Ans-5

Ans-5) Given : $C \rightarrow$ covariance matrix of vectors in $X = \{x_1, x_2 \dots x_N\}$
$e \rightarrow$ eigenvector of C with highest eigenvalue

To prove : vector f perpendicular to e for which f^tCf is maximized is eigenvector of C with second highest eigenvalue

Proof : We can apply two more constraints with our objective function.

1) $f^tf = 1$ ($\because f$ has unit magnitude)

2) $f^te = 0$ ($\because f$ is perpendicular to e)

Now our objective function that we need to maximize is,

$$J(f) = f^tCf - \lambda_2(f^tf - 1) - \phi(f^te)$$

(Using Lagrange multiplier)

Differentiating w.r.t f (& setting result equal to 0)

$$0 = Cf - \lambda_2 f - \phi e \quad —①$$

Multiplying with e^t on both sides

$$\lambda_2 e^tf + \phi e^te = e^tCf$$

$$\phi = 0 \quad —② \quad (\because e^tf = 0).$$

From ① & ②

$$\lambda_2 f = Cf$$

so, f is an eigenvector of C. (multiplying with f^t)

$$\lambda_2 = f^tCf$$

Since we choose to maximize f^tCf, we choose highest eigenvalue, which is not already taken. Hence, we get the second higest eigenvalue for λ_2. Thus, f is eigenvector with second highest eigenvalue.