Q6 (a) Given :

$u_i \rightarrow$ eigenvectors of A^TA
$v_i \rightarrow$ eigenvector of AA^T
$u_i^T u_j = 0 \quad \forall \; v_i^T v_j = 0$

To prove :

$A = U \Sigma V^T$

where

$U = [u_i \ldots]$ } given.
$V = [v_i \ldots]$

$\Sigma \leftarrow$ diagonal non-negative matrix

Solution :-

Eigenvalues of $AA^T = A^T A$

$\Rightarrow \quad A^T A u_i = \lambda_i u_i$ ——— (i)

$\Rightarrow \quad AA^T v_i = \lambda_i v_i$ ——— (ii)

From (i) & (ii)

$\hookrightarrow \quad \boxed{A v_i = \sqrt{\lambda_i}\, u_i}$ ——— (iii).

Given $\quad A = U \Sigma V^T$

$\Rightarrow \quad \boxed{U^T A V = \Sigma}$ ——— { because $u_i^T u_j$ = orthonormal & V = orthonormal }

$\hookrightarrow$ (iv)

From (iii) & (iv)

$$\left(\frac{A v_i}{\sqrt{\lambda_i}}\right)^T A v_j \Rightarrow \frac{1}{\lambda_i} v_i^T (A^T A) v_j$$

$$\Rightarrow \frac{v_i^T}{\sqrt{\lambda_i}} (\lambda_j v_j) \Rightarrow \frac{\lambda_j}{\sqrt{\lambda_i}} \; (v_i^T = v_j)$$

——— (v).

from (vi)

if $i \neq j$,

$v_i^T v_j = 0$.

if $i == j$

$\frac{\lambda_i}{\sqrt{\lambda_i}} \cdot \cancel{(v_i^T v_i)} \to 1$

non-zero.