

01. Propositional Logic

sets of numbers

\mathbb{N} : natural numbers ($\mathbb{Z}_{\geq 0}$)

\mathbb{Z} : integers

\mathbb{Q} : rational numbers

\mathbb{R} : real numbers

\mathbb{C} : complex numbers

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

basic properties of integers

closure (under addition and multiplication)

$$x + y \in \mathbb{Z} \wedge xy \in \mathbb{Z}$$

commutativity

$$a + b = b + a \wedge ab = ba$$

associativity

$$a + b + c = a + (b + c) = (a + b) + c$$

$$abc = a(bc) = (ab)c$$

distributivity

$$a(b + c) = ab + ac$$

trichotomy

$$(a < b) \vee (a > b) \vee (a = b)$$

transitive law

$$(a < b) \wedge (b < c) \implies (a < c)$$

definitions

even/odd

$$n \text{ is even} \leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k$$

$$n \text{ is odd} \leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k + 1$$

prime/composite

$$n \text{ is prime} \leftrightarrow n > 1 \text{ and } \forall r, s \in \mathbb{Z}^+, n = rs \rightarrow (r = n) \vee (s = r)$$

$$n \text{ is composite} \leftrightarrow n > 1 \text{ and } \exists r, s \in \mathbb{Z}^+ \text{ s.t. } n =$$

$$rs \text{ and } 1 < r < n \text{ and } 1 < s < n$$

divisibility (d divides n)

$$d \mid n \leftrightarrow \exists k \in \mathbb{Z} \mid n = kd$$

rationality

$$r \text{ is rational} \leftrightarrow \exists a, b \in \mathbb{Z} \mid r = \frac{a}{b} \text{ and } b \neq 0$$

floor/ceiling

$$\lfloor x \rfloor : \text{largest integer } y \text{ such that } y \leq x$$

$$\lceil x \rceil : \text{smallest integer } y \text{ such that } y \geq x$$

rules of inference

generalisation

$$p, \therefore p \vee q$$

specialisation

$$p \wedge q, \therefore p$$

elimination

$$p \vee q; \sim q, \therefore p$$

transitivity

$$p \rightarrow q; q \rightarrow r; \therefore p \rightarrow r$$

03. PROOFS

Proof by Exhaustion/Cases

- list out possible cases
 - Case 1: n is odd OR If $n = 9$, ...
 - Case 2: n is even OR If $n = 16$, ...
- therefore ...

Proof by Contradiction

- Suppose that ...
 - iproof ζ
 - ...but this contradicts ...
- Therefore the assumption that ... is false.
Hence

Proof by Contraposition

- Contrapositive statement: $\sim q \rightarrow \sim p$
- let $\sim q$
 - iproof ζ
 - hence $\sim p$
- $\therefore p \rightarrow q$

Proof by Construction

- Let $x = 3, y = 4, z = 5$.
- Then $x, y, z \in \mathbb{Z}_{\geq 1}$ and $x^2 + y^2 = 3^2 + 4^2 = 9 + 16 = 25 = 5^2$.
- Thus $\exists x, y, z \in \mathbb{Z}_{\geq 1}$ such that $x^2 + y^2 = z^2$.

Proof by Induction

- For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition "..."
- (base step) $P(1)$ is true because imanual method ζ
- (induction step)
 - let $k \in \mathbb{Z}_{\geq 1}$ s.t. $P(k)$ is true
 - Then ...
 - proof that $P(k + 1)$ is true - e.g.
 $P(k + 1) = P(k) + \text{term}_{k+1}$
 - So $P(k + 1)$ is true.
- Hence $\forall n \in \mathbb{Z}_{\geq 1} P(n)$ is true by MI.

INDUCTION

mathematical induction

to prove that $\forall n \in \mathbb{Z}_{\geq m} (P(n))$ is true,

- base step: show that $P(m)$ is true
- induction step: show that $\forall k \in \mathbb{Z}_{\geq m} (P(k) \Rightarrow P(k + 1))$ is true.
 - induction hypothesis: assumption that $P(k)$ is true

strong MI

to prove that $\forall n \in \mathbb{Z}_{\geq 0} (P(n))$ is true,

- base step: show that $P(0), P(1)$ are true
- induction step: show that $\forall k \in \mathbb{Z}_{\geq 0} (P(0) \cdots \wedge P(k + 1) \Rightarrow P(k + 2))$ is true.

justification:

- $P(0) \wedge P(1)$ by base case
- $P(0) \wedge P(1) \rightarrow P(2)$ by induction with $k = 0$
- $P(0) \wedge P(1) \wedge P(2) \rightarrow P(3)$ by induction with $k = 1$
- ...
- we deduce that $P(0), P(1), \dots$ are all true by a series of **modus ponens**

Proofs for Sets

Equality of Sets (A=B)

- (\Rightarrow)
 - Take any $z \in A$.
 - ...
 - $\therefore z \in B$.
- (\Leftarrow)
 - Take any $z \in B$.
 - ...

2.3. $\therefore z \in A$.

Element Method

- $A \cap (B \setminus C) = \{x : x \in A \wedge x \in (B \setminus C)\}$ (by def. of \cap)
- $= \{x : x \in A \wedge (x \in B \wedge x \notin C)\}$ (by def. of \setminus)
- ...
- $= (A \cap B) \setminus C$ (by def. of \setminus)

Other Proofs

iff ($A \leftrightarrow B$)

- (\Rightarrow) Suppose A .
 - ... iproof ζ ...
 - Hence $A \rightarrow B$
- (\Leftarrow) Suppose B .
 - ... iproof ζ ...
 - Hence $B \rightarrow A$

02. PREDICATE LOGIC

operations

- \sim : negation (not)
- \wedge : conjunction (and)
- \vee : disjunction (or) - coequal to \wedge
- \rightarrow : if-then

logical equivalence

- identical truth values in truth table
- definitions
- to show non-equivalence:
 - truth table method (only needs 1 row)
 - counter-example method

conditional statements

hypothesis \rightarrow *conclusion*

antecedent \rightarrow *consequent*

- vacuously true**: hypothesis is false
- implication law**: $p \rightarrow q \equiv \sim p \vee q$
- common statements for $p \rightarrow q$:
 - if p then q
 - q if p
 - p only if q
 - p iff q
 - p is sufficient for q
 - q is necessary for p
- contrapositive**: $\sim q \rightarrow \sim p$ statement \equiv contrapositive
- inverse**: $\sim p \rightarrow \sim q$ converse \equiv inverse
- converse**: $q \rightarrow p$
- r is a **necessary** condition for s: $\sim r \rightarrow \sim s$ and $s \rightarrow r$
- r is a **sufficient** condition for s: $r \rightarrow s$
- necessary & sufficient**: \leftrightarrow

valid arguments

- determining validity: construct truth table
 - valid \leftrightarrow conclusion is true when premises are true
- syllogism**: (argument form) 2 premises, 1 conclusion
- modus ponens**: $p \rightarrow q; p; \therefore q$
- modus tollens**: $p \rightarrow q; \sim q; \therefore \sim p$
- sound argument**: is valid & all premises are true

fallacies

converse error

$$p \rightarrow q$$

$$q$$

$$\therefore p$$

inverse error

$$p \rightarrow q$$

$$\sim p$$

$$\therefore \sim q$$

QUANTIFIED STATEMENTS

- truth set** of $P(x) = \{x \in D \mid P(x)\}$
- $P(x) \Rightarrow Q(x) : \forall x(P(x) \rightarrow Q(x))$
- $P(x) \Leftrightarrow Q(x) : \forall x(P(x) \leftrightarrow Q(x))$

relation between $\forall, \exists, \wedge, \vee$

- $\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$
- $\exists x \in D \mid Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$

relation between \sim, \forall, \exists

- $\sim \forall P(x) \leftrightarrow \exists x \sim P(x)$
- $\sim \exists P(x) \leftrightarrow \forall x \sim P(x)$

04. SETS

notation

- set roster notation [1]: $\{x_1, x_2, \dots, x_n\}$
- set roster notation [2]: $\{x_1, x_2, x_3, \dots\}$
- set-builder notation: $\{x \in \mathbb{U} : P(x)\}$
- replacement notation: $\{t(x) : x \in A\}$

definitions

- equal sets**: $A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$
 - $A = B \leftrightarrow (A \subseteq B) \wedge (A \supseteq B)$
 - order and repetition does not matter
- subset**: $A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$
- proper subset**: $A \subset B \leftrightarrow (A \subseteq B) \wedge (A \neq B)$
- power set** of A: $\mathcal{P}(A) = \{X \mid X \subseteq A\}$
 - $|\mathcal{P}(A)| = 2^{|A|}$, given that A is a finite set
 - $\mathcal{P}(\emptyset) = \{\emptyset\}$; $\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
 - $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
- cardinality** of a set, $|A|$: number of distinct elements
- singleton**: sets of size 1
- disjoint**: $A \cap B = \emptyset$

methods of proof for sets

- direct proof
- element method
- truth table

boolean operations

- union**: $A \cup B = \{x : x \in A \vee x \in B\}$
- intersection**: $A \cap B = \{x : x \in A \wedge x \in B\}$
- complement** (of B in A): $A \setminus B = \{x : x \in A \wedge x \notin B\}$
- complement** (of B): \bar{B} or $B^c = U \setminus B$
 - set difference law: $A \setminus B = A \cap \bar{B}$

05. RELATIONS

ordered pairs

- ordered pair**: (x, y)
 - $(x, y) = (x', y') \leftrightarrow x = x' \text{ and } y = y'$
- Cartesian product**:
 $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$
 - $|A \times B| = |A| \times |B|$
 - $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

- **ordered tuples** : expression of the form (x_1, x_2, \dots, x_n)
- defined recursively :
 $(x_1, x_2, \dots, x_{n+1}) = ((x_1, x_2, \dots, x_n), x_{n+1})$
- $(1, 2, 5) \neq (2, 1, 5)$ although $\{1, 2, 5\} = \{2, 1, 5\}$

relations

Let R be a relation from A to B and $(x, y) \in A \times B$. Then:
 xRy for $(x, y) \in R$ and $x\not R y$ for $(x, y) \notin R$

- a relation from A to B is a subset of $A \times B$.
- a **(binary) relation** on set A is a relation from A to A.
 - subset of A^2
- **inverse relation**: $xR^{-1}y \Leftrightarrow yRx$

operations on relations

- $S \circ R$ = undergo R relation then S relation
- $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

06. EQUIVALENCE RELATIONS AND PARTIAL ORDERS

reflexivity, symmetry, transitivity

Let A be a set and R be a relation on A .

$$\begin{array}{c} \text{reflexive} \\ \forall x \in A \ (xRx) \\ \text{symmetric} \\ \forall x, y \in A \ (xRy \Rightarrow yRx) \\ \text{transitive} \\ \forall x, y, z \in A \ (xRy \wedge yRz \Rightarrow xRz) \end{array}$$

- **equivalence relation**: a relation that is reflexive, symmetric and transitive
- **equivalence class**: the set of all things equivalent to x

equivalence classes

Let A be a set and R be an equivalence relation on A .

- $[x]_{\sim}$: **equivalence class** of x with respect to R
 - the set of all elements of A that x is related to
 $\forall x \in A, [x]_{\sim} = \{y \in A : xRy\}$
- A/\sim : The set of all equivalent classes
 $A/R = \{[x]_{\sim} : x \in A\}$

$xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap [y] \neq \emptyset$

partitions

- a **partition** of a set A is a set \mathcal{C} of *non-empty subsets* of A such that
 0. $\forall S \in \mathcal{C}, (\emptyset \neq S \subseteq A)$
 - \mathcal{C} is a set of nonempty subsets of A
 1. $\forall x \in A, \exists S \in \mathcal{C} (x \in S)$
 - every element of A is in some element of \mathcal{C}
 2. $\forall x \in A, \forall S_1, S_2 \in \mathcal{C} (x \in S_1 \wedge x \in S_2 \Rightarrow S_1 = S_2)$
 - if two items of \mathcal{C} have a nonempty intersection, then they are equal
- **components** : elements of a partition
- every partition comes from an equivalence relation

partial orders

- Let A be a set and R be a relation on A .
- R is **antisymmetric** if $\forall x, y \in A \ (xRy \wedge yRx \rightarrow x = y)$
 - includes vacuously true cases (e.g. $xRy \Leftrightarrow x < y$)
 - R is a **(non-strict) partial order** if R is reflexive, antisymmetric and transitive.
 - x and y are **comparable** if $\forall x, y \in A \ (xRy \vee yRx)$
 - R is a **(non-strict) total order** if R is a partial order and every pair of elements are comparable
 - a smallest element of A is an element $m \in A$ such that mRx for all $x \in A$

well-ordering principle

- every nonempty subset of $\mathbb{Z}_{\geq 0}$ has a smallest element.
- application: recursion has a base case

07. FUNCTIONS

definitions

- **function/map** from A to B : each element of A exactly f -related to one element of B.
 - **Important** : $(x, y) \in f \leftrightarrow y = f(x)$
 - (F1): every element in A f -related to at least one of B
 $\forall x \in A \exists y \in B \ (x, y) \in f$
 - (F2): every element in A f -related to at most one of B
 $\forall x \in A \exists y_1, y_2 \in B \ ((x, y_1) \in f \wedge (x, y_1) \in f \rightarrow y_1 = y_2)$
 - $f : A \rightarrow B$: " f is a function from A to B "
 - $f : x \rightarrow y$: " f maps x to y "
 - **domain** of f = A
 - **codomain** of f = B
 - **range/image** of f = $\{f(x) : x \in A\}$
 - * $\text{range}(f) \in \text{codomain}$
 - * if f is surjective: $\text{range}(f) \in \text{codomain} \in \text{range}(f)$
- **identity function** on A, $\text{id}_A : A \rightarrow A$
 - $\text{id}_A : x \rightarrow x$
 - $\text{range} = \text{domain} = \text{codomain} = A$
 - (P7.4.13) $f \circ \text{id}_A = f$ and $\text{id}_A \circ f = f$
- **well-defined function** : every element in the domain is assigned to exactly one element in the codomain

equality of functions

- same codomain and domain
- for all $x \in \text{codomain}$, same output

function composition

- $(g \circ f)(x) = g(f(x))$
- for $(g \circ f)$ to be well defined, codomain of f must be equal to the domain of g
- \times commutative $(g \circ f)(x) \neq (f \circ g)(x)$
- \checkmark **associative** - (T6.1.26) $f \circ (g \circ h) = (f \circ g) \circ h$

image & pre-image

- for $f : A \rightarrow B$
- if $X \subseteq B$, **image** of X,
 $f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\}$
 - if $Y \subseteq B$, **pre-image** of Y,
 $f^{-1}(Y) = \{x \in A : y = f(x) \text{ for some } y \in Y\}$

injection & surjection

- **surjective** (onto) : codomain = range
 - for every B , there is a A
 $\forall y \in B \exists x \in A \ (y = f(x))$
 - a function is **not** surjective iff
 $\exists y \in B \forall x \in A \ (y \neq f(x))$
- **injective** : one-to-one
 - for every B , at most one A
 $\forall x_1, x_2 \in A \ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$
 - a function is **not** injective iff
 $\exists x_1, x_2 \in A \ (f(x_1) = f(x_2) \wedge x_1 \neq x_2)$
- **bijective** : both surjective & injective

inverse

- $\forall x \in A, \forall y \in B (f(x) = y \Leftrightarrow g(y) = x)$
- **uniqueness** of inverses (P2.6.16)
 - if g, g' are inverses of $f : A \rightarrow B$, then $g = g'$

8. CARDINALITY

pigeonhole principle

For any function f from a finite set A with n elements to a finite set B with m elements if there is an injection $A \rightarrow B$, then $n \leq m$

dual pigeonhole principle

For any function f from a finite set A with n elements to a finite set B with m elements if there is an surjection $A \rightarrow B$, then $n \geq m$

T8.1.3

For any function f from a finite set A with n elements to a finite set B with m elements if there is a bijection $A \rightarrow B$, then $n = m$

- A function from a finite set to a smaller finite set cannot be injective.
- **presentation**:
 - There are m pigeons and n peigionholes
 - Thus, by Pigeonhole Principle, ...

same cardinality

- A set A is said to have the same cardinality (HSC) as a set B if there is a bijection $A \rightarrow B$
- reflexivity : A HSC A .
- symmetry : if A HSC B , then B HSC A .
- transitivity : if A HSC B , and B HSC C , then A HSC C .

finite sets

- A set A is finite if it HSC $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$
- n is the cadinality/size of A , denoted by $|A|$
- Let A and B be sets that HSC, then A is finite iff B is finite

9. COUNTABILITY

countable sets

- A set is countable if it is finite or has the same cardinality as \mathbb{N}
- \mathbb{Z} is countable
- $\mathbb{N} \times \mathbb{N}$ is countable

countability

- Let A and B be sets of same cardinality. A is countable iff B is countable
- Let A, B be sets such that $A \subseteq B$
 - If B is finite, then A is finite
 - If B is countable, then A is countable
- A set B is infinite if there is an injection f from some infinite set A to B
- A set B is uncountable if there is an injection f from some uncountable set A to B

uncountable sets

- No set A has the same cadinality as $\mathcal{P}(A)$
- Let A be countable infinite set, then $\mathcal{P}(A)$ is uncountable. Hence $\mathcal{P}(\mathbb{N})$ is uncountable

non-computability

- There is a subset S of \mathbb{N} s.t no program can, when given any input $n \in \mathbb{N}$
 - output T if $n \in S$; and
 - output F if $n \notin S$
- i.e no program can correctly determine whether a given input n belongs to S or not, for all possible inputs n.

10. COUNTING

rules

- **addition/sum rule**: Let A and B be **disjoint** finite sets
 $|A \cup B| = |A| + |B|$
- **difference rule**: Let X and Y be finite sets. Then $Y \setminus X$ is finite, and if $X \subseteq Y$
 $|Y \setminus X| = |Y| - |X|$
- **inclusion/exclusion rule 2 sets**:
 $|A \cup B| = |A| + |B| - |A \cap B|$
- **inclusion/exclusion rule 3 sets** : $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$
- **multiplication/product rule**: $|A \times B| = |A| \times |B|$
- **general multiplication rule**: Let A be set of size m , and for each $x \in A$, let B_x be set of size n . Then $\{(x, y) : x \in A \text{ and } y \in B_x\}$ is finite and has size mn
- **complement**: $P(\bar{A}) = 1 - P(A)$
- $|\mathcal{P}(A)| = 2^{|A|}$, given that A is a finite set

permutations

pick r elements from a set of size n without replacement where order matters

$P(n, r) = \frac{n!}{(n-r)!} \quad (\text{also } {}_n P_r, P_r^n)$

if $r > n$, 0 ways

permutations with indistinguishable objects

For n objects with n_k of type k indistinguishable from each other, the total number of distinguishable permutations
 $= \frac{n!}{n_1!n_2!...n_k!}$
E.g. num of permutatations for "EGG" = $\frac{3!}{2!} = 3$

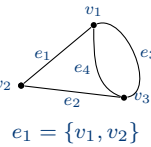
combinations

Let A be a set of size n . Number of subsets of A of size r is
 $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (also $C(n, r), {}_n C_r, C_{n,r}, {}^n C_r$)
 r -combinations from n elements with **repetition**
 $= \binom{r+n-1}{r}$

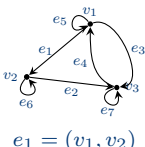
11. GRAPHS

types of graphs

undirected graph



directed graph



undirected graph

- denoted by $G = (V, E)$, comprising
 - nonempty set of *vertices/nodes*, $V = \{v_1, v_2, \dots, v_n\}$
 - a set of *edges*, $E = \{e_1, e_2, \dots, e_k\}$
- $e = \{v, w\}$ for an undirected edge E incident on vertices v and w

directed graph

- denoted by $G = (V, E)$, comprising
 - nonempty set V of *vertices*
 - a set E of *directed edges* (ordered pair of vertices)
- $e = (v, w)$: an directed edge E from vertex v to vertex w

simple graph

- undirected graph** with no loops or parallel edges

complete graph

- a complete graph on n vertices, $n > 0$, denoted K_n , is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices

subgraph of a graph

- H is a subgraph of $G \Leftrightarrow$
 - every vertex in H is also a vertex in G
 - every edge in H is also an edge in G
- a **proper** subgraph of G is a subgraph G of G such that $H \neq G$

paths and walks

- Let G be a graph; let v and w be vertices of G .
- path** (from v to w): links two vertices in a graph via a sequence of edges, with no repeated vertices ($\{x_0, x_1, \dots, x_l\}, \{x_0x_1, x_1x_2, \dots, x_{l-1}x_l\}$) where $v = x_0$ and $w = x_l$

- length** of walk: the number of edges, l
- each vertex in a path is in at most 2 edges in the path
- If there is a path P between u and v , and path G between v and w . Then there is a path between u and w

cycles

- circuit/cycle**: an undirected graph $G(V, E)$ where
 - $V = \{x_1, x_2, \dots, x_l\}$
 - $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{l-1}, x_l\}, \{x_l, x_1\}\}$
 - $n \in \mathbb{Z}_{\geq 3}$
- simple circuit/cycle**: does not have any other repeated vertex except the first and last
- An undirected graph is **cyclic** if it contains a loop/cycle
- An undirected graph is with no loop is cyclic iff it has two vertices between which there are two distinct paths

connectedness

- vertices v and w are connected $\Leftrightarrow \exists$ a path from v to w
- graph G is connected $\Leftrightarrow \forall$ vertices $v, w \in V, \exists$ a path from v to w

connected component

- a connected subgraph of the largest possible size
- graph H is a connected component of graph $G \Leftrightarrow$
 - H is a subgraph of G
 - H is connected
 - no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H
- P11.3.6 - Let G be an undirected graph, then every vertex v in G is in some connected component of G
- T11.3.7 - Let u, v be vertices in an undirected graph G . Then there is a path between u and v in $G \Leftrightarrow$ there is a connected component G that has both u and v in it

counting walks of length N

number of walks of length n from v_i to v_j
= the ij -th entry of A^n

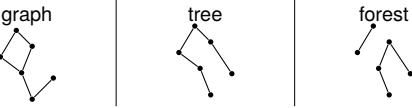
isomorphism

- graph isomorphism (\cong) is an equivalence relation.
- Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be two graphs.
 $G \cong G' \Leftrightarrow$ there exist bijections $g : V_G \rightarrow V_{G'}$ and $h : E_G \rightarrow E_{G'}$ that preserve the edge-edgepoint functions

of G and G' in the sense that $\forall v \in V_G$ and $e \in E_G$,
 v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

12. TREES

- tree** is a **connected acyclic undirected** graph
 - If G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.
- terminal vertex / leaf**: vertex with no child
- internal vertex / parent**: vertex that is not terminal



rooted trees

- rooted tree**: a tree in which there a distinguished vertex called the root.
- height** (of a rooted tree): length of a longest path between root and some vertex

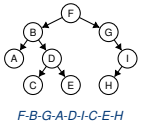
binary tree

- binary tree**: a rooted tree in which every parent has at most 2 children
 - at most one left child and at most one right child
- full binary tree**: a binary tree in which every parent has exactly 2 children
- (left/right) **subtree**: Given any parent v in a binary tree T , the binary tree whose root is the (left/right) child of v , whose vertices consist of the left child of v and all its descendants, and whose edges consist of all those edges of T that connect the vertices of the left subtree.

T10.6.1: Full Binary Tree Theorem

If T is a full binary tree with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices.

binary tree traversal



Breadth-First Search (BFS)

- starts at the root
- visits its adjacent vertices
- visits the next level

Depth-First Search (DFS)

- pre-order**
 - current vertex \rightarrow left subtree \rightarrow right subtree
- in-order**
 - left subtree \rightarrow current vertex \rightarrow right subtree
- post-order**
 - left subtree \rightarrow right subtree \rightarrow current vertex

spanning trees

- spanning tree** (for a graph G): a subgraph of G that contains every vertex of G and is a tree.
 - $w(e)$ - weight of edge e
 - $w(G)$ - total weight of G
- weighted graph**: each edge has an associated positive real number weight
 - total weight**: sum of the weights of all edges
- minimum spanning tree**: least possible total weight compared to all other spanning trees

Kruskal's algorithm

- For a connected weighted graph G with n vertices:
- initialise T to have all the vertices of G and no edges.
 - let E be the set of all edges in G ; let $m = 0$
 - while ($m < n - 1$)
 - find and remove the edge e in E of least weight
 - if adding e to the edge set of T does not produce a circuit:
 - add e to the edge set of T
 - set $m = m + 1$

Prim's algorithm

- For a connected weighted graph G with n vertices:
- pick any vertex v of G and let T be the graph with this vertex only
 - let V be the set of all vertices of G except v
 - for ($i = 0$ to $n - 1$)
 - find the edge e in G with the least weight of all the edges connected to T . let w be the endpoint of e .
 - add e and w to the edge and vertex sets of T
 - delete w from v

LOGICAL EQUIVALENCES			SET IDENTITIES		
commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$	commutative laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	associative laws	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
identity laws	$p \wedge \text{true} \equiv p$	$p \vee \text{false} \equiv p$	identity laws	$A \cap U = A$	$A \cup \emptyset = A$
idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$	idempotent laws	$A \cap A = A$	$A \cup A = A$
annihilators laws	$p \vee \text{true} \equiv \text{true}$	$p \wedge \text{false} \equiv \text{false}$	annihilators laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
negation laws	$p \vee \sim p \equiv \text{true}$	$p \wedge \sim p \equiv \text{false}$	complement laws	$A \cap \overline{A} = \emptyset$	$A \cup \overline{A} = U$
double negation law	$\sim(\sim p) \equiv p$	—	double complement law	$\overline{(\overline{A})} = A$	—
absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$	absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Implication law	$p \rightarrow q \equiv \sim p \vee q$	-	Set difference	$A \setminus B \equiv A \cap \overline{B}$	-

proven:

number theory

- the product of 2 consecutive odd numbers is always odd.
- the difference between 2 consecutive squares is always odd
- P3.2.4 - the square of any 2 even integers is even
- there is no greatest integer
- there are infinitely many prime numbers
- for all positive integers a and b , if $a|b$, then $a \leq b$.
- P3.2.8 - for all integers n , if n^2 is even then n is even
- all integers are rational numbers
- the sum of any 2 rational numbers is rational
- there exist irrational numbers p and q such that p^q is rational
- $\sqrt{2}$ is irrational.
- the only divisors of 1 are 1 and -1 .

divisibility

- L8.1.5 - Let $d, n \in \mathbb{Z}$ with $d \neq 0$. Then $d \mid n \Leftrightarrow n/d \in \mathbb{Z}$
- L8.1.9 - Let $d, n \in \mathbb{Z}$. If $d \mid n$, then $-d \mid n$ and $d \mid -n$ and $-d \mid -n$
- L8.1.10 - Let $d, n \in \mathbb{Z}$. If $d \mid n$ and $d \neq 0$, then $|d| \leq |n|$
- L8.2.5 - **Prime Divisor Lemma** (non-standard name):
 - Let $n \in \mathbb{Z}_{\geq 2}$. Then n has a prime divisor.
- P8.2.6 - **sizes of prime divisors**:
 - Let n be a composite positive integer. Then n has a prime divisor $p \leq \sqrt{n}$.

logic

- negation of a universal statement:
 - $\sim \forall P(x) \Leftrightarrow \exists x \sim P(x)$
- negation of an existential statement:
 - $\sim \exists P(x) \Leftrightarrow \forall x \sim P(x)$
- negation for more predicates :
 - $\sim \forall x \exists y Q(x, y) \Leftrightarrow \exists x \forall y \sim Q(x, y)$

sets

- P4.2.7 - $\emptyset \subseteq$ all sets
- T4.1.18 - there exists a unique set with no element. It is denoted by \emptyset .
- E4.3.7 - for all A, B : $(A \cap B) \cup (A \setminus B) = A$
- E4.3.9(1) - $(A \cap B) \subseteq A$

- E4.3.9(2) - $A \subseteq (A \cup B)$
- E4.3.10 - $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq (B \cap C)$
- T4.6 - $A \subseteq B \Leftrightarrow A \cup B = B$
- T5.3.11(1) - let A, B be disjoint finite sets. Then $|A \cup B| = |A| + |B|$
- T5.3.11(2) - let A_1, A_2, \dots, A_n be pairwise disjoint finite sets. Then $|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$
- T5.3.12 - **Inclusion-Exclusion Principle**:
 - for all finite sets A and B , $|A \cup B| = |A| + |B| - |A \cap B|$

relations

- E6.2.2 - The equality relation R on a set A has equivalence classes of the form $[x] = \{y \in A : x = y\} = \{x\}$ where $x \in A$
- L6.3.11 - Let R be an equivalence relation on a set A . Then A/R is a partition of A .
- If \mathcal{C} is a partition of A , then there is an equivalence relation of R on A such that $A/R = \mathcal{C}$.
- L6.3.5 - Let \sim be an equivalence relation a set A .
 - $x \in [x]$ for all $x \in A$
 - any equivalence class is non empty
- L6.3.6 - $\forall x, y \in A$ if $(x) \cap [y] \neq \emptyset$, then $[x] = [y]$
- - Consider a partial order \preceq on set A .
 - A smallest element is minimal.
 - There is at most one smallest element.
- T6.4 - $x \sim y \Leftrightarrow [x] = [y]$, where \sim is an equivalence relation
- P7.4.3 - if f is a bijection $A \rightarrow B$, then f^{-1} is a bijection $B \rightarrow A$

functions

- P7.4.13 - $f \circ \text{id}_A = f$ and $\text{id}_A \circ f = f$
- P7.4.3 - if f is a bijection $A \rightarrow B$, then f^{-1} is a bijection $B \rightarrow A$
- T7.6 - if f is surjective, and $g \circ f = \text{id}_A$, then g is injective
- E7.9 - Let $f : A \rightarrow B$. if f^{-1} is a function $B \rightarrow A$, then f^{-1} is bijective
- $\text{range}(f) \in \text{codomain}$
- if f is surjective: $\text{range}(f) \in \text{codomain} \in \text{range}(f)$

counting

- Tut8.7a - if $g \circ f$ is surjective, then g is surjective

- Tut8.7b - if $g \circ f$ is injective, then f is injective
- Tut8.9a - if there is a surjection $f : A \rightarrow B$ that is not an injection, then $n > m$
- Tut8.9b - if there is an injection $f : A \rightarrow B$ that is not a surjection, then $n < m$

graphs and trees

- - Let G be a graph.
 - - If G is connected, then any two distinct vertices of G can be connected by a path
 - T12.1.4 - A connected undirected graph G is a tree \Leftrightarrow removing any edge disconnects G
 - - If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G .
 - Tut11.5b - For all undirected graphs G with at least 1 vertex but no loop, either G or \overline{G} is connected
- P11.3.6 - Every vertex in G is in some **connected component** of G
- T12.1.4 - A connected undirected graph G is a **tree** \Leftrightarrow removing any edge disconnects G
- T12.1.6 - Let G be finite **tree** with > 1 vertex, $|E(G)| = |V(G)| - 1$
- T12.1.8 - Let G be a connected cyclic finite undirected graph, then $|E(G)| \geq |V(G)|$
- P12.2.3 - Let T be a finite **rooted tree** of heigh h in which every vertex has at most 2 children, then T has at most 2^h terminal vertices.
- P12.2.5 - Let T be a finite **rooted tree** in which every vertex has exactly 2 children. If T has exactly t terminal vertices, then T has exactly $t - 1$ internal vertices and thus exactly $2t - 1$ vertices in total
- Tut12.9 - Let T be a finite **tree** with at least 2 vertices. T has at least 2 vertices that are each in exactly one edge.
- - If T is a **full binary tree** with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices.

abbreviations

- L - lemma
- E - example
- P - proposition
- T - theorem