Reports of Recent Work

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1 Mutual information on random process

Random process driven by Gaussian random variables, {X} and {Y}, are generated as follows.

$$\begin{aligned} X_n &= \alpha X_{n-1} + \epsilon_n \\ Y_n &= \beta Y_{n-1} + \xi_{xy} f(X_{n-1}) + \eta_n \end{aligned} \tag{1}$$

where ϵ_n and η_n are independent Gaussian white noise, and α, β and ξ_{xy} are the self interaction strength and cross interaction strength between {X} and {Y}, respectively. $f(X)$ is a functional mapping between X and Y, which could be either linear or nonlinear.

1.1 Derivations

I will mainly discussed the correlation between X and Y, bonded by linear function, i.e. f(X) is a linear function, in terms of mutual information. By investigating the expression above, there is a one way interaction directly between X_n and Y_{n+1}. Therefore, I would like to write down the expression of mutual information between X_n and Y_{n+1}, as a function of their interaction strength, ξ_{xy}. X and Y stand for X_n and Y_{n+1} for convenience below.

$$\begin{aligned} I(X;Y) &= \sum_X \sum_Y p(x,y) log(\frac{p(x,y)}{p(x)p(y)}) \\ &= H(X) + H(Y) - H(X,Y) \end{aligned} \tag{2}$$

where p(x,y) is the joint probability distribution of X and Y, and p(x) and p(y) is the probability density function of variable X and Y. H is the entropy function. Since X and Y are linear combination of independent Gaussian variables, it is easy to conclude that X and Y are Gaussian distributed. By substuting p(x) into the function of entropy.

$$H(X) = \int p(x) log(p(x)) = \frac{1}{2}(1 + log(2\pi\sigma_x^2)) \tag{3}$$

where σ_x is the standard deviation of X. Similarly, the entropy of Y, H(Y), can be expressed as:

$$H(Y) = \frac{1}{2}(1 + log(2\pi\sigma_y^2)) \tag{4}$$

Define the correlation coefficient between X and Y is ρ, the joint entropy of X and Y can be expressed as:

$$H(X,Y) = \int p(x,y)log(p(x,y)) = 1 + log(2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}) \tag{5}$$

By substituting them into the expression of mutual information, we can obtain the expression of mutual information in the bivariate Gaussian distributed case as,

$$H(X,Y) = \int p(x,y)log(p(x,y)) = -\frac{1}{2}log(1-\rho^2) \tag{6}$$

Then, we need to calculate the correlation coefficient ρ,

$$\rho = \frac{Cov(x,y)}{\sigma_x\sigma_y} = \frac{E[XY] - \mu_x\mu_y}{\sigma_x\sigma_y} \tag{7}$$

Calculate the E(X), E(Y), σ_x and σ_y.

$$E[X_n] = E[\alpha X_{n-1} + \epsilon_n] = E[\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i] = \sum_{i=1}^{n} \alpha^{n-i}E[\epsilon_i] = 0 \tag{8}$$

$$E[Y_{n+1}] = E[\alpha Y_{n-1} + \xi_{xy}X_n + \eta_n] = E[\sum_{i=1}^{n+1} \beta^{n-i+1}\eta_i + \xi\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i] = 0 \tag{9}$$

$$\begin{aligned} D[x_n] &= D[\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i] = \sum_{i=1}^{n} D[\alpha^{n-i}\epsilon_i] \\ &= \sum_{i=1}^{n} \alpha^{2(n-i)}D[\epsilon_i] = \sum_{i=0}^{n} \alpha^{2i} = \frac{1-\alpha^{2n}}{1-\alpha^2} \end{aligned} \tag{10}$$

$$\begin{aligned} D[Y_n] &= D[\alpha Y_{n-1} + \xi_{xy}X_n + \eta_n] \\ &= \sum_{i=1}^{n+1} D[\beta^{n-i+1}\eta_i] + \xi^2\sum_{i=1}^{n} d[\alpha^{n-i}\epsilon_i] \\ &= \frac{1-\alpha^{2n}}{1-\alpha^2} + \xi^2\frac{1-\beta^{2n+2}}{1-\beta^2} \end{aligned} \tag{11}$$

By definition, standard deviations of X and Y are square roots of their deviation:

$$\begin{aligned} \sigma_x &= D[x]^{1/2} \\ \sigma_y &= D[y]^{1/2} \end{aligned} \tag{12}$$

Calculate E(XY),

$$\begin{aligned} D[XY] &= E[(\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i)(\sum_{i=1}^{n+1} \beta^{n-i+1}\eta_i + \xi\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i)] \\ &= E[(\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i)(\sum_{i=1}^{n+1} \beta^{n-i+1}\eta_i)] + \xi E[(\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i)(\sum_{i=1}^{n} \alpha^{n-i}\epsilon_i)] \end{aligned} \tag{13}$$

The first part of right side of the equation above is zero, since ϵ_i and η_i are independent. Meanwhile, $E[\epsilon_i \epsilon_j]$ is nonzero only if $i \neq j$.

$$\begin{aligned} D[XY] &= \xi(\sum_{i=0}^{n-1} \alpha^{2i} E[\epsilon_{2i}]) \\ &= \xi \frac{1-\alpha^{2n}}{1-\alpha^2} \end{aligned} \tag{14}$$

Substitute E(XY), μ_x, μ_y, σ_x and σ_y into ρ,

$$\rho = \xi\left(\frac{\frac{1-\alpha^{2n}}{1-\alpha^2}}{\frac{1-\beta^{2n+2}}{1-\beta^2} + \xi^2 \frac{1-\alpha^{2n}}{1-\alpha^2}}\right)^{1/2} \tag{15}$$

If $\alpha = \beta$, then,

$$I(X;Y) = -\frac{1}{2} log(\frac{1-\alpha^{2n+2}}{1-\alpha^{2n+2} + \xi^2 1 - \alpha^{2n}}) \tag{16}$$

If $\alpha << 1$, then,

$$I(X;Y) = \frac{1}{2} log(1+\xi^2) \tag{17}$$

1.2 Numerical Sample

Here, I wrote code to calculate this special case, and plotted the result. The theoretical value of mutual information as the function of interacting strength given above was also plotted.

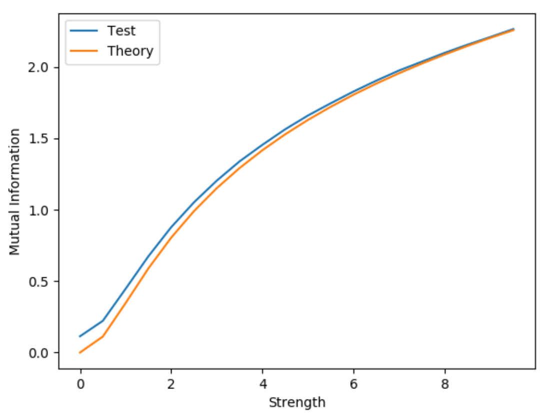

Figure 1: Relation between mutual information and interacting strength. The experimental result (blue curve) is almost identical to theoretical one(orange curve).

2 Two neuron system

In order to reduce the difficulty of analysis, I decided to start with two neuron system raster than one to multi neuronal system. By increasing the length of simulation period ten times, we can roughly simulate cases which is equivilant to one to ten neurons interaction neglecting spatial distribution of neurons. Here, I give the neuronal network setting applied in my simulation.

Driving rate	1.5 ms^{-1}
Driving strength	0.005
Synaptic strength(Excitatory)	0.005
Simulation time	60 s
Timing step	1/32 ms

2.1 Autocorrelating time scale

According to the requirement of wide-sense stationary process, the mean and autocovariane do not vary respect to time. Therefore, I ran two-neuron system to obtain 800 trials. The means and standard deviations of spike train and LFP at each time point were measured. Here, the standard deviation stands for autocovariance between X_n and itself. As for spike train of presynaptic neuron,

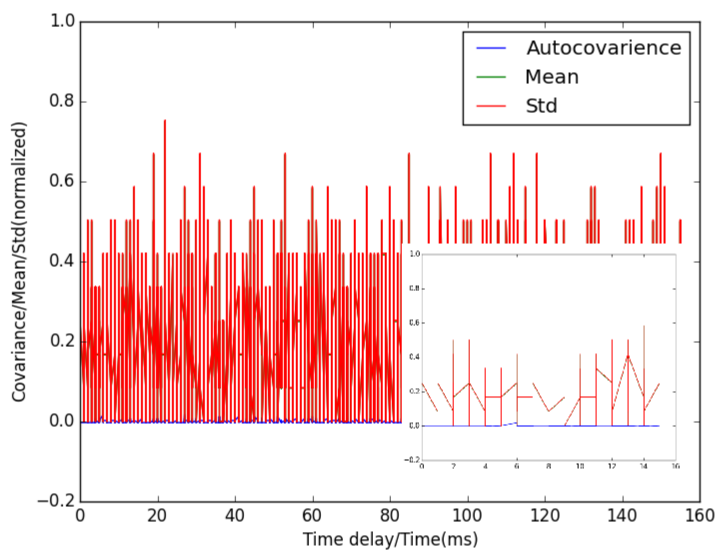

Figure 2: Stationary tests for spike train of presynaptic neuron. autocovariance(blue), mean value(green) and standard deviation(red) are plotted, respectively.

I take the time step as 0.5 milliseconds, which means that the element in spike train series is ”1” if there is a spiking events within 0.5 ms time window, and is ”0” otherwise. From Figure 2, I find that the auto correlation length of spike train is almost zero. In Figure 3, I take 40 milliseconds as the autocorrelation period.

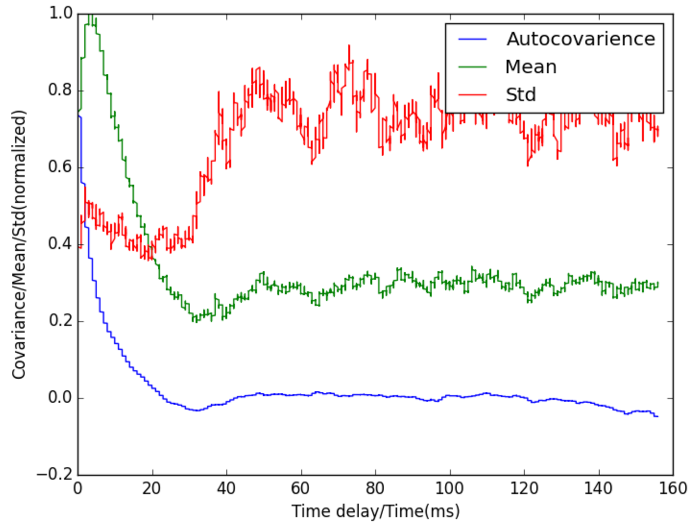

Figure 3: Stationary tests for LFP series of postsynaptic neuron. autocovariance(blue), mean value(green) and standard deviation(red) are plotted, respectively.

2.2 Choise of time step in MI calculation

2.3 Correlation interacting strength and mutual information

2.4 Proposal for theoretical explanations