Q1 kurtosis of a sum

Proof: Since X and Y are two independent random variables with zero mean and unit variance

$$E[X] = E[Y] = 0 \quad , \quad E[X^2] = E[Y^2] = 1$$

therefore $X+Y$ is also a random variable with zero mean

$$\text{kurt}(X+Y) = E[(X+Y)^4] - 3(E[(X+Y)^2])^2$$

$$= E[X^4 + 4X^3Y + 6X^2Y^2 + 4XY^3 + Y^4] - 3(E[X^2 + 2XY + Y^2])^2$$

using linearity of expectation, we obtain the following

$$\text{kurt}(X+Y) = E[X^4] + 4E[X^3Y] + 6E[X^2Y^2] + 4E[XY^3] + E[Y^4]$$

$$- 3(E(X^2) + 2E[XY] + E[Y^2])^2$$

Since X and Y are independent

$$\text{kurt}(X+Y) = E[X^4] + 4E[X^3]E[Y] + 6E[X^2][Y^2] + 4E[X]E[Y^3] + E[Y^4]$$

$$- 3(E(X^2) + 2E[X][Y] + E[Y^2])^2$$

Given $E[X] = E[Y] = 0$ and $E[X^2] = E[Y^2] = 1$

$$\text{kurt}(X+Y) = E[X^4] + 6E[X^2]E[Y^2] + E[Y^4] - 3(E[X^2] + E[Y^2])^2$$

$$= E[X^4] + 6 + E[Y^4] - 3(1+1)^2$$

$$= E[X^4] + E[Y^4] - 6$$

Now consider $\text{kurt}(X) = E[X^4] - 3(E[X^2])^2 = E[X^4] - 3$

$$\text{kurt}(Y) = E[Y^4] - 3(E[Y^2])^2 = E[Y^4] - 3$$

$$\text{kurt}(X) + \text{kurt}(Y) = E[X^4] + E[Y^4] - 6 = \text{kurt}(X+Y)$$

therefore we have shown that $\text{kurt}(X+Y) = \text{kurt}(X) + \text{kurt}(Y)$

$\blacksquare$

Q2 Negentropy and Mutual Information

Y_1, Y_2 potentially dependent random continous variables.

Both with same covariance matrix Σ

$$I(Y_1 : Y_2) = \iint p(y_1, y_2) \log \frac{p(y_1, y_2)}{p(y_1)p(y_2)} dy_1 dy_2 \quad (1)$$

$J(Y_1) = H(Y_g) - H(Y_1)$ (2) $J(Y_2) = H(Y_g) - H(Y_2)$ (3) where

entropy of Y_i is $H(Y_i) = -\int p(y_i) \log p(y_i) dy_i$ (4) for $i = 1, 2$

entropy of a Gaussian random variable $H(Y_g) = \frac{1}{2} \log(\det(2\pi e \Sigma))$ (5)

(a) $I(Y_1 ; Y_2) = H(Y_1) + H(Y_2) - H(Y_1, Y_2)$

proof: The joint entropy $H(Y_1, Y_2)$ is given

$$H(Y_1, Y_2) = -\iint p(y_1, y_2) \log p(y_1, y_2) dy_1 dy_2$$

According to (4) definition of entropy

$$H(Y_1) = -\int p(y_1) \log p(y_1) dy_1$$

$$H(Y_2) = -\int p(y_2) \log p(y_2) dy_2$$

According to sum rule $p(x) = \int p(x,y) dy$

$H(Y_1) = -\iint p(y_1, y_2) \log p(y_1) dy_1 dy_2$ Similarly

$H(Y_2) = -\iint p(y_2, y_1) \log p(y_2) dy_2 dy_1$

we can then show that

$H(Y_1) + H(Y_2) - H(Y_1, Y_2)$

$= -\iint p(y_1, y_2) \log p(y_1) dy_1 dy_2 - \iint p(y_1, y_2) \log p(y_2) dy_1 dy_2 + \iint p(y_1, y_2) \log p(y_1, y_2) dy_1 dy_2$

$= \iint p(y_1, y_2) \log \frac{p(y_1, y_2)}{p(y_1) p(y_2)} dy_1 dy_2 \quad = I(Y_1 ; Y_2)$

Therefore we have shown that $I(Y_1 ; Y_2) = H(Y_1) + H(Y_2) - H(Y_1, Y_2)$

$\blacksquare$

(b) Show $I(Y_1;Y_2) = C - \sum_{i=1}^{2} J(Y_i)$. for C only depends on $H(Y_g)$ and $H(Y_1, Y_2)$

Assume $I(Y_1;Y_2) = C - \sum_{i=1}^{2} J(Y_i)$, we want to find C

from (a) $I(Y_1;Y_2) = H(Y_1) + H(Y_2) - H(Y_1, Y_2)$

$$\sum_{i=1}^{2} J(Y_i) = J(Y_1) + J(Y_2)$$

$$= 2H(Y_g) - H(Y_1) - H(Y_2)$$

$$I(Y_1;Y_2) = C - \sum_{i=1}^{2} J(Y_i) = C - 2H(Y_g) + H(Y_1) + H(Y_2)$$

Hence $C - 2H(Y_g) = H(Y_1, Y_2)$

therefore we have shown that there exist a $C = 2H(Y_g) + H(Y_1, Y_2)$

that make $I(Y_1;Y_2) = C - \sum_{i=1}^{2} J(Y_i)$

∎

Q3 Under-constrained equations

$\Phi = [4 \ \ 1] \quad y = 1$

(a) Solve $\min_{x \in \mathbb{R}^2} \|x\|_2$ subject to $\Phi x = y$

① the objective function $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

the constraint equation $\Phi x = y$, i.e. $4x_1 + x_2 = 1$

Q4 Sparsity

(a) $\Phi = \begin{bmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix}$

Assum that there exist 2 distince 1-sparse solutions x, y

denote non-zero entry at j and k

$x = [x_1, x_2, x_3, x_4]$ where $x_j \neq 0$

$y = [y_1, y_2, y_3, y_4]$ where $y_k \neq 0$

Consider jth entry and k-th entry of ϕx and ϕy.

$(\phi x)_j = \phi_j x_j \neq 0$

$(\phi y)_k = \phi_k y_k \neq 0$

first and second row of Φ are orthogonal.

Assume $x \neq y$, $z = x+y$ result in $\Phi z = 2b$

z is not a 1-sparse solution since it has x_i and y_j

there 1-sparse solution to $\Phi x = b$ is unique

and any 1-sparse solution $x \in R^4$ can be recovered.

(b) Consider all possible combination of choosing 2 indices out of 4

to form distinct 2-sparse solutions.

The combinations are $x = [x_1, x_2, 0, 0]$ $x = [x_1, 0, x_3, 0]$

$x = [x_1, 0, 0, x_4]$ $x = [0, x_2, x_3, 0]$ $x = [0, x_2, 0, x_4]$

$x = [0, 0, x_3, x_4]$

Considering all possible choices, we can find up to 6 distinct 2-sparse solutions that satisfies $\Phi x = b$ ▨

Q5 Sparsity $\Phi = \begin{bmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix}$ 1-sparse solution $b = [0,3]^t$

(a) $x = [0, 0, 3, 0]^t$

x satisfies the given equation $\Phi x = b$

i.e. $$\begin{bmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

x has only one non-zero element

Therefore x is a 1-sparse solution. $\blacksquare$

(b) $x = [x_1, x_2, x_3, x_4]$

$$\Phi x = \begin{bmatrix} x_1 + (1/\sqrt{2})x_2 - (1/\sqrt{2})x_4 \\ (1/\sqrt{2})x_2 + x_3 + (1/\sqrt{2})x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

express x_1 and x_3 in term of x_2 and x_4 gives.

$$x_1 = (\tfrac{1}{\sqrt{2}})x_4 - (\tfrac{1}{\sqrt{2}})x_2$$
$$x_3 = 3 - (\tfrac{1}{\sqrt{2}})x_2 - (\tfrac{1}{\sqrt{2}})x_4$$

$$(3 - \tfrac{1}{\sqrt{2}}x_2 - \tfrac{1}{\sqrt{2}}x_4)(3 - \tfrac{1}{\sqrt{2}}x_2 - \tfrac{1}{\sqrt{2}}x_4)$$
$$9 - \tfrac{3}{\sqrt{2}}x_2 - \tfrac{3}{\sqrt{2}}x_4 - \tfrac{3}{\sqrt{2}}x_2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_2x_4$$
$$- \tfrac{3}{\sqrt{2}}x_4 + \tfrac{1}{2}x_2x_4 + \tfrac{1}{2}x_4^2$$

$$\|x\|_2^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$
$$= (\tfrac{1}{\sqrt{2}}x_4 - \tfrac{1}{\sqrt{2}}x_2)^2 + x_2^2 + (3 - \tfrac{1}{\sqrt{2}}x_2 - \tfrac{1}{\sqrt{2}}x_4)^2 + x_4^2$$
$$= \tfrac{1}{2}x_4^2 - \cancel{x_2x_4} + \tfrac{1}{2}x_2^2 + x_2^2 + 9 - \tfrac{6}{\sqrt{2}}x_2 - \tfrac{6}{\sqrt{2}}x_4 + \cancel{x_2x_4} + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_4^2$$
$$+ x_4^2$$
$$= 2x_2^2 + 2x_4^2 + 9 - \tfrac{6}{\sqrt{2}}x_2 - \tfrac{6}{\sqrt{2}}x_4$$

$$\frac{\partial \|x\|_2^2}{\partial x_2} = 4x_2 - \frac{6}{\sqrt{2}} = 0 \qquad x_2 = \frac{3}{2\sqrt{2}}$$

$$\frac{\partial \|x\|_2^2}{\partial x_4} = 4x_4 - \frac{6}{\sqrt{2}} = 0 \qquad x_4 = \frac{3}{2\sqrt{2}}$$

so $x_1 = \frac{1}{\sqrt{2}} \cdot \frac{3}{2\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \frac{3}{2\sqrt{2}} = 0$

$x_3 = 3 - \frac{1}{\sqrt{2}} \cdot \frac{3}{2\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{3}{2\sqrt{2}} = 3 - \frac{3}{2} = \frac{3}{2}$

$x = [0, \frac{3}{2\sqrt{2}}, \frac{3}{2}, \frac{3}{2\sqrt{2}}]$, the sparsity of x is 3.

$\blacksquare$

(c)