ENGN/COMP8535 Homework 2

Yuyang Han u7434576

Q1

(a) P is

```
[[0.         1.         0.         0.        ]
 [0.33333333 0.         0.33333333 0.33333333]
 [0.         0.         0.         1.        ]
 [1.         0.         0.         0.        ]]
```

(b) G is

```
[[0.0375     0.8875     0.0375     0.0375    ]
 [0.32083333 0.0375     0.32083333 0.32083333]
 [0.0375     0.0375     0.0375     0.8875    ]
 [0.8875     0.0375     0.0375     0.0375    ]]
```

(c) Use G and α, rank the webpages with fullrank algorithm. Using the power method to compute an approximation of π_∞ with 5 iterations from $\pi_0 = [0.25, 0.25, 0.25, 0.25]^T$ gets the following rank results.

```
[[0.32885526]
 [0.31353484]
 [0.11935062]
 [0.23825928]]
```

where the first page has the highest rank.

Q2 (a)

Here is the result of find the svd of X using python numpy library

```python
X = np.array([ [3, 2, 2],
               [2, 3, -2] ])

U, s, Vt = np.linalg.svd(X)

print('U:\n', U)

print('S:\n', np.diag(s))

print('V:\n', Vt.T)
```

✓ 0.3s

```
U:
 [[-0.70710678 -0.70710678]
 [-0.70710678  0.70710678]]
S:
 [[5. 0.]
 [0. 3.]]
V:
 [[-7.07106781e-01 -2.35702260e-01 -6.66666667e-01]
 [-7.07106781e-01  2.35702260e-01  6.66666667e-01]
 [-6.47932334e-17 -9.42809042e-01  3.33333333e-01]]
```

continue...

(a) Working out the svd of X by hand

Find SVD of $X = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$$X^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

the eigenvales and eigenvectors of X^TX can be found by setting the $\det(X^TX - \lambda I) = 0$

where we have $\det(X^TX - \lambda I) = -\lambda(\lambda-25)(\lambda-9) = 0$

the eigenvalues of X^TX are $\lambda_1 = 25$ $\lambda_2 = 9$ $\lambda_3 = 0$

Then we can find the eigenvectors x_1 x_2 x_3

- $\lambda_1 = 25$

$$(X^TX - 25I) = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$$ reduce to row-echlon form is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ we have $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

- $\lambda_2 = 9$

$$(X^TX - 9I)x_2 = 0 \quad \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$ reduce to row-echlon form is

$$\begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ solve for x_2 we have $x_2 = \begin{bmatrix} 1/4 \\ -1/4 \\ 1 \end{bmatrix}$

- $\lambda_3 = 0$

$$(X^TX)x_3 = 0 \quad \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$ reduce to row-echlon form is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solve for x_3 we have $X_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

the singular values are the square roots of the eigenvalues

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

columns of V are the normalized eigenvectors

therefore $V_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ 2\sqrt{2}/3 \end{bmatrix}$ $V_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$$V = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/6 & -2/3 \\ \sqrt{2}/2 & -\sqrt{2}/6 & 2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$

$$u_i = \frac{1}{\sigma_i} X v_i$$

So

$$u_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$u_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ 2\sqrt{2}/3 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

Thus

$$U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \quad V = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/6 & -2/3 \\ \sqrt{2}/2 & -\sqrt{2}/6 & 2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$

such that $X = U\Sigma V^T$

$\blacksquare$

(b) The best rank-one approximation to X is $\sigma_1 u_1 v_1^T$ where σ_1 is the first singular value, u_1 is the first left singular vector, and v_1^T is the first right singular vector of X. Below is the reconstructed X. using python.

```
[[2.50000000e+00 2.50000000e+00 2.29078674e-16]
 [2.50000000e+00 2.50000000e+00 2.29078674e-16]]
```

The best rank-1 appoximation to X is $\sigma_1 U_1 V_1^T$ where σ_1 is the first singular value. u_1 is the first left singular vector. v_1 is the first right singular vector

Hence X_1 is given by $\sigma_1 u_1 v_1^T$

$\sigma_1 = 5$ $\quad u_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ $\quad v_1^T = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right]$

$$X_1 = \begin{bmatrix} 2.5 & 2.5 & 0 \\ 2.5 & 2.5 & 0 \end{bmatrix}$$

(c)

$$X - X_1 = \begin{bmatrix} 0.5 & -0.5 & 2 \\ -0.5 & 0.5 & -2 \end{bmatrix}$$

$$\|X - X_1\| = \sqrt{0.5^2 + 0.5^2 + 0.5^2 + 0.5^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

therefore the approximation error under the Frobenius Norm is 3

Q3. $x_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $x_4 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ $x_5 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(a)

$$\mu = \frac{1}{5}\begin{bmatrix} 4+0+(-2)+4+2 \\ 3+1+2+(-1)+0 \end{bmatrix} = [8/5, 1]$$

$$\Delta x_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 2 \end{bmatrix} \quad \Delta x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.6 \\ 0 \end{bmatrix} \quad \Delta x_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1.6 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.6 \\ 1 \end{bmatrix}$$

$$\Delta x_4 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 1.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4 \\ -2 \end{bmatrix} \quad \Delta x_5 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -1 \end{bmatrix}$$

we form $X = [\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4, \Delta x_5] \in R^{d \times n}$

where $d = 2$ $n = 5$

then covariance matrix is given by $C = \frac{1}{n} X X^T$

$$C = \frac{1}{5}\begin{bmatrix} 2.4 & -1.6 & -3.6 & 2.4 & 0.4 \\ 2 & 0 & 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2.4 & 2 \\ -1.6 & 0 \\ -3.6 & 1 \\ 2.4 & -2 \\ 0.4 & -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} \frac{136}{5} & -4 \\ -4 & 10 \end{bmatrix}$$

the eigenvalues and eigenvectors of C are

eigenvalues : $[\,7.0212 \quad 2.2788\,]$

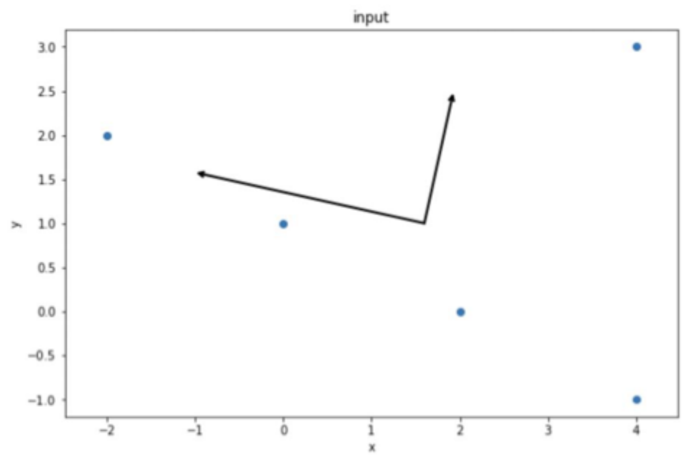
eigenvectors : $\begin{bmatrix} -0.9764 \\ 0.2159 \end{bmatrix}$ $\begin{bmatrix} 0.2159 \\ 0.9764 \end{bmatrix}$

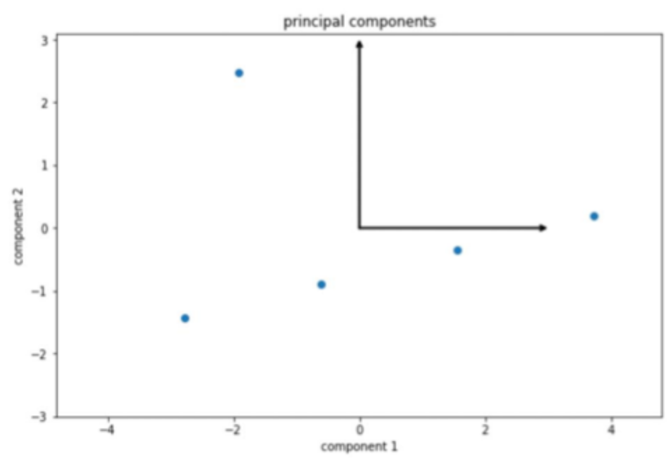
Then the first principal component is

$$u_1 = \begin{bmatrix} -0.9764 \\ 0.2159 \end{bmatrix}$$ since it is the eigenvector

corresponding to largest eigenvalue $\lambda_1 = 7.0212$

```
pca = PCA(n_components=2)
pca.fit(X)
✓ 0.2s

PCA(n_components=2)

print(pca.components_)

print(pca.explained_variance_)
✓ 0.2s

[[-0.97640184  0.21596167]
 [ 0.21596167  0.97640184]]
[7.02118114 2.27881886]
```





(b) Use PCA to compute lower-dimensional

$$y_j = u_1^T x_j = \begin{bmatrix} -0.9764 & 0.2159 \end{bmatrix} \Delta x_j$$

$y_1 = -1.9114$ $y_2 = 1.5622$ $y_3 = 3.731$

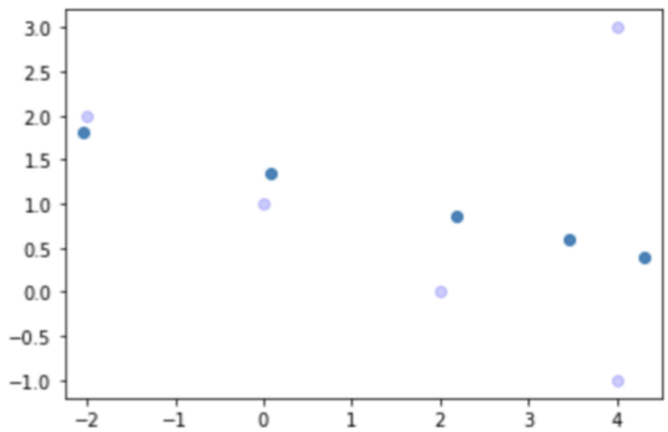
$y_4 = -2.7753$ $y_5 = -0.6065$

(b)

```
pca = PCA(n_components=1)
pca.fit(X)
X_1d = pca.transform(X)
print("original shape:    ", X.shape)
print("transformed shape:", X_1d.shape)

print(X_1d)
```

✓ 0.2s

```
original shape:    (5, 2)
transformed shape: (5, 1)
[[-1.91144107]
 [ 1.56224295]
 [ 3.7310083 ]
 [-2.77528776]
 [-0.60652241]]
```



Q4