CS1231
AY22/23 sem 2
github.com/NeoHW

01. Propositional Logic

sets of numbers

$\mathbb{N}$: natural numbers ($\mathbb{Z}_{\geq 0}$)
$\mathbb{Z}$: integers
$\mathbb{Q}$: rational numbers
$\mathbb{R}$: real numbers
$\mathbb{C}$: complex numbers
$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

basic properties of integers

closure (under addition and multiplication)
$$x + y \in \mathbb{Z} \wedge xy \in \mathbb{Z}$$
commutativity
$$a + b = b + a \wedge ab = ba$$
associativity
$$a + b + c = a + (b + c) = (a + b) + c$$
$$abc = a(bc) = (ab)c$$
distributivity
$$a(b + c) = ab + ac$$
trichotomy
$$(a < b) \vee (a > b) \vee (a = b)$$
transitive law
$$(a < b) \wedge (b < c) \implies (a < c)$$

definitions

even/odd
n is even $\leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k$
n is odd $\leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k + 1$
prime/composite
n is prime $\leftrightarrow n > 1$ and $\forall r, s \in \mathbb{Z}^+, n = rs \rightarrow (r = n) \vee (r = s)$
n is composite $\leftrightarrow n > 1$ and $\exists r, s \in \mathbb{Z}^+ s.t. n = rs$ and $1 < r < n$ and $1 < s < n$
divisibility (d divides n)
$d \mid n \leftrightarrow \exists k \in \mathbb{Z} \mid n = kd$
rationality
r is rational $\leftrightarrow \exists a, b \in \mathbb{Z} \mid r = \frac{a}{b}$ and $b \neq 0$
floor/ceiling
$\lfloor x \rfloor$: largest integer y such that $y \leq x$
$\lceil x \rceil$: smallest integer y such that $y \geq x$

rules of inference

generalisation	elimination
$p, \therefore p \vee q$	$p \vee q; \;\; \sim q, \therefore p$
specialisation	transitivity
$p \wedge q, \therefore p$	$p \rightarrow q; \; q \rightarrow r; \;\; \therefore p \rightarrow r$

03. PROOFS

Proof by Exhaustion/Cases

1. list out possible cases
   1.1. Case 1: n is odd OR If $n = 9$, ...
   1.2. Case 2: n is even OR If $n = 16$, ...
2. therefore ...

Proof by Contradiction

1. Suppose that ...
   1.1. ¡proof¿
   1.2. ...but this contradicts ...
2. Therefore the assumption that ... is false. Hence ....

Proof by Contraposition

1. Contrapositive statement: $\sim q \rightarrow \sim p$
2. let $\sim q$
   2.1. ¡proof¿
   2.2. hence $\sim p$
3. $\therefore p \rightarrow q$

Proof by Construction

1. Let $x = 3, y = 4, z = 5$.
2. Then $x, y, z \in \mathbb{Z}_{\geq 1}$ and $x^2 + y^2 = 3^2 + 4^2 = 9 + 16 = 25 = 5^2$.
3. Thus $\exists x, y, z \in \mathbb{Z}_{\geq 1}$ such that $x^2 + y^2 = z^2$.

Proof by Induction

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition "..."
2. (base step) $P(1)$ is true because ¡manual method¿
3. (induction step)
   3.1. let $k \in \mathbb{Z}_{\geq 1}$ s.t. $P(k)$ is true
   3.2. Then ...
   3.3. proof that $P(k + 1)$ is true - e.g. $P(k + 1) = P(k) + term_{k+1}$
   3.4. So $P(k + 1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geq 1} P(n)$ is true by MI.

INDUCTION

mathematical induction

to prove that $\forall n \in \mathbb{Z}_{\geq m}(P(n))$ is true,
- base step: show that $P(m)$ is true
- induction step: show that $\forall k \in \mathbb{Z}_{\geq m}(P(k) \Rightarrow P(k + 1))$ is true.
  - induction hypothesis: assumption that $P(k)$ is true

strong MI

to prove that $\forall n \in \mathbb{Z}_{\geq 0}(P(n))$ is true,
- base step: show that $P(0), P(1)$ are true
- induction step: show that $\forall k \in \mathbb{Z}_{\geq 0}(P(0) \cdots \wedge P(k + 1) \Rightarrow P(k + 2))$ is true.

justification:
- $P(0) \wedge P(1)$ by base case
- $P(0) \wedge P(1) \rightarrow P(2)$ by induction with $k = 0$
- $P(0) \wedge P(1) \wedge P(2) \rightarrow P(3)$ by induction with $k = 1$
- ...
- we deduce that $P(0), P(1), \ldots$ are all true by a series of **modus ponens**

Proofs for Sets

Equality of Sets (A=B)

1. $(\Rightarrow)$
   1.1. Take any $z \in A$.
   1.2. ...
   1.3. $\therefore z \in B$.
2. $(\Leftarrow)$
   2.1. Take any $z \in B$.
   2.2. ...
   2.3. $\therefore z \in A$.

Element Method

1. $A \cap (B \backslash C) = \{x : x \in A \wedge x \in (B \backslash C)\}$ (by def. of $\cap$)
2. $= \{x : x \in A \wedge (x \in B \wedge x \notin C)\}$ (by def. of $\backslash$)
3. ...
4. $= (A \cap B) \backslash C$ (by def. of $\backslash$)

Other Proofs

iff ($A \leftrightarrow B$)

1. $(\Rightarrow)$ Suppose A.
   1.1. ... ¡proof¿ ...
   1.2. Hence $A \rightarrow B$
2. $(\Leftarrow)$ Suppose B.
   2.1. ... ¡proof¿ ...
   2.2. Hence $B \rightarrow A$

02. PREDICATE LOGIC

operations

1 $\sim$: negation (not)
2 $\wedge$: conjunction (and)
2 $\vee$: disjunction (or) - coequal to $\wedge$
3 $\rightarrow$: if-then

logical equivalence

- identical truth values in truth table
- definitions
- to show non-equivalence:
  - truth table method (only needs 1 row)
  - counter-example method

conditional statements

hypothesis $\rightarrow$ conclusion
$$antecedent \rightarrow consequent$$

- **vacuously true** : hypothesis is false
- **implication law** : $p \rightarrow q \equiv \sim p \vee q$
- common statements for $p \rightarrow q$:
  - if p then q
  - q if p
  - p only if q
  - p iff q
  - p is sufficient for q
  - q is necessary for p
- **contrapositive** : $\sim q \rightarrow \sim p$
- **inverse** : $\sim p \rightarrow \sim q$
- **converse** : $q \rightarrow p$

statement $\equiv$ contrapositive
converse $\equiv$ inverse

- r is a **necessary** condition for s: $\sim r \rightarrow \sim s$ and $s \rightarrow r$
- r is a **sufficient** condition for s: $r \rightarrow s$
- **necessary** & **sufficient** : $\leftrightarrow$

valid arguments

- determining validity: construct truth table
  - valid $\leftrightarrow$ conclusion is true when premises are true
- **syllogism** : (argument form) 2 premises, 1 conclusion
- **modus ponens** : $p \rightarrow q; \; p; \; \therefore q$
- **modus tollens** : $p \rightarrow q; \; \sim q; \; \therefore \sim p$
- **sound argument** : is valid & all premises are true

fallacies

converse error	inverse error
$p \rightarrow q$	$p \rightarrow q$
q	$\sim p$
$\therefore p$	$\therefore \sim q$

QUANTIFIED STATEMENTS

- **truth set** of $P(x) = \{x \in D \mid P(x)\}$
- $P(x) \Rightarrow Q(x) : \forall x(P(x) \rightarrow Q(x))$
- $P(x) \Leftrightarrow Q(x) : \forall x(P(x) \leftrightarrow Q(x))$

relation between $\forall, \exists, \wedge, \vee$
- $\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$
- $\exists x \in D \mid Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$

relation between $\sim, \forall, \exists$
- $\sim \forall \; P(x) \leftrightarrow \exists x \sim P(x)$
- $\sim \exists \; P(x) \leftrightarrow \forall x \sim P(x)$

04. SETS

notation

- set roster notation [1]: $\{x_1, x_2, \ldots, x_n\}$
- set roster notation [2]: $\{x_1, x_2, x_3, \ldots\}$
- set-builder notation: $\{x \in \mathbb{U} : P(x)\}$
- replacement notation: $\{t(x) : x \in A\}$

definitions

- **equal sets** : $A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$
  - $A = B \leftrightarrow (A \subseteq B) \wedge (A \supseteq B)$
  - order and repetition does not matter
- **subset** : $A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$
- **proper subset** : $A \subsetneq B \leftrightarrow (A \subseteq B) \wedge (A \neq B)$
- **power set** of A : $\mathcal{P}(A) = \{X \mid X \subseteq A\}$
  - $|\mathcal{P}(A)| = 2^{|A|}$, given that A is a finite set
  - $\mathcal{P}(\emptyset) = \{\emptyset\}$; $\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
  - $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
- **cardinality** of a set, $| \; A \; |$: number of distinct elements
- **singleton** : sets of size 1
- **disjoint** : $A \cap B = \emptyset$

methods of proof for sets

- direct proof
- element method
- truth table

boolean operations

- **union:** $A \cup B = \{x : x \in A \vee x \in B\}$
- **intersection:** $A \cap B = \{x : x \in A \wedge x \in B\}$
- **complement** (of B in A): $A \backslash B = \{x : x \in A \wedge x \notin B\}$
- **complement** (of B): $\bar{B}$ or $B^c = U \backslash B$
  - set difference law: $A \backslash B = A \cap \bar{B}$

05. RELATIONS

ordered pairs

- **ordered pair** : (x, y)
  - $(x, y) = (x', y') \leftrightarrow x = x'$ and $y = y'$
- **Cartesian product** : $A \times B = \{(x, y) : x \in A$ and $y \in B\}$
  - $|A \times B| = |A| \times |B|$
  - $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

- **ordered tuples** : expression of the form $(x_1, x_2, \ldots, x_n)$
- defined recursively :
  $(x_1, x_2, \ldots, x_{n+1}) = ((x_1, x_2, \ldots, x_n), x_{n+1})$
- $(1, 2, 5) \neq (2, 1, 5)$ although $\{1, 2, 5\} = \{2, 1, 5\}$

relations

Let R be a relation from A to B and $(x, y) \in A \times B$. Then:
xRy for $(x, y) \in R$ and $x \not R y$ for $(x, y) \notin R$

- a relation from A to B is a subset of $A \times B$.
- a **(binary) relation** on set A is a relation from A to A.
  - subset of A^2
- **inverse relation**: $xR^{-1}y \Leftrightarrow yRx$

operations on relations

- $S \circ R =$ undergo R relation then S relation
- $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

06. EQUIVALENCE RELATIONS AND PARTIAL ORDERS

reflexivity, symmetry, transitivity

Let A be a set and R be a relation on A.

reflexive
$\forall x \in A\ (xRx)$
symmetric
$\forall x, y \in A\ (xRy \Rightarrow yRx)$
transitive
$\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$

- **equivalence relation**: a relation that is reflexive, symmetric and transitive
- **equivalence class**: the set of all things equivalent to x

equivalence classes

Let A be a set and R be an equivalence relation on A.

- $[x]_\sim$: **equivalence class** of x with respect to R
  - the set of all elements of A that x is related to
    $\forall x \in A, [x]_\sim = \{y \in A : xRy\}$
- $A/\sim$: The set of all equivalent classes
  $A/R = \{[x]_\sim : x \in A\}$

$xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap [y] \neq \emptyset$

partitions

- a **partition** of a set A is a set $\mathscr{C}$ of *non-empty subsets* of A such that
  0. $\forall S \in \mathscr{C},\ (\emptyset \neq S \subseteq A)$
     - $\mathscr{C}$ is a set of nonempty subsets of A
  1. $\forall x \in A,\ \exists S \in \mathscr{C}(x \in S)$
     - every element of A is in some element of $\mathscr{C}$
  2. $\forall x \in A,\ \forall S_1, S_2 \in \mathscr{C}(x \in S_1 \wedge x \in S_2 \Rightarrow S_1 = S_2)$
     - if two items of $\mathscr{C}$ have a nonempty intersection, then they are equal
- **components** : elements of a partition
- every partition comes from an equivalence relation

partial orders

Let A be a set and R be a relation on A.

- R is **antisymmetric** if $\forall x, y \in A\ (xRy \wedge yRx \rightarrow x = y)$
  - includes vacuously true cases (e.g. $xRy \Leftrightarrow x < y$)
- R is a **(non-strict) partial order** if R is reflexive, antisymmetric and transitive.
- x and y are **comparable** if $\forall x, y \in A\ (xRy \vee yRx)$
- R is a **(non-strict) total order** if R is a partial order and every pair of elements are comparable
- a smallest element of A is an element $m \in A$ such that mRx for all $x \in A$

well-ordering principle

- every nonempty subset of $\mathbb{Z}_{\geq 0}$ has a smallest element.
- application: recursion has a base case

07. FUNCTIONS

definitions

- **function/map** from A to B : each element of A exactly f-related to one element of B.
  - **Important** : $(x, y) \in f \leftrightarrow y = f(x)$
  - (F1): every element in A f-related to at least one of B
    $\forall x \in A\ \exists y \in B\ \ (x, y) \in f$
  - (F2): every element in A f-related to at most one of B
    $\forall x \in A\ \exists y_1, y_2 \in B\ \ ((x, y_1) \in f \wedge (x, y_2) \in f \rightarrow y_1 = y_2)$
  - $f : A \rightarrow B$: "f is a function from A to B"
  - $f : x \rightarrow y$: "f maps x to y"
  - **domain** of f = A
  - **codomain** of f = B
  - **range/image** of f = $\{f(x) : x \in A\}$
    $= \{y \in B \mid y = f(x) \text{ for some } x \in A\}$
    * range(f) $\in$ codomain
    * if f is surjective: range(f) $\in$ codomain $\in$ range(f)
- **identity function** on A, $\text{id}_A : A \rightarrow A$
  - $\text{id}_A : x \rightarrow x$
  - range = domain = codomain = A
  - (P7.4.13) $f \circ \text{id}_A = f$ and $\text{id}_A \circ f = f$
- **well-defined function** : every element in the domain is assigned to exactly one element in the codomain

equality of functions

- same codomain and domain
- for all $x \in$ codomain, same output

function composition

- $(g \circ f)(x) = g(f(x))$
- for $(g \circ f)$ to be well defined, codomain of f must be equal to the domain of g
- × commutative $(g \circ f)(x) \neq (f \circ g)(x)$
- ✓ **associative** - (T6.1.26) $f \circ (g \circ h) = (f \circ g) \circ h$

image & pre-image

for $f : A \rightarrow B$

- if $X \subseteq A$, **image** of X,
  $f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\}$
- if $Y \subseteq B$, **pre-image** of Y,
  $f^{-1}(Y) = \{x \in A : y = f(x) \text{ for some } y \in Y\}$

injection & surjection

- **surjective** (onto) : codomain = range
  - for every B, there is a A
    $\forall y \in B\ \exists x \in A\ (y = f(x))$
  - a function is **not** surjective iff
    $\exists y \in B\ \forall x \in A\ (y \neq f(x))$
- **injective** : one-to-one
  - All functions that have an inverse are bijective
  - for every B, at most one A
    $\forall x_1, x_2 \in A\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$
  - a function is **not** injective iff
    $\exists x_1, x_2 \in A\ (f(x_1) = f(x_2) \wedge x_1 \neq x_2)$
- **bijective** : both surjective & injective

inverse

- $\forall x \in A, \forall y \in B(f(x) = y \Leftrightarrow g(y) = x)$
- **uniqueness** of inverses (P2.6.16)
  - if g, g' are inverses of $f : A \rightarrow B$, then $g = g'$

8. CARDINALITY

pigeonhole principle

For any function f from a finite set A with n elements to a finite set B with m elements if there is an injection $A \rightarrow B$, then $n \leq m$

dual pigeonhole principle

For any function f from a finite set A with n elements to a finite set B with m elements if there is an surjection $A \rightarrow B$, then $n \geq m$

T8.1.3

For any function f from a finite set A with n elements to a finite set B with m elements if there is a bijection $A \rightarrow B$, then $n = m$

- A function from a finite set to a smaller finite set cannot be injective.
- **presentation:**
  - There are m pigeons and n pegionholes
  - Thus, by Pigeonhole Principle, ...

same cardinality

- A set A is said to have the same cardinality (HSC) as a set B if there is a bijection $A \rightarrow B$
- reflexivity : A HSC A.
- symmetry : if A HSC B, then B HSC A.
- transitivity : if A HSC B, and B HSC C, then A HSC C.

finite sets

- A set A is finite if it HSC $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$
- n is the cadinality/size of A, denoted by $|A|$
- Let A and B be sets that HSC, then A is finite iff B is finite

9. COUNTABILITY

countable sets

- A set is countable if it is finite or has the same cardinality as $\mathbb{N}$
- $\mathbb{Z}$ is countable
- $\mathbb{Z} \times \mathbb{Z}$ is countable
- $\mathbb{N} \times \mathbb{N}$ is countable

countability

- Let A and B be sets of same cardinality. A is countable iff B is countable
- Let A, B be sets such that $A \subseteq B$
  - If B is finite, then A is finite
  - If B is countable, then A is countable
- A set B is infinite if there is an injection f from some infinite set A to B
- A set B is uncountable if there is an injection f from some uncountable set A to B

uncountable sets

- No set A has the same cadinality as $\mathcal{P}(A)$
- Let A be countable infinite set, then $\mathcal{P}(A)$ is uncountable. Hence $\mathcal{P}(\mathbb{N})$ is uncountable

non-computability

- There is a subset S of $\mathbb{N}$ s.t no program can, when given any input $n \in \mathbb{N}$
  - output T if $n \in S$; and
  - output F if $n \notin S$

  i.e no program can correctly determine whether a given input n belongs to S or not, for all possible inputs n.

10. COUNTING

rules

- **addition/sum rule:** Let A and B be **disjoint** finite sets
  $|A \cup B| = |A| + |B|$
- **difference rule:** Let X and Y be finite sets. Then $Y \backslash X$ is finite, and if $X \subseteq Y$
  $|Y \backslash X| = |Y| - |X|$
- **inclusion/exclusion rule 2 sets:**
  $|A \cup B| = |A| + |B| - |A \cap B|$
- **inclusion/exclusion rule 3 sets :** $|A \cup B \cup C| =$
  $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$
- **multiplication/product rule:** $|A \times B| = |A| \times |B|$
- **general multiplication rule:** Let A be set of size m, and for each $x \in A$, let B_x be set of size n. Then $\{(x, y) : x \in A$ and $y \in B_x\}$ is finite and has size mn
- **complement:** $P(\bar{A}) = 1 - P(A)$
- $|\mathcal{P}(A)| = 2^{|A|}$, given that A is a finite set

permutations

pick r elements from a set of size n without replacement where order matters

$P(n, r) = \frac{n!}{(n-r)!}$ (also ${}_nP_r, P_r^n$)

if $r > n$, 0 ways

permutations with indistinguishable objects

For n objects with n_k of type k indistinguishable from each other, the total number of distinguishable permutations
$= \frac{n!}{n_1! n_2! \ldots n_k!}$
E.g. num of permuatations for "EGG" = $\frac{3!}{2!}$ = 3

combinations

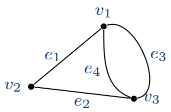
Let A be a set of size n. Number of subsets of A of size r is
$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (also $C(n, r)$, ${}_nC_r$, $C_{n,r}$, ${}^nC_r$)

r-combinations from n elements with **repetition**
$= \binom{r+n-1}{r}$

11. GRAPHS

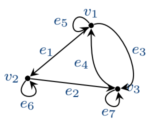
types of graphs

undirected graph



$e_1 = \{v_1, v_2\}$

directed graph



$e_1 = (v_1, v_2)$

undirected graph

- denoted by $G = (V, E)$, comprising
  - nonempty set of *vertices/nodes*, $V = \{v_1, v_2, \ldots, v_n\}$
  - a set of *edges*, $E = \{e_1, e_2, \cdots, e_k\}$
- $e = \{v, w\}$ for an undirected edge E incident on vertices v and w

directed graph

- denoted by $G = (V, E)$, comprising
  - nonempty set V of *vertices*
  - a set E of *directed edges* (ordered pair of vertices)
- $e = (v, w)$: an directed edge E from vertex v to vertex w

simple graph

- **undirected graph** with no loops or parallel edges

complete graph

- a complete graph on n vertices, $n > 0$, denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices

subgraph of a graph

H is a subgraph of $G \Leftrightarrow$
- every vertex in H is also a vertex in G
- every edge in H is also an edge in G
- a **proper** subgraph of G is a subgraph G of G such that $H \neq G$

paths and walks

Let G be a graph; let v and w be vertices of G.
- **path** (from v to w): links two vertices in a graph via a sequence of edges, with no repeated vertices ($\{x_0, x_1, \ldots, x_l\}, \{x_0x_1, x_1x_2, \ldots, x_{l-1}x_l\}$) where $v = x_0$ and $w = x_l$
  - **length** of walk: the number of edges, l
  - each vertex in a path is in at most 2 edges in the path
- If there is a path P between u and v, and path G between v and w. Then there is a path between u and w

cycles

- **circuit/cycle**: an undirected graph $G(V, E)$ where
  - $V = \{x_1, x_2, \ldots, x_l\}$
  - $E = \{\{x_1,x_2\},\{x_2,x_3\},\ldots,\{x_{l-1},x_l\},\{x_l,x_1\}\}$
  - $n \in \mathbb{Z}_{\geq 3}$
- **simple circuit/cycle**: does not have any other repeated vertex except the first and last
- An undirected graph is **cyclic** if it contains a loop/cycle
- An undirected graph is with no loop is cyclic iff it has two vertices between which there are two distinct paths

connectedness

- vertices v and w are connected $\Leftrightarrow \exists$ a path from v to w
- graph G is connected $\Leftrightarrow \forall$ vertices $v, w \in V, \exists$ a path from v to w

connected component

- a connected subgraph of the largest possible size
- graph H is a connected component of graph $G \Leftrightarrow$
  1. H is a subgraph of G
  2. H is connected
  3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H
- P11.3.6 - Let G be an undirected graph, then every vertex v in G is in some connected component of G
- T11.3.7 - Let u, v be vertices in an undirected graph G. Then there is a path between u and v in G $\leftrightarrow$ there is a connected component G that has both u and v in it

counting walks of length N

number of walks of length n from v_i to v_j
$=$ the ij-th entry of A^n

isomorphism

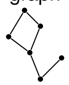
- graph isomorphism ($\cong$) is an equivalence relation.

Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be two graphs. $G \cong G' \Leftrightarrow$ there exist bijections $g : V_G \to V'_G$ and $h : E_G \to E'_G$ that preserve the edge-edgepoint functions of G and G' in the sense that $\forall v \in V_G$ and $e \in E_G$, v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

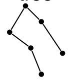
12. TREES

- **tree** is a **connected acyclic undirected** graph
  - If G is a connected graph with n vertices and $n-1$ edges, then G is a tree.
- **terminal vertex / leaf**: vertex with no child
- **internal vertex / parent**: vertex that is not terminal

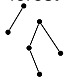
graph



tree



forest



rooted trees

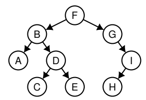
- **rooted tree**: a tree in which there a distinguished vertex called the root.
- **height** (of a rooted tree): length of a longest path between root and some vertex

binary tree

- **binary tree**: a rooted tree in which every parent has at most 2 children
  - at most one left child and at most one right child
- **full binary tree**: a binary tree in which every parent has exactly 2 children
- (left/right) **subtree**: Given any parent v in a binary tree T, the binary tree whose root is the (left/right) child of v, whose vertices consist of the left child of v and all its descendants, and whose edges consist of all those edges of T that connect the vertices of the left subtree.

T10.6.1: Full Binary Tree Theorem
If T is a full binary tree with k internal vertices, then T has a total of $2k+1$ vertices and has $k+1$ terminal vertices.

binary tree traversal



F-B-G-A-D-I-C-E-H

Breadth-First Search (BFS)
- starts at the root
- visits its adjacent vertices
- visits the next level

Depth-First Search (DFS)
- **pre-order**
  - current vertex $\to$ left subtree $\to$ right subtree
- **in-order**
  - left subtree $\to$ current vertex $\to$ right subtree
- **post-order**
  - left subtree $\to$ right subtree $\to$ current vertex

spanning trees

- **spanning tree** (for a graph G): a subgraph of G that contains every vertex of G and is a tree.
  - $w(e)$ - weight of edge e
  - $w(G)$ - total weight of G
- **weighted graph**: each edge has an associated positive real number weight
  - **total weight**: sum of the weights of all edges
- **minimum spanning tree**: least possible total weight compared to all other spanning trees

Kruskal's algorithm

For a connected weighted graph G with n vertices:

1. initialise T to have all the vertices of G and no edges.
2. let E be the set of all edges in G; let $m = 0$
3. while $(m < n-1)$
   - 3.1. find and remove the edge e in E of least weight
   - 3.2. if adding e to the edge set of T does not produce a circuit:
     - i. add e to the edge set of T
     - ii. set $m = m + 1$

Prim's algorithm

For a connected weighted graph G with n vertices:

1. pick any vertex v of G and let T be the graph with this vertex only
2. let V be the set of all vertices of G except v
3. for ($i = 0$ to $n - 1$)
   - 3.1. find the edge e in G with the least weight of all the edges connected to T. let w be the endpoint of e.
   - 3.2. add e and w to the edge and vertex sets of T
   - 3.3. delete w from v

LOGICAL EQUIVALENCES

commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
identity laws	$p \wedge true \equiv p$	$p \vee false \equiv p$
idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
annihilators laws	$p \vee true \equiv true$	$p \wedge false \equiv false$
negation laws	$p \vee \sim p \equiv true$	$p \wedge \sim p \equiv false$
double negation law	$\sim(\sim p) \equiv p$	—
absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
De Morgan's Laws	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$
Implication law	$p \to q \equiv \sim p \vee q$	-

SET IDENTITIES

commutative laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
associative laws	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
identity laws	$A \cap U = A$	$A \cup \emptyset = A$
idempotent laws	$A \cap A = A$	$A \cup A = A$
annihilators laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
complement laws	$A \cap \overline{A} = \emptyset$	$A \cup \overline{A} = U$
double **complement** law	$\overline{(\overline{A})} = A$	—
absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Set difference	$A \backslash B \equiv A \cap \overline{B}$	-

proven:

number theory

- - the product of 2 consecutive odd numbers is always odd.
- - the difference between 2 consecutive squares is always odd
- P3.2.4 - the square of any 2 even integers is even
- - there is no greatest integer
- - there are infinitely many prime numbers
- - for all positive integers a and b, if $a|b$, then $a \leq b$.
- P3.2.8 - for all integers n, if n^2 is even then n is even
- - all integers are rational numbers
- - the sum of any 2 rational numbers is rational
- - there exist irrational numbers p and q such that p^q is rational
- - $\sqrt{2}$ is irrational.
- - the only divisors of 1 are 1 and -1.

divisibility

- L8.1.5 - Let $d, n \in \mathbb{Z}$ with $d \neq 0$. Then $d \mid n \Leftrightarrow n/d \in \mathbb{Z}$
- L8.1.9 - Let $d, n \in \mathbb{Z}$. If $d \mid n$, then $-d \mid n$ and $d \mid -n$ and $-d \mid -n$
- L8.1.10 - Let $d, n \in \mathbb{Z}$. If $d \mid n$ and $d \neq 0$, then $|d| \leq |n|$
- L8.2.5 - **Prime Divisor Lemma** (non-standard name):
  - Let $n \in \mathbb{Z}_{\geq 2}$. Then n has a prime divisor.
- P8.2.6 - **sizes of prime divisors**:
  - Let n be a composite positive integer. Then n has a prime divisor $p \leq \sqrt{n}$.

logic

- negation of a universal statement:
  - $\sim\forall\, P(x) \leftrightarrow \exists x \sim P(x)$
- negation of an existential statement:
  - $\sim\exists\, P(x) \leftrightarrow \forall x \sim P(x)$
- negation for more predicates :
  - $\sim\forall x \exists y\, Q(x, y) \leftrightarrow \exists x \forall y \sim Q(x, y)$

sets

- P4.2.7 - $\emptyset \subseteq$ all sets
- T4.1.18 - there exists a unique set with no element. It is denoted by $\emptyset$.
- E4.3.7 - for all A, B: $(A \cap B) \cup (A \backslash B) = A$
- E4.3.9(1) - $(A \cap B) \subseteq A$
- E4.3.9(2) - $A \subseteq (A \cup B)$
- E4.3.10 - $A \subseteq B \wedge B \subseteq C \to A \subseteq (B \cap C)$
- T4.6 - $A \subseteq B \leftrightarrow A \cup B = B$
- T5.3.11(1) - let A, B be disjoint finite sets. Then $|A \cup B| = |A| + |B|$
- T5.3.11(2) - let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets. Then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$
- T5.3.12 - **Inclusion-Exclusion Principle**:
  - for all finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$

relations

- E6.2.2 - The equality relation R on a set A has equivalence classes of the form $[x] = \{y \in A : x = y\} = \{x\}$ where $x \in A$
- L6.3.11 - Let R be an equivalence relation on a set A. Then A/R is a partition of A.
- - If $\mathscr{C}$ is a partition of A, then there is an equivalence relation of R on A such that $A/R = \mathscr{C}$.
- L6.3.5 - Let $\sim$ be an equivalence relation a set A.
  - $x \in [x]$ for all $x \in A$
  - any equivalence class is non empty
- L6.3.6 - $\forall x, y \in A$ if $(x] \cap [y] \neq \emptyset$, then $[x] = [y]$
- - Consider a partial order $\preccurlyeq$ on set A.
  - A smallest element is minimal.
  - There is at most one smallest element.
- T6.4 - $x{\sim}y \leftrightarrow [x] = [y]$, where $\sim$ is an equivalence relation
- P7.4.3 - if f is a bijection $A \to B$, then f^{-1} is a bijection $B \to A$

functions

- P7.4.13 - $f \circ \text{id}_A = f$ and $\text{id}_A \circ f = f$
- P7.4.3 - if f is a bijection $A \to B$, then f^{-1} is a bijection $B \to A$
- T7.6 - if f is surjective, and $g \circ f = \text{id}_A$, then g is injective
- E7.9 - Let $f : A \to B$. if f^{-1} is a function $B \to A$, then f^{-1} is bijective
- range(f) $\in$ codomain
- if f is surjective: range(f) $\in$ codomain $\in$ range(f)

counting

- Tut8.7a - if $g \circ f$ is surjective, then g is surjective
- Tut8.7b - if $g \circ f$ is injective, then f is injective
- Tut8.9a - if there is a surjection $f : A \to B$ that is not an injection, then $n > m$
- Tut8.9b - if there is an injection $f : A \to B$ that is not a surjection, then $n < m$

graphs and trees

- - Let G be a graph.
  - - If G is connected, then any two distinct vertices of G can be connected by a path
  - T12.1.4 - A connected undirected graph G is a tree $\leftrightarrow$ removing any edge disconnects G
  - - If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.
  - Tut11.5b - For all undirected graphs G with at least 1 vertex but no loop, either G or $\overline{G}$ is connected
- P11.3.6 - Every vertex in G is in some **connected component** of G
- T12.1.4 - A connected undirected graph G is a **tree** $\leftrightarrow$ removing any edge disconnects G
- T12.1.6 - Let G be finite **tree** with $>$ 1 vertex, $|E(G)| = |V(G)| - 1$
- T12.1.8 - Let G be a connected cyclic finite undirected graph, then $|E(G)| \geq |V(G)|$
- P12.2.3 - Let T be a finite **rooted tree** of heigh h in which every vertex has at most 2 children, then T has at most 2^h terminal vertices.
- P12.2.5 - Let T be a finite **rooted tree** in which every vertex has exactly 2 children. If T has exactly t terminal vertices, then T has exactly $t - 1$ internal vertices and thus exactly $2t - 1$ vertices in total
- Tut12.9 - Let T be a finite **tree** with at least 2 vertices. T has at least 2 vertices that are each in exactly one edge.
- - If T is a **full binary tree** with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices.

abbreviations

- L - lemma
- E - example
- P - proposition
- T - theorem