Gardner's Minichess Variant is solved

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Abstract

A 5×5 board is the smallest board on which one can set up all kind of chess pieces as a start position. We consider Gardner's minichess variant in which all pieces are set as in a standard chessboard (from Rook to King). This game has roughly 9×10^{18} legal positions and is comparable in this respect with checkers. We weakly solve this game, that is we prove its game-theoretic value and give a strategy to draw against best play for White and Black sides. Our approach requires surprisingly small computing power. We give a human readable proof. The way the result is obtained is generic and could be generalized to bigger chess settings or to other games.

1 Introduction

Solving popular games like Othello, Checkers or Chess tantamount to the grail search in the field of computer games. The resolution of checkers [SBB+07] put a mark in the field in the sense that the space search of this game is enormous (5×10^{20}) and the difficulty to make correct move decisions fairly high.

The game of chess have always been recognized as the ultimate challenge in artificial intelligence. Since the early days of computer science chess and computers have interacted together [Pro12]. Nowadays computers have superhuman strength and the game is partially solved: endgame databases up to few pieces have been computed. The most famous ones being the Nalimov tables (6 pieces). Recently Lomonosov endgame tablebases [Ltd13] have been computed and give perfect play for 7 pieces (the size of the tablebase is 140 Terabytes). Nevertheless, the resolution of chess remains too difficult to be imagined: the number of legal positions is something around 10^{45} [All94] and decision complexity is very high (the amount of chess literature is a proof by itself).

Some studies have been done to resolve particular cases of chess on smaller board. Notably, 3×3, 3×4 and 4×4 (limited to 9 pieces on the board) chess variants have been solved by K. Kryukov [Kry04, Kry09, Kry11]. In these variants there is no starting position as in traditional chess. Positions are treated as puzzles. Each variant is strongly solved in the sense that the game-theoretic value of all legal position is determined together with the perfect play associated. The number of legal positions is roughly 3×10^{15} for the 4×4 variant [Kry11].

In this paper we study the variant called Gardner's Chess. It is played on a 5×5 board, the initial position is the initial position of chess but for the three pieces on the King side that are removed. The rules are the ones of classical chess without the two pawns move, en passant move and castling (pawns cannot promote to bishop). This variant has roughly 9×10^{18} legal positions. This variant has been played extensively notably in Italy by correspondence [Pri07]. The results of finished games were the following:

- White victory 40%
- draw 32%
- Black victory 28 %

2 Results

The game-theoretic value of Gardner's Chess is draw. We prove this by giving two oracles, one for White and one for Black. Both oracles can force draw versus best play. The intersection of the two oracles gives flawless games. Thus Gardner's chess is weakly solved.

The proof is surprisingly small and can be totally checked by a human. Oracles are given in appendix 3 for the White side and appendix 4 for the Black side. From this point of view our result strongly differs from the resolution of checkers despite the fact that space search and difficulty of decision are of the same order of magnitude in both games. Indeed, the proof of [SBB+07] is not checkable by human eyes: it has required an enormous computing power (hundreds of computers in parallel over a decade).

Most of our work was achieved with consumer-grade laptop computers. We have adapted the open source Stockfish chess engine [RCK+10] to play Gardner's Chess mainly by restricting the movements to the part of the board and changing the promotion ranks. Sources, executables for several environments and various files, including the oracles in PGN format as well as the list of the perfect openings for Gardner's Chess, can be found at the author's Minichess Resolution page: "http://membres-lig.imag.fr/prost/Minichessresolution/".

The main line of oracles were computed in a semi-automated way: we were mainly following the most equalizing line. It turns out that most of the deviations from the main line can be quickly decided. It is mainly due to the fact that in Gardner's chess pawns are immediately exchanged or blocked. Moreover, pieces cannot develop naturally since almost all free squares are controlled by pawns. Also the fact that promotion happens quickly leads to some very rapid checkmates that allow to prune the game tree.

Using these Oracles it is impossible to lose. Oracle for White (resp. for Black) does not examine alternative choices for White (resp. Black) decision nodes but indicates how to answer every possible Black (resp. White) "reasonable" move. Unreasonable moves, i.e. moves that obviously lead to a position where it is clear that Black (resp. White) cannot win can be dealt with our engine. We provide the maximal number of moves required to mate for our engine (not necessarily the distance to mate) Nevertheless, in these positions, from a human point of view, it is easy not to lose.

As a by-product of our study on Gardner's Chess the analysis of perfect openings shows the positions in which the evaluation of Stockfish is tricked. Indeed for some positions while showing largely "won" evaluation (up to +6) the position is completely equal. What is interesting is that these evaluation bugs can be found on a 8x8 board as well. Thus the analysis of these positions may help to improve the evaluation of Stockfish for standard chess games.

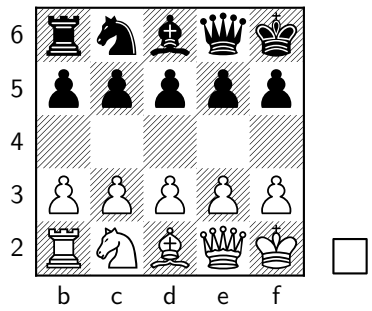
SOIT A complete description of the openings in gardner mini chess as well as a sample of tricky draws and difficult checkmates can be found at *"http://membres-lig.imag.fr/prost/Minichessresolution/"* SOIT in apendix

3 Gardner : Oracle for White Draw

We give an oracle for the White side of the Gardner variation. The objective is to force a draw versus the best play. Therefore, we give it as a tree of variations that needs no explanations on White nodes: it is maybe possible to find shorter draw (or even win) but our aim is to have an oracle the most readable from a human point of view: the definitive judgment on the leafs of this tree are clear.

Since there are no choices to be explored for White nodes we adopt the following convention to name sub-variations: first we note the depth in the oracle, then we enumerate deviations from the main line by enumerating Black (relevant) moves from left to right, pawns come first, after we enumerate moves of the pieces following the lexical order going from left to right and top to bottom. Thus the variation [3|1.3.7] is the one obtained by following the oracle until depth 3 and selecting as sub-variation move 1 as the first move for Black, then move 3 as second move for Black and 7 as the third move. We write $+-$ (resp. $-+$) when it is obvious that Black (resp. White) cannot win. We write $\sharp x\sharp_\bullet$ (resp. $\sharp x\sharp_\circ$) when there exists a forced checkmate of the Black King (res. White King) in x moves (though it is possible that shorter checkmates exist). Very often positions that look lost (because one side has a piece advantage for instance) can be fully

decided by our engine as forced checkmates. Justifying lines are written like this: △ **1 b4 c×b4 2 c×b4 d4**. Finally, the coordinate of the lower left square is **b2**. Hence the starting position is:



In this position the Black move identified by 1 is ... b4 and move number 6 is ... ♘b4, move number 7 is ... ♘c4.

We give the oracle as the following variation tree:

1 b4 c×b4 There are six deviations to consider:

- [1|1] **1... c4 2 d4**
  - [1|1.1] **2... e×d4 3 e×d4** here there are only two moves that do not lose a piece for nothing: (otherwise direct checkmates can be computed even though some are hard to find without computer assistance: 3 e×d4 ♗c5 ♯6♯•, 3 e×d4Ne5 ♯14♯•, 3 e×d4 ♕e3+ ♯6♯•,3 e×d4 ♕e4 ♯10♯•, 3 e×d4 ♕e5 ♯12♯•, 3 e×d4 ♗e5 ♯31♯•):
    - [1|1.1.1] **3... f4 4 ♕×e6+ ♔×e6** = White just has to move his King on e2-f2 and Black cannot break through. No matter what is the relative position of the two Kings, if black Knight takes on d4 or b4 White takes back with the Knight and the position is still blocked for Black and if Black plays ... ♗×b4 the position is ♯17♯• when kings on file e and ♯24♯• when kings are in file f. Finally, if Black plays ... ♘e5 white just takes it with d×e5 and if Black plays ... ♗e5 White just continue to move his king.
    - [1|1.1.2] **3... ♕×e2+ 4 ♔×e2** = for the same reason as line [1|1.1.1].
  - [1|1.2] **2... e4 3 f4** = Black is in zugzwang and must give a piece. Due to the blocked nature of the position he can do it without losing but he cannot break through e.g. **3... ♗e5 4 f×e5+ ♘×e5 5 d×e5+ ♕×e5 6 ♘d4** and White can simply moves back and forth with the Knight.
  - [1|1.3] **2... f4 3 e4** ♯38♯• Black is lost because he lacks space and he is soon losing a piece due to the pawns e4 d4.
  - [1|1.4] **2... ♘×b4 3 d×e5+** +− :
    - [1|1.4.1] **3... ♗×e5 4 ♖×b4** ♯20♯•.
    - [1|1.4.2] **3... ♕×e5 4 ♘×b4**
      - [1|1.4.2.1] **4... d4 5 c×d4** ♯10♯•.
      - [1|1.4.2.2] **4... f4 5 e×f4** ♯10♯•.
      - [1|1.4.2.3] **4... ♖c6 5 ♘×c6** ♯10♯•.
      - [1|1.4.2.4] **4... ♗c5 5 ♘c2** = White blocks the position on the dark squares with ♘d4 and ♖b4 (and moves his Rook between b2-b4 if Black does not move. △ **5... f4 6 ♘d4 b4** (other moves leads to a loss for Black) **7 e×f4 ♗×d4+ 8 c×d4 ♕×e2+** (other moves lead to direct checkmate).
      - [1|1.4.2.5] **4... ♗×b4 5 ♖×b4** ♯25♯•.

· [1|1.4.2.6] **4. . . ♕e6 5 e4** ♯17♯•.
· [1|1.4.2.7] **4. . . ♕d4 5 e×d4** ♯3♯•.
· [1|1.4.2.8] **4. . . ♕e4 5 f×e4** ♯9♯•.
· [1|1.4.2.9] **4. . . ♕f4 5 e×f4** ♯4♯•.
· [1|1.4.2.9] **4. . . ♕×d3 5 e×f4** ♯4♯•.
· [1|1.4.2.10] **4. . . ♕×b3 5 ♗×b3+** ♯5♯•.
· [1|1.4.2.11] **4. . . ♔e6 5 ♘c2** +− similar to line [1|1.4.2.4].

∗ [1|1.4.3] **3. . . ♔×e5 4 c×d4** ♯13♯•

– [1|1.5] **2. . . ♘×d4 3 ♘×d4 ♘×d4** ♯29♯•.

– [1|1.6] **2. . . ♗×b4 3 d×e5+** ♯29♯•.

– [1|1.7] **2. . . ♗c5 3 b×c5** ♯8♯•.

- [1|2] **1. . . d4 2 b×c5** ♯47♯• △ after **2. . . ♗×c5 3 f4** both the pressure on the b file and on diagonal b2 f6 are to strong to be sustained by by Black.
- [1|3] **1. . . e4 2 b×c5** ♯28♯• The point is that on **2. . . ♗×c5 3 d4** the threat of ♖×b5 combined with the lack of space for Black is too hard to be met. Other moves just lose a piece at least.
- [1|4] **1. . . f4 2 b×c5** ♯24♯• △ 2... Bxc5 3.d4 the threat of ♖×b5.
- [1|5] **1. . . ♘×b4 2 c×b4** ♯24♯• White is a piece up for nothing.
- [1|6] **1. . . ♘d4 2 b×c5** ♯17♯• White is a piece up for nothing.

2 c×b4 d4 we only consider the two other pawns move for Black since all Knight and Bishop moves lose a piece and end up in a position where clearly Black cannot win (2. . . ♘d4 ♯23♯•, 2. . . ♘×b4 ♯18♯•, 2. . . ♗×b4 ♯15♯•).

- [2|1] **2. . . e4 3 ♗c3+** ♯15♯•.
- [2|2] **2. . . f4 3 b×c5** ♯29♯• the idea is that the b6 pawn is lost and Black is lacking space, △ **3. . . ♗×c5 4 d4 ♗d6 5 ♖×b5**.

3 e4 f4 other moves lead directly to forced checkmate.

- [3|1] **3. . . f×e4 4 f×e4** ♯8♯• The threat of ♕f3+ is too strong.
- [3|2] **3. . . ♘×b4** ♯15♯•.
- [3|3] **3. . . ♗×b4** ♯15♯•.
- [3|4] **3. . . ♕b3** ♯12♯•.
- [3|5] **3. . . ♕c4** ♯8♯•.
- [3|6] **3. . . ♕c6** ♯8♯•.

4 ♗×f4 e×f4 again alternatives lead to forced checkmate since they lose a piece for nothing.

- [4|1] **4. . . ♘×b4 5 ♘×b4 ♗×b4**, otherwise ♘c7, **6 ♗×e5+** ♯23♯•
- [4|2] **4. . . ♗×b4 5 ♘×b4 ♘×b4**, otherwise ♘c7, **6 ♗×e5+** ♯23♯•
- [4|3] **4. . . ♗c5** ♯8♯•.
- [4|4] **4. . . ♕b3** ♯11♯•.

- [4|5] **4. . . ♕c4** ♯8♯•.
- [4|6] **4. . . ♕d5** ♯5♯•.
- [4|7] **4. . . ♕f5** ♯8♯•.

5 ♕d2 ♗e5

- [5|1] **5. . . ♘e5 6 ♕×f4+** ♯2♯•.
- [5|2] **5. . . ♘×b4 6 ♘×b4** +− d5 cannot be protected and the threat of ♕×f4+ forbids **6. . . ♗×b4**.

6 ♔e2 = White just moves his King on e2-f2 and Black cannot untangle by . . . ♘e5 because of ♕×f4 and must otherwise give a piece and cannot win.

4 Gardner : Oracles for Black Draw

We give, for each of the White seven legal first move, an oracle from the Black point of view that forces the draw. So here we give no explanations for Black decision nodes and we explore all reasonable moves (as explained erlier) at White decision nodes. These oracles are sometimes much simpler than the White oracle for draw since, rather curiously, it is sometimes more difficult for White to achieve draw. It means that often even slight deviations from the main line directly lead to positions that can be decided as forced checkmates.

4.1 White moves b4

1 b4 c×b4

- [1|1] **2 c4 b×c4** ♯15♯∘ the b4 c4 pawn duo is too strong.
- [1|2] **2 d4 b×c3**
  - [1|2.1] **3 d×e5+** ♯17♯∘.
  - [1|2.2] **3 e4** ♯9♯∘.
  - [1|2.3] **3 f4** ♯10♯∘.
  - [1|2.4] **3 ♖b3** ♯17♯∘.
  - [1|2.5] **3 ♖b4** ♯12♯∘.
  - [1|2.6] **3 ♖×b5** ♯12♯∘.
  - [1|2.7] **3 ♘a4** ♯9♯∘.
  - [1|2.8] **3 ♗×c3 b4**
    - [1|2.8.1] **4 d×e5+ ♗×e5** there is only one White move to consider since the other ones lead to checkmate (**5 e4** ♯13♯∘, **5 f4** ♯15♯∘, **5 ♖b3** ♯15♯∘, **5 ♖×b4** ♯♯∘, **5 ♘×b4** ♯22♯∘, **5 ♘d4** ♯10♯∘, **5 ♕d2** ♯8♯∘, **5 ♕d3** ♯15♯∘, **5 ♕c4** ♯10♯∘, **5 ♕b5** ♯8♯∘, **5 ♗×b4** ♯17♯∘, **5 ♗d2** ♯11♯∘, **5 ♗d4** ♯29♯∘) **5 ♗×e5+ ♕×e5** here again White move is forced otherwise direct checkmate can be found (**6 e4** ♯10♯∘, **6 f4** ♯8♯∘, **6 ♖×b4** ♯11♯∘, **6 ♘×b4** ♯22♯∘, **6 ♘d4** ♯19♯∘, **6 ♕d2** ♯10♯∘, **6 ♕b5** ♯7♯∘, **6 ♕c4** ♯6♯∘, **6 ♕d3** ♯7♯∘) **6 ♖b3 f4** , once again among the 12 possible variations only one do not lead to checkmate (**7 e4** ♯16♯∘, **7 e×f4** ♯37♯∘, **7 ♖b2** ♯6♯∘, **7 ♖×b4** ♯16♯∘, **7 ♖c3** ♯6♯∘, **7 ♖d3** ♯9♯∘, **7 ♘×b4** ♯13♯∘, **7 ♘d4** ♯8♯∘, **7 ♕d3** ♯29♯∘, **7 ♕c4** ♯5♯∘, **7 ♕b5** ♯8♯∘,) **7 ♕d2**
    - [1|2.8.2] **4 e4 d×e4** ♯19♯∘.
    - [1|2.8.3] **4 f4 e×f4** ♯30♯∘.

* [1|2.8.4] **4 ♖b3 f4** only four variations need consideration as the other ones are losing straight away (**5 d×e5+** ♯♯$_\circ$, ATTENTION 5eme nulle? **5 e4** ♯19♯$_\circ$, **5 ♖b2** ♯33♯$_\circ$, **5 ♖×b4** ♯27♯$_\circ$, **5 ♗×b4** ♯22♯$_\circ$, **5 ♘×b4** ♯27♯$_\circ$, **5 ♕b5** ♯10♯$_\circ$, **5 ♕c4** ♯8♯$_\circ$, **5 ♕d2** ♯20♯$_\circ$)
  - [1|2.8.4.1] **5 e×f4 e×f4** = △ **6 ♕d3 ♕f5 7 ♕e2 ♕e6** with repetition. White cannot play ♕d2 due to . . . b×c3. Once queen have been exchanged the position is blocked and Black can just move his King ad lib.
  - [1|2.8.4.2] **5 ♗b2 f×e3+ 6 ♕×e3 e×d4 7 ♗×d4+ e×d4 8 ♕×d4+ ♗e5** −+ Black can play . . . ♗c3 on any queen moves and blocks the position.
  - [1|2.8.4.3] **5 ♗d2 e×d4 6 e×d4 ♕×e2+ 7 ♔×e2** −+ White cannot untangle and Black may just move his King around.
  - [1|2.8.4.4] **5 ♕d3 e×d4** = similar as line [1|2.8.4.1].
* [1|2.8.5] **4 ♖×b4 ♖×b4** = △ **5 d×e5+ ♗×e5 6 ♗×b4 ♘×b4 7 ♘×b4 d4**
* [1|2.8.6] **4 ♘×b4 ♘×b4** ♯26♯$_\circ$.
* [1|2.8.7] **4 ♗×b4 ♘×b4** ♯24♯$_\circ$.
* [1|2.8.8] **4 ♗d2 b3** ♯23♯$_\circ$.
* [1|2.8.9] **4 ♕d2 b×c3** ♯18♯$_\circ$.
* [1|2.8.10] **4 ♕d3 e4** ♯26♯$_\circ$.
* [1|2.8.11] **4 ♕c4 d×c4** ♯10♯$_\circ$.
* [1|2.8.12] **4 ♕b5 ♖×b5** ♯9♯$_\circ$.

– [1|2.9] **3 ♕d3** ♯10♯$_\circ$.

– [1|2.10] **3 ♕c4** ♯10♯$_\circ$.

– [1|2.11] **3 ♕×b5** ♯7♯$_\circ$.

- [1|3] **2 e4 b×c3** −+ since **3 ♗×c3 b4** is ♯20♯$_\circ$ and other moves lose a piece and ends in a losing position for White.

- [1|4] **2 f4 b×c3**

– [1|4.1] **3 d4** ♯10♯$_\circ$.

– [1|4.2] **3 e4** ♯9♯$_\circ$.

– [1|4.3] **3 f×e5+ ♗×e5** ♯15♯$_\circ$.

– [1|4.4] **3 ♖b3** ♯15♯$_\circ$.

– [1|4.5] **3 ♖b4** ♯15♯$_\circ$.

– [1|4.6] **3 ♖×b5** ♯14♯$_\circ$.

– [1|4.7] **3 ♘b4** ♯10♯$_\circ$.

– [1|4.8] **3 ♘d4** ♯14♯$_\circ$.

– [1|4.9] **3 ♗×c3 b4**
* [1|4.9.1] **4 d4** ♯33♯$_\circ$.
* [1|4.9.2] **4 e4** ♯12♯$_\circ$.
* [1|4.9.3] **4 f×e5+ ♗×e5** only two moves do not lead to direct checkmate (**5 e4** ♯12♯$_\circ$, **5 ♖b3** ♯19♯$_\circ$, **5 ♖×b4** ♯27♯$_\circ$, **5 ♘×b4** ♯26♯$_\circ$, **5 ♘d4** ♯10♯$_\circ$, **5 ♗×b4** ♯14♯$_\circ$, **5 ♗d4** ♯23♯$_\circ$, **5 ♗d2** ♯11♯$_\circ$, **5 ♕d2** ♯14♯$_\circ$, **5 ♕f3** ♯12♯$_\circ$, **5 ♔f3** ♯11♯$_\circ$)
  - [1|4.9.3.1] **5 ♗×e5+ ♕×e5** (**6 e4** ♯10♯$_\circ$, **6 ♖b3** ♯♯$_\circ$, ATTENTION variante nulle ? **6 ♖×b4** ♯12♯$_\circ$, **6 ♘×b4** ♯15♯$_\circ$, **6 ♘d4** ♯14♯$_\circ$, **6 ♕d2** ♯8♯$_\circ$, **6 ♕f3** ♯8♯$_\circ$, **6 ♔f3** ♯8♯$_\circ$,) **6 d4 ♕e4 7 ♖b3 f4** = since the only moves that not lose for White are either **8 e×f4 ♕×e2+ 9 ♔×e2 ♔f5** and Black and White King move ad lib., or **8 ♕d2 f3 9 ♘×b4 ♖×b4 10 ♖×b4 ♘×b4 11 ♕×b4 ♕c2+** and perpetual check or **12 ♕c5 ♕b3 13 ♕d6 ♕×e3+ 14 ♔×e3** stalemate.

· [1|4.9.3.2] **5 d4 ♗d6 6 ♖b3** (other moves lose directly because of the threat . . . b3) **6. . . ♕e4** −+ since White can't do nothing to untangle and Black may just move his Rook on b5 b6.

∗ [1|4.9.4] **4 ♖b3 d4** there are two moves to consider the other ones lead to direct checkmate (**5 e4** ♯10♯∘, **5 ♖b2** ♯10♯∘, **5 ♖×b4** ♯20♯∘, **5 ♘×b4** ♯13♯∘, **5 ♘×d4** ♯14♯∘, **5 ♗b2** ♯14♯∘, **5 ♗×b4** ♯15♯∘, **5 ♗×d4** ♯14♯∘, **5 ♗d2** ♯11♯∘, **5 ♕d2** ♯11♯∘, **5 ♕f3** ♯10♯∘, **5 ♔f3** ♯9♯∘)

· [1|4.9.4.1] **5 e×d4 e×d4 6 ♕×e6+**, other White moves lose straightforwardly since the Bisop is lost, **6. . . ♔×e6 7 ♗×d4**, otherwise Black just moves his Rook on b5 b6 and White cannot break through **7. . . ♘×d4 8 ♘×d4+ ♔d5** = the f4 pawn is going to fall and White cannot win this position.

· [1|4.9.4.2] **5 f×e5+ ♗×e5 6 ♖×b4** other moves lose the Rook and lead to quick White defeat **6. . . ♖×b4 7 ♗×b4 ♘×b4 8 ♘×b4 d×e3+ 9 ♕×e3 ♕d6 10 ♘c2 ♕c6** = the best for White is to repeat moves with **11 ♘b4 ♕d6**.

∗ [1|4.9.5] **4 ♖×b4 ♖×b4** there are two moves to consider the other ones lead to direct checkmate (**5 d4** ♯20♯∘, **5 e4** ♯12♯∘, **5 ♘d4** ♯12♯∘, **5 ♗×b4** ♯20♯∘, **5 ♗b2** ♯9♯∘, **5 ♗d4** ♯14♯∘, **5 ♗d2** ♯11♯∘, **5 ♗×e5+** ♯18♯∘, **5 ♕d2** ♯16♯∘, **5 ♕f3** ♯12♯∘, **5 ♔f3** ♯12♯∘)

· [1|4.9.5.1] **5 f×e5+ ♗×e5 6 ♗×b4 ♘×b4 7 ♘×b4 d4** = last pawns will soon be exchanged and White cannot force any advantage △ **8 e4 ♕b3 9 ♘d5+ ♔e6 10 ♕f3 f×e4**.

· [1|4.9.5.2] **5 ♘×b4 ♘×b4 6 f×e5+ ♗×e5 7 ♗×b4 d4** = for the same reasons as line [1|4.9.5.1].

∗ [1|4.9.6] **4 ♗×b4** ♯26♯∘.

∗ [1|4.9.7] **4 ♗d4** ♯22♯∘.

∗ [1|4.9.8] **4 ♗×e5+ ♗×e5** there is only one line to consider since other variations lead to checkmate (**5 d4** ♯24♯∘, **5 e4** ♯10♯∘, **5 ♖b3** ♯21♯∘, **5 ♖×b4** ♯13♯∘, **5 ♘×b4** ♯14♯∘, **5 ♘d4** ♯14♯∘, **5 ♕d2** ♯9♯∘, **5 ♕f3** ♯9♯∘, **5 ♔f3** ♯9♯∘) **5 f×e5+ ♕×e5 6 d4 ♕e4** = the position is blocked and Black can just move his King to e6 f6 White can't remove his Rook from the b file and if he tries to break through the ending will be the same as in line [1|4.9.3.1].

∗ [1|4.9.9] **4 ♗d2 e×f4** ♯23♯∘.

∗ [1|4.9.10] **4 ♘×b4 ♖×b4** there are only to variants to consider since the other ones lead to checkmate (**5 d4** ♯20♯∘, **5 e4** ♯12♯∘, **5 ♖b3** ♯10♯∘, **5 ♖c2** ♯24♯∘, **5 ♖d2** ♯21♯∘, **5 ♗×b4** ♯31♯∘, **5 ♗d2** ♯9♯∘, **5 ♗d4** ♯12♯∘, **5 ♗×e5+** ♯18♯∘, **5 ♕c2** ♯20♯∘, **5 ♕d2** ♯17♯∘, **5 ♕f3** ♯22♯∘, **5 ♔f3** ♯21♯∘)

· [1|4.9.10.1] **5 f×e5+ ♗×e5 6 ♖×b4 ♘×b4 7 ♗×b4 d4** = this endgame is completly draw since a couple of pawns will be excanged and the remaing ones are mutually blocked.

· [1|4.9.10.2] **5 ♖×b4 ♗×b4 6 f×e5+ ♘×e5 7 ♗×b4 ♕b6** = .

∗ [1|4.9.11] **4 ♘d4** ♯19♯∘

∗ [1|4.9.12] **4 ♕d2** ♯22♯∘.

∗ [1|4.9.13] **4 ♕f3 d4** there is only one move to consider otherwise White is directly checkmated (**5 e×d4** ♯20♯∘, **5 e4** ♯10♯∘, **5 ♖b3** ♯10♯∘, **5 ♖×b4** ♯20♯∘, **5 ♘×b4** ♯10♯∘, **5 ♘×d4** ♯10♯∘, **5 ♗×b4** ♯19♯∘, **5 ♗d2** ♯22♯∘, **5 ♗×d4** ♯15♯∘, **5 ♕e2** ♯10♯∘, **5 ♕e4** ♯8♯∘, **5 ♕d5** ♯9♯∘, **5 ♕×c6** ♯11♯∘, **5 ♔e2** ♯9♯∘) **5 f×e5+ ♘×e5 6 ♗×d4 b3** −+ △ **7 ♔e2 b×c2♕+ 8 ♖×c2 ♕b3** White can hold the balance due to the pin on the Knight and of the threat e4 which forces Black to move back his Queen to e6.

∗ [1|4.9.14] **4 ♔f3** ♯22♯∘.

– [1|4.10] **3 ♕f3** ♯8♯∘.

– [1|4.11] **3 ♔f3** ♯8♯∘.

• [1|5] **2 ♖b3 d4** ♯17♯∘.

- [1|6] **2 ♖×b4 ♘×b4** ♯21♯$_\circ$.
- [1|7] **2 ♘×b4 ♘×b4**
  - [1|7.1] **3 c×b4 d4**
    - [1|7.1.1] **4 e×d4 e×d4** two variations are woth investigations since other moves lose directly (**5 ♖b3** ♯9♯$_\circ$, **5 ♖c2** ♯13♯$_\circ$, **5 ♗c3** ♯11♯$_\circ$, **5 ♗e3** ♯29♯$_\circ$, **5 ♗f4** ♯15♯$_\circ$, **5 ♕e3** ♯8♯$_\circ$, **5 ♕e4** ♯10♯$_\circ$, **5 ♕e5** ♯7♯$_\circ$)
      - [1|7.1.1.1] **5 f4 ♕×e2+ 6 ♔×e2 ♖c6** = the position is totally blocked on dark squarres and White can only play his King or his Rook on b2 b3.
      - [1|7.1.1.2] **5 ♕×e6+ ♔×e6** = for the same reasons as line [1|7.1.1.1].
    - [1|7.1.2] **4 e4 f4** only one move does not lose and lead to a type of drawn position already seen in line [1|1.2] of the White oracle (**5 ♖b3** ♯6♯$_\circ$, **5 ♖c2** ♯16♯$_\circ$, **5 ♗c3** ♯8♯$_\circ$, **5 ♗e3** ♯10♯$_\circ$, **5 ♕e3** ♯9♯$_\circ$) **5 ♗×f4 e×f4 6 ♕c4 ♕e5** = White can just move his Rook on b2 b3 or his King other over moves are loosing (he cannot lose the control of the c file).
    - [1|7.1.3] **4 f4 e×f4** Only two variations do not lose (**5 e4** ♯29♯$_\circ$, **5 ♖b3** ♯12♯$_\circ$, **5 ♖c2** ♯17♯$_\circ$, **5 ♗c3** ♯2♯$_\circ$, **5 ♕f3** ♯19♯$_\circ$, **5 ♔f3** ♯12♯$_\circ$)
      - [1|7.1.3.1] **5 e×d4 ♕×e2+ 6 ♔×e2 ♔e6** = Black King will seat on d5 and White cannot get through.
      - [1|7.1.3.1] **5 e×f4 ♕×e2+ 6 ♔×e2 ♔e6** = same as line above, the Black King seats on d5 and Black may just move his Rook between b6 c6.
    - [1|7.1.4] **4 ♖b3 ♕×b3** ♯12♯$_\circ$.
    - [1|7.1.5] **4 ♖c2 ♕b3** ♯16♯$_\circ$.
    - [1|7.1.6] **4 ♗c3 d×c3** ♯11♯$_\circ$.
  - [1|7.2] **3 c4 b×c4** ♯12♯$_\circ$.
  - [1|7.3] **3 d4 ♖c6** ♯♯$_\circ$. Attention nulle, RQ si on remplace l'item par 3. d4 e4 only one possible white move 4 cxb4 exf3 A ETUDIER otherwise 4.c4 M17,4.f4 M10,4.Rxb4 M9, 4. Rb3 M9, 4.Qxb5 M8, 4.Rc M8, 4.Qc4 M8
  - [1|7.4] **3 e4 ♘c6** ♯33♯$_\circ$.
  - [1|7.5] **3 f4 ♘c6** ♯33♯$_\circ$.
  - [1|7.6] **3 ♖b3 d4** ♯18♯$_\circ$.
  - [1|7.7] **3 ♖×b4 ♗×b4** ♯25♯$_\circ$.
  - [1|7.8] **3 ♖c2 ♘×c2** ♯11♯$_\circ$.
- [1|8] **2 ♘d4 b×c3** ♯20♯$_\circ$.

2 c×b4 d4

- [2|1] **3 e×d4 e×d4** only three variations are worth of consideration (**4 ♖b3** ♯10♯$_\circ$, **4 ♘e3** ♯29♯$_\circ$, **4 ♗c3** ♯15♯$_\circ$, **4 ♗e3** ♯19♯$_\circ$, **4 ♗f4** ♯11♯$_\circ$)
  - [2|1.1] **4 f4 ♕×e2+ 5 ♔×e2 f4** = since only Kings can move without losing a piece and leading to a lost position (♖b3 is possible but changes nothing to the evaluation of the position).
  - [2|1.2] **4 ♕×e6+ ♔×e6 5 f4** = similar as line [2|1.1].
  - [2|1.3] **4 ♘×d4 ♘×d4** other white moves leads to mate (**5 ♕e3** ♯22♯$_\circ$, **5 ♕×e6** ♯16♯$_\circ$, **5 ♗e3** ♯10♯$_\circ$), **5 ♗c3**
- [2|2] **3 f4 e×f4** there are only three variations that not lose for White (**4 e4** ♯24♯$_\circ$, **4 ♖b3** ♯14♯$_\circ$, **4 ♗c3** ♯11♯$_\circ$, **4 ♕f3** ♯23♯$_\circ$, **4 ♔f3** ♯15♯$_\circ$):

  - [2|2.1] **4 e×d4 ♕×e2+ 5 ♔×e2** = .
  - [2|2.2] **4 e×f4 ♕×e2+ 5 ♔×e2** = see variation [2|1.2].
  - [2|2.3] **4 ♘×d4 ♘×d4 5 e×d4** (5 ♗c3 f×e3+ 6 ♕×e3 ♕×e3+ 7 ♔×e3 ♗e5 followed by exchanges to a completly drawn endgame does not change the assesment of the position) **5. . . ♕×e2+ 6 ♔×e2 ♔e6** = the Black King will seat on d5.

- [2|3] **3 ♖b3 ♕×b3** ♯11♯∘.
- [2|4] **3 ♘×d4 ♘×d4**
  - [2|4.1] **4 e×d4 e×d4** we only consider two moves since the other ones leafd to forced checkmate (**5 ♖b3** ♯9♯∘, **5 ♖c2** ♯13♯∘, **5 ♗c3** ♯11♯∘, **5 ♗e3** ♯19♯∘, **5 ♗f4** ♯15♯∘, **5 ♕e3** ♯8♯∘, **5 ♕e4** ♯10♯∘, **5 ♕e5** ♯7♯∘)
    - [2|4.1.1] **5 f4 ♕×e2+ 6 ♔×e2** = see variation [1|7.1.1.1]
    - [2|4.1.2] **5 ♕×e6+ ♔×e6** = see variation [1|7.1.1.2]
  - [2|4.2] **4 e4** ♯11♯∘.
  - [2|4.3] **4 f4** ♯9♯∘.
  - [2|4.4] **4 ♖b3** ♯9♯∘.
  - [2|4.5] **4 ♖c2** ♯10♯∘.
  - [2|4.6] **4 ♗c3** ♯11♯∘.
- [2|5] **3 ♗c3 d×c3** ♯8♯∘

3 e4 f4

- [3|1] **4 ♖b3 ♕×b3** ♯7♯∘.
- [3|1] **4 ♘×d4 ♘×d4** ♯16♯∘.
- [3|1] **4 ♘e3 f×e3** ♯18♯∘.
- [3|1] **4 ♗c3 d×c3** ♯9♯∘.
- [3|1] **4 ♗e3** ♯18♯∘.
- [3|1] **4 ♕e3** ♯12♯∘.

4 ♗×f4 e×f4

- [4|1] **5 e5** ♯16♯∘.
- [4|1] **5 ♖b3** ♯7♯∘.
- [4|1] **5 ♘×d4** ♯9♯∘.
- [4|1] **5 ♘e3** ♯7♯∘.
- [4|1] **5 ♕e3** ♯7♯∘.

5 ♕d2 ♗e5 = since the only non losing moves for White are limited to the King and Queen moves over the d2 e2 and f2 squares.

4.2 White moves c4

1 c4 b×c4 The pin on the b file leads to forced mate ♯27♯∘.

4.3 White moves d4

1 d4 e4

- [1|1] **2 b4 c4 3 f4** otherwise White is checkmated quickly (**3 f×e4** ♯9♯$_\circ$, **3 ♖b3** ♯8♯$_\circ$, **3 ♕×c4**♯16♯$_\circ$, **3 ♕d3** ♯9♯$_\circ$) **3. . . ♗×b4**
  - [1|1.1] **4 c×b4 ♕d6** = despite his extra piece White cannot win since he is blocked by his own pawns on dark squares.
  - [1|1.2] **4 ♖b3** ♯8♯$_\circ$.
  - [1|1.3] **4 ♖×b4 ♘×b4** ♯21♯$_\circ$.
  - [1|1.4] **4 ♘×b4 ♘×b4** = △ **5 ♖×b4 ♕d6** and White may only move his Rook, on **5 b×c4 ♕d6** is similar to [1|1.1].
  - [1|1.5] **4 ♕×c4** ♯15♯$_\circ$.
  - [1|1.6] **4 ♕d3** ♯10♯$_\circ$.
  - [1|1.7] **4 ♕f3** ♯11.♯$_\circ$
- [1|2] **2 c4 b×c4** there are two variations to consider for White (3.b4 M15,3.bxc4 M19,3.Nb4 M14,3.Bb4 M10,3.fxe4 M10,3.f4 M17,3.Qd3 M8,3.Qxc4 M?):
  - [1|2.1] **3 d×c5 ♗×c5 4 ♗c3 ♕e5** (otherwise if 4.b4 M20,4.Nb4 M12,4.Nd4 M15,4.Bb4 M14,4.Qd3 M8,4.Qxc4 M11,4.f4 M12) white move is still forced and he can't take the queen 5.Qd2 f4 otherwise (5.Bxe5+ M19 5.bxc4 M16, 5.fxe4 M13, 5.b4 M9,5.Nd4 M12,5.Nb4 M12,5.f4 M7,5.Bd4 M9,5.Qxc4 M9,5.Qc3 M7) white move still forced 6.Bxe5 Nxe5 otherwise (6.Ke2 M12,6.fxe4 M13,6.b4 M17) and white moves still forced 7.b4 fxe3 (otherwise 7.Nd4 M16, 7.Qc3 M12,7.fxe4 M20) white moves still forced 8.Nxe3 Nd3 (otherwise 8.Qxe3 M14, 8.Ke2 M1) white still have only one move 9.Ke2 Nf4 (otherwise 9.Qxd3 M6) now only one legal move10.Kf2 Nf4 and draw by repetition.
  - [1|2.2] **3 ♗c3**
- [1|3] **2 d×c5 ♗×c5**
  - [1|3.1] **3 b4 e×f3** XXXXX A ETUDIER
    - [1|3.1.1] **4 ♕×f3 ♗d6** There is two possibilities (If 5.Ke2 M33,5.Rb3 M?(bug mat plus de 50 empeche de conclure),5.Qe2 M21,5.e4 M21, 5.Qf4 M17, 5.Qxd5 M9,5.Qe4 M6,5.Qxf5 M8)
      5.c4 bxc4 A ETUDIER
      5.Nd4 Nxd4 One possible move 6.cxd4 Rc6 (6.exd4 M39, 6. c4 M12, 6. Qxd5 M9, 6. Rb3 M9, 6. Qe2 M9, 6. Rc2M8, 6. Qf4 M8). Two possible moves 7.Ke2 or 7.Qe2 (7.e4 M14,7.Rb3 M31))
    - [1|3.1.2] **4 b×c5 f×e2♕ 5 ♔×e2 ♘e5** If promotion 6.cxb6 Qxb6 otherwise if 6. Rb3 Rc6, if 6. Rb4 Rc6 other moves leads to mate 6.Kf2 M,6.e4 M
    - Otherwise 4.Qd3 M12, 4.Kxf3 M 23,4.Rb3 M11,4.c4 M11,4.Nd4 M12,4.Qc4 M8 and 4.Qxb5 M10
  - [1|3.2] **3 c4 b4** ♯23♯$_\circ$.
  - [1|3.3] **3 f×e4** ♯14♯$_\circ$.
  - [1|3.4] **3 f4 ♕d6** ♯14♯$_\circ$.
  - [1|3.5] **3 ♘b4 f4** ♯16♯$_\circ$.
  - [1|3.6] **3 ♘d4 ♘×d4**
    - [1|3.6.1] **4 b4** ♯12♯$_\circ$.
    - [1|3.6.2] **4 c4** ♯9♯$_\circ$.

    - [1|3.6.3] **4 c×d4 e×f3** ????
    - [1|3.6.4] **4 e×d4 ♗d6** we consider only two variations since other moves lead to forced checkmate (**5 ♖c2 f4** ♯23♯$_\circ$, **5 b4 f4** ♯18♯$_\circ$, **5 ♗e3 f4** ♯22♯$_\circ$, **5 ♕e3 f4** ♯22♯$_\circ$, **5 c4 b×c4** ♯23♯$_\circ$, **5 ♔e3 f4+** ♯23♯$_\circ$, **5 ♗f4** ♯11♯$_\circ$, **5 ♕×b5** ♯10♯$_\circ$, **5 ♕c4** ♯8♯$_\circ$, **5 ♕d3** ♯4♯$_\circ$, **5 ♗f4** ♯11♯$_\circ$)
        - [1|3.6.4.1] **5 f×e4 ♕×e4** = if White exchanges Queen on e4 then with . . . f×e4 Black closes the position and with . . . ♖c6 White cannot do anything. If White does not exchange Queens then Black may just play his King (on **6 b4 f4** is ♯28♯$_\circ$).
        - [1|3.6.4.2] **5 f4 e3+** = since Black follows with . . . ♕e4 and blocks the position.
    - [1|3.6.5] **4 f×e4** ♯10♯$_\circ$.
    - [1|3.6.6] **4 f4** ♯9♯$_\circ$.
    - [1|3.6.7] **4 ♖c2** ♯9♯$_\circ$.
    - [1|3.6.8] **4 ♕×b5** ♯8♯$_\circ$.
    - [1|3.6.9] **4 ♕c4** ♯8♯$_\circ$.
    - [1|3.6.10] **4 ♕d3** ♯7♯$_\circ$.
  - [1|3.7] **3 ♕×b5** ♯11♯$_\circ$.
  - [1|3.8] **3 ♕c4** ♯9♯$_\circ$.
  - [1|3.9] **3 ♕d3** ♯9♯$_\circ$.

- [1|4] **2 f×e4 f×e4** ♯12♯$_\circ$.

- [1|5] **2 f4 c4**
  - [1|5.1] **3 b4**
  - [1|5.2] **3 b×c4 d×c4** we only consider two moves since the other ones lead to forced checkmate (**4 d5 ♕×d5** ♯22♯$_\circ$, **4 ♖×b5 ♖×b5** ♯10♯$_\circ$, **4 ♖b3** ♯9♯$_\circ$, **4 ♕×c4** ♯8♯$_\circ$, **4 ♕d3** ♯9♯$_\circ$, **4 ♕f3** ♯11♯$_\circ$)
    - [1|5.2.1] **4 ♖b4 ♘×b4 5 c×b4** (5 d5 ♯8♯$_\circ$, 5 ♘×b4 ♯23♯$_\circ$, 5 ♕×c4 ♯5♯$_\circ$, 5 ♕d3 ♯7♯$_\circ$,5 ♕f3 ♯6♯$_\circ$) **5. . . ♕d5** = because the position is totally blocked and Black just moves his King to e6 f6. The only way to untangle for White is to sacrifice the Queen on c4 which lead to quick checkmate.
    - [1|5.2.2] **4 ♘×b4 ♗c5** ♯12♯$_\circ$ this surprising move lead to direct checkmate since White is completly blocked and will eventually, due to his lack of space, have to gite his Queen within a few moves.
  - [1|5.3] **3 ♘b4 ♘×b4** = The draw is tricky to understand at first sight but becomes clear with the following variation (otherwise White is quickly lost) **4 c×b4 ♗×f4** . From here the idea is to build a blockade on dark squares. There are only two variations to consider (other variation lead to quick mate).
    - After **5 e×f4 ♖c6** (in order to be able to take with the Rook in the case of b×c4) it is done and Black just moves his Queen on d6 and his King on e6 f6.
    - On **5 b×c4** (other White moves are quickly losing for White) **5. . . b×c4 6 e×f4 ♖b5** another blockade is built on dark squares and White cannot break through.
  - [1|5.4] **3 ♕d3** ♯12♯$_\circ$.
  - [1|5.5] **3 ♕×c4 b×c4** ♯15♯$_\circ$.
  - [1|5.5] **3 ♕f3** ♯8♯$_\circ$.

- [1|6] **2 ♘b4** ♯21♯$_\circ$.

- [1|7] **2 ♕×b5** ♯12♯$_\circ$.

- [1|8] **2 ♕c4** ♯11♯∘.
- [1|9] **2 ♕d3** ♯10♯∘.

4.4 White moves e4

1 e4 f4 most of the moves are straightforwardly losing for White.

- [1|1] **2 b4 c×b4** there is just one move to consider since the other ones lead to direct checkmate for White (**3 c4** ♯15♯∘, **3 d4** ♯27♯∘, **3 e×d5** ♯27♯∘, **3 ♖b3** ♯17♯∘, **3 ♖×b4** ♯34♯∘, **3 ♘×b4** ♯19♯∘, **3 ♘d4** ♯14♯∘, **3 ♘e3** ♯17♯∘, **3 ♗e3** ♯14♯∘, **3 ♗×f4** ♯16♯∘, **3 ♕e3** ♯17♯∘) **3 c×b4 d4** This position is draw for the same reason as position [1|1.2] of the White oracle (see section 3). White is in zugzwang and must give a piece, the only non losing way to do it is by **4 ♗×f4 e×f4 5 ♕d2**
- [1|2] **2 c4 d4** White has only two drawing lines (**3 b4** ♯22♯∘, **3 d×c4** ♯19♯∘, **3 d4** ♯15♯∘, **3 ♘b4** ♯12♯∘, **3 ♘d4** ♯15♯∘, **3 ♘e3** ♯16♯∘, **3 ♗c3** ♯20♯∘, **3 ♗b4** ♯12♯∘, **3 ♗e3** ♯13♯∘, **3 ♗×f4** ♯12♯∘, **3 ♕e3** ♯14♯∘)
  - [1|2.1] **3 b×c4 ♖×b2** = is a tricky draw in which White appear to be losing but can hold. The mainline is the following **4 c×d5 ♕×d5 5 e×d5 ♘d4 6 ♘×d4 c×d4** at this point Black will regain the Queen and the bishop by force (otherwise White get mated) and end up in a ending like this one **7 ♗b4 ♖×e2+ 8 ♔×e2 ♗×b4** and the position is a curious draw (clearly White cannot win which is enough for our oracle).
  - [1|2.2] **3 c×d5 ♕×d5** here again only two moves can save White (**4 b4** ♯11♯∘, **4 d4** ♯10♯∘, **4 ♘b4** ♯9♯∘, **4 ♘d4** ♯8♯∘, **4 ♘e3** ♯11♯∘, **4 ♗b4** ♯7♯∘, **4 ♗c3** ♯11♯∘, **4 ♗e3** ♯7♯∘, **4 ♗×f4** ♯9♯∘, **4 ♕e3** ♯11♯∘, **4 ♕e4** ♯11♯∘, **4 ♕×e5+** ♯3♯∘)
    - [1|2.2.1] **4 b×c4 ♖×b2** = see line [1|2.1].
    - [1|2.2.2] **4 d×c4 ♕e6** −+ since White is restricted by Black pawns that completly control the dark squares and cannot move his Knight, hence his Rook. Black may just move his Queen between e6 f5. △ **5 b4 c×b4** ♯22♯∘.
- [1|3] **2 d4** ♯25♯∘.
- [1|4] **2 ♘b4** ♯17♯∘.
- [1|5] **2 ♘d4** ♯25♯∘.
- [1|6] **2 ♘e3 f×e3+ 3 ♗×e3** is forced otherwise White is checkmated (**3 ♕×e3** ♯25♯∘ , **3 ♔×e3** ♯22♯∘) **3. . . d4** −+ White cannot win △ **4 ♗c2 d×c3 5 ♗×c3 b4 6 ♗d2 ♘d4 7 ♕e3 ♖c6** White is in zugzwang and must give another piece.
- [1|7] **2 ♗e3 d4**
  - [1|7.1] **3 b4 d×c3** ♯11♯∘ Black either promotes c pawn or is a Rook up (and **4 ♖b2 ♕×b2 5 ♗×c5 ♘d4** is not helping).
  - [1|7.2] **3 c4 f×e3+** White move is forced (otherwise **4 ♕×e3** ♯13♯∘) **4 ♘×e3 d×e3+** = because on each recapture by White Black closes the position with . . . b4 and white cannot break through since f4 leads to a quick defeat.
  - [1|7.3] **3 c×d4 c×d4** = △ **4 ♗×f4 e×f4 5 b4** (otherwise White is a piece down and will lose) **5. . . ♗e5** and the position is completely blocked on the dark squares. Other moves than **4 ♗×f4** are direct checkmate (**4 b4** ♯17♯∘, **4 ♘b4** ♯9♯∘, **4 ♘×d4** ♯14♯∘, **4 ♗d2** ♯14♯∘, **4 ♗×d4 ♘×d4** ♯12♯∘, **4 ♕d2** ♯15♯∘).
  - [1|7.4] **3 ♘b4 f×e3+** ♯9♯∘.

  - [1|7.5] **3 ♘×d4 f×e3+** ♯17♯$_\circ$.
  - [1|7.6] **3 ♗d2** ♯25♯$_\circ$.
  - [1|7.7] **3 ♗×d4 c×d4** ♯13♯$_\circ$.
  - [1|7.8] **3 ♗×f4 d×c3** ♯16♯$_\circ$.
  - [1|7.9] **3 ♕d2** ♯28♯$_\circ$.
- [1|8] **2 ♗×f4 e×f4** ♯22♯$_\circ$.
- [1|9] **2 ♕e3 f×e3+** ♯25♯$_\circ$.

2 e×d5 ♕×d5

- [2|1] **3 b4 c×b4** there is only one move that does not lose straight away (**4 c4** ♯16♯$_\circ$, **4 d4** ♯15♯$_\circ$, **4 ♖b3** ♯12♯$_\circ$, **4 ♖×b4** ♯14♯$_\circ$, **4 ♘×b4** ♯13♯$_\circ$, **4 ♘d4** ♯12♯$_\circ$, **4 ♘e3** ♯14♯$_\circ$, **4 ♗e3** ♯10♯$_\circ$, **4 ♗×f4** ♯10♯$_\circ$, **4 ♕e3** ♯9♯$_\circ$, **4 ♕e4** ♯14♯$_\circ$, **4 ♕×e5+** ♯5♯$_\circ$) **4 c×b4**
- [2|2] **3 c4 b×c4** by transposition we have reach line [1|2.2].
- [2|3] **3 d4 e×d4** ♯21♯$_\circ$.
- [2|4] **3 ♘b4 c×b4** ♯11♯$_\circ$.
- [2|5] **3 ♘d4 e×d4** ♯16♯$_\circ$.
- [2|6] **3 ♘e3 f×e3+** ♯52♯$_\circ$.
- [2|7] **3 ♗e3 f×e3+** ♯44♯$_\circ$.
- [2|8] **3 ♗×f4 e×f4** ♯15♯$_\circ$.
- [2|9] **3 ♕e3 f×e3** ♯10♯$_\circ$.
- [2|10] **3 ♕×e5+ ♘×e5** ♯2♯$_\circ$.

3 ♕e4 ♕×e4 only two moves do not lose for White (**4 b4** ♯7♯$_\circ$, **4 c4** ♯6♯$_\circ$, **4 d4** ♯7♯$_\circ$, **4 ♘b4** ♯10♯$_\circ$, **4 ♘d4** ♯6♯$_\circ$, **4 ♘e3** ♯7♯$_\circ$, **4 ♗e3 ♕×e3** checkmate, **4 ♗×f4** ♯8♯$_\circ$).

- [3|1] **4 d×e4 b4**

4 f×e4 b4

- [4|1] **5 d4** ♯14♯$_\circ$.
- [4|2] **5 ♘×b4** ♯21♯$_\circ$.
- [4|3] **5 ♘d4** ♯13♯$_\circ$.
- [4|4] **5 ♗e3** ♯7♯$_\circ$.
- [4|5] **5 ♗×f4** ♯7♯$_\circ$.
- [4|6] **5 ♔e2 ♔e6 6 ♔f2** (or 6 ♔f3) **6. . . ♔f6** = since other moves than King loses (see previous lines) or rejoin the mainline (c4).
- [4|7] **5 ♔f3 ♔e6** same as line [4|6].

5 c4 ♘d4 = the position is totally locked on dark squares and White cannot progress.

4.5 White moves f4

1 f4 e×f4

- [1|1] **2 b4 f3** all White moves loses directly (**3 b×c5** ♯14♯∘, **3 c4** ♯11♯∘, **3 d4** ♯11♯∘, **3 e4** ♯9♯∘, **3 ♖b3** ♯9♯∘, **3 ♘d4** ♯11♯∘, **3 ♔×f3** ♯24♯∘) but for **3 ♕×f3 ♘e5** , here again all White moves but one leads to White loss (**4 c4** ♯14♯∘, **4 d4** ♯14♯∘, **4 e4** ♯11♯∘ **4 ♖b3** ♯13♯∘, **4 ♘d4** ♯19♯∘, **4 ♕e2** ♯23♯∘, **4 ♕f4** ♯9♯∘, **4 ♕×f5+** ♯2♯∘, **4 ♕e4** ♯7♯∘, **4 ♕×d5** ♯6♯∘, **4 ♔e2** ♯20♯∘) **4 b×c5 ♘×d3+ 5 ♔e2 ♘×c5** we have
  - [1|1.1] **6 ♖b4 ♘e4** only one move does not lead to checkmate (**7 c4** ♯12♯∘, **7 ♖b3** ♯21♯∘, **7 ♖b2** ♯37♯∘, **7 ♔d3** ♯29♯∘, **7 ♕f2** ♯13♯∘, **7 ♕f4** ♯15♯∘) **7 ♘d4 ♘×d2 8 ♔×d2 ♕e4** = △ 9.Qxe4 fxe4 10.Rxb5 Rxb5 11.Nxb5 and the ending Knight vs. Bishop is draw or △ 9.Rxb5 Qxf3 10.Rxb6 Qf2+ 11.Kd3 Ke6 Nc6+ and perpetual check.
  - [1|1.2] **6 ♘d4 ♕e4** Because of the threat to the White king, White's move is forced **7 ♕×e4 f×e4** −+ White is blocked and must take the b pawn with his Rook if he looks for any progress after the rook exchanges Black easily draw. If **8 ♘×b4 ♘c2 9 ♖b3 ♗c5** White is in zugzwang and lose.

  the other moves lead to forced checkmates: **6 ♘b4** ♯37♯∘ **6 c4** ♯14♯∘, **6 e4** ♯11♯∘, **6 ♖b3** ♯12♯∘, **6 ♖×b5** ♯14♯∘, **6 ♕f2** ♯21♯∘, **6 ♕f4** ♯12♯∘, **6 ♕e4** ♯8♯∘, **6 ♕×d5** ♯10♯∘, **6 ♕×f5+** ♯9♯∘.
- [1|2] **2 c4 ♗e5** ♯26♯∘.
- [1|3] **2 d4 f×e3+**
  - [1|3.1] **3 ♘×e3 c×d4** ♯26♯∘.
  - [1|3.2] **3 ♗×e3 ♕e4** White's move is forced (**4 ♕f3** ♯♯∘, **4 ♕d2** ♯♯∘, **4 b4** ♯35♯∘, **4 c4** ♯16♯∘, **4 ♘b4** ♯15♯∘, **4 ♗d2** ♯♯∘, **4 ♗f4** ♯♯∘, **4 ♕d3** ♯♯∘, **4 ♕c4** ♯♯∘, **4 ♕×b5** ♯♯∘) **4 d×c5 ♗×c5** only two moves do not lose for white (**5 c4** ♯4♯∘, **5 ♘b4** ♯14♯∘, **5 ♘d4** ♯14♯∘ **5 ♕d2** ♯5♯∘, **5 ♕d3** ♯5♯∘, **5 ♕c4** ♯5♯∘, **5 ♕×b5** ♯5♯∘, **5 ♕f3** ♯32♯∘, **5 ♗c4** ♯15♯∘)
    - [1|3.2.1] **5 b4 ♗d6** the only non losing move is **6 ♗×b6 ♕f4+** = perpertual check on d2 f4.
    - [1|3.2.2] **5 ♗×c5 ♕f4+** = perpertual check on d2 f4.
  - [1|3.3] **3 ♕×e3** ♯41♯∘.
  - [1|3.4] **3 ♔f3 ♕e4** checkmate.
- [1|4] **2 e4 f×e4** ♯18♯∘.
- [1|5] **2 e×f4 ♕×e2+ 3 ♔×e2 d4**
  - [1|5.1] **4 b4 ♔e6**
    - [1|5.1.1] **5 b×c5**
    - [1|5.1.2] **5 c4**
    - [1|5.1.3] **5 c×d4**
    - [1|5.1.4] **5 ♖b3**
    - [1|5.1.5] **5 ♘×d4**
    - [1|5.1.6] **5 ♘e3** ♯32♯∘.
    - [1|5.1.7] **5 ♗e3** ♯40♯∘.
    - [1|5.1.8] **5 ♔f3**
    - [1|5.1.9] **5 ♔f2 d×c3 6 ♗×c3** otherwise due to promotion White is quickly checkmated **6. . . c×b4**
  - [1|5.2] **4 c4 b4** = Position is totally blocked and Black just moves his King on e6 f6.

- [1|5.3] **4 c×d4 c×d4** = for the same reason as line [1|5.2].
- [1|5.4] **4 ♘b4 ♘×b4** = Black blocks the position as in lines [1|5.2] and [1|5.3].
- [1|5.5] **4 ♘×d4 ♘×d4+** = Black blocks the position.
- [1|5.6] **4 ♘e3 d×e3+** −+
- [1|5.7] **4 ♗e3** ♯8♯∘.
- [1|5.8] **4 ♔f2 b4** = the position is blocked and Black king just moves between squares e6 f6.
- [1|5.9] **4 ♔f3 b4** = the position is blocked and Black king just moves between squares e6 f6.

- [1|6] **2 ♘b4** ♯20♯∘ White gives a piece.
- [1|7] **2 ♘d4** ♯24♯∘ White gives a piece.
- [1|8] **2 ♕f3** ♯24♯∘ after **2. . . ♘e5** the threat on c3 and f3 cannot be parried.
- [1|9] **2 ♔f3 ♘e5+** ♯17♯∘.

2 e×f4 ♕×e2+ 3 ♔×e2 d4

- [4|1] **4 b4 ♔e6**
  - [4|1.1] **5 b×c5 ♗×c5**
  - [4|1.2] **5 c4 b×c4** only one move does not lose straightforwardly (**6 b×c6** ♯10♯∘, **6 b5** ♯19♯∘, **6 ♖b3** ♯7♯∘, **6 ♘×d4** ♯♯∘, **6 ♘e3** ♯12♯∘, **6 ♗c3** ♯8♯∘, **6 ♗e3** ♯11♯∘, **6 ♔f2** ♯12♯∘, **6 ♔f3** ♯11♯∘) **6 d×c4 c×b4** only three moves are worth considering (**7 c5** ♯20♯∘, **7 ♖×b4** ♯12♯∘, **7 ♘×b4** ♯24♯∘, **7 ♘×d4+** ♯22♯∘, **7 ♗c3** ♯12♯∘, **7 ♗×b4** ♯20♯∘, **7 ♗e3** ♯16♯∘, **7 ♔f3** ♯21♯∘, **7 ♔f2** ♯22♯∘)
    - [4|1.2.1] **7 ♘e3 d×e3 8 ♗×e3 ♘e5** = .
    - [4|1.2.2] **7 ♔d3 ♗c5** only one move does not lose (**8 ♖×b4** ♯24♯∘, **8 ♘×b4** ♯24♯∘, **8 ♘×d4+** ♯24♯∘, **8 ♘e3** ♯21♯∘, **8 ♗×b4** ♯24♯∘, **8 ♗c3** ♯11♯∘, **8 ♗e3** ♯21♯∘, **8 ♔e2** ♯18♯∘) **8 ♖b3 ♔d6 9 ♔e2 ♔e6** = White is blocked and cannot do anything concrete in this position.
    - [4|1.2.3] **7 ♖b3 ♗c5** = this move simply transposes to variation [4|1.2.2]
  - [4|1.3] **5 c×d4 c×d4** = the position is totally blocked Black just moves his King on e6 d5.
  - [4|1.4] **5 ♖b3 c4** only one move otherwise 6.dxc4 M24,6.cxd4 M13,6.Nxd4 M10 **6 ♖b2 d×c3** only one move otherwise 7.dxc4 M10, 7.Ke3 M6, 7.Be3 M6, **7 ♗×c3 c×d3+** only one move otherwise 8.Kf3 M15, 8.Ke3 M19,8.Kd2 M13 **8 ♔×d3 ♗×f4** only one move otherwise 9.Rb3 M? 9.Bd4 M?, 9.Ke2 M? **9 ♘d4 ♔d5**
  - [4|1.5] **5 ♘×d4**
  - [4|1.6] **5 ♘e3**
  - [4|1.7] **5 ♗e3**
  - [4|1.8] **5 ♔f2 d×c3** White has only one move due to the threat of queening the c pawn (**6 b×c5** ♯4♯∘, **6 d4** ♯5♯∘, **6 ♖b3** ♯4♯∘, **6 ♘d4** ♯10♯∘, **6 ♘e3** ♯10♯∘, **6 ♗e3** ♯7♯∘, **6 ♔e2** ♯6♯∘, **6 ♔e3** ♯6♯∘, **6 ♔f3** ♯5♯∘) **6 ♗×c3 c×b4** there are four moves to consider since the other ones lead to direct checkmate (**7 d4** ♯13♯∘, **7 ♖b3** ♯16♯∘, **7 ♖×b4** ♯23♯∘, **7 ♘d4+** ♯24♯∘, **7 ♘e3** ♯11♯∘, **7 ♗f6** ♯18♯∘, **7 ♗e5** ♯13♯∘, **7 ♔e2** ♯12♯∘, **7 ♔e3** ♯13♯∘, **7 ♔f3** ♯13♯∘)
    - [4|1.8.1] **7 ♗×b4 ♘×b4** here also White's move is forced otherwise it is checkmate(**8 d4** ♯11♯∘, **8 ♖b3** ♯10♯∘, **8 ♖×b4** ♯18♯∘, **8 ♘d4** ♯13♯∘, **8 ♘e3** ♯13♯∘, **8 ♔e2** ♯12♯∘, **8 ♔e3** ♯13♯∘, **8 ♔f3** ♯12♯∘) **8 ♘×b4 ♗×b4** at this point only the Bishop's capture does not lead to checkmate(**9 d4** ♯11♯∘, **9 ♖b3** ♯12♯∘, **9 ♖c2** ♯11♯∘, **9 ♖d2** ♯7♯∘, **9 ♖e2+** ♯12♯∘, **9 ♔e2** ♯11♯∘, **9 ♔e3** ♯10♯∘, **9 ♔f3** ♯11♯∘) **9 ♖×b4 ♔d5** = this Rook endgame is a draw.
    - [4|1.8.2] **7 ♗d2**

* [4|1.8.3] **7 ♗d4 ♘×d4** only the recapture of the Knight does not lead to White checkmate (**8 ♖b3** ♯8♯∘, **8 ♖×b4** ♯10♯∘, **8 ♘×b4** ♯10♯∘, **8 ♘e3** ♯9♯∘, **8 ♔e3** ♯8♯∘) **8 ♘×d4+ ♔d5**
  - [4|1.8.3.1] **9 ♖b3** ♯9♯∘.
  - [4|1.8.3.2] **9 ♖×b4** ♯10♯∘.
  - [4|1.8.3.3] **9 ♘b3** ♯31♯∘.
  - [4|1.8.3.4] **9 ♘×b5** ♯10♯∘.
  - [4|1.8.3.5] **9 ♘c2** ??? vrifier.
  - [4|1.8.3.6] **9 ♘c6** ♯11♯∘.
  - [4|1.8.3.7] **9 ♘e2** ♯22♯∘.
  - [4|1.8.3.8] **9 ♘e6** ♯12♯∘.
  - [4|1.8.3.9] **9 ♘f3** ♯23♯∘.
  - [4|1.8.3.10] **9 ♘×f5** ??????? vrifier.
  - [4|1.8.3.11] **9 ♖c2** ♯11♯∘
  - [4|1.8.3.12] **9 ♖d2** ♯12♯∘.
  - [4|1.8.3.13] **9 ♖e2** ♯12♯∘.
  - [4|1.8.3.14] **9 ♔e2** ♯11♯∘.
  - [4|1.8.3.15] **9 ♔e3** ♯11♯∘.
  - [4|1.8.3.16] **9 ♔f3** ♯11♯∘.
* [4|1.8.4] **7 ♘×b4 ♗×b4** all moves but one lead to White checkmate (**8 d4** ♯10♯∘, **8 ♖b3** ♯17♯∘, **8 ♖×b4** ♯11♯∘, **8 ♖c2** ♯11♯∘, **8 ♖d2** ♯10♯∘, **8 ♖e2+** ♯13♯∘, **8 ♗d2** ♯11♯∘, **8 ♗d4** ♯7♯∘, **8 ♗e5** ♯11♯∘, **8 ♗f6** ♯12♯∘, **8 ♔e2** ♯9♯∘, **8 ♔e3** ♯10♯∘, **8 ♔f3** ♯9♯∘) **8 ♗×b4 ♘×b4** = after **9 ♖×b4** we have the same drawn Rook endgame as varid41.8.1 every other move lead to White checkmate(**9 d4** ♯8♯∘, **9 ♖b3** ♯11♯∘, **9 ♖c2** ♯7♯∘, **9 ♖d2** ♯12♯∘, **9 ♖e2+** ♯12♯∘, **9 ♔e3** ♯12♯∘, **9 ♔e2** ♯12♯∘, **9 ♔f3** ♯11♯∘).

– [4|1.9] **5 ♔f3 c×b4**

- [4|2] **4 c×d4 c×d4** = see line [1|5.3].
- [4|3] **4 ♘b4 c×b4** ♯24♯∘.
- [4|4] **4 ♘×d4 c×d4** ♯31♯∘.
- [4|5] **4 ♘e3 d×e3** ♯31♯∘.
- [4|6] **4 ♗e3 d×e3** ♯8♯∘.
- [4|7] **4 ♔f2**
- [4|8] **4 ♔f3**

4 c4 b4 = see line [1|5.2]

4.6 White moves ♘b4

1 ♘b4 c×b4 ♯21♯∘ White is a piece down.

4.7 White moves ♘d4

1 ♘d4 e×d4 ♯25♯∘ White is a piece down.

5 Conclusion

The game-theoretical value of Gardner's chess has been proved to be a draw. The proof was done in a semi-automated way in which humans were guiding the engine. The authors were 'pushing' lines for which it was thought that the exact distance to checkmate could be computed and backtracked once leaves were showing perfect distance to checkmate. This meta-algorithm leads to a very asymmetric way of selecting moves. For instance, when a position is thought to be decidable as a White win, very few time is spent on White decision nodes (since we 'know' the game to be won more or less no matter what). The idea is that enormous time and energy can be saved when the game theoretic value of a position, rather than the most precise move or the shortest path to checkmate, is looked for. Indeed, when a game is thought to be winning, e.g. for White, one has only to provide one forced line (even if it is not the 'best' one) and thus can avoid exhaustive search at White decision nodes. It can be seen as a form of meta-negascout [Fis81]. Nevertheless it is very different in the sense that the process is very asymmetric and guided by the fact that the overall evaluation of the position is known.

This procedure can be fully automated and tuned to some given degree of precision (basically what is the threshold after which a position is considered as decided). For future works we plan to implement it and test it for larger chess variants in order to compute their game theoretic values. Other games could also be considered.

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