Reports of Recent Works

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1 Mutual information between Gaussian random variables

Here I am going to derive the expression for mutual information between two variables generated by random processes analytically. Let's define X_n and Y_n as:

$$\begin{aligned} X_n &= \alpha X_{n-1} + \epsilon_n \\ Y_n &= \beta Y_{n-1} + \xi_{xy} f(X_{n-1}) + \eta_n \end{aligned} \tag{1}$$

where ϵ_n and η_n are independent random variables with normal distribution, and α, β and ξ_{xy} are the self interaction strength and cross interaction strength between {X} and {Y}, respectively. $f(X)$ is a functional mapping, which could be either linear or nonlinear.

1.1 Derivations

I will mainly discussed the first order correlation between X_n and Y_{n+1}, connected by linear function, i.e. $f(X) = X$ is a linear function, in terms of mutual information.

$$\begin{aligned} X_n &= \alpha X_{n-1} + \epsilon_n \\ Y_{n+1} &= \beta Y_n + \xi f(X_n) + \eta_{n+1} \end{aligned} \tag{2}$$

I would like to write down the expression of mutual information as a function of ξ. X and Y stand for X_n and Y_{n+1} for convenience below.

$$\begin{aligned} I(X;Y) &= \sum_X \sum_Y p(x,y) log \frac{p(x,y)}{p(x)p(y)} \\ &= H(X) + H(Y) - H(X,Y) \end{aligned} \tag{3}$$

where p(x,y) is the joint probability distribution of X and Y, and p(x) and p(y) is the probability density function of variable X and Y. H is entropy. Since ϵ_i and η_i are independent Gaussian random variables, it is easy to conclude that X and Y are Gaussian distributed. By substuting p(x) into the function of entropy.

$$H(X) = \int p(x) log(p(x)) = \frac{1}{2}(1 + log(2\pi\sigma_x^2)) \tag{4}$$

where σ_x is the standard deviation of X. The entropy of Y, H(Y), can be expressed in the same way. Define the correlation coefficient between X and Y is ρ, the joint entropy of X and Y can be expressed as:

$$H(X,Y) = \int p(x,y) log(p(x,y)) = 1 + log(2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}) \tag{5}$$

By substituting them into the expression of mutual information, we can obtain the expression of mutual information in the bivariate Gaussian distributed case as,

$$I(X,Y) = -\frac{1}{2}log(1-\rho^2) \tag{6}$$

Then, we need to calculate the correlating coefficient ρ,

$$\rho = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{E[XY] - \mu_x \mu_y}{\sigma_x \sigma_y} \tag{7}$$

Calculate the E(X), E(Y), σ_x and σ_y.

$$E[X_n] = E[\alpha X_{n-1} + \epsilon_n] = E[\sum_{i=1}^{n} \alpha^{n-i} \epsilon_i] = \sum_{i=1}^{n} \alpha^{n-i} E[\epsilon_i] = 0 \tag{8}$$

$$E[Y_{n+1}] = E[\beta Y_n + \xi X_n + \eta_{n+1}] = E[\sum_{i=1}^{n+1} \beta^{n+1-i} \eta_i + \xi \sum_{i=1}^{n} \beta^{n-i} \sum_{j=1}^{i} \alpha^{i-j} \epsilon_j] = 0 \tag{9}$$

$$D[X] = D[\sum_{i=1}^{n} \alpha^{n-i} \epsilon_i] = \sum_{i=1}^{n} \alpha^{2(n-i)} D[\epsilon_i] = \frac{1-\alpha^{2n}}{1-\alpha^2} \tag{10}$$

$$\begin{aligned} D[Y] &= D[\sum_{i=1}^{n+1} \beta^{n+1-i} \eta_i + \xi \sum_{i=1}^{n} \beta^{n-i} \sum_{j=1}^{i} \alpha^{i-j} \epsilon_j] \\ &= \sum_{i=1}^{n+1} D[\beta^{n-i+1} \eta_i] + \xi^2 D[\sum_{i=1}^{n} \beta^{n-i} \sum_{j=1}^{i} \alpha^{i-j} \epsilon_j] \end{aligned} \tag{11}$$

If $\alpha = \beta$,

$$D[Y] = \frac{1-\alpha^{2n+1}}{1-\alpha^2} + \xi^2 \sum_{i=0}^{n-1} (j+1)^2 \alpha^{2i} \tag{12}$$

If $\alpha \neq \beta$,

$$D[Y] = \frac{1-\beta^{2n+1}}{1-\beta^2} + \xi^2 \sum_{i=1}^{n} (\frac{\beta^i - \alpha^i}{\beta - \alpha})^2 \tag{13}$$

By definition, standard deviations of X and Y are square roots of their deviation. Eventually, we need to calculate E(XY).

$$E[XY] = E[(\sum_{i=1}^{n} \alpha^{n-i} \epsilon_i)(\sum_{i=1}^{n+1} \beta^{n+1-i} \eta_i + \xi \sum_{i=1}^{n} \beta^{n-i} \sum_{j=1}^{i} \alpha^{i-j} \epsilon_j)] \tag{14}$$

The first part of right side of the equation above is zero, since ϵ_i and η_i are independent. And $E[\epsilon_i \epsilon_j] = \delta_{ij}$. So when $\alpha = \beta$,

$$E[XY] = \frac{\xi}{1-\alpha^2}(\frac{1-\alpha^{2n}}{1-\alpha^2} - n\alpha^{2n}) \tag{15}$$

When $\alpha \neq \beta$,

$$E[XY] = \frac{\xi}{1-\alpha/\beta}\left(\frac{1-(\alpha\beta)^n}{1-\alpha\beta} - \frac{1-\alpha^{2n}}{1-\alpha^2}\frac{\alpha}{\beta}\right) \tag{16}$$

When $n >> 1$, we can substitute E(XY), μ_x, μ_y, σ_x and σ_y into ρ. For $\alpha = \beta$,

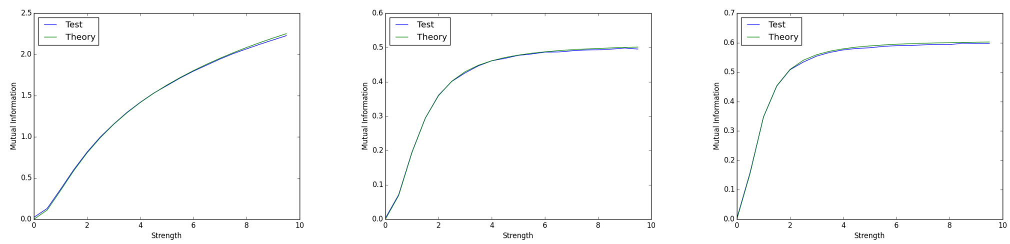
$$\rho^2 = \frac{\xi^2}{(1-\alpha^2)^2 + \xi^2(1+\alpha^2)} \tag{17}$$

for $\alpha \neq \beta$,

$$\rho^2 = \frac{\xi^2(1-\beta^2)}{(1-\alpha\beta)^2(1-\alpha^2) + \xi^2(1-(\alpha\beta)^2)} \tag{18}$$

1.2 Numerical Sample

Here, I presented results from numerical simulations. Both the theoretical and numerical value of the mutual information as the function of interacting strength are ploted.



(a) $\alpha = 0.01$, $\beta = 0.01$. (b) $\alpha = 0.01$, $\beta = 0.6$. (c) $\alpha = 0.5$, $\beta = 0.6$.

2 Two neuron system

In order to reduce the difficulty of analysis, I decided to start with two neuron system raster than one to multi neuronal system. By increasing the length of simulation period ten times, we can roughly simulate cases which is equivilant to one to ten neurons interaction neglecting spatial distribution of neurons. Here, I give the neuronal network setting applied in my simulation.

Driving rate	1.5 ms^{-1}
Driving strength	0.005
Synaptic strength(Excitatory)	0.005
Simulation time	60 s
Timing step	1/32 ms

2.1 Autocorrelating time scale

According to the requirement of wide-sense stationary process, the mean and autocovariane do not vary respect to time. Therefore, I ran two-neuron system to obtain 800 trials. The means and standard deviations of spike train and LFP at each time point were measured. Here, the standard deviation stands for autocovariance between X_n and itself. As for spike train of presynaptic neuron, I take the time step as 0.5 milliseconds, which means that the element in spike train series is "1" if there is a spiking events within 0.5 ms time window, and is "0" otherwise. From Figure **??**, I find that the auto correlation length of spike train is almost zero. In Figure **??**, the mean and standard deviation of signal reach their stationary state, respectively. Therefore, I take data in the first 40 milliseconds away, and calculate its autocovariance. The length of autocorrelation is around 20 milliseconds.

2.2 Choise of time step in MI calculation

After running test under different timing steps, I decided to choose 0.5 ms as the proper step. If smaller steps are taken, '1' elements is far too less than '0'. In this way, the signal peak of mutual information would be barried into noise level.

2.3 Correlation interacting strength and mutual information