The Math Behind Escher

1°) Transform coordinate plane to complex plane

Use either of these:

- Rectangular Form: $z = x + iy$
- Polar ": $z = r.\cos(\theta) + r.\sin(\theta)\, i$
- Exponential : $z = r.e^{i\theta}$

Note: the rectangular form is the one that works best in Python, the others have rounding err.

2°) Apply Log to the complex plane → L(z)

called L(w) in paper

This is the term 'Lnr' in Python
(natural log of the radius = Lnr)

Depending on the complex representation, some tricks apply:

↳ On the exponential repres. of complex:

$$z = |r| e^{i\theta}$$

$$\ln(z) = \ln(|r|) + i\theta$$ ⟹ this is the one used by the shader code

But Python has rounding errors with this one!

3°) Escher transform

From the paper: $h(w) = w^{\alpha}$
where 'w' is the complex plane

$$\boxed{\alpha = \frac{2\pi i + \ln(256)}{2\pi i} = 1 - \frac{\ln(256)}{2\pi i}}$$

$$\left(\frac{(2\pi i + \ln(256))(-i)}{2\pi i \quad (-i)} \longrightarrow \frac{2\pi \cdot 1}{2\pi \cdot 1} - \frac{\ln(256)\, i}{2\pi \cdot 1} = 1 - \frac{\ln(256)}{2\pi}\right.$$

$(-i)(i) = 1$

↓

4:) Combine all steps → Complex + Log + transform + Exp

$h(w) = w^{\alpha}$ / trick: $e^{L(x)} = x$

$L(a^{b}) = b\,L(a)$

$h(w) = e^{L(w^{\alpha})}$

$$\boxed{h(w) = e^{\alpha L(w)}}$$

$L(w)$ is the log of the complex plane we had before

↳ 3 operations → 1°) $L(w)$

2°) $\alpha L(w)$

3°) $e^{(\cdot)}$

Note that if we do $e^{L(w)} = w$ → what we had started with

↳ so the term α is the rotation

Inverse

$\hookrightarrow$ Need to do : $1/\alpha$

with:

$$\boxed{\alpha = \frac{2\pi i + Ln(256)}{2\pi i} = 1 - \frac{Ln(256)}{2\pi i}}$$

$$\left(\frac{(2\pi i + Ln(256))(-i)}{2\pi i \quad (-i)} \longrightarrow \quad (-i)(i) = 1 \qquad \frac{2\pi \cdot 1}{2\pi \cdot 1} - \frac{Ln(256)\, i}{2\pi \cdot 1} = 1 - \frac{Ln(256)}{2\pi}\right)$$

$$\Rightarrow \boxed{\frac{1}{\alpha} = \frac{2\pi i}{2\pi i + Ln(256)} \overset{*}{=} \frac{4\pi^2 + 2\pi (i) Ln(256)}{Ln^2(256) + 4\pi^2}}$$

Forward: $z = w^{\text{"}\alpha\text{"}}$ $\longrightarrow$ from straigth to Escher

$z = e^{\alpha L(w)}$ (notation is weird)

Backward: from Escher to straight $\rightarrow$ from z to w

$$z = e^{\alpha L(w)}$$

$$L(z) = \alpha L(w)$$

$$\frac{1}{\alpha} L(z) = L(w)$$

$$\boxed{e^{\frac{1}{\alpha} \cdot L(z)} = w}$$

We get z $\rightarrow$ the transfm image

1°) Apply $L(z)$

2°) $\frac{1}{\alpha} L(z)$

3°) $e^{(\ )}$

⊛

$$1/\alpha = \frac{1}{\frac{2\pi(i) + \ln(256)}{2\pi(i)}} = \frac{2\pi(i)}{2\pi(i) + \ln(256)} =$$

if we want to rationalize further, to remove $2\pi(i) + \ln(\,)$ from denom. multiply by its conjugate $\rightarrow \frac{\ln(256) - 2\pi(i)}{\ln(256) - 2\pi(i)}$

$$= \frac{2\pi(i)\left(\ln(256) - 2\pi(i)\right)}{\left(2\pi(i) + \ln(256)\right)\left(\ln(256) - 2\pi(i)\right)}$$

Simplify $2\pi(i)\left(\ln(256) - 2\pi(i)\right)$

Complex conjugate rule: $(ai)(b + ci) = -ac + abi$

$a = 2\pi$, $b = \ln(256)$, $c = -2\pi$

$= \underbrace{-2\pi(-2\pi)} + 2\pi\ln(256)(i)$

$= 4\pi^2 + 2\pi\ln(256)(i)$

Simplify denom: $(2\pi(i) + \ln(256))(\ln(256) - 2\pi(i)) = \ln^2(256) + 4\pi^2$

Complex arithmetic rule: $(a + bi)(a - bi) = a^2 + b^2$

Rewrite in std complex form: $\frac{4\pi^2}{\ln^2(256) + 4\pi^2} + \frac{2\pi\ln(256)}{\ln^2(256) + 4\pi^2}(i)$