COMP/ENGN8535 Homework Assignment #1

General Instructions

- Due Date: Monday 13 March 2023, 23:59pm (AEDT).
- Full marks for this homework = 100 points. Estimated time needed to complete this homework is about 8 hours including self-study time.
- Please complete as many problems as you can, and answer the questions clearly and concisely. In all your solutions, you need to show the key steps for how you reach your answers. The clarity of your answers may affect the final marks.
- Submit your answers in a single "pdf" file, with file name U4056789-HW-1.pdf (Note: replace the U4056789 with your own university ID) to Wattle before the deadline. It is a good idea to also include a cover page with your full name and university ID so that we can identify your submission easily when marking.
- In preparing your PDF submission, both typed and handwritten-and-scanned answers are acceptable, though the former is preferred due to its clarity.
- Late submissions will not be accepted and incur a Mark of 0.

1 2

3 4

5 6

Academic Integrity: Your homework must be completed individually and independently. You must comply with the University Policy on Academic Integrity. Plagiarism will lead to severe consequences, including academic misconduct. You are allowed to discuss the problems and solution ideas with other students, but your final solutions must be your own work and written in your own language.

1 3 5

2 4 6

Problems

1. (20 points) The *Frobenius norm* of a matrix $H \in \mathbb{R}^{n\times m}$ is defined as

$$\|H\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} H_{ij}^2\right)^{1/2}.$$

$$P = \begin{array}{cc} & m \\ n & \begin{bmatrix} H_{11} & H_{12} & H_{1m} \\ H_{21} & & \\ H_{n1} & & H_{nm} \end{bmatrix} \end{array} \begin{array}{cc} & n \\ m & \begin{bmatrix} H_{11} & H_{21} \cdots & H_{n1} \\ H_{1m} & \cdots & H_{nm} \end{bmatrix} \end{array}$$

Prove the following:

For i in range (n):

For j in range(m):

$H_{11}^2 + H_{12}^2 \cdots + H_{1m}^2$

$P_{22} = H_{21}^2 + H_{22}^2 + H \cdots H_{2m}^2$

$$\begin{bmatrix} P_{11} & \\ & P_{22} \end{bmatrix}$$

$H_{11}^2 + H_{12}^2 + H_{13}^2 \cdots H_{1m}^2$

$\|H\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} H_{ij}^2}$

推导

(a) (5 points) $\|H\|_F$ is equal to the square root of the matrix trace of HH^T.

$tr(HH^T) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} H_{ij}^2}$

$$\|H\|_F = \sqrt{\mathbf{tr}(HH^T)}$$

$n \times m \otimes m \times n$ $\quad n \times n$

(b) (5 points) Any orthogonal matrix $P \in \mathbb{R}^{n \times n}$ or $Q \in \mathbb{R}^{m \times m}$ preserves the Frobenius norm, that is,

$$\|H\|_F = \|PH\|_F = \|HQ\|_F$$

$n \times n \;\; n \times m = n \times m$

$n \times m \;\; m \times m = n \times m$

(c) (10 points) Prove that $\|H\|_F$ is equal to the root sum square of H's singular values, that is,

$$\|H\|_F = \sqrt{\sum_i \sigma_i^2}$$

$tr\ PH(PH)^T = tr\ HQ(HQ)^T$

$PHH^TP^T = HQQ^TH$

2. (20 points) (a) (10 points) Work out the singular value decomposition (SVD) for the following matrix by hand. List the **key steps** of your calculation.

SVD formula

$A =$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) (10 points) Consider the matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$Av = \lambda_1 v \quad \lambda_1 = 2$

$Aw = \lambda_2 w$

and vectors

$$v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

$24 + 8 = 32$

8

2

$$w = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

only v is eigenvector

from $Av = \lambda_1 v \quad \lambda = 2$

Which of v, w is/are an eigenvector(s) of A and what are their eigenvalues? You should solve this without much calculation.

3. (20 points) A symmetric real matrix A is *positive definite* (PD) if and only if $x^T A x > 0$ for every vector $x \neq \mathbf{0}$.

Using this definition to determine whether or not a matrix is positive definite is however difficult. An alternative strategy is to use Sylvester's criterion, which states that a symmetric matrix is PD if and only if all its upper-left sub-matrices have a positive determinant.

including itself

symmetric $A^T = A$ square matrix

[1]

in A $\det[1] = 1$ positive

(a) (10 points) Show whether or not the following two matrices are PD. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ A is PD

and $B = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. $\operatorname{Det}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = 6 - 1 = 5$ B is PD as well.

(b) (10 points) Find the range of values for (real number) b such that the following real matrix is a PD matrix. List the **key steps** how you reach your answer.

$$\det \begin{pmatrix} 3 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 1 \end{pmatrix} > 0$$

4. (10 points) (a) (5 points) Let X and Y be real-valued random variables. Suppose X and Y are statistically independent, *i.e.* $P(X, Y) = P(X)P(Y)$.

$P(X|Y) = \frac{P(X,Y)}{P(Y)}$

Show that $P(X|Y) = P(X)$, in other words, the probability of X given Y is the same as probability of X. This is equivalent to say, knowing the value of Y gives no information about the probability of X.

(b) (5 points) Let A and B be real-valued random variables. Suppose $0 < P(B) < 1$, show that if $P(A|B) = P(A)$, then A and B are statistically independent.

5. (20 points) Tom goes to the hospital to do the yearly body checkup, and the doctor has a bad news and a good news. The bad news is that Tom tested positive for a serious disease, and the test is 99% accurate (*i.e.*, the probability of testing positive given that someone has the disease is 0.99, as is the probability of testing negative given that they don't have the disease). The good news is that, according to the latest research, this is a rare disease in Australia, striking only one in 100,000 people. What are the chances that Tom actually has the disease? (Show your calculations as well as giving the final result.)

6. (10 points) (a) (3 points) Given $z = x(y + 4)$ subject to $x + y = 8$. Use the Lagrange multiplier method to find the local optimum value of z, as well as the optimal x and y. Is the above solution a minimum or a maximum?

(b) (7 points) Use the Lagrange Multiplier method to prove that a triangle with maximum area that has a given perimeter of 2 must be equilateral. Can the solution be found using the Lagrange method ?

END.