $(n > 0)$ has $n - 1$ edges.
- L10.5.3 - If G is any connected graph, C is any circuit in G, and one of the edges of C is removed from G, then the graph that remains is still connected.
- L10.5.4 - If G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.
- T10.6.1 - If T is a full binary tree with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices.
- T10.6.2 - For non-negative integers h, if T is any binary tree with height h and t terminal vertices, then $t \leq 2^h$.
- P10.7.1 -
  1. Every connected graph has a spanning tree.
  2. Any two spanning trees for a graph have the same number of edges

abbreviations

- L - lemma
- E - example
- P - proposition
- T - theorem