

# 1. ALGEBRA

## 1.1. Arithmetic Operations

$$a(b+c) = ab+ac$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{db}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

## 1.2. Exponents and Radicals

$$x^m x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n + y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

## 1.3. Factoring Special Polynomials

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

## 1.4. Binomial Theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y +$$

$$(x-y)^3 = x^3 - 3x^2y +$$

$$3xy^2 + y^3$$

$$3xy^2 - y^3$$

## 1.5. Quadratic Formula

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## 1.6. Inequalities and Absolute Value

If  $a < b$  and  $b < c$ , then  $a < c$

If  $a < b$ , then  $a + c < b + c$

If  $a < b$  and  $c > 0$ , then  $ca < cb$

If  $a < b$  and  $c < 0$ , then  $ca > cb$

If  $a > 0$ , then

$$|x| = a \quad \text{means} \quad x = a \quad \text{or} \quad x = -a$$

$$|x| < a \quad \text{means} \quad -a < x < a$$

$$|x| > a \quad \text{means} \quad x > a \quad \text{or} \quad x < -a$$

# 2. GEOMETRY

## 2.1. Geometric Formulas

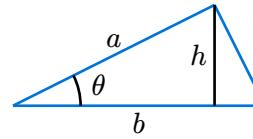
Formulas for:

- Area  $A$
- Circumference  $C$
- Volume  $V$

### Triangle

$$A = \frac{1}{2}bh$$

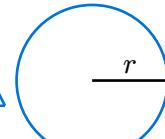
$$A = \frac{1}{2}ab \sin(\theta)$$



### Circle

$$A = \pi r^2$$

$$C = 2\pi r$$



### Sector of Circle

$$A = \frac{1}{2}r^2\theta$$

$$s = r\theta \text{ (theta in radians)}$$



## 2.2. Trigonometric Functions

Function	Definition	Reciprocal
$\sin(x)$	$\frac{\text{opposite}}{\text{hypotenuse}}$	$\frac{1}{\sin(x)} = \csc(x)$
$\cos(x)$	$\frac{\text{adjacent}}{\text{hypotenuse}}$	$\frac{1}{\cos(x)} = \sec(x)$
$\tan(x)$	$\frac{\sin(x)}{\cos(x)} = \frac{\text{opposite}}{\text{adjacent}}$	$\frac{1}{\tan(x)} = \cot(x)$
$\csc(x)$	$\frac{\sin(x)}{\cos(x)} = \frac{\text{hypotenuse}}{\text{opposite}}$	$\frac{1}{\csc(x)} = \sin(x)$
$\sec(x)$	$\frac{\sin(x)}{\cos(x)} = \frac{\text{hypotenuse}}{\text{adjacent}}$	$\frac{1}{\sec(x)} = \cos(x)$
$\cot(x)$	$\frac{\cos(x)}{\sin(x)} = \frac{\text{adjacent}}{\text{opposite}}$	$\frac{1}{\cot(x)} = \tan(x)$

# Contents

1. ALGEBRA .....	1
1.1. Arithmetic Operations .....	1
1.2. Exponents and Radicals .....	1
1.3. Factoring Special Polynomials .....	1
1.4. Binomial Theorem .....	1
1.5. Quadratic Formula .....	1
1.6. Inequalities and Absolute Value .....	1
2. GEOMETRY .....	1
2.1. Geometric Formulas .....	1
2.2. Trigonometric Functions .....	1
3. Matrix .....	16
3.1. Matrix Vector Product .....	16
3.2. Matrix Multiplication .....	16
3.3. Transpose .....	17
4. Matrix Factorization .....	17
4.1. LU Decomposition .....	17
4.2. QR Decomposition .....	17
4.3. Cholesky Decomposition .....	17
4.4. Singular Value Decomposition (SVD) .....	17
5. Vectors .....	20
5.1. Real Coordinate Spaces .....	20
5.2. Vector Operations .....	20
5.2.1. Vector Addition .....	20
5.2.2. Vector Subtraction .....	21
5.2.3. Scalar Multiplication .....	22
5.3. Unit Vector .....	23
5.4. Parametric Representation of line .....	24
5.5. Vector Spaces .....	27
6. Matrices .....	28
6.1. Matrix-Vector Products .....	28
6.2. Null Space .....	29
6.2.1. Column Space .....	30
6.2.2. Dimension of a Subspace .....	31
6.2.3. Nullity .....	31
6.2.4. Rank .....	31

6.2.5. Matrix Representation of Systems of Equations .....	31
6.3. Matrix Multiplication .....	32
7. Linear Combinations .....	33
8. Span .....	33
9. Linear Independence .....	36
10. Subspace .....	38
11. Basis .....	41
11.1. Vector Dot Product .....	42
11.1.1. Magnitude (Length) .....	42
11.1.2. Properties .....	42
11.1.2.1. Commutative .....	42
11.1.2.2. Distributive .....	42
11.1.2.3. Associativity .....	43
11.1.3. Cauchy-Schwarz Inequality .....	43
11.1.4. Vector Triangle Inequality .....	45
11.2. Angles Between Vectors .....	45
11.2.1. Plane in $\mathbb{R}^3$ .....	48
11.2.2. Point Distance to Plane .....	50
11.2.3. Distance Between Planes .....	50
11.2.4. Cross Product .....	50
11.2.5. Proof: Relationship Between Cross Product and Sin of Angle .....	51
11.2.6. Dot and Cross Products .....	51
11.3. Row Echelon Form (REF) .....	51
11.3.1. Solution Types in Linear Systems: Unique, Infinite, or None .....	53
11.3.2. Special Cases .....	54
12. Matrices .....	55
12.1. Matrix-Vector Products .....	55
12.2. Null Space .....	56
12.2.1. Column Space .....	57
12.2.2. Dimension of a Subspace .....	58
12.2.3. Nullity .....	58
12.2.4. Rank .....	58
12.2.5. Matrix Representation of Systems of Equations .....	58
12.3. Matrix Multiplication .....	59
12.4. Linear Transformation .....	60
12.4.1. Functions .....	60
12.5. Vector Transformation .....	61
12.6. Linear Transformation .....	62
12.7. Matrix Vector Products .....	63
12.8. Linear transformations as matrix vector products .....	66
12.9. Image of a subset under transformation .....	67
12.10. Image of a transformation .....	68
12.11. Preimage of a set .....	70
12.11.1. Kernel of a Transofrmation .....	71
12.11.2. Kernel and Null Space .....	72
12.12. Sum and Scalar Multiples of Linear Transformation .....	72
12.12.1. Sum .....	72
12.12.2. Scalar Multiplication .....	72
12.13. Projection .....	73

12.14. Composition of Linear Transformations .....	77
12.15. Matrix Product .....	79
12.16. Matrix Product Associativity .....	80
12.17. Eigenvectors .....	80
12.17.1. Transformation .....	80
12.18. Eigenvalues .....	81
12.18.1. LU Decomposition .....	81
12.19. Solving Systems of Linear Equations .....	82
12.19.1. Gaussian Elimination .....	86
12.19.2. Substitution .....	87
12.19.3. Addition or Subtraction Method .....	88
13. Cheatsheet .....	90
13.1. Limits .....	90
13.2. Derivatives .....	91
13.3. Integrals .....	92
13.4. Differential .....	92
13.5. Product Rule → Integration by Parts .....	95
13.6. Chain Rule → $u$ -Substitution .....	95
13.7. Quotient Rule → Algebraic Manipulation / Substitution .....	95
14. Limits & Continuity .....	96
15. Properties of Limits .....	96
15.1. Continuous .....	96
15.1.1. Addition, Subtraction, Multiplication, Division .....	96
15.1.2. Constant .....	96
15.2. Non-continuous .....	96
15.3. Composite Functions .....	97
15.4. Limits by Direct Substitution .....	99
15.4.1. Limits of Piecewise Functions .....	99
15.4.2. Absolute Value .....	99
15.5. Limits by Factoring .....	99
15.6. Limits by Rationalizing .....	99
15.7. Continuity & Differentiability at a Point .....	99
15.8. Power Rule .....	100
15.9. Constant Rule .....	101
15.10. Constant Multiple Rule .....	101
15.11. Sum Rule .....	101
15.12. Difference Rule .....	101
15.13. Square Root .....	102
15.14. Derivative of a Polynomial .....	102
15.15. Sin .....	103
15.16. Cos .....	104
15.17. $e^x$ .....	104
15.18. $\ln(x)$ .....	105
15.19. Product Rule .....	105
15.20. Quotient Rule .....	106
15.20.1. $\tan(x)$ .....	106
15.20.2. $\cot(x)$ .....	107
15.20.3. $\sec(x)$ .....	108

15.20.4. $\csc(x)$ .....	109
15.21. Chain Rule .....	110
15.22. Implicit Differentiation .....	111
15.23. Derivatives of Inverse Functions .....	112
15.23.1. Derivative Inverse Sin .....	115
15.23.2. Derivative Inverse Cos .....	115
15.23.3. Derivative Inverse Tan .....	115
15.24. Inverse Functions .....	115
15.25. L'Hôpital's Rule .....	117
15.26. Mean Value Theorem .....	119
15.27. Extreme Value Theorem .....	120
15.27.1. Critical points .....	120
15.27.2. Global vs. Local Extrema .....	121
15.27.3. First and Second Derivative Tests .....	121
15.27.3.1. First Derivative Test .....	121
15.27.3.2. Second Derivative Test .....	122
15.28. Differentiation Rules .....	123
16. Probability Theory .....	129
16.1. Probability Axioms .....	129
16.1.1. Non-Negativity .....	129
16.1.2. Normalization .....	129
16.1.3. Additivity .....	129
16.2. Rules .....	129
16.2.1. Complement Rule .....	129
16.2.2. Multiplication Rule .....	129
16.2.3. Addition Rule .....	130
16.2.4. Conditional Probability .....	130
16.2.5. Law of Total Probability .....	130
16.2.6. Law of Large Numbers .....	131
16.2.7. Central Limit Theorem .....	131
16.3. Bayes Theorem .....	131
17. Descriptive Statistics .....	131
17.1. Central Tendency .....	131
17.1.1. Mean .....	131
17.1.2. Median .....	132
17.1.3. Mode .....	132
17.2. Dispersion .....	132
17.2.1. Range .....	132
17.2.2. Variance .....	133
17.2.3. Standard deviation .....	133
17.2.4. Interquartile Range (IQR) .....	134
18. Probability Distributions .....	134
18.1. Gaussian (Normal) distribution .....	134
18.2. t-Distribution .....	135
18.3. Binomial distribution .....	135
18.4. Poisson distribution .....	136
18.5. Exponential distribution .....	136
19. Functions .....	137
19.1. PDF (Probability Density Function) .....	137

19.2. PMF (Probability Mass Function) .....	137
19.3. CDF (Cumulative Distribution Function) .....	137
19.4. PPF (Percent-Point Function) .....	138
19.5. SF (Survival Function) .....	138
20. Error Metrics .....	138
20.1. MAE (Mean Absolute Error) .....	138
20.2. MSE (Mean Squared Error) .....	139
20.3. RMSE (Root Mean Squared Error) .....	139
20.4. MAPE (Mean Absolute Percentage Error) .....	139
20.5. R-squared .....	139
20.6. Adj R-squared .....	140
20.7. MSLE (Mean Squared Logarithmic Error) .....	140
20.8. Cross-Entropy Loss (Log Loss) .....	140
21. Hypothesis Testing .....	141
21.1. Hypotheses .....	141
21.1.1. Null ( $H_0$ ) .....	141
21.1.2. Alternative ( $H_1$ or $H_a$ ) .....	141
21.2. Error Types .....	141
21.2.1. Type I ( $\alpha$ ) .....	141
21.2.2. Type II ( $\beta$ ) .....	141
21.3. t-Tests .....	141
21.3.1. One-sample .....	141
21.3.2. Independent .....	143
21.3.3. Paired .....	143
21.4. Chi-square tests .....	143
21.4.1. Goodness of Fit Test .....	143
21.4.2. Test of independence .....	144
21.5. ANOVA (Analysis of Variance) .....	144
21.5.1. One-way .....	144
21.5.2. Two-way .....	145
22. Regression Analysis: .....	145
22.1. Simple linear regression .....	145
22.2. Multiple regression .....	147
22.3. Logistic Regression .....	147
22.4. Model diagnostics .....	147
22.4.1. p-Values .....	147
22.4.2. F-Statistic .....	148
22.4.3. Confidence Intervals (CI) .....	149
23. Correlation .....	149
23.1. Pearson .....	149
23.2. Spearman's Rank .....	149
24. Non-Parametric Statistics .....	150
24.1. Mann-Whitney U .....	150
24.2. Wilcoxon Signed-Rank .....	150
24.3. Kolmogorov-Smirnov .....	150
24.4. Kruskal-Wallis .....	150
25. Time Series .....	151
25.1. SMA (Simple Moving Averages) .....	151
25.2. WMA (Weighted Moving Average) .....	152

25.3. Exponential Smoothing .....	152
25.4. Seasonal Decomposition .....	153
25.5. ARMA (AutoRegressive Moving Average) .....	154
25.6. ARIMA (AutoRegressive Integrated Moving Average) .....	154
26. Causes of variation .....	156
26.1. Common .....	156
26.2. Special .....	156
27. Design of Experiments (DOE): .....	156
28. Control Charts: .....	156
28.1. P-charts (Proportion) .....	156
28.2. NP-charts (Number Proportion) .....	157
28.3. C-charts (Count) .....	157
28.4. U-charts (Unit) .....	158
28.5. $\bar{X}$ -chart .....	159
28.6. R-chart .....	159
29. Process Capability Analysis .....	159
29.1. $C_p$ (Process Capability Index) .....	159
29.2. $C_{pk}$ (Process Capability Index with Centering) .....	160
29.3. $C_{pm}$ (Taguchi Capability Index) .....	161
29.4. $P_p$ (Process Performance Index) .....	161
29.5. $P_{pk}$ (Process Performance Index with Centering) .....	162
30. Inventory Management .....	163
30.1. Newsvendor .....	163
30.2. ABC Analysis .....	165
30.3. Fill Rate .....	166
30.4. (OCT) Order Cycle Time .....	167
30.5. ROP (Reorder Point) .....	167
30.6. XYZ Analysis .....	167
30.7. EOQ (Economic Order Quantity) .....	168
30.7.1. Perfect Order Rate .....	169
30.8. Safety Stock .....	169
31. Queuing Theory .....	170
31.1. M/M/1 .....	170
32. Network Optimization .....	174
32.1. Shortest Path .....	174
32.2. Maximum Flow .....	174
32.3. Netwrok Flow Optimization .....	174
32.4. Ford-Fulkerson .....	178
33. Optimization .....	182
33.1. Mathematical Programs .....	182
33.1.1. Formulation .....	182
33.1.2. Transformation .....	183
33.1.3. Constraints .....	183
33.1.4. Feasable Solutions .....	183
33.1.5. Feasable Region and Optimal Solution .....	184
33.1.6. Binding Constraint .....	184
33.1.7. Strict and Weak Inequalities .....	184
33.1.8. Graphical Approach .....	184
33.1.9. Types of LP .....	185

33.1.10. LP Formulation .....	185
33.2. IP (Integer Programming) .....	194
33.3. Selection and Logical Relations on Binary Variables $x_i \in \{0, 1\}$ .....	194
33.3.1. Cardinality Constraints .....	194
33.3.2. Logical OR .....	194
33.3.3. Conditional Selection: “If X, then Y” .....	195
33.3.4. Conditional OR: “If not X, then select Y and Z” .....	196
33.3.5. XOR / Exclusive Conditions .....	196
33.4. Constraint Activation and Relaxation via Binary Indicators $z_i \in \{1, 0\}$ .....	197
33.4.1. Base Case: At Least One of Two Constraints Satisfied .....	197
33.4.2. General Formulation for $m$ Constraints with At Least $k$ Satisfied .....	197
33.5. Fixed Charge Constraints (Setup Costs) .....	198
33.6. Facility Location .....	198
33.6.1. Set Covering Problem .....	199
33.6.2. Maximum Covering Problem .....	199
33.7. Machine Scheduling .....	205
33.7.1. Single Machine Serial Production .....	206
33.7.2. Multiple Parallel Machines .....	208
33.7.3. Flow Shop .....	209
33.7.4. Job Shop .....	209
33.8. Vehicle Routing .....	209
33.8.1. Traveling Salesperson .....	209
33.9. Gradient Descent .....	211
33.10. Monte Carlo .....	212



# Linear Algebra

Vector	$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$	
Scalar Vector Multiplication	$c \in \mathbb{R} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $c\vec{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$	
Dot Product	$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i$ $= x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ $= \vec{x}^T \vec{y}$	
Cross Product ( $\mathbb{R}^3$ )	$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\vec{c} = \vec{a} \times \vec{b}$ $\vec{c} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$	Returns a vector orthogonal to the two vectors
Vector Space	<p>Closure under addition  <math>\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V</math></p> <p>Closure under scalar multiplication  <math>\vec{v} \in V \wedge c \in \mathbb{R} \Rightarrow c\vec{v} \in V</math></p> <p>Commutativity of addition  <math>\vec{u} + \vec{v} = \vec{v} + \vec{u}</math></p> <p>Associativity of addition  <math>(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})</math></p> <p>Additive identity  <math>\exists \mathbf{0} \in V \mid \vec{v} + \mathbf{0} = \vec{v}</math></p> <p>Additive inverse  <math>\forall \vec{v} \in V \exists -\vec{v} \in V \mid \vec{v} + (-\vec{v}) = \mathbf{0}</math></p> <p>Scalar multiplication (compatibility)  <math>a(b\vec{v}) = (ab)\vec{v}</math></p> <p>Distributivity over vector addition  <math>a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}</math></p>	

	<p>Distributivity over scalar addition  <math>(a + b)\vec{v} = a\vec{v} + b\vec{v}</math></p> <p>Multiplicative identity  <math>1\vec{v} = \vec{v}</math></p>	
Subspace	<p>Non-emptiness  <math>\mathbf{0} \in V</math></p> <p>Closure under addition  If <math>\vec{u}, \vec{v} \in V</math>, then <math>\vec{u} + \vec{v} \in V</math></p> <p>Closure under scalar multiplication  If <math>\vec{v} \in V, c \in \mathbb{R}</math>, then <math>c\vec{v} \in V</math></p>	A subspace is a subset of a vector space that is itself a vector space, satisfying the same axioms as the original. If $V$ is a vector space in $\mathbb{R}^n$ , then the subspace $U$ is always contained in $\mathbb{R}^n$ , meaning $U \subseteq \mathbb{R}^n$
Vector Addition	$\vec{u} = (u_1, u_2, \dots, u_n)$ $\vec{v} = (v_1, v_2, \dots, v_n)$ $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$	
Dot Product	$\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i$	
Orthogonality	$\vec{u} \cdot \vec{v} = \mathbf{0}$	Angle between the two vectors is $90^\circ$
Angle between vectors	$\Theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\ \vec{u}\ _2 \cdot \ \vec{v}\ _2}\right)$	
$L_1$ Norm (Manhattan)	$\ \vec{u}\ _1 = \sum_{i=1}^n  u_i $	
$L_2$ Norm (Euclidean)	$\ \vec{u}\ _2 = \sqrt{\sum_{i=1}^n u_i^2}$	
$L_1$ Distance (Manhattan)	$d(\vec{u}, \vec{v}) = \sum_{i=1}^n  u_i - v_i $	
$L_2$ Distance (Euclidean)	$d(\vec{u}, \vec{v}) = \sqrt{\sum_{i=1}^n (u_i - v_i)^2}$	
Projection	$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$	
Linear Independence		<p>A set of vectors is linearly independent if no vector in the set can be written as a linear combination of the others</p> <p>A set of vectors <math>\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}</math> is linearly independent if the only solution to the equation</p> $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \mathbf{0}$ <p>is <math>c_1 = c_2 = \dots = c_n = 0</math></p>
Transformation	$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ $T(\vec{v}) = A\vec{v}$ <ul style="list-style-type: none"> <li>• Additivity</li> </ul>	<ul style="list-style-type: none"> <li>• Surjective (onto)</li> </ul> <p>Every element in <math>B</math> is the image of at least one element in <math>A</math>. The transformation covers the entire codomain.</p>

	$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ <ul style="list-style-type: none"> <li>• Homogeneity</li> </ul> $T(c\vec{u}) = cT(\vec{u})$	$\text{Range}(T) = B$ <ul style="list-style-type: none"> <li>• Injective (one-to-one)</li> </ul> <p>Different inputs in <math>A</math> map to different outputs in <math>B</math>. The transformation is information-preserving – doesn't collapse distinct vectors together</p> $T(\vec{x}_1) = T(\vec{x}_2) \Rightarrow x_1 = x_2$ <p>Or equivalently:</p> $\ker(T) = \{\mathbf{0}\}$
Domain	$T : V \rightarrow W$ $\text{Domain}(T) = V$	Set of all <b>input vectors</b> $V$ that the <b>transformation</b> acts on
Codomain	$T : V \rightarrow W$ $\text{Codomain}(T) = W$	Set of all possible <b>output vectors</b> $W$ to which elements of the domain $V$ are mapped under the <b>transformation</b>
Transpose	$\det(A) = \det(A^T)$ $(AB)^T = B^T A^T$ $(A^T)^{-1} = (A^{-1})^T$	
Image	$T : V \rightarrow W$ $\text{Im}(T) = \{\vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V\}$	<p>The image of a <b>transformation</b> <math>T : V \rightarrow W</math> is the set of all possible outputs <math>T(v)</math> for <math>v \in V</math>:</p> <ul style="list-style-type: none"> <li>• It is a subspace of the codomain <math>W</math></li> <li>• If <math>T</math> is represented by a matrix <math>A</math>, the image of <math>T</math> is the column space of <math>A</math></li> </ul>
Preimage	$T : V \rightarrow W$ $T^{-1}(\vec{w}) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{w}\}$	The preimage of a transformation refers to the set of all elements in the domain that map to a particular element or subset in the codomain
Span	$\text{Span}(\{v_1, v_2, \dots, v_k\}) = \left\{ \sum_{i=1}^n c_i \vec{v}_i \mid c_i \in \mathbb{R} \right\}$	The span of a set of vectors is the collection of all possible linear combinations of those vectors
Composition	$T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad T_2 : \mathbb{R}^m \rightarrow \mathbb{R}^p$ $T_1(\vec{v}) = A\vec{v} \quad T_2(\vec{v}) = B\vec{v}$ $T \circ S(\vec{v}) = T_2(T_1(\vec{v}))$ $T_2 \circ T_1(\vec{v}) = B(A\vec{v}) = (BA)\vec{v}$ <ul style="list-style-type: none"> <li>• Additivity</li> </ul> $(T_3 \circ T_2) \circ T_1 = T_3 \circ (T_2 \circ T_1)$ <ul style="list-style-type: none"> <li>• Homogeneity</li> </ul> $T_2 \circ T_1(c\vec{u}) = c(T_2 \circ T_1)(\vec{u})$ <ul style="list-style-type: none"> <li>• Identity Transformation</li> </ul>	

	$I \circ T = T$ $T \circ I = T$					
Column Space (Range)	$\text{Col}(A) = \{Ax \mid \vec{x} \in \mathbb{R}^n\}$ Or equivalently $A = [\vec{c}_1 \ \vec{c}_2 \ \dots \ \vec{c}_3]$ $\text{Col}(A) = \text{span}(\vec{c}_1, \vec{c}_2, \dots, \vec{c}_3)$	The column space (or range) of a <b>matrix</b> $A$ is the set of all linear combinations of its columns				
Determinant	$\det(A)$	The determinant of a square matrix $A$ measure of the "scale factor" by which the matrix $A$ transforms a space <ul style="list-style-type: none"> <li>• <math>\det(A) \neq 0</math> <ul style="list-style-type: none"> <li>‣ <math>A</math> does not collapse the space</li> <li>‣ <math>A</math> has full rank</li> <li>‣ <math>A</math>'s columns are linearly independent</li> <li>‣ <math>A</math> is invertable</li> </ul> </li> <li>• <math>\det(A) = 0</math> <ul style="list-style-type: none"> <li>‣ <math>A</math> collapses the space into lower dimension</li> <li>‣ <math>A</math> does not have full rank</li> <li>‣ <math>A</math>'s columns are linearly dependent</li> <li>‣ <math>A</math> is non-invertable (singular)</li> </ul> </li> </ul>				
Invertibility	$\det(A) \neq 0 \implies \text{Invertible}$ $\det(A) = 0 \implies \text{Non-Invertible}$ $AA^{-1} = A^{-1}A = I_n$ $(AB)^{-1} = B^{-1}A^{-1}$ $(A^T)^{-1} = (A^{-1})^T$					
Basis	Linear Independence $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \mathbf{0}$ $\Rightarrow c_1 = c_2 = \dots = c_k = 0$ Spanning $\forall \vec{v} \in V, \exists c_1, \dots, c_k \in \mathbb{R} \text{ s.t. } \vec{v} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$	<ul style="list-style-type: none"> <li>• A basis of a <b>vector space</b> <math>V</math> is a set of linearly independent vectors that span the space</li> <li>• Every vector in <math>V</math> can be uniquely written as a linear combination of the basis vectors</li> </ul> <p>E.g.:</p>				
Dimension	$\dim(V)$	Number of linearly independent vectors ( <b>basis</b> ) in a <b>vector space</b> $V$ $V \subseteq \mathbb{R}^n$ <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td><math>\dim(V) = 0</math></td> <td><math>V = \{\mathbf{0}\}</math></td> </tr> <tr> <td><math>\dim(V) = 1</math></td> <td><math>V</math> is a <b>line</b> through the origin in <math>\mathbb{R}^n</math></td> </tr> </table>	$\dim(V) = 0$	$V = \{\mathbf{0}\}$	$\dim(V) = 1$	$V$ is a <b>line</b> through the origin in $\mathbb{R}^n$
$\dim(V) = 0$	$V = \{\mathbf{0}\}$					
$\dim(V) = 1$	$V$ is a <b>line</b> through the origin in $\mathbb{R}^n$					

		<table border="1"> <tr> <td><math>\dim(V) = 2</math></td><td><math>V</math> is a <b>plane</b> through the origin in <math>\mathbb{R}^n</math></td></tr> <tr> <td><math>\dim(V) = k</math></td><td><math>V</math> is a <math>k</math>-dimensional <b>flat</b> subspace of <math>\mathbb{R}^n</math></td></tr> <tr> <td><math>\dim(V) = n</math></td><td><math>V = \mathbb{R}^n</math></td></tr> </table>	$\dim(V) = 2$	$V$ is a <b>plane</b> through the origin in $\mathbb{R}^n$	$\dim(V) = k$	$V$ is a $k$ -dimensional <b>flat</b> subspace of $\mathbb{R}^n$	$\dim(V) = n$	$V = \mathbb{R}^n$
$\dim(V) = 2$	$V$ is a <b>plane</b> through the origin in $\mathbb{R}^n$							
$\dim(V) = k$	$V$ is a $k$ -dimensional <b>flat</b> subspace of $\mathbb{R}^n$							
$\dim(V) = n$	$V = \mathbb{R}^n$							
		<ul style="list-style-type: none"> <li>• <math>\mathbb{R}^2</math> has dimension 2:           <ul style="list-style-type: none"> <li>▶ A basis: <math>\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}</math></li> </ul> </li> <li>• <math>\mathbb{R}^3</math> has dimension 3:           <ul style="list-style-type: none"> <li>▶ A basis: <math>\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}</math></li> </ul> </li> </ul>						
Rank	$\text{Rank}(A) = \dim(\text{Col}(A)) = \dim(\text{Row}(A))$	<ul style="list-style-type: none"> <li>• The rank of a <b>matrix</b> <math>A</math> is the <b>dimension</b> of its column space (or row space)</li> <li>• Number of linearly independent columns (or rows)</li> </ul>						
Eigen	$Ax = \lambda x, \quad x \neq \mathbf{0}$	<p>Set of all nonzero vectors <math>\vec{x}</math> such that when the transformation represented by matrix <math>A</math> is applied to <math>\vec{x}</math>, the result is a scaled version of <math>\vec{x}</math> itself</p> <p>These vectors lie along directions that are preserved by the transformation:</p> <ul style="list-style-type: none"> <li>• <math> \lambda  &gt; 1</math>: stretched</li> <li>• <math>0 &lt;  \lambda  &lt; 1</math>: shrunk</li> <li>• <math>\lambda &lt; 0</math>: flipped</li> <li>• <math>\lambda = 1</math>: stay the same</li> </ul>						
Null Space (kernel)	$\text{Null}(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \mathbf{0} \}$	The null space of a matrix $A$ is the set of all input vectors that get mapped to the zero vector when you multiply them by $A$						
Identity Matrix	$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$							
Matrix Inverse	$A \cdot A^{-1} = I$							
RREF	<ol style="list-style-type: none"> <li>1. Row Swapping (Interchange)  <math>R_1 \leftrightarrow R_2</math></li> <li>2. Row Scaling (Multiplication)  <math>R_1 \rightarrow \frac{1}{3}R_1</math></li> <li>3. Row Addition (Replacement)  <math>R_1 \rightarrow R_1 - 2R_2</math></li> </ol>							

### 3. Matrix

$$m \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix})$$

#### 3.1. Matrix Vector Product

$$m \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

#### 3.2. Matrix Multiplication

$$A = m \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad B = n \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

$$A = m \begin{bmatrix} [a_{11} & a_{12} & \dots & a_{1n}] \\ [a_{21} & a_{22} & \dots & a_{2n}] \\ \vdots \\ [a_{m1} & a_{m2} & \dots & a_{mn}] \end{bmatrix} \quad B = n \begin{bmatrix} [b_{11}] & [b_{12}] & \dots & [b_{1p}] \\ [b_{21}] & [b_{22}] & \dots & [b_{2p}] \\ \vdots & \vdots & \ddots & \vdots \\ [b_{n1}] & [b_{n2}] & \dots & [b_{np}] \end{bmatrix}$$

#### A: Row Representation

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

$$r_i = [a_{i1} \ a_{i2} \ \dots \ a_{in}], \text{ for } i = 1, 2, \dots, m$$

#### B: Column Representation

$$B = [c_1 \ c_2 \ \dots \ c_p]$$

$$c_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}, \text{ for } j = 1, 2, \dots, p$$

$$C = m \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & \dots & r_1 \cdot c_p \\ r_2 \cdot c_1 & r_2 \cdot c_2 & \dots & r_2 \cdot c_p \\ \vdots & \vdots & \ddots & \vdots \\ r_m \cdot c_1 & r_m \cdot c_2 & \dots & r_m \cdot c_p \end{bmatrix}$$

### 3.3. Transpose

$$A^T$$

$$A = \underset{m}{\textcolor{blue}{m}} \begin{bmatrix} & & & & \underset{n}{n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad A^T = \underset{n}{\textcolor{red}{n}} \begin{bmatrix} & & & & \underset{m}{m} \\ a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

$$A = \underset{m}{\textcolor{blue}{m}} \begin{bmatrix} & & & & \underset{n}{n} \\ a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{ij} & \dots & a_{mn} \end{bmatrix} \quad A^T = \underset{n}{\textcolor{red}{n}} \begin{bmatrix} & & & & \underset{m}{m} \\ a_{11} & a_{21} & \dots & a_{i1} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{i2} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{1j} & a_{2j} & \dots & a_{ij} & \dots & a_{mj} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{in} & \dots & a_{mn} \end{bmatrix}$$

## 4. Matrix Factorization

### 4.1. LU Decomposition

$$A = LU$$

where:

- $L$ : lower triangular matrix (entries above the diagonal are zero)
- $U$ : upper triangular matrix (entries below the diagonal are zero)

1.  $m \times n$  Matrix (with  $m \geq n$ )

$$L = \underset{m}{\textcolor{blue}{m}} \begin{bmatrix} & & & & \underset{n}{n} \\ l_{11} & l_{12} & \dots & l_{1n} \\ l_{21} & l_{22} & \dots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \dots & l_{mn} \end{bmatrix} \quad U = \underset{n}{\textcolor{red}{n}} \begin{bmatrix} & & & & \underset{n}{n} \\ u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix}$$

2.  $m \times n$  Matrix (with  $m < n$ )

see QR Decomposition

### 4.2. QR Decomposition

### 4.3. Cholesky Decomposition

### 4.4. Singular Value Decomposition (SVD)

$$A = U\Sigma V^T$$

$$U = \underset{m}{\textcolor{blue}{m}} \begin{bmatrix} & & & & \underset{n}{n} \\ u_{11} & u_{12} & \dots & u_{1m} \\ u_{21} & u_{22} & \dots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mm} \end{bmatrix} \quad \Sigma = \underset{m}{\textcolor{blue}{m}} \begin{bmatrix} & & & & \underset{n}{n} \\ \varepsilon_{11} & \varepsilon_{12} & \dots & \varepsilon_{1m} \\ \varepsilon_{21} & \varepsilon_{22} & \dots & \varepsilon_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n1} & \varepsilon_{n2} & \dots & \varepsilon_{nm} \end{bmatrix} \quad V^T = \underset{n}{\textcolor{red}{n}} \begin{bmatrix} & & & & \underset{n}{n} \\ v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix}$$

	$\left[ \begin{array}{ccc c} 1 & 2 & -1 & 2 \\ 2 & 3 & 1 & 5 \\ 3 & 4 & -2 & 4 \end{array} \right]$
$R_2 \rightarrow R_2 - 2R_1$ $R_3 \rightarrow R_3 - 3R_1$	$\left[ \begin{array}{ccc c} 1 & 2 & -1 & 2 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 1 & -2 \end{array} \right]$
$R_2 \rightarrow -R_2$	$\left[ \begin{array}{ccc c} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & -2 & 1 & -2 \end{array} \right]$
$R_1 \rightarrow R_1 - 2R_2$ $R_3 \rightarrow R_3 + 2R_2$	$\left[ \begin{array}{ccc c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -5 & -4 \end{array} \right]$
$R_3 \rightarrow \frac{1}{-5}R_3$	$\left[ \begin{array}{ccc c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & \frac{4}{5} \end{array} \right]$
$R_1 \rightarrow R_1 - 5R_3$ $R_2 \rightarrow R_2 + 3R_3$	$\left[ \begin{array}{ccc c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{5} \\ 0 & 0 & 1 & \frac{4}{5} \end{array} \right]$

$$A = \begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix}$$

### 1. Null Space

The **null-space** of  $A$ , denoted  $N(A)$ , consists of all vectors  $\vec{x} \in \mathbb{R}^3$  such that  $A\vec{x} = 0$ . The set of all such vectors is the **pre-image** of the zero vector under the transformation defined by  $A$ . In other words,  $N(A) = \{x \in \mathbb{R}^3 \mid A\vec{x} = 0\}$ , which represents the set of vectors that  $A$  maps to zero.

$$N(A) = \{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \mathbf{0}\}$$

To find the null space  $N(A)$  of the matrix  $A$ , we can use the **row-reduced echelon form (RREF)**. By augmenting the matrix  $A$  with a zero column and performing row operations, we reduce it to the form:

$$\begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

	$\left[ \begin{array}{ccc c} 2 & -1 & -3 & 0 \\ -4 & 2 & 6 & 0 \end{array} \right]$
--	---

$R_1 \rightarrow \frac{R_1}{2}$ $R_2 \rightarrow \frac{R_2}{4}$	$\left[ \begin{array}{ccc c} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ -1 & \frac{1}{2} & \frac{3}{2} & 0 \end{array} \right]$
$R_2 \rightarrow R_2 - R_1$	$\left[ \begin{array}{ccc c} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

$$\left[ \begin{array}{ccc} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - \frac{1}{2}x_2 - \frac{3}{2}x_3 = 0$$

$$x_1 = \frac{1}{2}x_2 + \frac{3}{2}x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$N(a) = \text{span} \left( \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

The dimension of the **null-space** is the number of vectors in this basis, which is 2. This is important because the dimension of the null space gives us insight into how many degrees of freedom exist in the system of equations  $Ax = 0$

## 2. Column Space

$$\begin{aligned} C(A) &= \text{span} \left( \left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\} \right) \\ &= \text{span} \left( \left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\} \right) \end{aligned}$$

## 3. Basis

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

## 4. Rank

Number of vector in the basis of our column space

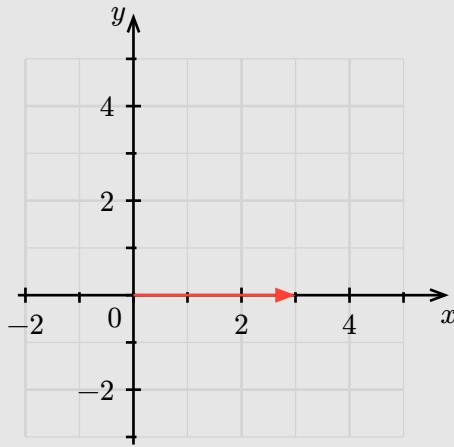
$$\text{Rank}(A) = 1$$

## 5. Vectors

Vector = Magnitude + Direction

A car is moving:

$\underbrace{3 \text{ MPH}}_{\text{Magnitude}}$        $\underbrace{\text{East}}_{\text{Direction}}$   
 $\underbrace{(\text{Speed} \rightarrow \text{Scalar})}_{\text{Velocity (Vector)}}$



$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

### 5.1. Real Coordinate Spaces

N-dimensional Real Coordinate Space

$$\mathbb{R}^n$$

$$\vec{x} \in \mathbb{R}^n$$

All possible real-valued n-tuples

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

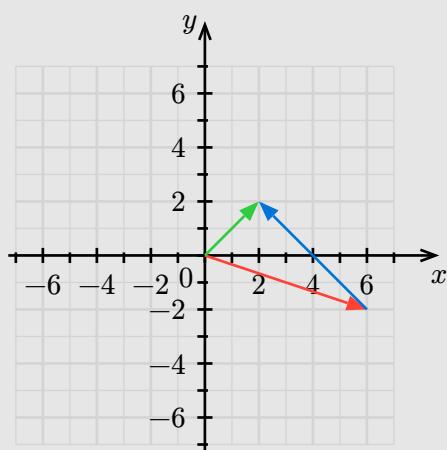
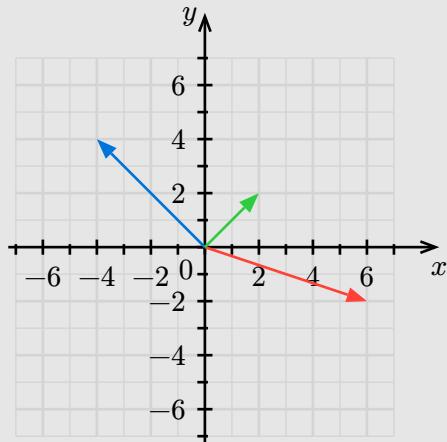
### 5.2. Vector Operations

#### 5.2.1. Vector Addition

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

$$\vec{a} + \vec{b} = \begin{bmatrix} 6 + -4 \\ -2 + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

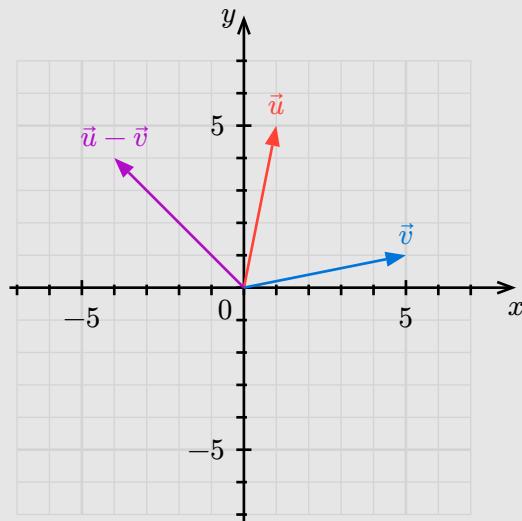


### 5.2.2. Vector Subtraction

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{u} - \vec{v} = \begin{bmatrix} 1 - 5 \\ 5 - 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

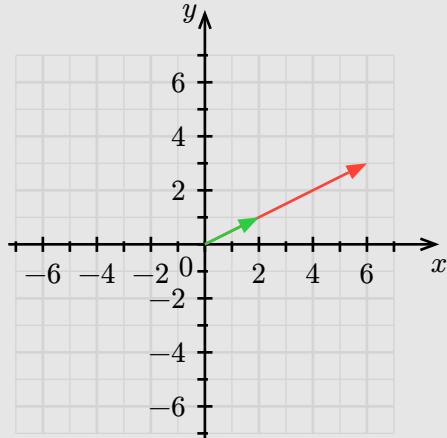


### 5.2.3. Scalar Multiplication

$$c \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \times x_1 \\ c \times x_2 \\ c \times x_3 \end{bmatrix}$$

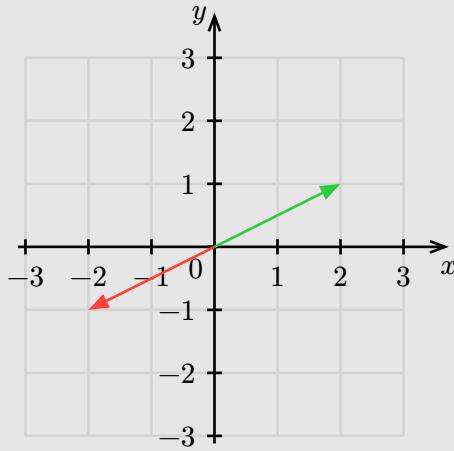
$$\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$3\vec{a} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$



$$\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$-1\vec{a} = -1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \cdot 2 \\ -1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



### 5.3. Unit Vector

A vector that has a magnitude (or length) of exactly 1

For a vector  $\vec{v}$  in  $n$ -dimensional space, a unit vector  $\hat{v}$  is defined as:

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

Where:

- $\|\vec{v}\|$  is the **magnitude** (or **norm**) of the vector  $\vec{v}$ , computed as:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

#### Key Properties

- **Magnitude:**

$$\|\hat{v}\| = 1$$

- **Direction:** A unit vector points in the same direction as the original vector  $\vec{v}$

Finding unit vector (vector of magnitude 1) with given direction

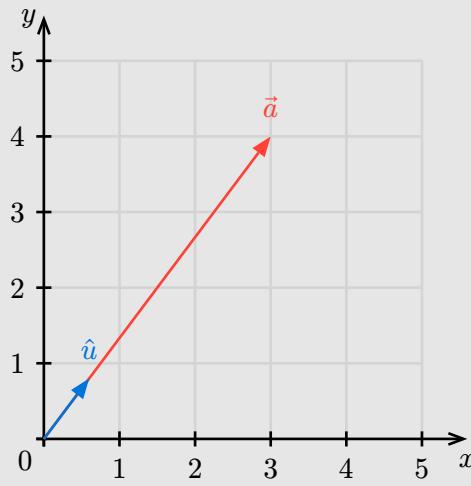
$$\vec{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Magnitude

$$\|\vec{a}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\hat{u} = \left( \frac{3}{\|\vec{a}\|}, \frac{4}{\|\vec{a}\|} \right) = \left( \frac{3}{5}, \frac{4}{5} \right)$$

$$\|\hat{u}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$$



#### 5.4. Parametric Representation of line

Set  $L$  of all points (i.e., line) equal to the set of all vectors  $\vec{x}$  plus some scalar  $t$  times the vector  $\vec{v}$  such that  $t$  can be any real number ( $\mathbb{R}$ )

$$L = \{\vec{x} + t\vec{v} \mid t \in \mathbb{R}\}$$

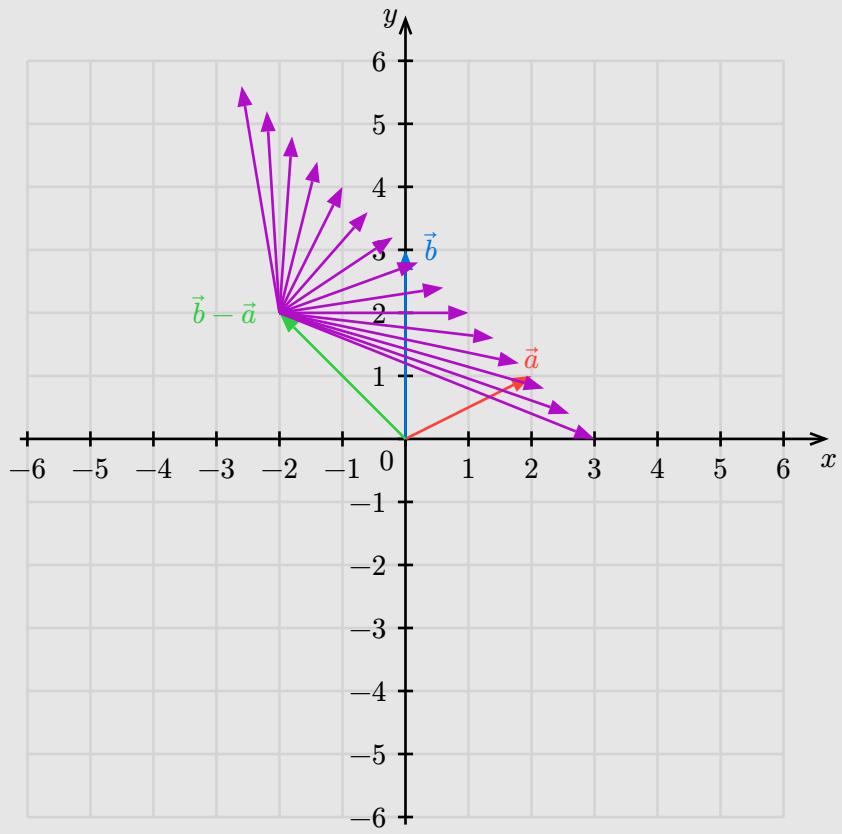
$$\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$t = 1$$

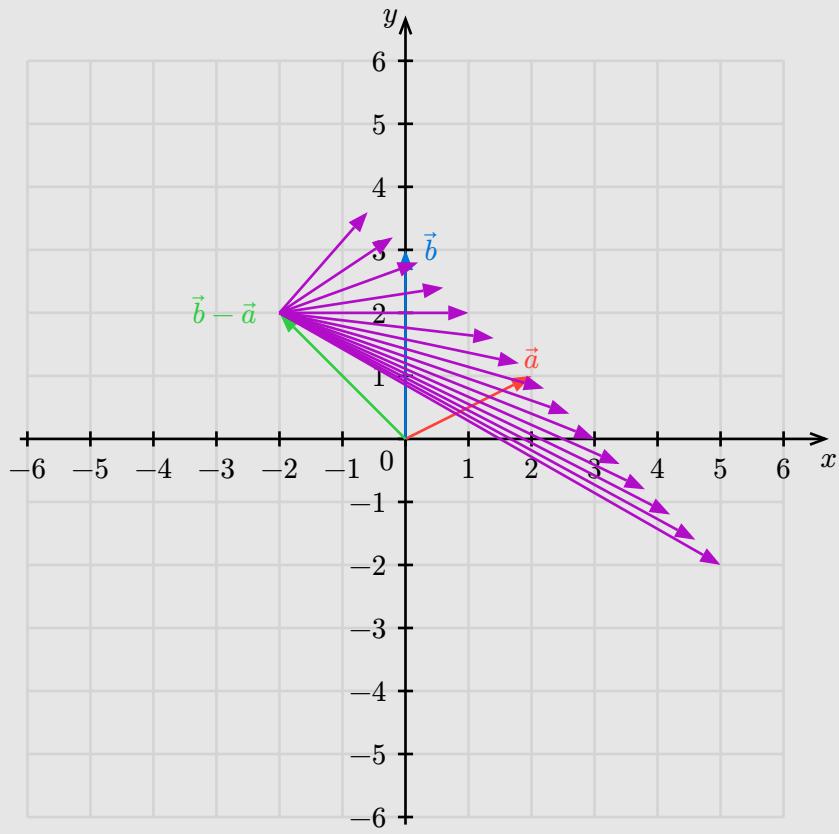
The line  $L$  can be defined as:

$$\begin{aligned} \vec{b} + t(\vec{b} - \vec{a}) &= \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 1 \left( \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 5 \end{bmatrix} \end{aligned}$$



The line  $L$  can also be defined as:

$$\begin{aligned}
 \vec{a} + t(\vec{b} - \vec{a}) &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \left( \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \\
 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ 3 \end{bmatrix}
 \end{aligned}$$



Generalization

$$P_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad P_2 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$L = P_1 + t(P_1 - P_2) \mid t \in \mathbb{R}$$

$$\vec{P}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \vec{P}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\vec{P}_1 - \vec{P}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

- $L$  starts at  $P_1$  and moves toward  $P_2$  as  $t$  decreases

$$L = P_1 + t(P_1 - P_2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$x = -1 + t(-1)$$

$$y = 2 + t(-1)$$

- $L$  starts at  $P_2$  and moves toward  $P_1$  as  $t$  increases

$$L = P_2 + t(P_1 - P_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$x = 0 + t(-1)$$

$$y = 3 + t(-1)$$

- $L$  starts at  $P_1$  and moves toward  $P_2$  as  $t$  increases

$$L = P_1 + t(P_2 - P_1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = -1 + t(1)$$

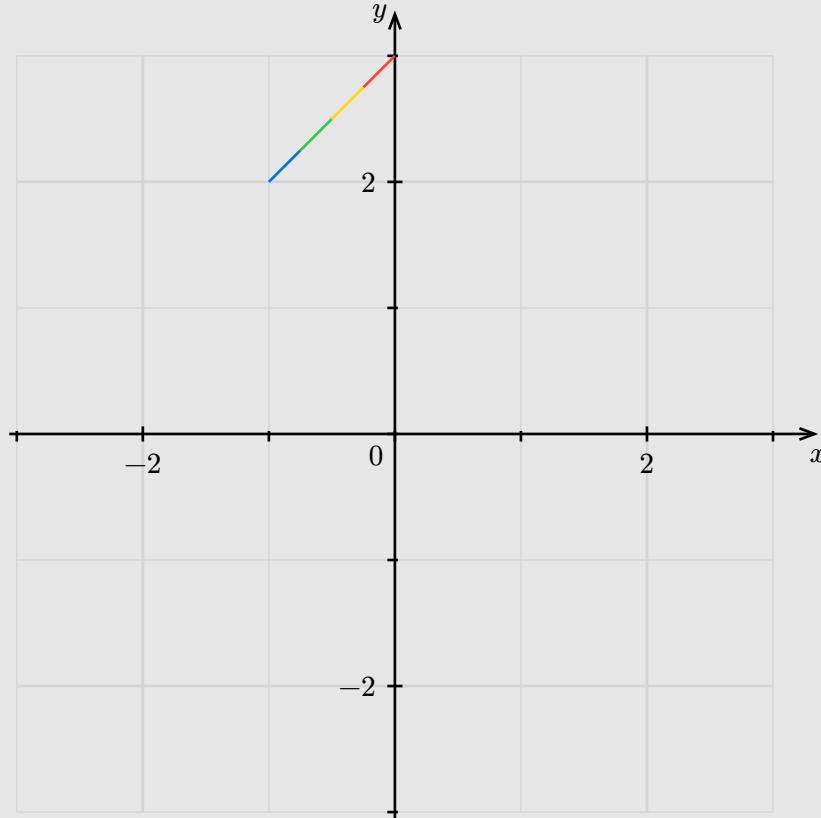
$$y = 2 + t(1)$$

- $L$  starts at  $P_2$  and moves toward  $P_1$  as  $t$  decreases

$$L = P_2 + t(P_1 - P_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = 0 + t(1)$$

$$y = 3 + t(1)$$



## 5.5. Vector Spaces

Let's say your factory can produce up to 300 units of product 1, 500 units of product 2, and 400 units of product 3. The set of all possible production combinations forms a vector space:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$0 \leq x_1 \leq 300$$

$$0 \leq x_2 \leq 500$$

$$0 \leq x_3 \leq 400$$

## 6. Matrices

$m \times n$  matrix  $\mathbf{A}$

- $m$ : rows
- $n$ : columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

### 6.1. Matrix-Vector Products

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{A}\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

For the dot product to be defined, the number of columns in the matrix  $\mathbf{A}$  (which is  $n$ ) must match the number of elements in the vector  $\vec{x}$  (also  $n$ ).

The result of multiplying matrix  $\mathbf{A}$  and vector  $\vec{x}$  will be a column vector with dimensions  $m \times 1$ , where  $m$  is the number of rows in the matrix  $\mathbf{A}$

$$(m \times n) \cdot (n \times 1) = m \times 1$$

1. As Row vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{a}^T = [a_1, a_2, \dots, a_n]$$

$$\vec{b}^T = [b_1, b_2, \dots, b_n]$$

$$A = \begin{bmatrix} [a_1, a_2, \dots, a_n] \\ [b_1, b_2, \dots, b_n] \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}^T \\ \vec{b}^T \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} \vec{a} \cdot \vec{x} \\ \vec{b} \cdot \vec{x} \end{bmatrix}$$

## 2. As Column Vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A = \begin{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\vec{x} = x_1 \vec{a} + x_2 \vec{b}$$

## 6.2. Null Space

The null space (or kernel) of a matrix  $A$  is the set of all vectors  $x$  that satisfy the equation:

$$A\vec{x} = \mathbf{0}$$

Where:

- $A$ :  $m \times n$  matrix
- $\vec{x}$ :  $n$ -dimensional vector
- $\mathbf{0}$ : zero vector in  $\mathbb{R}^m$

$$N(A) = N(\text{rref}(A)) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

We want to find the null space of  $A$ , which consists of all vectors  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$  that satisfy:

$$A\vec{x} = \mathbf{0}$$

This expands to the following system of linear equations:

$$\begin{cases} 1x_1 + 1x_2 + 1x_3 + 1x_4 = 0 \\ 1x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 4x_1 + 3x_2 + 2x_3 + 1x_4 = 0 \end{cases}$$

This can be represented as the augmented matrix:

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array} \right]$$

### 6.2.1. Column Space

The **columns space** (or range) of matrix  $A$  is span of its columns vectors

If the matrix  $A$  has columns  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ , then the column space of  $A$  is defined as:

$$\text{Col}(A) = \{ \vec{y} \in \mathbb{R}^m \mid \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n \}$$

or equivalently,

$$\text{Col}(A) = \text{span}(\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\})$$

Consider the simple example of a  $2 \times 2$  matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

The matrix has two columns:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

The column space, denoted  $\text{Col}(A)$ , is the span of these two vectors:

$$\text{Col}(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix}\right\}\right)$$

### Finding the Column Space

We observe that the two columns  $\vec{a}_1$  and  $\vec{a}_2$  are **linearly dependent**:

$$\vec{a}_2 = k\vec{a}_1$$

This means that  $\vec{a}_2$  is a scalar multiple of  $\vec{a}_1$ , the the two columns are **linearly dependent**. As a result, the column space is spanned by just one vector,  $\vec{a}_1$ , because any linear combination of  $\vec{a}_1$  and  $\vec{a}_2$  can be reduced to a multiple of  $\vec{a}_1$ .

Therefore, the column space of  $A$  is:

$$\text{Col}(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}\right)$$

which represents all vectors of the form:

$$c \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c \\ 3c \end{bmatrix} \quad \text{for any scalar } c$$

In other words, the column space is a line in  $\mathbb{R}^2$  through the origin in the direction of

### Rank of $A$

The rank of  $A$ , which is the **dimension of its column space**, is 1 because there is only one linearly independent column

This means the column space is the span of the columns of  $A$ , or all vectors that can be formed by taking linear combinations of the columns of  $A$ .

#### 6.2.2. Dimension of a Subspace

Number of elements in a basis for the subspace

#### 6.2.3. Nullity

##### Dimension of the Null Space

$$\dim(N(A))$$

The nullity of  $A$ : number of non-pivot columns (i.e., free variables) in the rref of  $A$

#### 6.2.4. Rank

##### Dimension of the column space

$$\text{rank}(A) = \dim(C(A))$$

#### 6.2.5. Matrix Representation of Systems of Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m &= b_n \end{aligned}$$

Coefficient Matrix ( $A$ ):

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Variable Vector (x):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Constant Vector (b):

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

The system of equations:

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 100 \\ 4x_1 + 2x_2 + 1x_3 &= 80 \\ 1x_1 + 5x_2 + 2x_3 &= 60 \end{aligned}$$

Can be represented as a matrix equation:

$$\begin{bmatrix} 2 & 3 & 5 \\ 4 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 80 \\ 60 \end{bmatrix}$$

### 6.3. Matrix Multiplication

$m \times n$  matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

$n \times p$  matrix:

$$B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix}$$

Compute Each Element of Result Matrix  $C$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Let  $A$  be an  $n \times m$  matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Let  $B$  an  $p \times n$  matrix:

$$B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

Calculate Each Element of  $C$

$$\begin{aligned} c_{11} &= (1 \cdot 7) + (2 \cdot 9) + (3 \cdot 11) = 58 \\ c_{12} &= (1 \cdot 8) + (2 \cdot 10) + (3 \cdot 12) = 64 \\ c_{21} &= (4 \cdot 7) + (5 \cdot 9) + (6 \cdot 11) = 138 \\ c_{22} &= (4 \cdot 8) + (5 \cdot 10) + (6 \cdot 12) = 154 \end{aligned}$$

$C$  is a  $m \times p$  matrix

$$C = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

## 7. Linear Combinations

Set of vector

$$v_1, v_2, \dots, v_n \in \mathbb{R}^n$$

Where

- $v_1, v_2, \dots, v_n$ : set of vectors
- $\mathbb{R}^n$ : set of all ordered tuples of  $n$  real numbers

Linear combination of those vector

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$$c_1, c_2, \dots, c_n \in \mathbb{R}$$

Where:

- $c_1, c_2, \dots, c_n$ : constants or weights

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$0\vec{a} + 0\vec{b}$$

$$3\vec{a} + 2\vec{b}$$

## 8. Span

Represents the subspace of the vector space that is “covered” by these vectors through their linear combinations

If you have a set of vectors  $v_1, v_2, \dots, v_n$ , the span of these vectors is the set of all vectors that can be written as:

$$\text{Span}(v_1, v_2, \dots, v_n) = \{c_1 v_1 + c_2 v_2 + \dots + c_n v_n \mid c_1, c_2, \dots, c_n \in \mathbb{R}\}$$

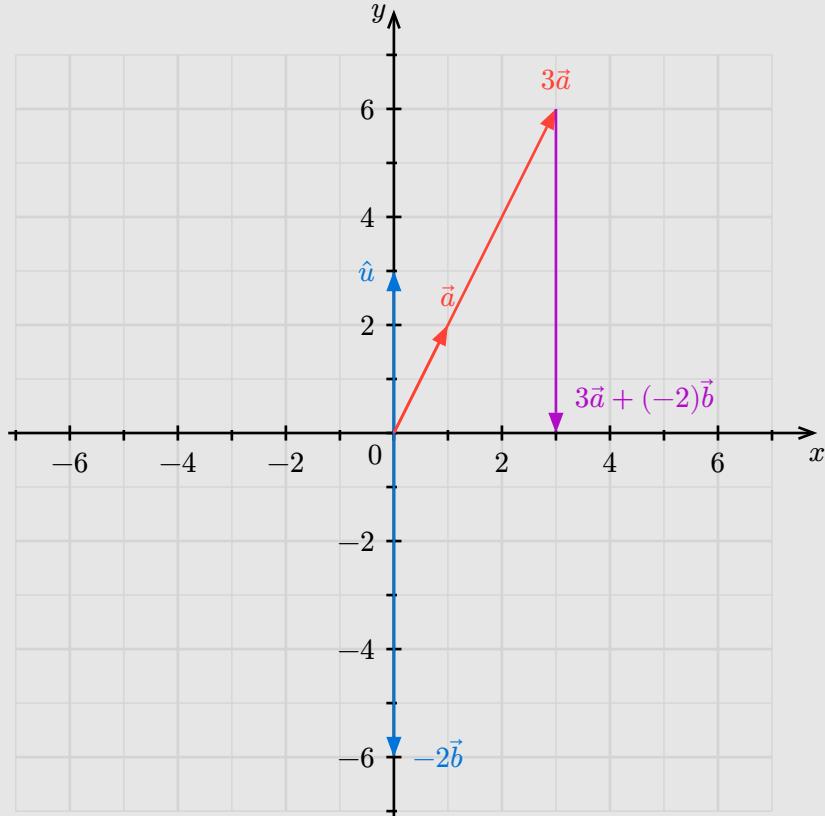
Any vector in  $\mathbb{R}^2$  can be represented by a linear combination with some combination of these vectors

### 1. Spanning $\mathbb{R}^2$

$$\text{Span}(\vec{a}, \vec{b}) = \mathbb{R}^2$$

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$3\vec{a} + (-2)\vec{b} = \begin{bmatrix} 3-0 \\ 6-6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



Any point  $\vec{x}$  can be represented as a linear combination of  $\vec{a}$  and  $\vec{b}$

1. Define the vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. Express  $\vec{x}$  as a linear combinations

$$c_1 \vec{a} + c_2 \vec{b} = \vec{x}$$

Which expands to

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3. Set up the system of equations

$$1c_1 + 0c_2 = x_1 \quad (1)$$

$$2c_1 + 3c_2 = x_2 \quad (2)$$

4. Express  $c_1$ : From equation (1), we can directly express  $c_1$

$$c_1 = x_1$$

5. Substitute  $c_1$  into equation (2)

$$2x_1 + 3c_2 = x_2$$

Rearranging gives:

$$3c_2 = x_2 - 2x_1$$

6. Solve for  $c_2$  : Dividing both sides by 3 yields

$$c_2 = \frac{x_2 - 2x_1}{3}$$

7. Example with specific values: Let's say we want to find  $c_1$  and  $c_2$  when  $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Substitute  $x_1 = 2$  and  $x_2 = 2$

$$c_1 = x_1 = 2$$

$$c_2 = \frac{2 - 2 \cdot 2}{3} = -\frac{2}{3}$$

8. Final linear combination: Now, substituting  $c_1$  and  $c_2$  back into the linear combination

$$2\vec{a} - \frac{2}{3}\vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Verifying

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

9. This shows that

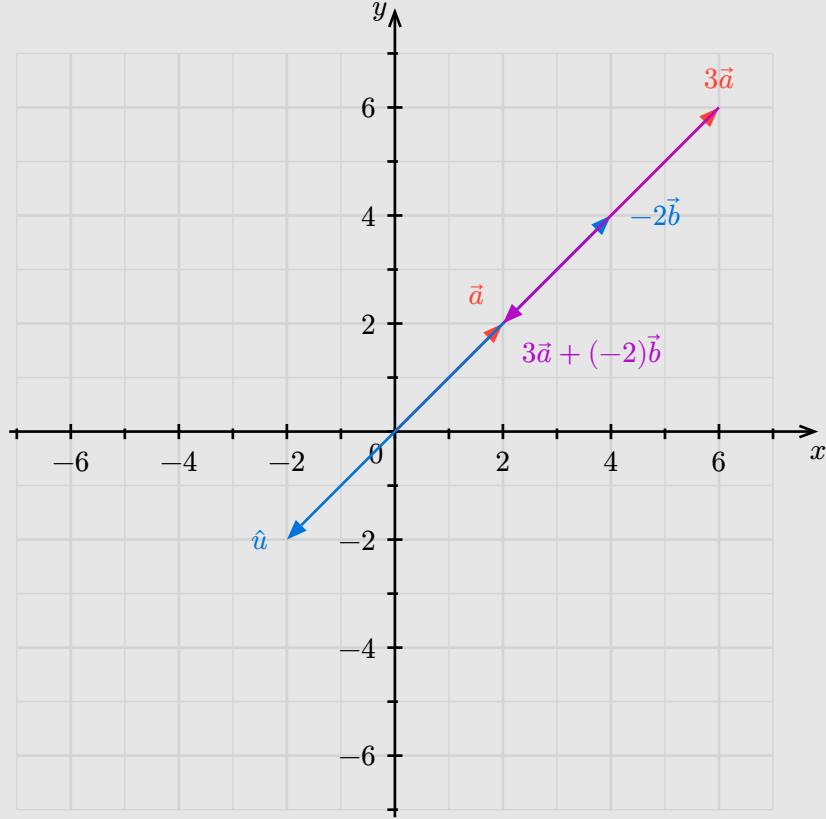
$$2\vec{a} - \frac{2}{3}\vec{b} = \vec{x}$$

## 2. Spanning Line in $\mathbb{R}^2$

Any linear combination of  $\vec{a}$  and  $\vec{b}$  will produce vectors that lie along the same line. This is the line through the origin in the direction of  $\vec{a}$  (or  $\vec{b}$ ), with all points on the line being scalar multiples of  $\vec{a}$

$$\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$3\vec{a} + (-2)\vec{b} = \begin{bmatrix} 6 & -4 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



## 9. Linear Independence

### 1. Definition of Linear Independence

The set of vectors

$$S = \{v_1, v_2, \dots, v_n\}$$

is said to be **linearly independent** if the only solution to the equation

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \mathbf{0}$$

is  $c_1 = c_2 = \dots = c_n = 0$ . In other words, no vector in the set can be written as a linear combination of the others.

If at least one constant  $c_i$  is non-zero, the set is linearly dependent.

**Example 1:** Testing for Linear Independence

**Problem:** Is the following set of vectors **linearly dependent**?

$$S = \{\vec{v}_1, \vec{v}_2\}$$

Where:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

For a set of vectors to be **linearly independent**, the only solution to the equation:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \mathbf{0}$$

must be  $c_1 = 0$  and  $c_2 = 0$

In this case:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{0}$$

If not only the zero solution exists (i.e., if  $c_1$  or  $c_2$  can be non-zero), the set is **linearly dependent**.

### Step 1. Set up the system of equations:

1.  $2c_1 + 3c_2 = 0$
2.  $1c_1 + 2c_2 = 0$

### 2. Eliminate one variable

$$2 \times (1c_1 + 2c_2 = 0) \Rightarrow 2c_1 + 4c_2 = 0$$

Now the system is

$$\begin{aligned} 2c_1 + 3c_2 &= 0 \\ 2c_1 + 4c_2 &= 0 \end{aligned}$$

### 3. Subtract the equations

$$(2c_1 + 3c_2) - (2c_1 + 4c_2) = 0$$

Simplifies to:

$$(2c_1 - 2c_1) + (4c_2 - 3c_2) = 0$$

So:

$$c_2 = 0$$

### 4. Substitute back to find $c_1$

Now that we know  $c_2 = 0$ , substitute this value into one of the original equations. Let's use the second equation:

$$1c_1 + 2c_2 = 0$$

Substitute  $c_2 = 0$ :

$$\begin{aligned} 1c_1 + 2(0) &= 0 \\ c_1 &= 0 \end{aligned}$$

### Conclusion:

Since  $c_1 = 0$  and  $c_2 = 0$ , the set of vectors  $S$  is **linearly independent**. These vectors span  $\mathbb{R}^2$ .

---

### Example 2: Testing for Linear Dependence

**Problem:** Is the following set of vectors **linearly dependent**?

$$S = \{\vec{v}_1 \vec{v}_2\}$$

Where:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

The span of this set is the collection of all vectors that can be formed by linear combinations of  $\vec{v}_1$  and  $\vec{v}_2$ :

$$c_1 v_1 + c_2 v_2$$

Since  $v_2 = 2v_1$ , the linear combination becomes:

$$\begin{aligned} c_1 v_1 + c_2 (2v_1) &= (c_1 + 2c_2)v_1 \\ c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} &= \\ c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} &= \\ (c_1 + 2c_2) \begin{bmatrix} 2 \\ 3 \end{bmatrix} &= \\ c_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \end{aligned}$$

Thus, any linear combination of these vectors is just a scalar multiple of  $v_1$ . The span is a single line in  $\mathbb{R}^2$ , and the vectors are **linearly dependent**.

For any two **colinear** vectors in  $\mathbb{R}^2$ , their span reduces to a single line.

One vector in the set can be represented by some combination of other vectors in the set

## 2. General Rule

In  $\mathbb{R}^n$ , if you have more than  $n$  vectors, at least one vector must be linearly dependent on the others, meaning the set cannot be linearly independent.

## 10. Subspace

$V$  is a linear subspace of  $\mathbb{R}^n$ :

- **Non-emptiness:**  $V$  contains the **0** vector

$$\mathbf{0} \in V$$

- **Closure under addition:** If  $u$  and  $v$  are any vectors in the subspace  $V$ , then their sum  $u + v$  must also be in  $V$ .

$$\text{If } u, v \in V, \text{ then } u + v \in V$$

- **Closure under scalar multiplication:** If  $u$  is any vector in  $V$  and  $c$  is any scalar (real number), then the product  $cu$  must also be in  $V$ .

$$\text{If } u \in V \text{ and } c \in V, \text{ then } cu \in V$$

### Example 1: Subspace

**Problem:** Is  $V$  a subspace of  $\mathbb{R}^2$

$$V = \{\mathbf{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- **Non-emptiness** 

$$\mathbf{0} \in V$$

- **Closure under addition** 

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- **Closure under scalar multiplication** 

$$c \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

### Conclusion

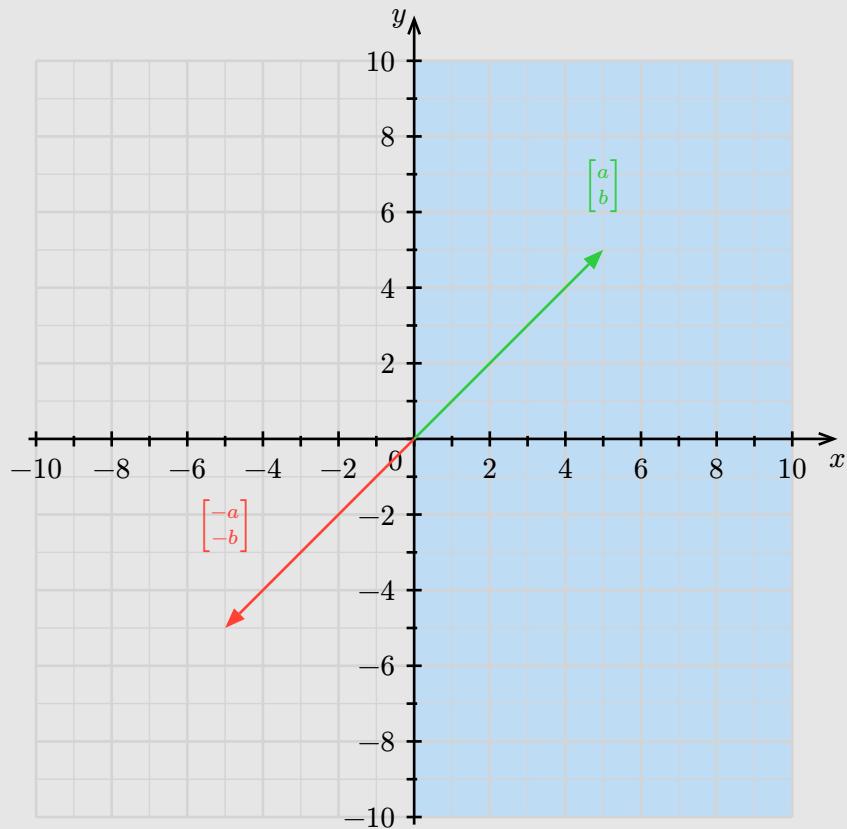
The subset  $V$  of  $\mathbb{R}^3$  is a **subspace**

---

**Example 2:** Not Subspace

**Problem:** Is  $S$  a subspace of  $\mathbb{R}^2$  

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1 \geq 0 \right\}$$



- Non-emptiness ✓

$$\mathbf{0} \in S$$

- Closure under addition ✓

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

$$a \geq 0$$

$$b \geq 0$$

$$a+b \geq 0$$

- Closure under scalar multiplication ✗

$$-1 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix}$$

## Conclusion

### Span and Subspace

The span of any set of vectors is a valid subspace

$$U = \text{Span}(v_1, v_2, \dots, v_n) = \text{Valid Subspace of } \mathbb{R}^n$$

- Non-emptiness

$$0v_1 + 0v_2 + \dots + 0v_n = \mathbf{0}$$

- **Closure under addition**

$$\vec{X} = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

$$\vec{Y} = b_1 v_1 + b_2 v_2 + \dots + b_n v_n$$

$$\begin{aligned}\vec{X} + \vec{Y} &= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \dots + (a_n + b_n)v_n \\ &= c_1 v_1 + c_2 v_2 + \dots + c_n v_n\end{aligned}$$

- **Closure under scalar multiplication**

$$\vec{X} = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

$$\begin{aligned}b\vec{X} &= bc_1 v_1 + bc_2 v_2 + \dots + bc_n v_n \\ &= c_1 v_1 + c_2 v_2 + \dots + c_n v_n\end{aligned}$$

## 11. Basis

Non-redundant set of vectors that span  $\mathbb{R}^n$

A basis of a vector space is a set of vectors that satisfies two conditions:

1. **Linear Independence:** No vector in the set can be written as a linear combination of the others. This means that **the only way to combine the vectors to get the zero vector is by using all zero coefficients.**

The vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  are linearly independent if the only solution to the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n = \mathbf{0}$$

is  $c_1 = c_2 = \dots = c_n = 0$ , where  $c_i$  are scalar coefficients.

2. **Spanning:** The set of vectors can be linearly combined to form any vector in the vector space. In other words, **every vector in the vector space can be expressed as a linear combination of the basis vectors.**

The set  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  spans the vector space  $V$  if any vector  $v \in V$  can be expressed as a linear combination of the basis vectors:

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

for some scalars  $c_1, c_2, \dots, c_n$ .

Consider the vector space  $\mathbb{R}^2$  (the 2-dimensional Euclidean space). A common basis for  $\mathbb{R}^2$  is  $\{e_1, e_2\}$ , where:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This set is a basis because:

- **Linear Independence:** The only solution to  $c_1 e_1 + c_2 e_2 = \mathbf{0}$  is  $c_1 = c_2 = 0$ .
- **Spanning:** Any vector  $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$  can be written as  $\vec{v} = x e_1 + y e_2$

This means  $\{e_1, e_2\}$  is a basis for  $\mathbb{R}^2$ , and the dimension of  $\mathbb{R}^2$  is 2.

## 11.1. Vector Dot Product

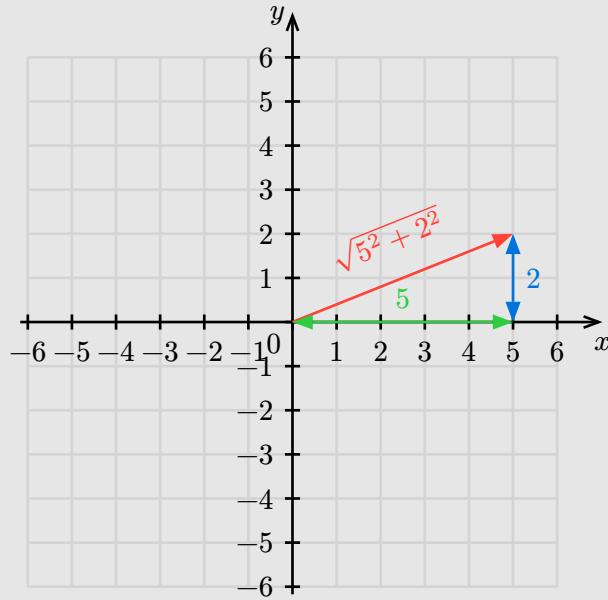
$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

### 11.1.1. Magnitude (Length)

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

$$\vec{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\|\vec{a}\| = \sqrt{5^2 + 2^2}$$



$$\vec{a} \cdot \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1^2 + a_2^2 + \dots + a_n^2$$

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$$

$$\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$$

### 11.1.2. Properties

#### 11.1.2.1. Commutative

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

#### 11.1.2.2. Distributive

$$(\vec{v} + \vec{w}) \cdot \vec{x} = \vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x}$$

### 11.1.2.3. Associativity

$$(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$$

### 11.1.3. Cauchy-Schwarz Inequality

$$\begin{aligned} |\vec{u} \cdot \vec{v}| &\leq \|\vec{u}\| \|\vec{v}\| \\ |\vec{u} \cdot \vec{v}| &= \|\vec{u}\| \|\vec{v}\| \quad \text{when } \vec{u} = c\vec{v} \end{aligned}$$

Where:

- $\vec{u} \cdot \vec{v}$ : dot product of vectors  $\vec{u}$  and  $\vec{v}$
- $\|\vec{u}\|$  and  $\|\vec{v}\|$ : magnitudes (lengths) of vectors  $\vec{u}$  and  $\vec{v}$

#### Step 1: Understand the dot product

The dot product of two vectors  $\vec{u} = [u_1, u_2, \dots, u_n]$  and  $\vec{v} = [v_1, v_2, \dots, v_n]$  is calculated as:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

The magnitude (or norm) of a vector  $\vec{u}$  is:

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

#### Step 2: Define a new function

We introduce a parameter  $t \in \mathbb{R}$  and define a new vector:

$$w(t) = \vec{u} - t\vec{v}$$

Now, consider the dot product of this new vector with itself, which is always non-negative because it represents the square of the magnitude of  $w(t)$ :

$$w(t) \cdot w(t) \geq 0$$

This inequality makes sense because the dot product of any vector with itself is the square of its magnitude, and **a square is always non-negative**.

#### Step 3: Expand the dot product

Expand  $w(t) \cdot w(t)$ :

$$w(t) \cdot w(t) = (\vec{u} - t\vec{v}) \cdot (\vec{u} - t\vec{v})$$

Now, we apply the distributive property of the dot product, which behaves similarly to the distributive property of multiplication. We expand each term:

$$(\vec{u} - t\vec{v}) \cdot (\vec{u} - t\vec{v}) = \vec{u} \cdot \vec{u} - t(\vec{u} \cdot \vec{v}) - t(\vec{v} \cdot \vec{u}) + t^2(\vec{v} \cdot \vec{v})$$

Since the dot product is commutative ( $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ ), we can rewrite this as:

$$\vec{u} \cdot \vec{u} - 2t(\vec{u} \cdot \vec{v}) + t^2(\vec{v} \cdot \vec{v})$$

This simplifies to:

$$\|\vec{u}\|^2 - 2t(\vec{u} \cdot \vec{v}) + t^2 \|\vec{v}\|^2$$

We've now expressed the result of expanding the dot product as a quadratic expression in  $t$ , where

- $\|\vec{u}\|^2$  is a constant term,
- $-2t(\vec{u} \cdot \vec{v})$  is the linear term in  $t$
- $t^2 \|\vec{v}\|^2$  is the quadratic term

**Step 4:** Treat as a quadratic equation

Now that we have the quadratic expression:

$$\|\vec{v}\|^2 t^2 - 2(\vec{u} \cdot \vec{v})t + \|\vec{u}\|^2 \geq 0$$

We recognize this as a standard quadratic inequality of the form  $at^2 + bt + c \geq 0$ , where:

- $a = \|\vec{v}\|^2$
- $b = -2(\vec{u} \cdot \vec{v})$
- $c = \|\vec{u}\|^2$

For any quadratic expression  $at^2 + bt + c$  to always be non-negative, its discriminant must be less than or equal to zero. The discriminant of a quadratic equation  $at^2 + bt + c = 0$  is given by:

$$\Delta = b^2 - 4ac$$

Substituting in the values of  $a$ ,  $b$ , and  $c$  from our expression:

$$\Delta = (-2(\vec{u} \cdot \vec{v}))^2 - 4 \cdot \|\vec{v}\|^2 \cdot \|\vec{u}\|^2$$

Simplifying:

$$\Delta = 4(\vec{u} \cdot \vec{v})^2 - 4 \|\vec{v}\|^2 \|\vec{u}\|^2$$

**Step 5:** Apply the discriminant condition

For the quadratic inequality to hold, the discriminant must be less than or equal to zero:

$$\Delta = 4(\vec{u} \cdot \vec{v})^2 - 4 \|\vec{v}\|^2 \|\vec{u}\|^2 \leq 0$$

Divide by 4:

$$\Delta = (\vec{u} \cdot \vec{v})^2 \leq \|\vec{v}\|^2 \|\vec{u}\|^2$$

Take the square root of both sides:

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

**Step 1:** Compute the dot product  $\vec{u} \cdot \vec{v}$ 

$$\vec{u} \cdot \vec{v} = (1)(3) + (2)(4) = 3 + 8 = 11$$

**Step 2:** Compute the norms of  $\vec{u}$  and  $\vec{v}$ 

- The norm  $\|\vec{u}\|$  is:

$$\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$$

- The norm  $\|\vec{v}\|$  is:

$$\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

**Step 3:** Verify the Cauchy-Schwarz inequality

The inequality states:

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Substitute the values:

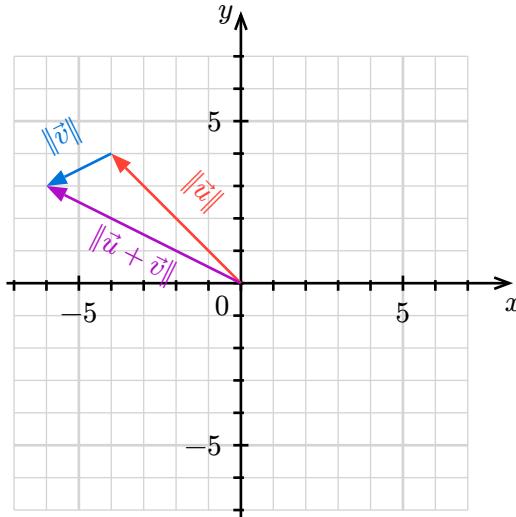
$$|11| \leq 5\sqrt{5} = 11.18$$

Since  $11 \leq 11.18$ , the inequality holds.

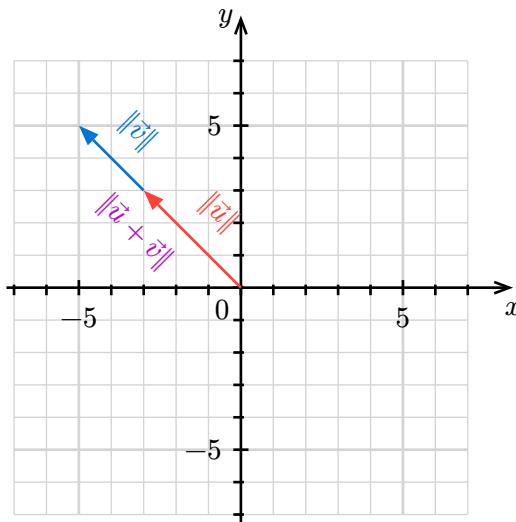
The Cauchy-Schwarz inequality is satisfied for the vectors  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

#### 11.1.4. Vector Triangle Inequality

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$



$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

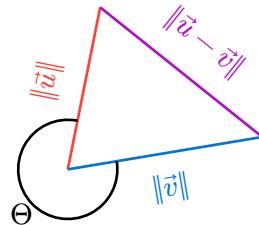
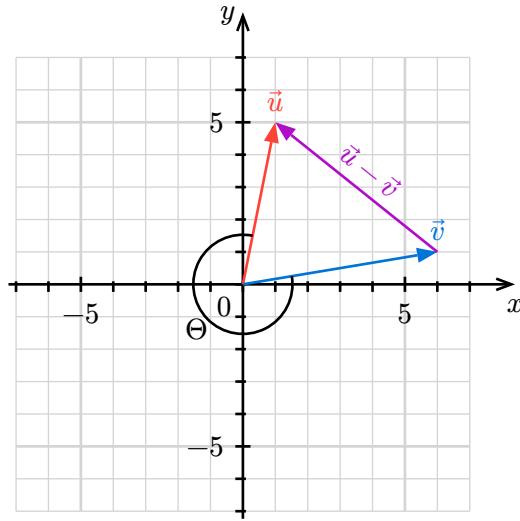


$$\vec{u} = c\vec{v} \quad c > 0$$

#### 11.2. Angles Between Vectors

The scalar  $\|\vec{u}\|$  is the length of the vector  $\vec{u}$

Say  $\vec{u}, \vec{v} \in \mathbb{R}^n$



Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$$

Where:

- $a, b$  and  $c$ : lengths of the sides of a triangle
- $C$ : angle opposite the side  $c$

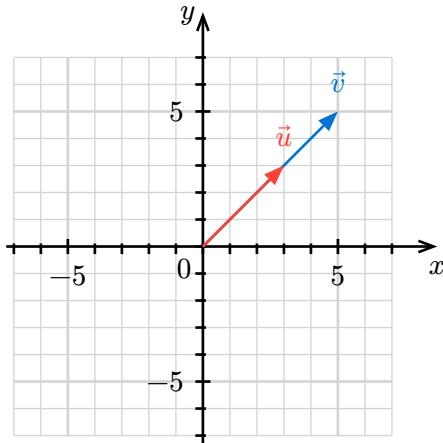
$$\begin{aligned}
 \|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 \|\vec{u}\| \|\vec{v}\| \cdot \cos(\Theta) \\
 (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) &= \\
 \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} &= \\
 \|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 \|\vec{u}\| \|\vec{v}\| \cdot \cos(\Theta)
 \end{aligned}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cdot \cos(\Theta)$$

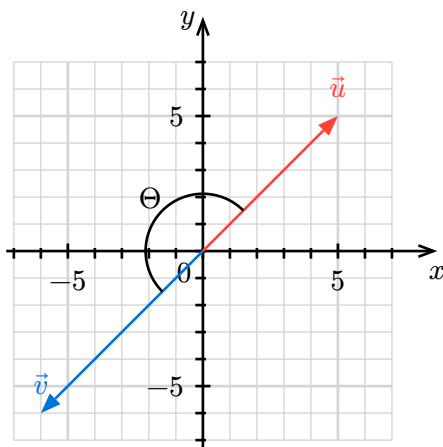
$$\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \cos(\Theta)$$

$$\Theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$$

So, if  $\vec{u}$  is a scalar multiple of  $\vec{v}$  ( $\vec{u} = c\vec{v}$ ) where  $c > 0$ , then  $\Theta = 0^\circ$



And, if  $\vec{u}$  is a scalar multiple of  $\vec{v}$  ( $\vec{u} = c\vec{v}$ ) where  $c < 0$ , then  $\Theta = 180^\circ$



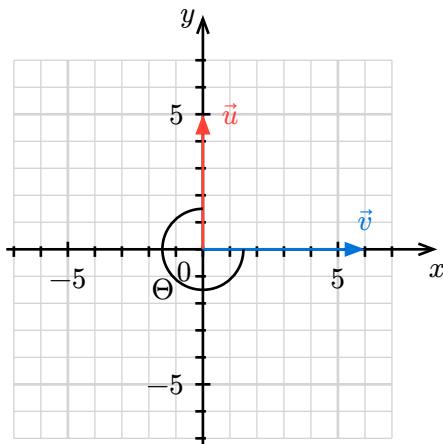
$\vec{u}$  and  $\vec{v}$  are perpendicular if the angle  $\Theta$  between them is  $90^\circ$

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \|\vec{u}\| \|\vec{v}\| \cdot \cos(90^\circ) \\ \vec{u} \cdot \vec{v} &= 0\end{aligned}$$

If  $\vec{u}$  and  $\vec{v}$  are perpendicular, then  $\vec{u} \cdot \vec{v} = 0$

If  $\vec{u}$  and  $\vec{v}$  are non-zero and  $\vec{u} \cdot \vec{v} = 0$ , then they are perpendicular

If  $\vec{u} \cdot \vec{v} = 0$  then  $\vec{u}$  and  $\vec{v}$  are **orthogonal**.



### 11.2.1. Plane in $\mathbb{R}^3$

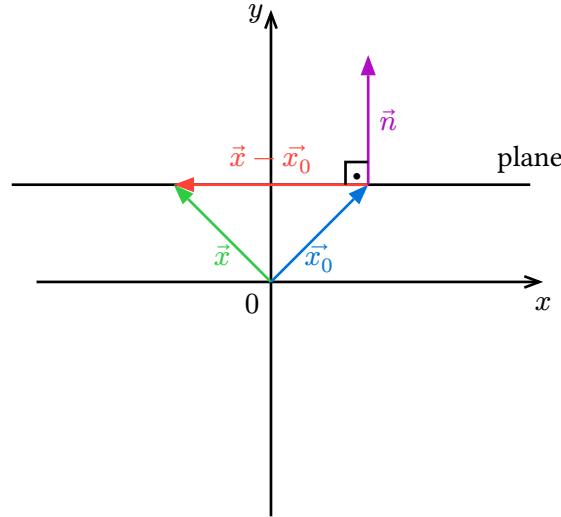
Plane: Each point  $(x, y, z)$  on the satisfies the equation

$$ax + by + cz = d$$

Normal Vector: vector that is perpendicular (orthogonal) to a plane, line, or curve, at a specific point

#### 1. Plane

If a plane is defined by the equation  $ax + by + cz = d$ , the vector  $\vec{n} = \langle a, b, c \rangle$  is a normal vector to the plane because it is perpendicular to any vector that lies in the plane



$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad \vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\vec{x} - \vec{x}_0 = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

$\vec{x} - \vec{x}_0$  is a vector that lies on the plane, then  $\vec{n}$  is normal if:

$$\vec{n} \cdot \vec{x} - \vec{x}_0 = 0$$

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Find the equation of the plane given the point on the plane  $\vec{x}_0$  and a the normal vector  $\vec{n}$

$$\vec{n} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

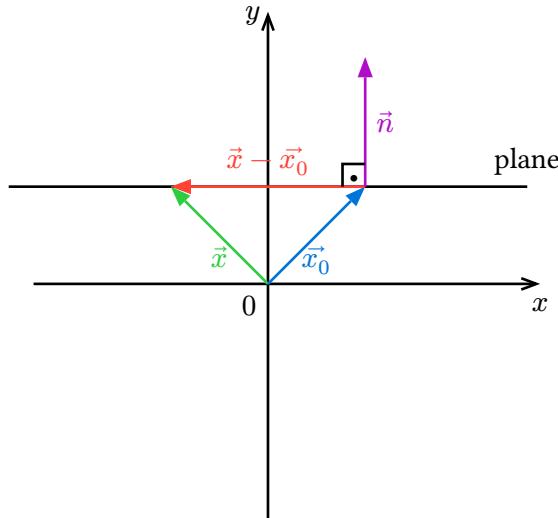
$$\vec{x} - \vec{x}_0 = \begin{bmatrix} x - 1 \\ y - 2 \\ z - 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y - 2 \\ z - 3 \end{bmatrix} = 0$$

$$(x - 1) + 3(y - 2) - 2(z - 3) = 0$$

$$x - 1 + 3y - 6 - 2z + 6 = 0$$

$$x + 3y - 2z = 1$$



The normal vector to a plane can be directly obtained from the coefficients of  $x$ ,  $y$ , and  $z$  in the plane equation of the form:

$$Ax + By + Cz = D$$

$$\vec{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Find the equation of the normal vector  $\vec{n}$  given the equation for the plane:

$$-3x + \sqrt{2}y + 7z = \pi$$

$$\vec{n} = -3\hat{i} + \sqrt{2}\hat{j} + 7\hat{k}$$

$$\vec{n} = \begin{bmatrix} -3 \\ \sqrt{2} \\ 7 \end{bmatrix}$$

## 2. Curve

For a curve described by a function  $y = f(x)$ , the normal vector at a point on the curve is perpendicular to the tangent line at that point. If the tangent vector has slope  $f'(x)$ , the normal vector will have a slope of  $-\frac{1}{f'(x)}$

### 11.2.2. Point Distance to Plane

$$d = \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}$$

Given a the equation of the plane:

$$1x - 2y + 3z = 5$$

And a point **not** on the plane:

$$(2, 3, 1)$$

Find the shortest path (normal vector) from the plane to the point

$$\begin{aligned} d &= \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{1 \cdot 2 - 2 \cdot 3 + 3 \cdot 1 - 5}{\sqrt{1^2 + 2^2 + 3^2}} \\ &= \frac{2 - 6 + 3 - 5}{\sqrt{1 + 4 + 9}} \\ &= -\frac{6}{\sqrt{14}} \end{aligned}$$

### 11.2.3. Distance Between Planes

#### 11.2.4. Cross Product

Only defined in  $\mathbb{R}^3$

Returns a vector orthogonal to the two vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\vec{c} = \vec{a} \times \vec{b}$$

$$\vec{c} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

$$\vec{c} = \vec{a} \times \vec{b}$$

$$\vec{c} = \begin{bmatrix} -7 \cdot 4 - 1 \cdot 2 \\ 1 \cdot 5 - 1 \cdot 4 \\ 1 \cdot 2 - (-7) \cdot 5 \end{bmatrix} = \begin{bmatrix} -30 \\ 1 \\ 37 \end{bmatrix}$$

$\vec{c}$  is orthogonal to both  $\vec{a}$  and  $\vec{b}$

Proof:  $\vec{c}$  is orthogonal to  $\vec{a}$  and  $\vec{b}$

When the dot product of two vectors is equal to 0, it means that the two vectors are perpendicular (or orthogonal) to each other

1. Orthogonal to Vector  $\vec{a}$

$$\vec{c} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1$$

$$\cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_2a_1b_3} + \cancel{a_3a_1b_2} - \cancel{a_3a_2b_1} = 0$$

2. Orthogonal to Vector  $\vec{b}$

$$\vec{c} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$b_1a_2b_3 - b_1a_3b_2 + b_2a_3b_1 - b_2a_1b_3 + b_3a_1b_2 - b_3a_2b_1$$

$$\cancel{b_1a_2b_3} - \cancel{b_1a_3b_2} + \cancel{b_2a_3b_1} - \cancel{b_2a_1b_3} + \cancel{b_3a_1b_2} - \cancel{b_3a_2b_1} = 0$$

#### 11.2.5. Proof: Relationship Between Cross Product and Sin of Angle

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\vec{c} = \vec{a} \times \vec{b}$$

$$\vec{c} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\Theta)$$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin(\Theta)$$

#### 11.2.6. Dot and Cross Products

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\Theta)$$

$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \cos(\Theta)$$

$$\Theta = \arccos()$$

### 11.3. Row Echelon Form (REF)

Visual Structure:

1. Pivot (leading 1): The leading entry of each non-zero row is 1

2. Zeros below pivots: Every pivot has zeros below it in its column
3. Rightward movement of pivots: Each leading 1 in a lower row is further to the right than in the row above it
4. Rows of all zeros (if any) are at the bottom of the matrix

$$\left[ \begin{array}{cccc|c} 1 & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & 1 & a_{23} & a_{24} & b_2 \\ 0 & 0 & 1 & a_{34} & b_3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Elementary row operations:

1. **Row Swapping:** Swap two rows
2. **Row Multiplication:** Multiply a row by a non-zero scalar
3. **Row Addition / Subtraction:** Add or subtract a multiple of one row from another row

Consider the system of linear equations:

$$\begin{aligned} 2x + y + z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + 1y + 2z &= -3 \end{aligned}$$

The augmented matrix for this system is:

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array} \right]$$

**Step 1:** Make the leading entry of the first row a 1

We divide the first row by 2 (row multiplication):

$$R_1 \rightarrow \frac{1}{2}R_1 = \left[ \begin{array}{ccc|c} 1 & 0.5 & 0.5 & 4 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array} \right]$$

**Step 2:** Eliminate the entries below the first pivot. We now want the entries below the first pivot (1 in the first column) to become zeros. We use row addition:

1.  $R_2 \rightarrow R_2 + 3R_1$
2.  $R_3 \rightarrow R_3 + 2R_1$

This gives:

$$\left[ \begin{array}{ccc|c} 1 & 0.5 & 0.5 & 4 \\ 0 & 0.5 & 3.5 & 1 \\ 0 & 2 & 3 & 5 \end{array} \right]$$

**Step 3:** Make the leading entry of the second row a 1

We divide the second row by 0.5 (row multiplication):

$$R_2 \rightarrow \frac{1}{0.5}R_2 = \left[ \begin{array}{ccc|c} 1 & 0.5 & 0.5 & 4 \\ 0 & 1 & 7 & 2 \\ 0 & 2 & 3 & 5 \end{array} \right]$$

**Step 4:** Eliminate the entry below the second pivot

We now want the entry below the second pivot (1 in the second column) to become zero. We use row addition:

$$R_3 \rightarrow R_3 - 2R_2 = \left[ \begin{array}{ccc|c} 1 & 0.5 & 0.5 & 4 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & -11 & 1 \end{array} \right]$$

**Step 5:** Make the leading entry of the third row a 1

We divide the third row by  $-11$  (row multiplication):

$$R_3 \rightarrow -\frac{1}{11}R_3 = \left[ \begin{array}{ccc|c} 1 & 0.5 & 0.5 & 4 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 1 & -\frac{1}{11} \end{array} \right]$$

**Step 6:** Back-substitute to solve for the variables

From the third row:

$$z = -\frac{1}{11}$$

Substitute  $z$  into the second row:

$$\begin{aligned} y + 7z &= 2 \\ y + 7\left(-\frac{1}{11}\right) &= 2 \\ y &= 2 + \frac{7}{11} \\ y &= \frac{29}{11} \end{aligned}$$

Substitute  $y$  and  $z$  into the first row:

$$\begin{aligned} x + 0.5y + 0.5z &= 4 \\ x + 0.5\left(\frac{29}{11}\right) + 0.5\left(-\frac{1}{11}\right) &= 4 \\ x &= 4 - \frac{14}{11} + \frac{1}{11} \\ x &= \frac{31}{11} \end{aligned}$$

**Final Solution:**

$$x = \frac{31}{11} \quad y = \frac{29}{11} \quad z = -\frac{1}{11}$$

**11.3.1. Solution Types in Linear Systems: Unique, Infinite, or None****1. Unique Solution**

$$\left[ \begin{array}{cccc|c} 1 & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & 1 & a_{23} & a_{24} & b_2 \\ 0 & 0 & 1 & a_{34} & b_3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

## 2. No Solution

$$0 = a$$

## 3. No Unique Solution (Infinite Number of Solutions)

Column 2 and 4 indicate free variables  $x_2$  and  $x_4$  because they have no pivot entries

$$\left[ \begin{array}{cccc|c} 1 & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & 0 & 1 & a_{24} & b_2 \\ 0 & 0 & 1 & a_{34} & b_3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

### 11.3.2. Special Cases

Rows of all zeros appear in row echelon form (REF) in the following situations:

#### 1. Dependent Equations

Some equations are multiples or linear combinations of others

$$\begin{aligned} 2x + 4y &= 8 \\ x + 2y &= 4 \end{aligned}$$

Augmented matrix:

$$\left[ \begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 2 & 4 \end{array} \right]$$

After row reduction:

$$\left[ \begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

#### 2. Underdetermined Systems

Number of variables is greater than the number of independent equations

$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y + z &= 5 \end{aligned}$$

Augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 5 \end{array} \right]$$

After row reduction:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

#### 3. Inconsistent Systems

No solution exists

Rows of zeros on the left side (coefficients of the variables) and a non-zero entry on the right side (augmented column)

$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 7 \end{aligned}$$

Augmented matrix:

$$\left[ \begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 7 \end{array} \right]$$

After row reduction:

$$\left[ \begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right]$$

## 12. Matrices

$m \times n$  matrix  $A$

- $m$ : rows
- $n$ : columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

### 12.1. Matrix-Vector Products

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

For the dot product to be defined, the number of columns in the matrix  $A$  (which is  $n$ ) must match the number of elements in the vector  $\vec{x}$  (also  $n$ ).

The result of multiplying matrix  $A$  and vector  $\vec{x}$  will be a column vector with dimensions  $m \times 1$ , where  $m$  is the number of rows in the matrix  $A$

$$(m \times n) \cdot (n \times 1) = m \times 1$$

1. As Row vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{a}^T = [a_1, a_2, \dots, a_n]$$

$$\vec{b}^T = [b_1, b_2, \dots, b_n]$$

$$A = \begin{bmatrix} [a_1, a_2, \dots, a_n] \\ [b_1, b_2, \dots, b_n] \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}^T \\ \vec{b}^T \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} \vec{a} \cdot \vec{x} \\ \vec{b} \cdot \vec{x} \end{bmatrix}$$

## 2. As Column Vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A = \begin{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\vec{x} = x_1 \vec{a} + x_2 \vec{b}$$

## 12.2. Null Space

The null space (or kernel) of a matrix  $A$  is the set of all vectors  $x$  that satisfy the equation:

$$A\vec{x} = \mathbf{0}$$

Where:

- $A$ :  $m \times n$  matrix
- $\vec{x}$ :  $n$ -dimensional vector
- $\mathbf{0}$ : zero vector in  $\mathbb{R}^m$

$$N(A) = N(\text{rref}(A)) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

We want to find the null space of  $A$ , which consists of all vectors  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$  that satisfy:

$$A\vec{x} = \mathbf{0}$$

This expands to the following system of linear equations:

$$\begin{cases} 1x_1 + 1x_2 + 1x_3 + 1x_4 = 0 \\ 1x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 4x_1 + 3x_2 + 2x_3 + 1x_4 = 0 \end{cases}$$

This can be represented as the augmented matrix:

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array} \right]$$

### 12.2.1. Column Space

The **columns space** (or range) of matrix  $A$  is span of its columns vectors

If the matrix  $A$  has columns  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ , then the column space of  $A$  is defined as:

$$\text{Col}(A) = \{ \vec{y} \in \mathbb{R}^m \mid \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n \}$$

or equivalently,

$$\text{Col}(A) = \text{span}(\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\})$$

Consider the simple example of a  $2 \times 2$  matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

The matrix has two columns:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

The column space, denoted  $\text{Col}(A)$ , is the span of these two vectors:

$$\text{Col}(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix}\right\}\right)$$

### Finding the Column Space

We observe that the two columns  $\vec{a}_1$  and  $\vec{a}_2$  are **linearly dependent**:

$$\vec{a}_2 = k\vec{a}_1$$

This means that  $\vec{a}_2$  is a scalar multiple of  $\vec{a}_1$ , the the two columns are **linearly dependent**. As a result, the column space is spanned by just one vector,  $\vec{a}_1$ , because any linear combination of  $\vec{a}_1$  and  $\vec{a}_2$  can be reduced to a multiple of  $\vec{a}_1$ .

Therefore, the column space of  $A$  is:

$$\text{Col}(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}\right)$$

which represents all vectors of the form:

$$c \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c \\ 3c \end{bmatrix} \quad \text{for any scalar } c$$

In other words, the column space is a line in  $\mathbb{R}^2$  through the origin in the direction of

### Rank of $A$

The rank of  $A$ , which is the **dimension of its column space**, is 1 because there is only one linearly independent column

This means the column space is the span of the columns of  $A$ , or all vectors that can be formed by taking linear combinations of the columns of  $A$ .

#### 12.2.2. Dimension of a Subspace

Number of elements in a basis for the subspace

#### 12.2.3. Nullity

##### Dimension of the Null Space

$$\dim(N(A))$$

The nullity of  $A$ : number of non-pivot columns (i.e., free variables) in the rref of  $A$

#### 12.2.4. Rank

##### Dimension of the column space

$$\text{rank}(A) = \dim(C(A))$$

#### 12.2.5. Matrix Representation of Systems of Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m &= b_n \end{aligned}$$

Coefficient Matrix ( $A$ ):

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Variable Vector (x):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Constant Vector (b):

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

The system of equations:

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 100 \\ 4x_1 + 2x_2 + 1x_3 &= 80 \\ 1x_1 + 5x_2 + 2x_3 &= 60 \end{aligned}$$

Can be represented as a matrix equation:

$$\begin{bmatrix} 2 & 3 & 5 \\ 4 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 80 \\ 60 \end{bmatrix}$$

### 12.3. Matrix Multiplication

$m \times n$  matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

$n \times p$  matrix:

$$B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix}$$

Compute Each Element of Result Matrix  $C$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Let  $A$  be an  $n \times m$  matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Let  $B$  an  $p \times n$  matrix:

$$B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

Calculate Each Element of  $C$

$$\begin{aligned} c_{11} &= (1 \cdot 7) + (2 \cdot 9) + (3 \cdot 11) = 58 \\ c_{12} &= (1 \cdot 8) + (2 \cdot 10) + (3 \cdot 12) = 64 \\ c_{21} &= (4 \cdot 7) + (5 \cdot 9) + (6 \cdot 11) = 138 \\ c_{22} &= (4 \cdot 8) + (5 \cdot 10) + (6 \cdot 12) = 154 \end{aligned}$$

$C$  is a  $m \times p$  matrix

$$C = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

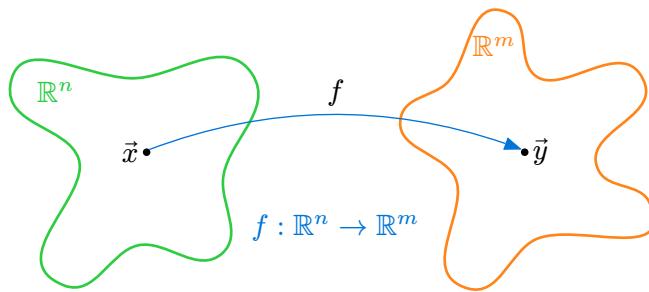
## 12.4. Linear Transformation

### 12.4.1. Functions

A function  $f$  that maps elements from a set  $X$  (the domain) to a set  $Y$  (the codomain):

$$f : X \rightarrow Y$$

- Domain: The set  $X$  contains all possible inputs for the function  $f$
- Codomain: The set  $Y$  is the space where all possible outputs of  $f$  reside, though not every element in  $Y$  must be an output of  $f$



If

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

is defined by

$$\begin{aligned} f(x) &= x^2 \\ f : x &\mapsto x^2 \end{aligned}$$

then:

- **Domain:**  $X = \mathbb{R}$ , any real number  $((\infty, \infty))$
- **Codomain:**  $Y = \mathbb{R}$ , any real number  $((\infty, \infty))$
- **Range:** the subset of the codomain  $(\mathbb{R})$ ,  $[0, \infty)$

## 12.5. Vector Transformation

A function  $f$  that maps an  $n$ -dimensional vector in  $\mathbb{R}^n$  to an  $m$ -dimensional vector in  $\mathbb{R}^m$ :

### 1. Function Definition

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

This means  $f$  takes as input a vector in  $\mathbb{R}^n$  (an  $n$ -dimensional space of real numbers) and maps it to a vector in  $\mathbb{R}^m$  (an  $m$ -dimensional space of real numbers).

### 2. Input Vector $\vec{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where } x_1, x_2, \dots, x_n \in \mathbb{R}$$

Here,  $\vec{x}$  is an  $n$ -dimensional vector, and each component  $x_i$  is a real number

### 3. Input Vector $\vec{y}$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad \text{where } y_1, y_2, \dots, y_m \in \mathbb{R}$$

The output  $\vec{y}$  is an  $m$ -dimensional vector, and each component  $y_i$  is also a real number

## Summary

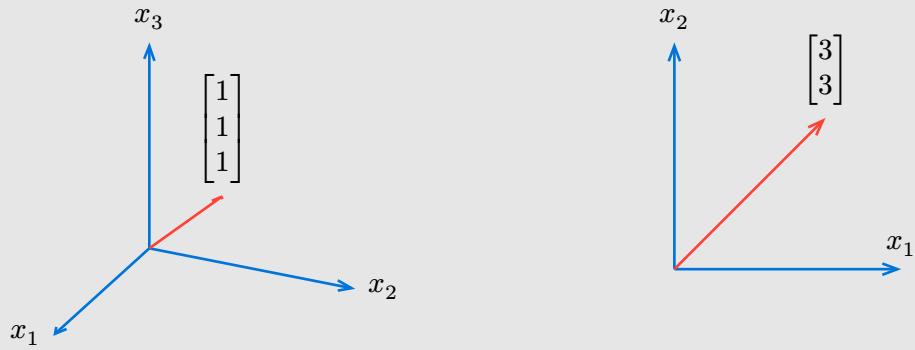
The function  $f$  takes an  $n$ -dimensional vector of real numbers as input and produces an  $m$ -dimensional vector of real numbers as output

$$f(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3)$$

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_3 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$



## 12.6. Linear Transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\vec{a}, \vec{b} \in \mathbb{R}^n$$

For a transformation **linear** it must satisfy two conditions:

1. Additivity (or linearity of addition)

$$T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$$

Let's consider a linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$

Now let's take two vectors in  $\mathbb{R}^2$ :

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Then the additivity property can be verified as follows:

1. First, find  $T(\vec{a}) + T(\vec{b})$  separately:

$$T(\vec{a}) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 1 \\ 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$T(\vec{b}) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 3 \\ 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$T(\vec{a}) + T(\vec{b}) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 + 6 \\ 6 + 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

2. Next, find  $T(\vec{a} + \vec{b})$ :

$$\vec{a} + \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$T(\vec{a} + \vec{b}) = T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 4 \\ 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

Since  $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$ , this confirms the additivity (linearity of addition) property of the transformation  $T$

2. Homogeneity (or linearity of scalar multiplication):

$$T(c\vec{a}) = cT(\vec{a})$$

Let's consider a linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$

Now let's take one vectors in  $\mathbb{R}^2$ :

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and a scalar  $c = 3$

Then the homogeneity property can be verified as follows:

1. First, find  $cT(\vec{a})$

$$T(\vec{a}) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 1 \\ 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$cT(\vec{a}) = 3 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

2. Then, find  $T(c\vec{a})$  by  $c$  to get  $c\vec{a}$

$$c\vec{a} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 \\ 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$T(c\vec{a}) = T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 3 \\ 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

Since  $T(c\vec{a}) = cT(\vec{a})$ , this confirms the homogeneity (linearity of scalar multiplication) property of the transformation  $T$

## 12.7. Matrix Vector Products

Matrix product with vector is always a linear transformation

$$\begin{aligned} T : \mathbb{R}^n &\rightarrow \mathbb{R}^m \\ T(\vec{x}) &= A\vec{x} \end{aligned}$$

$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

1. Additivity (or linearity of addition)

$$T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$$

$$\begin{aligned} A \cdot (\vec{a} + \vec{b}) &= A \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \dots + (a_n + b_n)v_n \\ &= a_1 v_1 + b_1 v_1 + a_2 v_2 + b_2 v_2 + \dots + a_n v_n + b_n v_n \\ &= (a_1 v_1 + a_2 v_2 + \dots + a_n v_n) + (b_1 v_1 + b_2 v_2 + \dots + b_n v_n) \\ &= A \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

1. Calculate  $A(\mathbf{u} + \mathbf{v})$

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} (2 \cdot 1) + (1 \cdot 6) \\ (0 \cdot 4) + (3 \cdot 6) \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 14 \\ 18 \end{bmatrix}} \end{aligned}$$

2. Calculate  $A\mathbf{u} + A\mathbf{v}$

$$\begin{aligned}
 A\mathbf{u} &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} (2 \cdot 1) + (1 \cdot 2) \\ (0 \cdot 1) + (3 \cdot 2) \end{bmatrix} \\
 &= \begin{bmatrix} 4 \\ 6 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 A\mathbf{v} &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
 &= \begin{bmatrix} (2 \cdot 3) + (1 \cdot 4) \\ (0 \cdot 3) + (3 \cdot 4) \end{bmatrix} \\
 &= \begin{bmatrix} 10 \\ 12 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 A\mathbf{u} + A\mathbf{v} &= \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 10 \\ 12 \end{bmatrix} \\
 &= \boxed{\begin{bmatrix} 14 \\ 18 \end{bmatrix}}
 \end{aligned}$$

2. Homogeneity (or linearity of scalar multiplication):

$$T(c\vec{a}) = cT(\vec{a})$$

$$\begin{aligned}
 A \cdot (c\vec{a}) &= \begin{bmatrix} & & & \\ v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} \\
 &= ca_1 v_1 + ca_2 v_2 + \dots + ca_n v_n \\
 &= \underbrace{c(a_1 v_1 + a_2 v_2 + \dots + a_n v_n)}_{A\vec{a}}
 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad c = 5$$

1. Calculate  $A(c\mathbf{v})$

$$cv = 5 \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$$

$$\begin{aligned}
 A(cv) &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 15 \\ 20 \end{bmatrix} \\
 &= \begin{bmatrix} (2 \cdot 15) + (1 \cdot 20) \\ (0 \cdot 15) + (3 \cdot 20) \end{bmatrix} \\
 &= \boxed{\begin{bmatrix} 50 \\ 60 \end{bmatrix}}
 \end{aligned}$$

2. Calculate  $c(A\mathbf{v})$

$$\begin{aligned} A\mathbf{v} &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} (2 \cdot 3) + (1 \cdot 4) \\ (0 \cdot 3) + (3 \cdot 4) \end{bmatrix} \\ &= \begin{bmatrix} 10 \\ 12 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} c(A\mathbf{v}) &= 5 \cdot \begin{bmatrix} 10 \\ 12 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 50 \\ 60 \end{bmatrix}} \end{aligned}$$

## 12.8. Linear transformations as matrix vector products

The  $n \times n$  matrix  $I_n$ :

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$I_n \vec{x} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

### Standard Basis

$$I_n \vec{x} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{bmatrix}$$

$\{e_1, e_2, \dots, e_n\}$  is the standard basis for  $\mathbb{R}^n$

$$\begin{aligned}
I_n \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} &= a_1 \vec{e}_1 + a_2 \vec{e}_2 + \dots + a_n \vec{e}_n \\
&= a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + a_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_n \end{bmatrix}
\end{aligned}$$

### 12.9. Image of a subset under transformation

$$\vec{x}_0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \quad \vec{x}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\begin{aligned}
L_0 &= \{\vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) \mid 0 \leq t \leq 1\} \\
L_1 &= \{\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1) \mid 0 \leq t \leq 1\} \\
L_2 &= \{\vec{x}_2 + t(\vec{x}_0 - \vec{x}_2) \mid 0 \leq t \leq 1\}
\end{aligned}$$

The triangle  $T$  can be defined as the set of these points:

$$S = \{L_0, L_1, L_2\}$$

Let's define a transformation

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

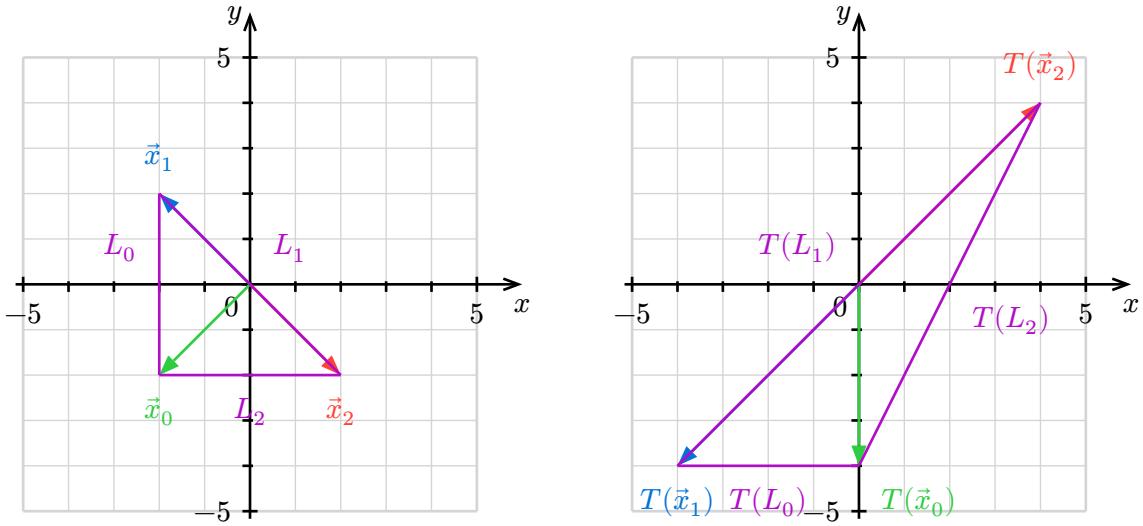
$$T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned}
T(L_0) &= \{T(\vec{x}_0 + t(\vec{x}_1 - \vec{x}_0)) \mid 0 \leq t \leq 1\} \\
&= \{T(\vec{x}_0) + T(t(\vec{x}_1 - \vec{x}_0)) \mid 0 \leq t \leq 1\} \\
&= \{T(\vec{x}_0) + tT(\vec{x}_1 - \vec{x}_0) \mid 0 \leq t \leq 1\} \\
&= \{T(\vec{x}_0) + tT(\vec{x}_1) - T(\vec{x}_0) \mid 0 \leq t \leq 1\}
\end{aligned}$$

$$T(\vec{x}_0) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$T(\vec{x}_1) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$T(\vec{x}_2) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$



$T(L_0)$  is the image of  $L_0$  under  $T$

$T(S)$  is the image of  $S$  under  $T$

## 12.10. Image of a transformation

The **image** of a transformation  $T$  is defined as:

$$\text{im}(T) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$$

or, equivalently,

$$T(\mathbb{R}^n)$$

$T(\mathbb{R}^n)$  is the image of  $\mathbb{R}^2$  under  $T$

This is the set of all possible outputs when  $T$  is applied to vectors in  $\mathbb{R}^n$

### Understanding $T(\mathbb{R}^n)$

#### 1. Whole space transformation

The image of  $\mathbb{R}^n$  under  $T$  is the complete set of transformed vectors, often denoted as  $\text{im}(T)$

#### 2. Subset Transformation

For any subset  $V \subseteq \mathbb{R}^n$ , the image of  $V$  under  $T$  is the set of transformed vectors from  $V$

### Matrix representation of $T$

If  $T$  is represented by a  $m \times n$  matrix  $A$ , then:

$$T(\vec{x}) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

where:

- $A$  is the matrix associated with  $T$
- $\vec{x} \in \mathbb{R}^n$  represents a vector in the input space

### Transformation in terms of columns of $A$

If  $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ , then for  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ :

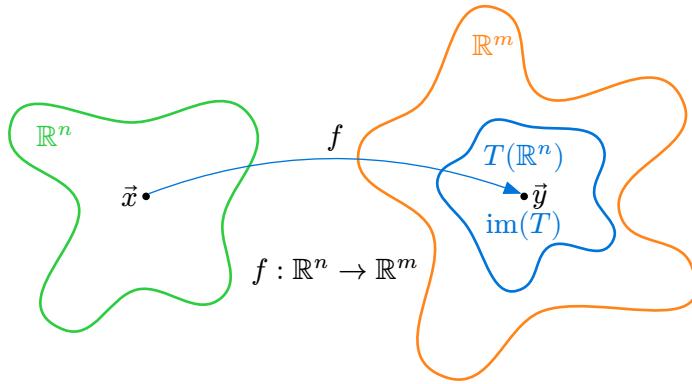
$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$$

## Column Space of $A$

The image of  $T$  (or  $\text{im}(T)$ ) is the column space of  $A$ :

$$C(A) = \text{span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$$

This is the set of all possible linear combinations of the columns of  $A$ , and thus represents all possible outputs of the transformation  $T$



Suppose we have a matrix  $A$ :

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Matrix  $A$  defines the transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that for any vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ , the image under  $T$  is:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

## Calculating Transformation

To see what  $T$  does to vectors in  $\mathbb{R}^2$ , let's compute a few specific examples:

1. for  $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. for  $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

## Image of $T$

The image of  $T$ ,  $\text{im}(T)$ , is the set of all linear combinations of the vectors  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ :

$$\text{im}(T) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

Thus, any vector in the image of  $T$  can be written as:

$$y = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

where  $x_1, x_2 \in \mathbb{R}$ .

### Column space interpretation

The image of  $T$  is all the vectors in  $\mathbb{R}^2$  that can be formed as linear combinations of  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .

## 12.11. Preimage of a set

The preimage of a set  $S$  under a function  $T$ , denoted  $T^{-1}(S)$ , is the set of all elements in the domain of  $T$  the domain of  $T$  that map to elements in  $S$  under the transformation  $T$ .

If  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a function from a set  $\mathbb{R}^n$  to a set  $\mathbb{R}^m$ , and  $S \subseteq \mathbb{R}^m$  is a subset of a target space, the the preimage of  $S$  under  $T$  is:

$$T(-1)(S) = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) \in S\}$$

This means that  $T(-1)(S)$  consists of all elements in  $\mathbb{R}^n$  that, when transformed by  $T$ , end up in  $S$ .

For any subset  $S \subseteq \mathbb{R}^m$ , the preimage  $T^{-1}(S)$  collects all points in the domain that end up in  $S$  after applying  $T$ . If  $S$  is a single point, the preimage will be the set of all points in the domain that map to that specific point (this could be empty, a single point, or even a set of points, depending on the function).

Consider the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

This transformation  $T$  maps any vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  in  $\mathbb{R}^2$  to:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_2 \end{bmatrix}$$

Now, let's find the preimage of a subset  $S \subseteq \mathbb{R}^2$ . Suppose we want the preimage of the set  $S = \left\{ \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

The primeage of  $S$  under  $T$ , denoted  $T^{-1}(S)$ , consists of all vectors  $\vec{x} \in \mathbb{R}^2$  such that  $T(\vec{x}) \in \left\{ \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ . To find this, we solve for  $\vec{x}$  in both cases:  $T(\vec{x}) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$  and  $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

**Preimage of  $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$**

$$A\vec{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Solving each component:

1.  $2x_1 = 4 \Rightarrow x_1 = 2$
2.  $3x_2 = 6 \Rightarrow x_2 = 2$

Thus, the preimage of  $S$  is the single point:

$$T^{-1}\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$$

**Preimage of  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$**

$$A\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solving each component:

1.  $2x_1 = 2 \Rightarrow x_1 = 1$
2.  $3x_2 = 3 \Rightarrow x_2 = 1$

Thus, the preimage of  $S$  is the single point:

$$T^{-1}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

**Preimage of the set  $S$**

Since  $S = \left\{ \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ , the preimage of  $S$  is the union of the preimage of each vector in  $S$ :

$$T^{-1}(S) = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

### 12.11.1. Kernel of a Transformation

The kernel of a transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , denoted  $\ker(T)$ , is the set of all vectors in  $\mathbb{R}^n$  that  $T$  maps to the zero vector in  $\mathbb{R}^m$ . Formally, we define the kernel as:

$$\ker(T) = \left\{ \vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0} \right\}$$

The kernel consists of all vectors that are “annihilated” by  $T$ , resulting in the zero vector after applying  $T$ .

The transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

so that  $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2x_1 \\ 3x_2 \end{bmatrix}$  for any  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

To find the kernel of  $T$ , we need to find all the vectors  $\vec{x} \in \mathbb{R}^2$  that satisfy:

$$T(\vec{x}) = \vec{0} \Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This leads to the system of equations:

1.  $3x_1 = 0 \rightarrow x_1 = 0$

2.  $3x_2 = 0$  arrow.double  $x_2 = 0$

Thus, the only solution is  $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

### 12.11.2. Kernel and Null Space

The kernel of  $T$

$$\ker(T) = \text{Null}(A)$$

## 12.12. Sum and Scalar Multiples of Linear Transformation

### 12.12.1. Sum

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad S : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$(T + S) : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \quad B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \end{bmatrix}$$

$$\begin{aligned} (T + S)(\vec{x}) &= T(\vec{x}) + S(\vec{x}) \\ &= A\vec{x} + B\vec{x} \\ &= x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n + x_1\vec{b}_1 + x_2\vec{b}_2 + \dots + x_n\vec{b}_n \\ &= x_1(\vec{a}_1 + \vec{b}_1) + x_2(\vec{a}_2 + \vec{b}_2) + \dots + x_n(\vec{a}_n + \vec{b}_n) \\ &= \begin{bmatrix} \vec{a}_1 + \vec{b}_1 & \vec{a}_2 + \vec{b}_2 & \dots & \vec{a}_n + \vec{b}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= (A + B)\vec{x} \end{aligned}$$

### 12.12.2. Scalar Multiplication

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$cT : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\begin{aligned} (cT)(\vec{x}) &= c(T(\vec{x})) \\ &= c(x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n) \\ &= x_1c\vec{a}_1 + x_2c\vec{a}_2 + \dots + x_n c\vec{a}_n \\ &= \begin{bmatrix} c\vec{a}_1 & c\vec{a}_2 & \dots & c\vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= cA\vec{x} \end{aligned}$$

### 12.13. Projection

The projection of a vector  $\vec{x}$ , onto a line  $L$ , denoted as  $\text{Proj}_L(\vec{x})$ , is a vector that lies on the line  $L$ , such that the difference between  $\vec{x}$  and its projection,  $\text{Proj}_L(\vec{x}) - \vec{x}$ , is orthogonal to  $L$

$\text{Proj}_L(\vec{x})$  can be seen as the “shadow” cast by  $\vec{x}$  onto  $L$  when light shines perpendicularly to  $L$ .

$$\text{Proj}_L(\vec{x}) = c\vec{v} = \left( \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$$

Equivalently

$$\text{Proj}_L(\vec{x}) = c\vec{v} = \left( \frac{\vec{x} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$$

Where:

- $\vec{v}$  is a direction vector for the line  $L$
- $c = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$  is a scalar

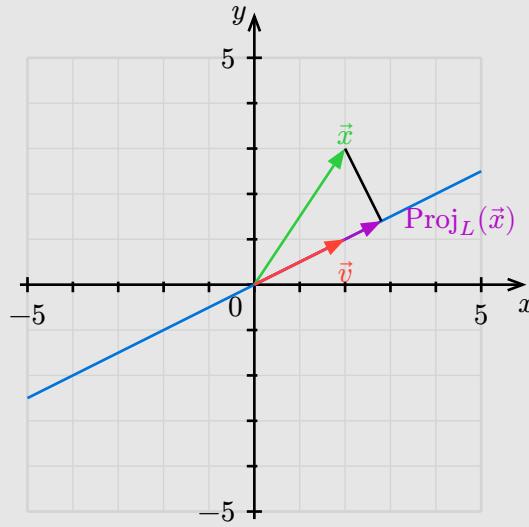
$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

1. Define the line  $L$  as all vectors of the form  $c\vec{v}$ , where  $c$  is a scalar:

$$\begin{aligned} L &= \{c\vec{v} \mid c \in \mathbb{R}\} \\ &= \left\{ c \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid c \in \mathbb{R} \right\} \end{aligned}$$

2. Compute the projection using

$$\begin{aligned} \text{Proj}_L(\vec{x}) &= \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \\ &= \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{7}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2.8 \\ 1.4 \end{bmatrix} \end{aligned}$$



## Projection as a Transformation

As a matrix vector product

$$L = \{c\vec{v} \mid c \in \mathbb{R}\}$$

$$\text{Proj}_L : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

If  $\vec{v}$  is a unit vector:

$$\|\vec{v}\| = 1$$

Then

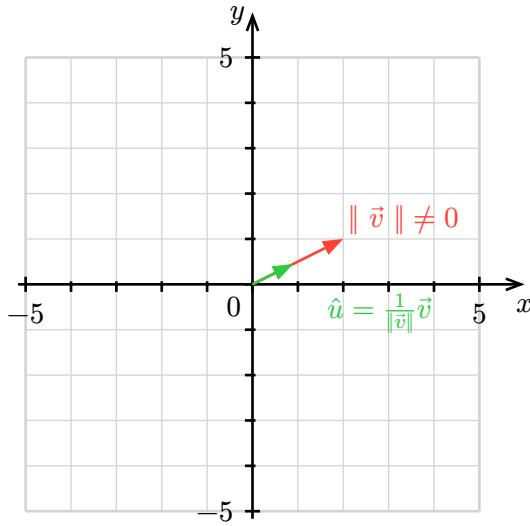
$$\begin{aligned} \text{Proj}_L(\vec{x}) &= \left( \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \\ &= \left( \frac{\vec{x} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} \end{aligned}$$

If we redefine our line  $L$  as all the scalar multiples of our unit vector  $\hat{u}$ :

$$L = \{c\hat{u} \mid c \in \mathbb{R}\}$$

Simplifies to:

$$\boxed{(\vec{x} \cdot \hat{u})\hat{u}}$$



### Projection as a Linear Transformation

Let  $\hat{u} \in \mathbb{R}^n$  be a unit vector:

$$\hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \text{where } \|\hat{u}\| = 1$$

The projection matrix  $A$  is:

$$A = \hat{u}\hat{u}^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} [u_1 \quad u_2 \quad \dots \quad u_n]$$

Expands to:

$$A = \begin{bmatrix} u_1u_1 & u_1u_2 & \dots & u_1u_n \\ u_2u_1 & u_2u_2 & \dots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \dots & u_nu_n \end{bmatrix}$$

For any  $\vec{x} \in \mathbb{R}^n$ , the projection of  $\vec{x}$  onto the line spanned by  $\hat{u}$  is:

$$\text{Proj}_L(\vec{x}) = A\vec{x} = (\hat{u} \cdot \vec{x})\hat{u}$$

Consider a vector  $\vec{v}$  in  $\mathbb{R}^2$ :

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

#### 1. Construct the Unit Vector

$$\|\vec{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

This unit vector  $\hat{u}$  defines the line  $L$ , which consists of all the scalar multiples of  $\vec{v}$ :

$$L = \{c\vec{v} \mid c \in \mathbb{R}\}$$

## 2. Derive the Projection Matrix

The projection of any vector  $\vec{x}$  onto the line  $L$  is given by:

$$\text{Proj}_L(\vec{x}) = A\vec{x}$$

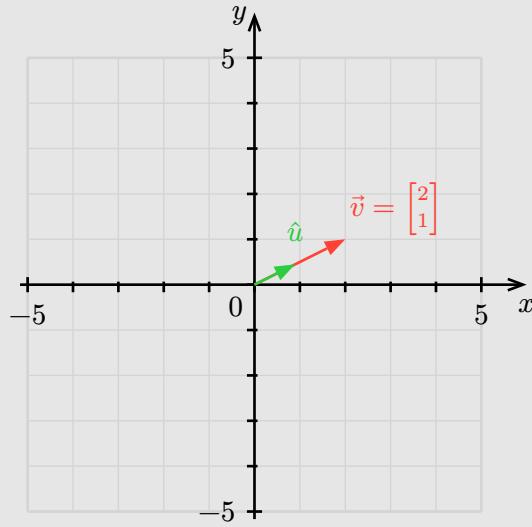
Where  $A$  is the projection matrix. To construct  $A$  we use the formula:

$$\begin{aligned} A &= \hat{u}\hat{u}^T \\ &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 \\ u_2 \cdot u_1 & u_2 \cdot u_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{2}{\sqrt{5}}\right)^2 & \frac{1}{\sqrt{5}} \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \frac{1}{\sqrt{5}} & \left(\frac{1}{\sqrt{5}}\right)^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \end{aligned}$$

## 3. Applying the Projection

To project any vector  $\vec{x}$  onto  $L$ , we multiply  $\vec{x}$  by the matrix  $A$ :

$$\begin{aligned} \text{Proj}_L(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \vec{x} \end{aligned}$$



1. Additivity of Projections (Linearity with respect to addition)

$$\begin{aligned}
 \text{Proj}_L(\vec{a} + \vec{b}) &= ((\vec{a} + \vec{b}) \cdot \hat{u}) \hat{u} \\
 &= (\vec{a} \cdot \hat{u} + \vec{b} \cdot \hat{u}) \hat{u} \\
 &= (\vec{a} \cdot \hat{u}) \hat{u} + (\vec{b} \cdot \hat{u}) \hat{u} \\
 &= \text{Proj}_L(\vec{a}) + \text{Proj}_L(\vec{b})
 \end{aligned}$$

2. Homogeneity of Projections (Linearity with respect to scalar multiplication)

$$\begin{aligned}
 \text{Proj}_L(c\vec{a}) &= (c\vec{a} \cdot \hat{u}) \hat{u} \\
 &= c(\vec{a} \cdot \hat{u}) \hat{u} \\
 &= c\text{Proj}_L(\vec{a})
 \end{aligned}$$

### General Properties of $A$

1. Idempotence

$$A^2 = A$$

2. Symmetry

$$A^T = A$$

3. Rank

$$\text{rank}(A) = 1$$

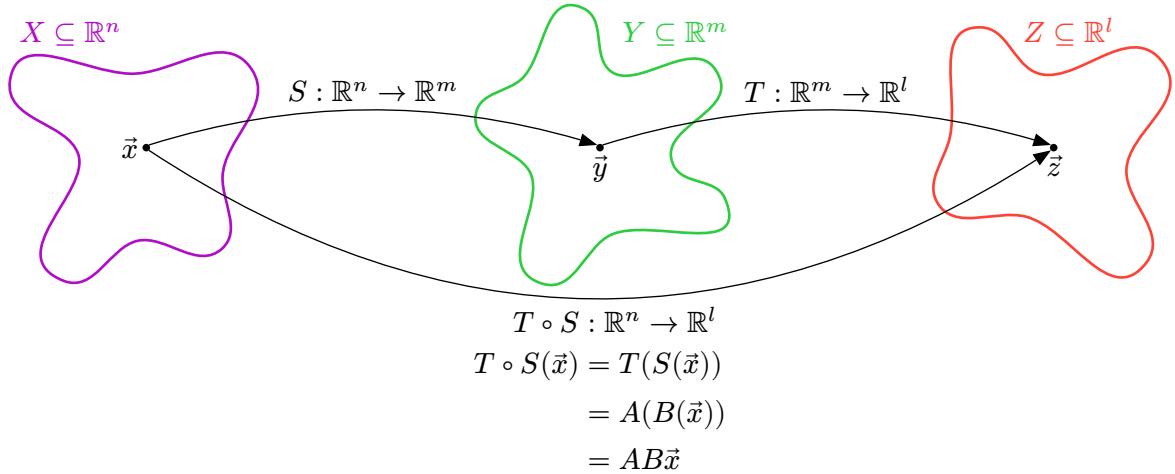
Because  $\hat{u}\hat{u}^T$  projects onto a one-dimensional subspace spanned by  $\hat{u}$

### 12.14. Composition of Linear Transformations

$$S : \textcolor{violet}{X} \rightarrow \textcolor{green}{Y} \quad T : \textcolor{green}{Y} \rightarrow \textcolor{red}{Z}$$

$$T \circ S : \textcolor{violet}{X} \rightarrow \textcolor{red}{Z}$$

$$S(\vec{x}) = \underbrace{A}_{m \times n} \vec{x} \quad T(\vec{x}) = \underbrace{B}_{l \times m} \vec{x}$$



Consider two linear transformations  $T$  and  $S$ , where:

- $T$  maps  $\mathbb{R}^m \rightarrow \mathbb{R}^l$
- $S$  maps  $\mathbb{R}^n \rightarrow \mathbb{R}^m$

The composition  $T \circ S$  is a linear transformation mapping  $\mathbb{R}^n \rightarrow \mathbb{R}^l$  defined by:

$$T \circ S(\vec{x}) = T(S(\vec{x}))$$

### Key Properties of $T \circ S$

#### 1. Additivity

$$\begin{aligned}
 T \circ S(\vec{x} + \vec{y}) &= T(S(\vec{x} + \vec{y})) \\
 &= T(S(\vec{x}) + S(\vec{y})) \\
 &= T(S(\vec{x})) + T(S(\vec{y})) \\
 &= T \circ S(\vec{x}) + T \circ S(\vec{y})
 \end{aligned}$$

#### 2. Homogeneity

$$\begin{aligned}
 T \circ S(c\vec{x}) &= T(S(c\vec{x})) \\
 &= T(cS(\vec{x})) \\
 &= cT(S(\vec{x})) \\
 &= c(T \circ S)(\vec{x})
 \end{aligned}$$

### Matrix Representation of $T \circ S$

Let  $S$  be represented by the matrix  $A$  ( $m \times n$ ), and let  $T$  be represented by the matrix  $B$  ( $l \times m$ )

For a vector  $\vec{x} \in \mathbb{R}^n$

$$\begin{aligned}
 T \circ S(\vec{x}) &= T(S(\vec{x})) = T(A\vec{x}) \\
 &= \underbrace{B}_{l \times m} \left( \underbrace{A}_{m \times n} \vec{x} \right) \\
 &= \underbrace{C}_{l \times n} \vec{x}
 \end{aligned}$$

The composition  $T \circ S$  is therefore represented by the matrix  $C = A \cdot B$ , where  $C$  is of size  $l \times n$

### Column-Wise Interpretation

The matrix  $A$  can be decomposed column-wise:

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

where  $\vec{a}_i$  is the  $i$ -th column of  $A$  and  $I_n$  is the identity matrix in  $\mathbb{R}^n$ , and its columns are the standard basis vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ .

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \overbrace{e_1}^1 & \overbrace{e_2}^2 & & \overbrace{e_n}^n \end{bmatrix}$$

To compute  $C$ :

1. For each  $\vec{e}_i$  in the basis of  $\mathbb{R}^n$ ,  $A\vec{e}_i = \vec{a}_i$ , the  $i$ -th column of  $A$

$$\begin{aligned} C &= \begin{bmatrix} B(Ae_1) & B(Ae_2) & \dots & B(Ae_n) \end{bmatrix} \\ &= \begin{bmatrix} B\left(A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) & B\left(A \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right) & \dots & B\left(A \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) \end{bmatrix} \\ &= \begin{bmatrix} B\vec{a}_1 & B\vec{a}_2 & \dots & B\vec{a}_n \end{bmatrix} \end{aligned}$$

The composition  $T \circ S$  is the linear map represented by  $C = B \cdot A$

Each column of  $C$  reflects how  $T$  transforms the action of  $S$  on a standard basis vector

### 12.15. Matrix Product

$$\underbrace{A}_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \underbrace{B}_{n \times p} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

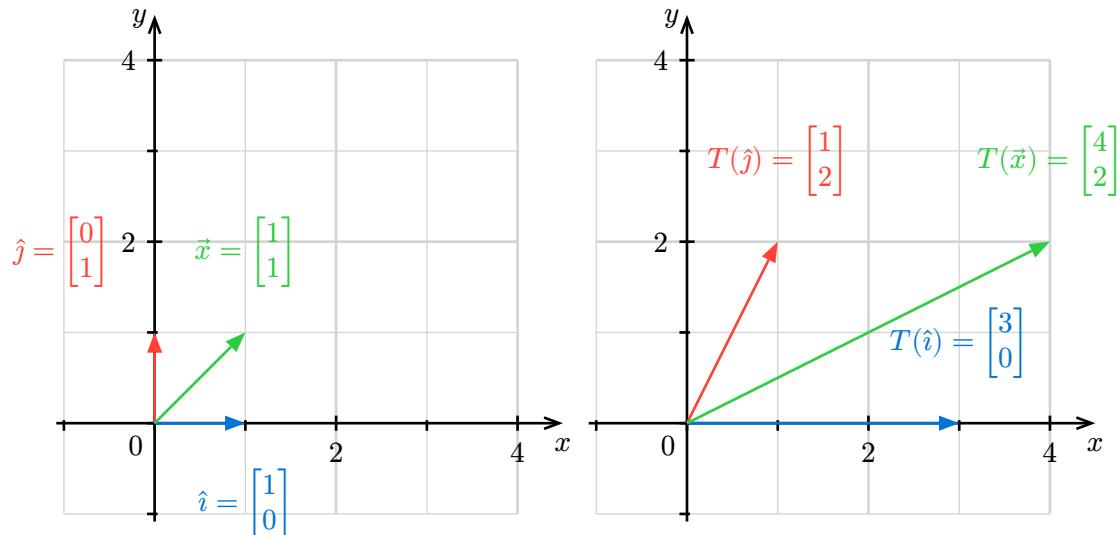
$$\underbrace{A}_{2 \times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \underbrace{B}_{3 \times 2} = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

$$\begin{aligned}
 AB &= \left[ A \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} \quad A \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \right] \\
 &= \left[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \right] \\
 &= \left[ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \right]
 \end{aligned}$$

## 12.16. Matrix Product Associativity

## 12.17. Eigenvectors

### 12.17.1. Transformation



$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{aligned}
 T(\hat{i}) &= A\hat{i} \\
 &= \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 3 \\ 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 T(\hat{j}) &= A\hat{j} \\
 &= \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
T(\vec{x}) &= A\vec{x} \\
&= \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 2 \end{bmatrix}
\end{aligned}$$

## 12.18. Eigenvalues

### 12.18.1. LU Decomposition

Given a matrix  $A$ , LU decomposition aims to express  $A$  as:

$$A = LU$$

Where:

- $L$ : Lower triangular matrix (all elements above the diagonal are zero)
- $U$ : Upper triangular matrix (all elements below the diagonal are zero)

1. Solve  $Ly = b$  Using Forward Substitution

$$Ly = b$$

Where:

- $L$ : Lower triangular matrix (all elements above the diagonal are zero)
- $y$ : Intermediate vector we are solving for
- $b$ : Right-hand side vector

$$\begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- First row:  $L_{11}y_1 = b_1$ , so  $y_1 = \frac{b_1}{L_{11}}$
- Second row:  $L_{21}y_1 + L_{22}y_2 = b_2$ , substitute  $y_1$  into this equation and solve for  $y_2$ :

$$y_2 = \frac{b_2 - L_{21}y_1}{L_{22}}$$

- Third row:  $L_{31}y_1 + L_{32}y_2 + L_{33}y_3 = b_3$ , substitute  $y_1$  and  $y_2$  into this equation, solve for  $y_3$ :

$$y_3 = \frac{b_3 - L_{31}y_1 - L_{32}y_2}{L_{33}}$$

2. Solve  $Ux = y$  Using Backward Substitution

$$Ux = y$$

Where:

- $U$ : Upper triangular matrix (all elements below the diagonal are zero)
- $y$ : Vector of unknowns (solution)
- $b$ : Vector computed from the forward substitution step

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- Third row:  $U_{33}x_3 = y_3$ , so  $x_3 = \frac{y_3}{U_{33}}$
- Second row:  $U_{22}x_2 + U_{23}x_3 = y_2$ , substitute  $x_3$  from the previous step and solve for  $x_2$ :

$$x_2 = \frac{y_2 - U_{23}x_3}{U_{22}}$$

- First row:  $U_{11}x_1 + U_{12}x_2 + U_{13}x_3 = y_1$ , substitute  $x_2$  and  $x_3$  from the previous step and solve for  $x_1$ :

$$x_1 = \frac{y_1 - U_{12}x_2 - U_{13}x_3}{U_{11}}$$

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 6 & 18 & 5 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 12 \\ 31 \end{pmatrix}$$

1. Factor  $A$  into  $L$  and  $U$ :

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

2. Solve  $Ax = b$

- $y_1 = \frac{b_1}{L_{11}} = \frac{5}{1} = 5$
- $y_2 = \frac{b_2 - L_{21}y_1}{L_{22}} = \frac{12 - 2 \times 5}{1} = 2$
- $y_3 = \frac{b_3 - L_{31}y_1 - L_{32}y_2}{L_{33}} = \frac{31 - 4 \times 5 - 3 \times 2}{1} = 5$

So,

$$y = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

3. Solve  $Ux = y$

- $x_3 = \frac{y_3}{U_{33}} = \frac{5}{1} = 5$
- $x_2 = \frac{y_2 - U_{23}x_3}{U_{22}} = \frac{2 - 1 \times 5}{1} = -3$
- $x_1 = \frac{y_1 - U_{12}x_2 - U_{13}x_3}{U_{11}} = \frac{5 - 3 \times 5 - 1 \times 5}{1} = -5$

So,

$$x = \begin{pmatrix} -5 \\ -3 \\ 5 \end{pmatrix}$$

## 12.19. Solving Systems of Linear Equations

Linear Equation

$$y = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

1. Consistency

Whether a system of linear equations has at least one solution

### **Consistent System**

$$\begin{aligned}x + y &= 3 \\x - y &= 1\end{aligned}$$

This system has a unique solution

$$(x, y) = (2, 1)$$

### **Inconsistent System**

$$\begin{aligned}x + y &= 3 \\x + y &= 5\end{aligned}$$

This system is inconsistent (equations contradict each other, no solution can satisfy both)

## 2. Independence

Whether the equations in the system provide unique and non-redundant information about the variables

### **Independent Equations**

$$\begin{aligned}x + y &= 3 \\x - y &= 1\end{aligned}$$

Neither equation can be derived from the other (they provide unique information and intersect at a single point)

### **Dependent Equations**

$$\begin{aligned}x + y &= 3 \\2x + 2y &= 6\end{aligned}$$

Second equation is just a multiple of the first equation (they describe the same line)

## 3. Recognizing Systems with No Solution or Infinite Solutions

$$\begin{aligned}3x + 2y &= 6 & \text{(Equation 1)} \\6x + 4y &= 12 & \text{(Equation 2)}\end{aligned}$$

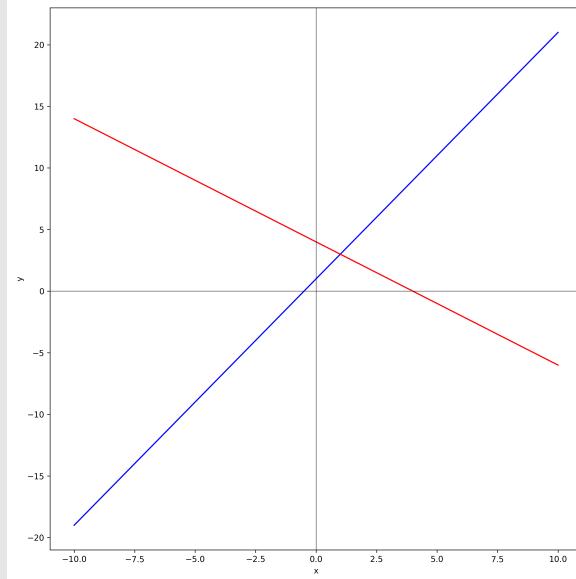


Figure 1: Unique Solution (Consistent and Independent)

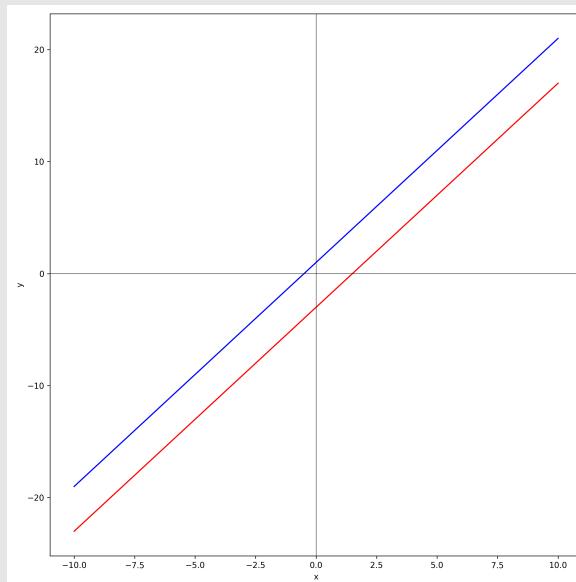


Figure 2: No Solution (Inconsistent)

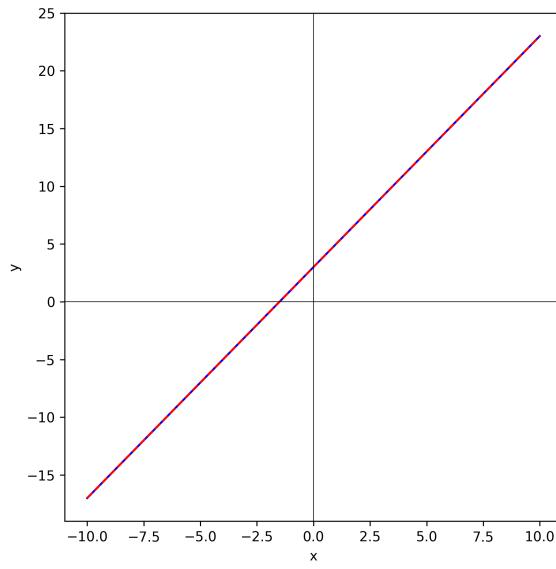


Figure 3: Infinitely Many Solutions (Consistent and Dependent)

## 2. Matrix Representation

System of Equations

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
 \end{aligned}$$

Matrix Representation

Coefficient vector (A)

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Variable vector (x)

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Constant vector (b)

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Matrix equation

$$Ax = b$$

```

from scipy.linalg import solve

X = np.array([
    [1, 1, 1],
    [2, -1, 3],
    [3, 4, -1]
])

Y = np.array([6, 14, 1])

intersection_point = solve(X, Y)

```

### 12.19.1. Gaussian Elimination

Convert a matrix into its row echelon form (REF) or reduced row echelon form (RREF)

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
 \end{aligned}$$

1. Create an augmented matrix

$$A = \left( \begin{array}{cccc|c}
 a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
 a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{2m1} & a_{m2} & \dots & a_{mn} & b_m
 \end{array} \right)$$

2. Forward Elimination

Eliminate the element in the  $i$ -th of the  $k$ -th column ( $k > i$ )

$$R_k \leftarrow R_k - \frac{a_{ki}}{a_{ii}} R_i$$

Where

- $a_{ii}$ : Pivot element
- $R_k$ :  $k$ -th row
- $R_i$ :  $i$ -th row

3. Back Substitution

4. Reduced Row Echelon Form (RREF)

$$\begin{aligned}
 2x_1 + 3x_2 &= 5 \\
 4x_1 + 5x_2 &= 5
 \end{aligned}$$

1. Create an augmented matrix

$$\left( \begin{array}{cc|c}
 2 & 3 & 5 \\
 4 & 5 & 6
 \end{array} \right)$$

2. Forward Elimination

$$R_k \leftarrow R_k - \frac{a_{ki}}{a_{ii}} R_i$$

$$R_k \leftarrow R_k - \frac{a_{21}}{a_{11}} R_i$$

$$R_2 \leftarrow R_2 - \frac{4}{2} R_1$$

$$R_2 \leftarrow R_2 - 2 \times R_1$$

$$\left( \begin{array}{cc|c} 2 & 3 & 5 \\ 4 - 2 \times 2 & 5 - 2 \times 3 & 6 - 2 \times 5 \end{array} \right)$$

Simplifies to:

$$\left( \begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -1 & -4 \end{array} \right)$$

System is now:

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ -1x_2 &= -4 \end{aligned}$$

### 3. Back Substitution

$$\begin{aligned} -1x_2 &= -4 \\ x_2 &= 4 \end{aligned}$$

Substitute:

$$\begin{aligned} 2x_1 + 3(4) &= 5 \\ 2x_1 + 12 &= 5 \\ x_1 &= -3.5 \end{aligned}$$

Solution:

$$\begin{aligned} x_1 &= -3.5 \\ x_2 &= 4 \end{aligned}$$

## 12.19.2. Substitution

$$x + y = 10 \quad (\text{Equation 1})$$

$$2x - y = 5 \quad (\text{Equation 2})$$

1. Solve Equation 1 for  $y$

$$y = 10 - x$$

2. Substitute into Equation 2

$$2x - (10 - x) = 5$$

3. Solve for x:

$$\begin{aligned}2x - 10 + x &= 5 \\3x - 10 &= 5 \\3x &= 15 \\x &= 5\end{aligned}$$

4. Find  $y$  using value of  $x$

$$\begin{aligned}y &= 10 - x \\y &= 10 - 5 \\y &= 5\end{aligned}$$

### 12.19.3. Addition or Subtraction Method

$$\begin{aligned}3x + 2y &= 12 && \text{(Equation 1)} \\2x - 2y &= 4 && \text{(Equation 2)}\end{aligned}$$

1. Add the equations

$$\begin{aligned}(3x + 2y) + (2x - 2y) &= 12 + 4 \\5x &= 16 \\x &= \frac{16}{5} \\x &= 3.2\end{aligned}$$

2. Substitute

$$\begin{aligned}3(3.2) + 2y &= 12 \\9.6 + 2y &= 12 \\2y = 12 - 9.6y &= \frac{2.4}{2} \\y &= 1.2\end{aligned}$$

# Calculus I

## 13. Cheatsheet

### 13.1. Limits

Epsilon-Delta

For every distance  $\varepsilon$  around  $L$ , there's a  $\delta$ -range around  $a$  that keeps  $f(x)$  within  $\varepsilon$  of  $L$ .

$$\lim_{x \rightarrow x_0} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

The limit of  $f(x)$  as  $x$  approaches  $x_0$  equals  $L$  if and only if, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that, whenever  $0 < |x - x_0| < \delta$ , implies that  $|f(x) - L| < \varepsilon$

---

Limit Type	Name	Quantifiers
$x \rightarrow x_0, f(x) \rightarrow L$	Epsilon-Delta	$\forall \varepsilon, \exists \delta$
$x \rightarrow x_0, f(x) \rightarrow \infty$	M-Delta	$\forall M, \exists \delta$
$x \rightarrow \infty, f(x) \rightarrow L$	epsilon-N	$\forall \varepsilon, \exists N$
$x \rightarrow \infty, f(x) \rightarrow \infty$	M-N	$\forall M, \exists N$

Finite  $\rightarrow$  Finite (**Epsilon-Delta**)

$$\lim_{x \rightarrow x_0} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Finite  $\rightarrow$  Infinity (**M-Delta**)

$+\infty$

$$\lim_{x \rightarrow x_0} f(x) = +\infty \iff \forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \Rightarrow f(x) > M$$

$-\infty$

$$\lim_{x \rightarrow x_0} f(x) = -\infty \iff \forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \Rightarrow f(x) < -M$$

Infinity  $\rightarrow$  Finite (**Epsilon-N**)

$+\infty$

$$\lim_{x \rightarrow +\infty} f(x) = L \iff \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } x > N \Rightarrow |f(x) - L| < \varepsilon$$

$-\infty$

$$\lim_{x \rightarrow +\infty} f(x) = L \iff \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } x < -N \Rightarrow |f(x) - L| < \varepsilon$$

Infinity  $\rightarrow$  Infinity (**M-N**)

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty \iff \forall M > 0, \exists N > 0 \text{ s.t. } x < -N \Rightarrow f(x) > M$$


---

## 13.2. Derivatives

$$\frac{df}{dx}(x) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Let  $f(x) = x^2$

**a. Find  $f'(x)$**

$$f'(x) = \boxed{2x}$$

**b. Prove a**

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2x + h) \\ &= \lim_{h \rightarrow 0} (2x + \cancel{0}) \\ &= \boxed{2x} \end{aligned}$$

**c. Prove b**

$$\lim_{x \rightarrow x_0} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

p.f.:

Let  $\varepsilon > 0$

Choose  $\delta = \varepsilon$

Suppose  $0 < |h - 0| < \delta$

Check

$$\left| \frac{(x+h)^2 - x^2}{h} - 2x \right|$$

$$\begin{aligned}
&= \left| \frac{x^2 + 2xh + h^2 - x^2}{h} - 2x \right| \\
&= \left| \frac{h(2x + h)}{h} - 2x \right| \\
&= |2x + h - 2x| \\
&= |h| < \delta = \varepsilon
\end{aligned}$$

### 13.3. Integrals

### 13.4. Differential

If you have a function:

$$y = f(x)$$

Then the differential  $du$  is defined as:

$$dy = f'(x) \cdot dx$$

This means:

- $f'(x) = \frac{dy}{dx}$  is the derivative
- $dx$  is a small change in  $x$
- $dy$  is the corresponding small change in  $y$

So:

$$dy = \frac{dy}{dx} \cdot dx$$

Let's say“

$$y = x^2$$

Then:

$$\frac{dy}{dx} = 2x$$

$$\frac{d}{dx} \left( \int f(x) dx \right) = f(x)$$

$$\int \left( \frac{d}{dx} f(x) \right) dx = f(x)$$

Rule	$\frac{d}{dx}$ Rule	$\frac{d}{dx}$ Example	$\int$ Rule	$\int$ Example
Constant	$\frac{d}{dx}[c] = 0$	$\frac{d}{dx}[7] = 0$	$\int c dx = cx + C$	$\int 7 dx = 7x + C$
Power	$\frac{d}{dx}[x^n] = nx^{n-1}$	$\frac{d}{dx}[x^4] = 4x^3$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ ( $n \neq -1$ )	$\int x^3 dx = \frac{x^4}{4} + C$
Constant Multiple	$\frac{d}{dx}[cf] = cf'$	$\frac{d}{dx}[3x^2] = 3 \cdot 2x = 6x$	$\int cf dx = c \int f dx$	$\int 3x^2 dx = 3 \cdot \frac{x^3}{3} = x^3 + C$
Sum	$\frac{d}{dx}[f + g] = f' + g'$	$\frac{d}{dx}(x^2 + x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) = 2x + 1$	$\int (f + g) dx = \int f dx + \int g dx$	$\int (x^2 + x) dx = \int x^2 dx + \int x dx = \frac{x^3}{3} + \frac{x^2}{2} + C$
Difference	$\frac{d}{dx}[f - g] = f' - g'$	$\frac{d}{dx}(x^2 - x) = \frac{d}{dx}(x^2) - \frac{d}{dx}(x) = 2x - 1$	$\int (f - g) dx = \int f dx - \int g dx$	$\int (x^2 - x) dx = \int x^2 dx - \int x dx = \frac{x^3}{3} - \frac{x^2}{2} + C$
Product	$\frac{d}{dx}[fg] = f'g + fg'$	$\frac{d}{dx}[x \sin x] = 1 \cdot \sin x + x \cdot \cos x$	Integration by Parts $\int u dv = uv - \int v du$	
Quotient	$\frac{d}{dx} \left[ \frac{f}{g} \right] = \frac{f'g - fg'}{g^2}$	$\frac{d}{dx} \left[ \frac{x^2}{x+1} \right] = \frac{2x(x+1) - x^2(1)}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$	Algebraic Manipulation / Substitution	
Chain	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$	$\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x$	Integration by Substitution $\int f(g(x))g'(x) dx = \int f(u) du$	

Rule	$\frac{d}{dx}$ Rule	$\frac{d}{dx}$ Example	$\int$ Rule	$\int$ Example
Exponential	$\frac{d}{dx}[e^x] = e^x$	$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + C$	$\int e^x dx = e^x + C$
	$\frac{d}{dx}[a^x] = a^x \ln(a)$	$\frac{d}{dx}[2^x] = 2^x \ln 2$	$\int a^x dx = \frac{a^x}{\ln a} + C$	$\int 2^x dx = \frac{2^x}{\ln 2} + C$
Logarithmic	$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$	$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$	$\int \ln(x) dx$ $= x \ln x - x + C$	$\int \ln(x) dx$ $= x \ln x - x + C$
	$\frac{d}{dx}[\log_a(x)] = \frac{1}{x \ln(a)}$	$\frac{d}{dx} \log_2(x) = \frac{1}{x \ln(2)}$	$\int \log_a(x) dx =$ $= \frac{x \ln x}{\ln a} - \frac{x}{\ln a} + C$	$\int \log_{10}(x) dx$ $= \frac{x \ln x}{\ln 10} - \frac{x}{\ln 10} + C$
Sin	$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\int \sin(x) dx$ $= -\cos x + C$	$\int \sin(x) dx$ $= -\cos x + C$
Cos	$\frac{d}{dx}[\cos(x)] = -\sin(x)$	$\frac{d}{dx}[\cos(x)] = -\sin(x)$	$\int \cos(x) dx$ $= \sin x + C$	$\int \cos(x) dx$ $= \sin x + C$
Tan	$\frac{d}{dx}[\tan(x)]$ $= \sec^2(x)$	$\frac{d}{dx}[\tan(x)]$ $= \sec^2(x)$	$\int \tan(x) dx$ $= -\ln \cos x  + C$	$\int \tan(x) dx$ $= -\ln \cos x  + C$

### 13.5. Product Rule —> Integration by Parts

$$\int u \, dv = uv - \int v \, du$$

Given a function:

$$f(x) = x \cdot e^x$$

Integrate (by parts):

$$\int x \cdot e^x \, dx$$

**Step 1:** Choose  $u$  and  $dv$

We choose:

- $u = x$  (easy to **differentiate**)
- $dv = e^x$  (easy to **integrate**)

**Step 2:** Compute  $du$  and  $v$

- $u = x \rightarrow du = dx$
- $dv = e^x \, dx \rightarrow v = \int e^x \, dx = e^x$

**Step 3:** Plug into formula

$$\begin{aligned} \int x \cdot e^x \, dx &= u \cdot v - \int v \cdot du \\ &= x \cdot e^x \, dx - \int e^x \cdot dx \end{aligned}$$

**Step 4:** Compute the remaining integral:

$$\int e^x \, dx = e^x$$

**Step 5:** Finish the expression:

$$\begin{aligned} \int x \cdot e^x &= x \cdot e^x - e^x + C \\ &= e^x(x - 1) + C \end{aligned}$$

### 13.6. Chain Rule —> $u$ -Substitution

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du$$

### 13.7. Quotient Rule —> Algebraic Manipulation / Substitution

## 14. Limits & Continuity

## 15. Properties of Limits

### 15.1. Continuous

#### 15.1.1. Addition, Subtraction, Multiplication, Division

$$\lim_{x \rightarrow c} (f(x) * g(x)) = \lim_{x \rightarrow c} f(x) * \lim_{x \rightarrow c} g(x)$$
$$* \in \{+, -, \times, \div\}$$

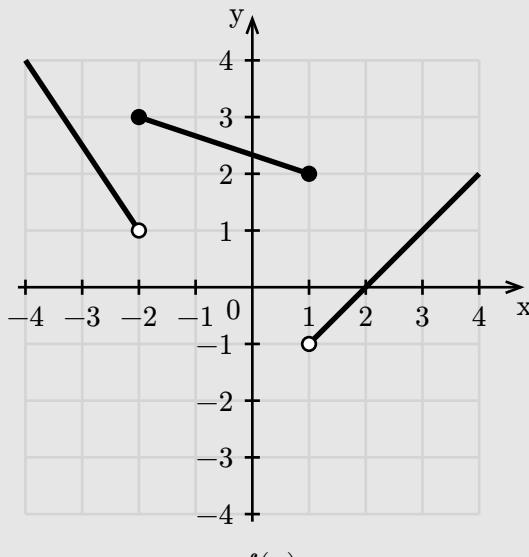
#### 15.1.2. Constant

$$\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x)$$

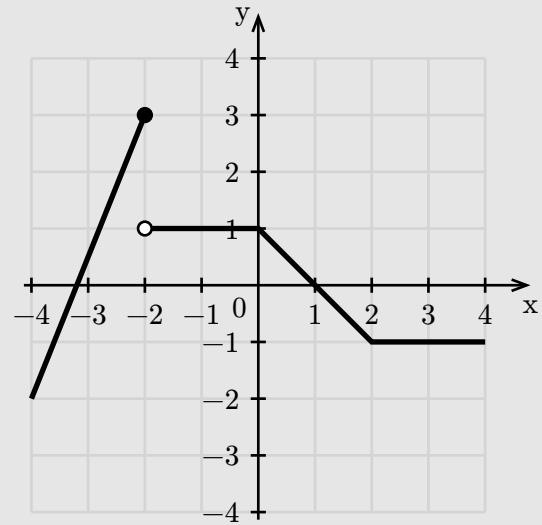
### 15.2. Non-continuous

Even though the limit for either function may not exist, their  $*$  can exist as long as

$$\lim_{x \rightarrow c^-} (f(x) * g(x)) = \lim_{x \rightarrow c^+} (f(x) * g(x))$$
$$* \in \{+, -, \times, \div\}$$



$f(x)$



$g(x)$

**Problem 1:** Limit Exists

$$\lim_{x \rightarrow -2} (f(x) + g(x))$$

1. Left-hand limit ( $x \rightarrow -2^-$ )

$$\lim_{x \rightarrow -2^-} f(x) = 1 \quad \lim_{x \rightarrow -2^-} g(x) = 3$$

Adding these

$$\begin{aligned} \lim_{x \rightarrow -2^-} (f(x) + g(x)) &= \lim_{x \rightarrow -2^-} f(x) + \lim_{x \rightarrow -2^-} g(x) \\ &= 1 + 3 \\ &= 4 \end{aligned}$$

2. Right-hand limit ( $x \rightarrow -2^+$ )

$$\lim_{x \rightarrow -2^+} f(x) = 3 \quad \lim_{x \rightarrow -2^+} g(x) = 1$$

Adding these

$$\begin{aligned} \lim_{x \rightarrow -2^+} (f(x) + g(x)) &= \lim_{x \rightarrow -2^+} f(x) + \lim_{x \rightarrow -2^+} g(x) \\ &= 3 + 1 \\ &= 4 \end{aligned}$$

3. Since both the left-hand and right-hand limits agree

$$\lim_{x \rightarrow -2} (f(x) + g(x)) = 4$$

**Problem 2:** Limit Does Not Exist

$$\lim_{x \rightarrow 1} (f(x) + g(x))$$

1. Left-hand limit ( $x \rightarrow 1^-$ )

$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \lim_{x \rightarrow 1^-} g(x) = 0$$

Adding these

$$\begin{aligned} \lim_{x \rightarrow 1^-} (f(x) + g(x)) &= \lim_{x \rightarrow 1^-} f(x) + \lim_{x \rightarrow 1^-} g(x) \\ &= 2 + 0 \\ &= 2 \end{aligned}$$

2. Right-hand limit ( $x \rightarrow 1^+$ )

$$\lim_{x \rightarrow 1^+} f(x) = -1 \quad \lim_{x \rightarrow 1^+} g(x) = 0$$

Adding these

$$\begin{aligned} \lim_{x \rightarrow 1^+} (f(x) + g(x)) &= \lim_{x \rightarrow 1^+} f(x) + \lim_{x \rightarrow 1^+} g(x) \\ &= -1 + 0 \\ &= -1 \end{aligned}$$

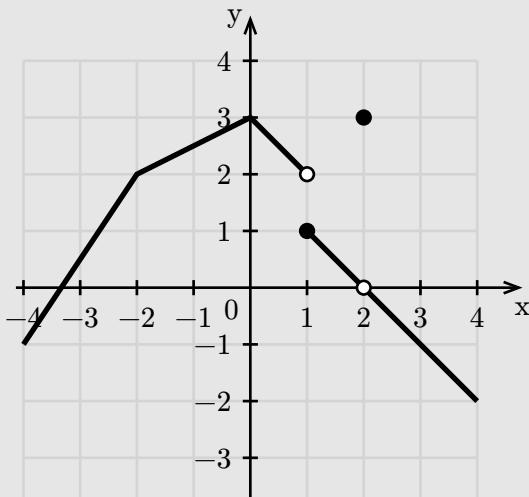
3. Since both the left-hand and right-hand limits do not agree, the limit  $\lim_{x \rightarrow c} (f(x) + g(x))$  does not exist

### 15.3. Composite Functions

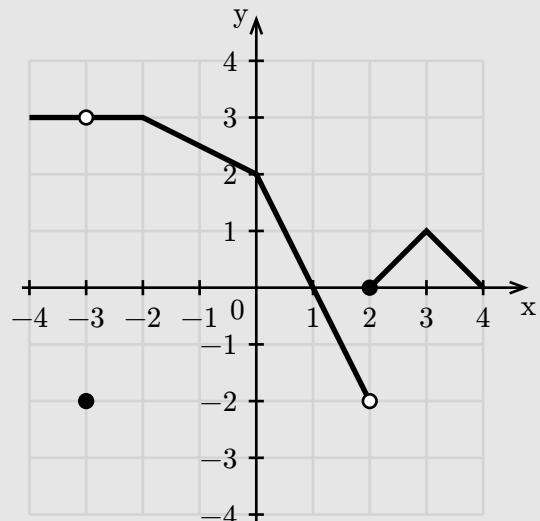
$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right)$$

For this to hold true, two important conditions must be satisfied:

- **Inner limit exists:** The limit  $\lim_{x \rightarrow c}$  must exist and equal some value  $L$ . That is, as  $x$  gets arbitrarily close to  $c$ ,  $g(x)$  approaches a well-defined number  $L$
- **Continuity of the outer function:** The function  $f$  must be continuous at the point  $L$ . Continuity ensures that  $f$  behaves predictably near  $L$ , without any jumps, gaps, or undefined points



$f(x)$



$g(x)$

**Problem 1:** Inner Limit & Continuity Exist

$$\lim_{x \rightarrow -3} f(g(x))$$

1. Inner limit  $\lim_{x \rightarrow -3} g(x)$

Observing  $g(x)$ , as  $x \rightarrow -3$ ,  $g(x) \rightarrow 3$ . The inner limit  $L = 3$  exists

$$\lim_{x \rightarrow -3} g(x) = 3$$

2. Continuity of  $f(x)$  at  $x = 3$

Observing  $f(x)$ ,  $f(3) = -1$ . Since  $f(x)$  is continuous at  $x = 3$ , the composite limit holds

$$\begin{aligned} f\left(\lim_{x \rightarrow -3} g(x)\right) &= f(3) \\ &= -1 \end{aligned}$$

**Problem 2:** Inner Limit Does Not Exist

$$\lim_{x \rightarrow 2} f(g(x))$$

1. Inner limit  $\lim_{x \rightarrow 2} g(x)$

Observing  $g(x)$ , as  $x \rightarrow 2$ ,  $g(x) \rightarrow 2$ . The inner limit does not exist

**Problem 3:** Continuity Does Not Exist

$$\lim_{x \rightarrow 0.5} f(g(x))$$

1. Inner limit  $\lim_{x \rightarrow 0.5} g(x)$

Observing  $g(x)$ , as  $x \rightarrow 0.5$ ,  $g(x) \rightarrow 1$ . The inner limit  $L = 1$  exists

$$\lim_{x \rightarrow 0.5} g(x) = 1$$

2. Continuity of  $f(x)$  at  $x = 1$

Observing  $f(x)$ ,  $f(1)$  is not continuous. Since  $f(x)$  is not continuous at  $x = 1$ , the composite limit does not hold

## 15.4. Limits by Direct Substitution

Limit exists

$$\begin{aligned}\lim_{x \rightarrow -1} (6x^2 + 5x - 1) &= 6(-1)^2 + 5(-1) - 1 \\ &= 6 - 5 - 1 \\ &= 0\end{aligned}$$

Limit does not exist (Undefined)

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x}{\ln(x)} &= \frac{1}{\ln(1)} \\ &= \frac{1}{0}\end{aligned}$$

### 15.4.1. Limits of Piecewise Functions

### 15.4.2. Absolute Value

## 15.5. Limits by Factoring

## 15.6. Limits by Rationalizing

## 15.7. Continuity & Differentiability at a Point

Piecewise function:

$$f(x) = \begin{cases} x^2 & \text{if } x < 3 \\ 6x - 9 & \text{if } x \geq 3 \end{cases}$$

### 1. Check for Continuity

- Value of  $f(3)$

$$f(3) = 6(3) - 9 = \boxed{9}$$

- Left-Hand Limit (LHL)

$$\lim_{x \rightarrow 3^-} f(x) = 3^2 = \boxed{9}$$

- Right-Hand Limit (RHL)

$$\lim_{x \rightarrow 3^+} f(x) = 6(3) - 9 = \boxed{9}$$

Since  $f(3) = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x)$ ,  $f(x)$  is **continuous** at  $x = 3$

### 2. Check for Differentiability

- Left-Hand Derivative (LHD)

$$\begin{aligned}
\lim_{x \rightarrow 3^-} \frac{f(x) - f(3)}{x - 3} &= \frac{x^2 - 3^2}{x - 3} \\
&= \frac{x^2 - 9}{x - 3} \\
&= \frac{(x + 3)(x - 3)}{x - 3} \\
&= x + 3 \\
&= \boxed{6}
\end{aligned}$$

- Right-Hand Derivative

$$\begin{aligned}
\lim_{x \rightarrow 3^+} \frac{f(x) - f(3)}{x - 3} &= \frac{(6x - 9) - 3^2}{x - 3} \\
&= \frac{6x - 9 - 9}{x - 3} \\
&= \frac{6x - 18}{x - 3} \\
&= \frac{6(x - 3)}{x - 3} \\
&= \boxed{6}
\end{aligned}$$

Since the left-hand and right-hand derivatives are equal,  $f(x)$  is **differentiable** at  $x = 3$

**Conclusion:**  $f(x)$  is both continuous & differentiable at  $x = 3$

## 15.8. Power Rule

$$\begin{aligned}
f(x) &= x^n, \quad n \neq 0 \\
f'(x) &= nx^{n-1}
\end{aligned}$$

$$\begin{aligned}
f(x) &= x^3 \\
f'(x) &= 3x^2
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(\sqrt[3]{x^2}) &= \frac{d}{dx}((x^2)^{\frac{1}{3}}) \\
&= \frac{d}{dx}(x^{2 \times \frac{1}{3}}) \\
&= \frac{d}{dx}(x^{\frac{2}{3}}) \\
&= \frac{d}{dx}\left(\frac{2}{3}x^{-\frac{1}{3}}\right)
\end{aligned}$$

### 15.9. Constant Rule

$$\frac{d}{dx}[\textcolor{red}{k}] = 0$$

$$\frac{d}{dx}[\textcolor{red}{-3}] = 0$$

### 15.10. Constant Multiple Rule

$$\begin{aligned}\frac{d}{dx}[\textcolor{red}{k}f(x)] &= \textcolor{red}{k}\frac{d}{dx}[f(x)] \\ &= \textcolor{red}{k}f'(x)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[2x^5] &= 2\frac{d}{dx}[x^5] \\ &= 2 \cdot 5x^4 \\ &= 10x^4\end{aligned}$$

### 15.11. Sum Rule

$$\begin{aligned}\frac{d}{dx}[f(x) + g(x)] &= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \\ &= f'(x) + g'(x)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[x^3 + x^{-4}] &= \frac{d}{dx}[x^3] + \frac{d}{dx}[x^{-4}] \\ &= 3x^2 + (-4x^{-5}) \\ &= 3x^2 - 4x^{-5}\end{aligned}$$

### 15.12. Difference Rule

$$\begin{aligned}\frac{d}{dx}[f(x) - g(x)] &= \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \\ &= f'(x) - g'(x)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[x^4 - x^3] &= \frac{d}{dx}[x^4] - \frac{d}{dx}[x^3] \\ &= 4x^3 - 3x^2\end{aligned}$$

### 15.13. Square Root

$$\begin{aligned}\frac{d}{dx} \sqrt[4]{x} &= \frac{d}{dx} x^{\frac{1}{4}} \\ &= \frac{1}{4} x^{\frac{1}{4}-1} \\ &= \frac{1}{4} \cdot x^{-\frac{3}{4}} \\ &= \frac{1}{4} \cdot \frac{1}{x^{3/4}} \\ &= \frac{1}{4x^{3/4}}\end{aligned}$$

### 15.14. Derivative of a Polynomial

$$f(x) = 2x^3 - 7x^2 + 3x - 100$$

$$f'(x) = 2 \cdot 3x^2 - 7 \cdot 2x + 3 + 0$$

$$h(x) = 3f(x) + 2g(x)$$

Evaluate  $\frac{d}{dx}h(x)$  at  $x = 9$

$$\begin{aligned}\frac{d}{dx}(h(x)) &= \frac{d}{dx}(3f(x) + 2g(x)) \\ &= \frac{d}{dx}3f(x) + \frac{d}{dx}2g(x) \\ &= 3\frac{d}{dx}f(x) + 2\frac{d}{dx}g(x)\end{aligned}$$

Evaluate  $h'(9)$

$$h'(9) = 3f'(9) + 2g'(9)$$

$$\begin{aligned}g(x) &= \frac{2}{x^3} - \frac{1}{x^2} \\ \frac{d}{dx}(g(x)) &= \frac{d}{dx}(2x^{-3} - 1x^{-2}) \\ g'(x) &= 2 \cdot (-3)x^{-4} - (-2)x^{-3} \\ &= -6x^{-4} + 2x^{-3}\end{aligned}$$

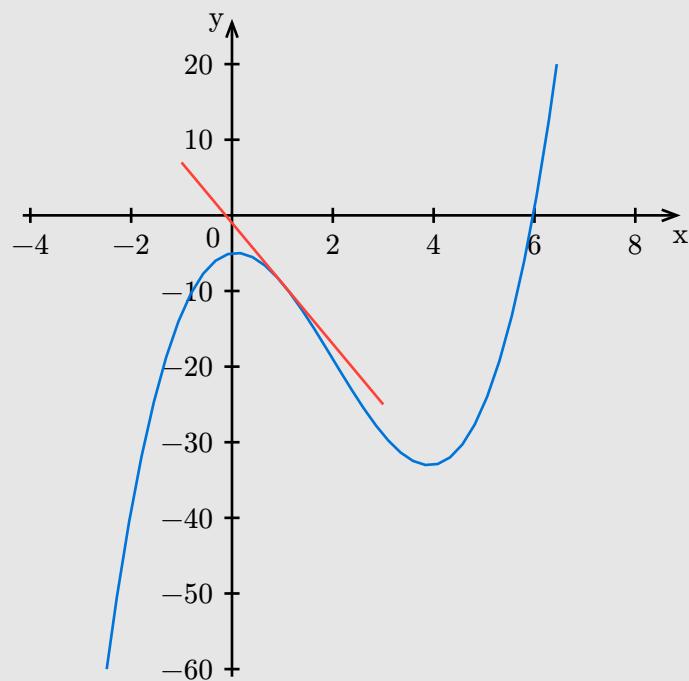
$$\begin{aligned}
 g'(2) &= -6 \cdot 2^{-4} + 2 \cdot 2^{-3} \\
 &= -\frac{6}{2^4} + \frac{2}{2^3} \\
 &= -\frac{3}{8} + \frac{2}{8} \\
 &= -\frac{1}{8}
 \end{aligned}$$

-9 -8

$$\begin{aligned}
 f(x) &= x^3 - 6x^2 + x - 5 \\
 y &= mx + b \\
 f'(1) &= -8 \quad y = -8 \quad x + b \\
 -9 &= -8 \cdot 1 + b \\
 -9 &= -8 + b \\
 -9 + 8 &= -8 + 8 + b \\
 b &= -1
 \end{aligned}$$

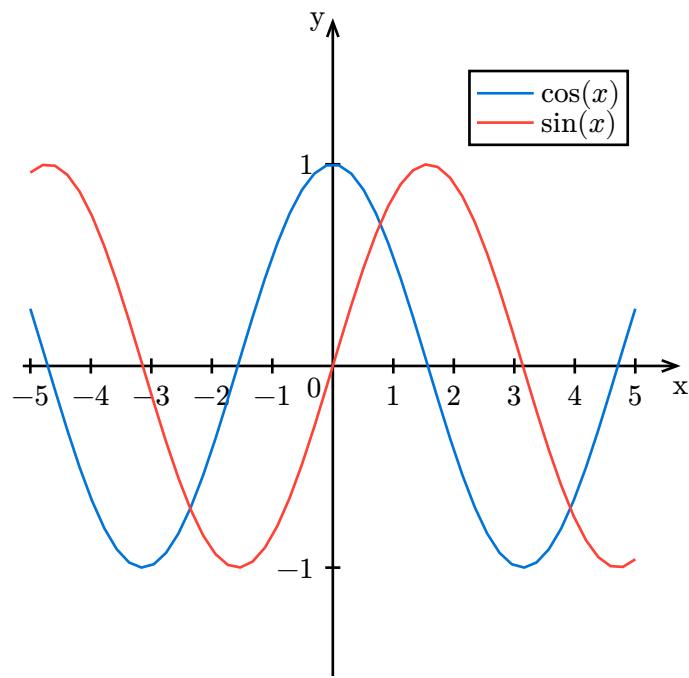
Tangent line to  $f(x)$  at  $x = 1$

$$y = -8x - 1$$



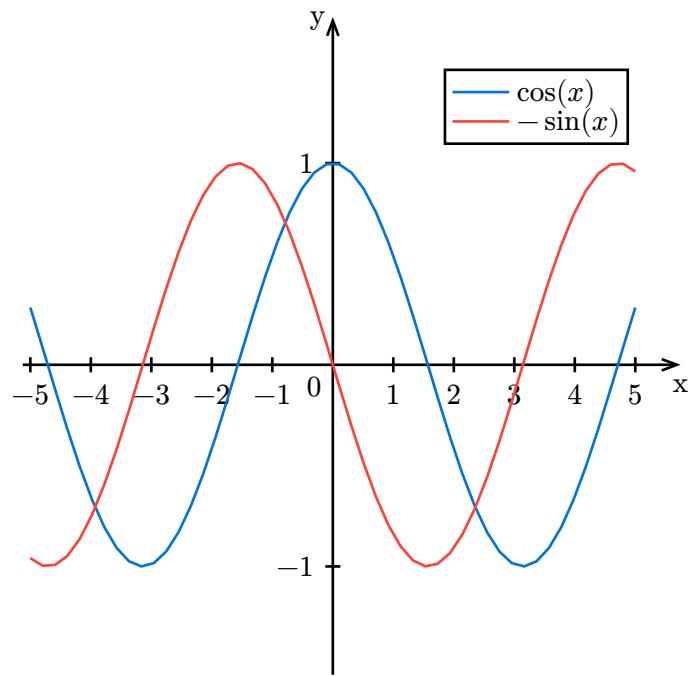
### 15.15. Sin

$$\frac{d}{dx} \sin(x) = \cos(x)$$



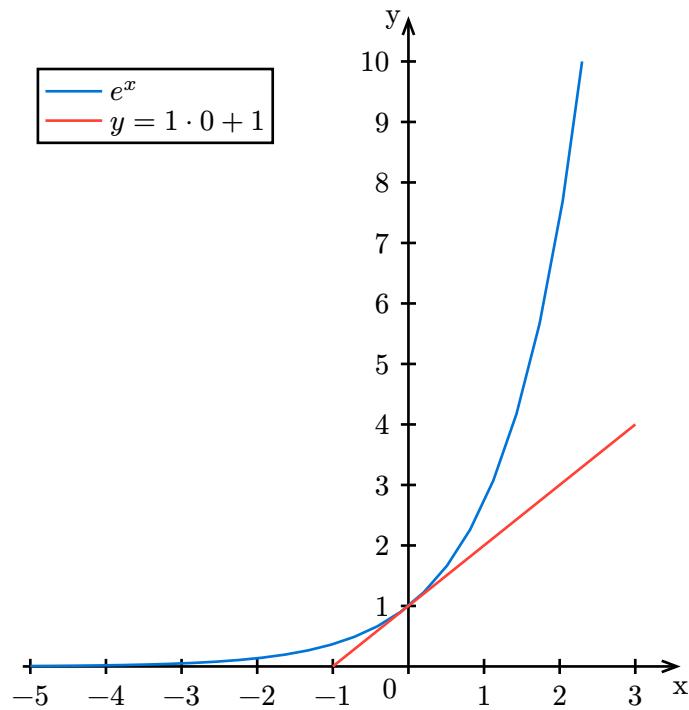
### 15.16. Cos

$$\frac{d}{dx} \cos(x) = -\sin(x)$$



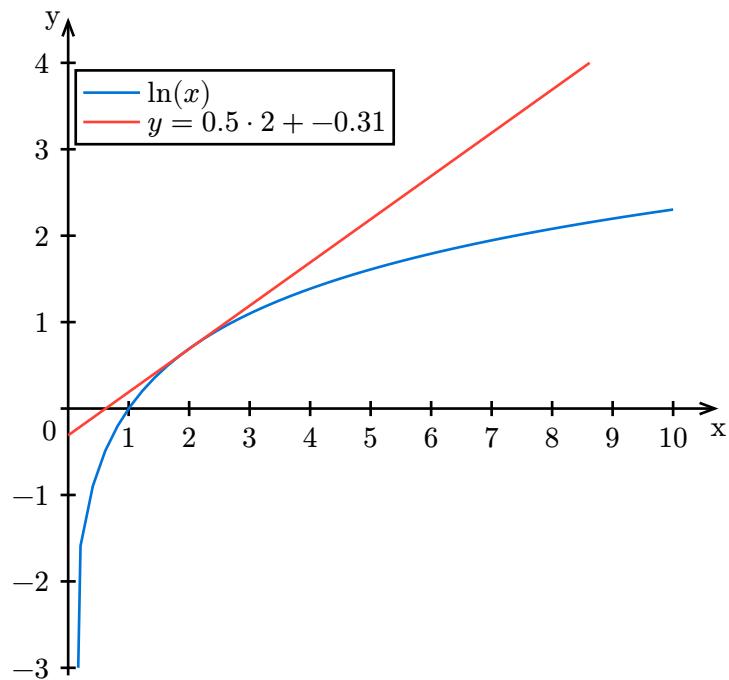
### 15.17. $e^x$

$$\frac{d}{dx} e^x = e^x$$



### 15.18. $\ln(x)$

$$\ln(x) = \frac{1}{x}$$



### 15.19. Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}[x^2 \sin(x)]$$

$$\begin{array}{ll} f(x) = x^2 & g(x) = \sin(x) \\ f'(x) = 2x & g'(x) = \cos(x) \end{array}$$

$$\frac{d}{dx} [x^2 \sin(x)] = 2x \sin(x) + x^2 \cos(x)$$

### 15.20. Quotient Rule

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

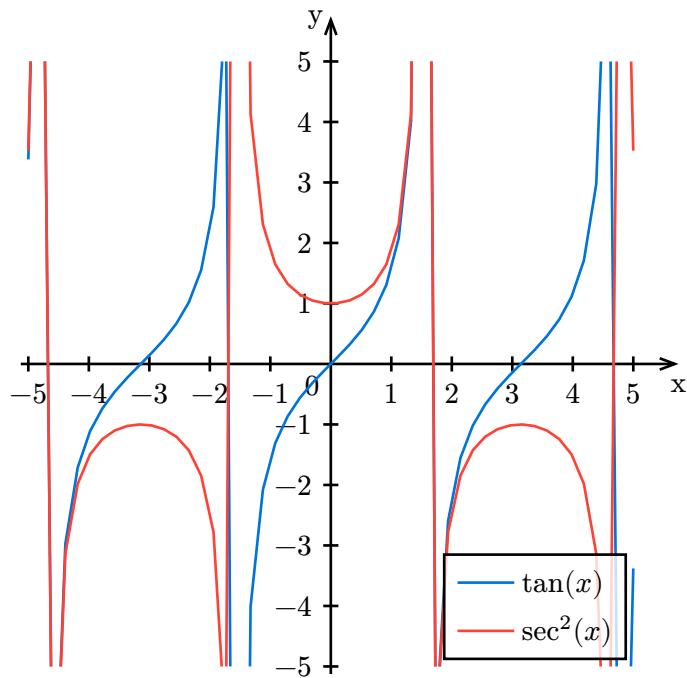
$$\frac{d}{dx} \left[ \frac{x^2}{\cos(x)} \right]$$

$$\begin{array}{ll} f(x) = x^2 & g(x) = \cos(x) \\ f'(x) = 2x & g'(x) = -\sin(x) \end{array}$$

$$\frac{d}{dx} \left[ \frac{x^2}{\cos(x)} \right] = \frac{2x \cos(x) - x^2(-\sin(x))}{[\cos(x)]^2}$$

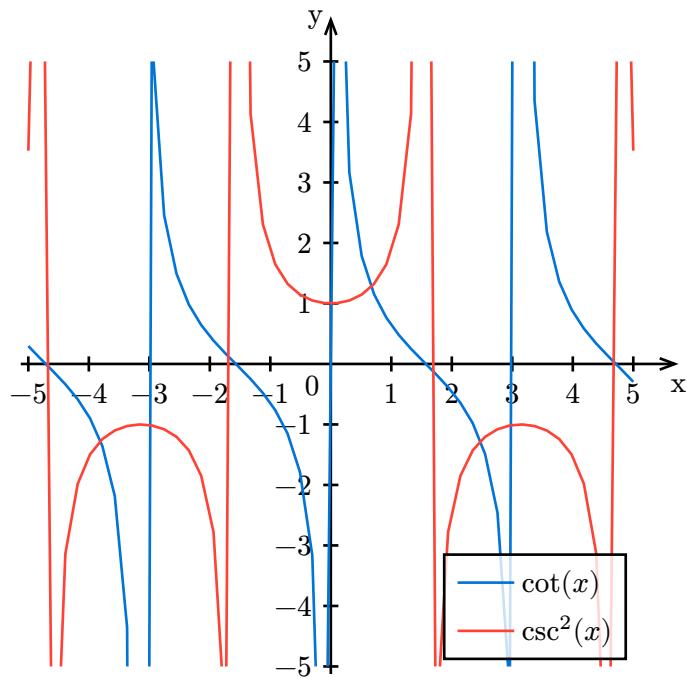
#### 15.20.1. $\tan(x)$

$$\begin{aligned} \frac{d}{dx} [\tan(x)] &= \frac{d}{dx} \left[ \frac{\sin(x)}{\cos(x)} \right] \\ &= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot -\sin(x)}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \boxed{\sec^2(x)} \end{aligned}$$



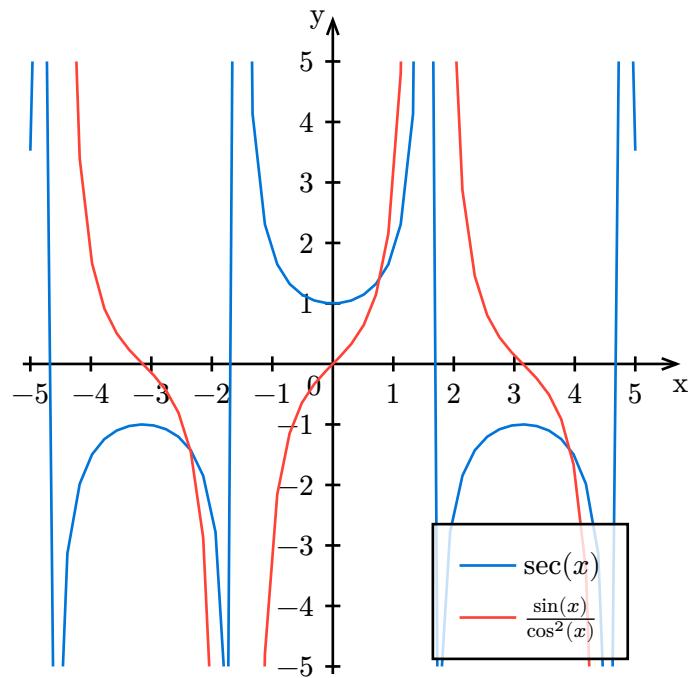
### 15.20.2. $\cot(x)$

$$\begin{aligned}
 \frac{d}{dx}[\cot(x)] &= \frac{d}{dx} \left[ \frac{\cos(x)}{\sin(x)} \right] \\
 &= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot -\sin(x)}{\cos^2(x)} \\
 &= \frac{-\sin^2(x) - \cos^2(x)}{\cos^2(x)} \\
 &= -\frac{1}{\sin^2(x)} \\
 &= -\csc^2(x)
 \end{aligned}$$



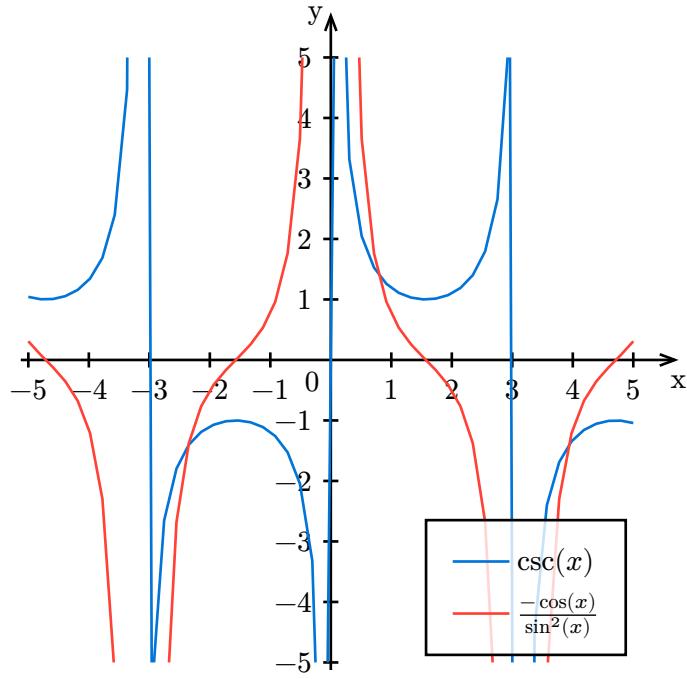
### 15.20.3. $\sec(x)$

$$\begin{aligned}
 \frac{d}{dx}[\sec(x)] &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] \\
 &= \frac{0 \cdot \cos(x) - 1 \cdot -\sin(x)}{\cos^2(x)} \\
 &= \frac{0 + 1 \cdot \sin(x)}{\cos^2(x)} \\
 &= \frac{\sin(x)}{\cos^2(x)} \\
 &= \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} \\
 &= \boxed{\tan(x) \cdot \sec(x)}
 \end{aligned}$$



#### 15.20.4. $\csc(x)$

$$\begin{aligned}
 \frac{d}{dx}[\csc(x)] &= \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] \\
 &= \frac{0 \cdot \sin(x) - 1 \cdot \cos(x)}{\sin^2(x)} \\
 &= \frac{0 - 1 \cdot \cos(x)}{\sin^2(x)} \\
 &= \boxed{\frac{-\cos(x)}{\sin^2(x)}} \\
 &= -\frac{\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)} \\
 &= \boxed{\cot(x) \cdot \csc(x)}
 \end{aligned}$$



### 15.21. Chain Rule

Compute the derivative of a composite function

$$\begin{aligned} h'(x) &= \frac{d}{dx}[\mathbf{f}(\mathbf{g}(x))] \\ &= \mathbf{f}'(\mathbf{g}(x)) \cdot \mathbf{g}'(x) \end{aligned}$$

More generally:

$$y = f_1(f_2(f_3(\dots f_n(x)\dots)))$$

$$\frac{dy}{dx} = f'_1(f_2(f_3(\dots f_n(x)\dots))) \cdot f'_2(f_3(\dots f_n(x)\dots)) \cdot f'_3(\dots f_n(x)\dots) \cdot \dots \cdot f'_n(x)$$

$$\frac{d}{dx}[\mathbf{f}(\mathbf{g}(x))] = \mathbf{f}'(\mathbf{g}(x)) \cdot \mathbf{g}'(x)$$

$$\frac{d}{dx}[e^{\sin(x)}]$$

$$\frac{d}{dx}[e^x] = \frac{1}{x} \quad \frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\mathbf{f}'(\mathbf{g}(x)) = \frac{1}{\sin(x)} \quad \mathbf{g}'(x) = \cos(x)$$

$$\frac{d}{dx} \left[ \ln \left( \underbrace{\frac{g(x)}{\sin(x)}}_{\mathbf{f}(\mathbf{g}(x))} \right) \right] = \frac{1}{\sin(x)} \cdot \cos(x)$$

$$f(x) = \cos^3(x) = (\cos(x))^3$$

$$f(x) = v(u(x))$$

$$f'(x) = v'(u(x)) \cdot u'(x)$$

$$\begin{aligned}f'(x) &= \frac{d\mathbf{v}}{d\mathbf{u}} \cdot \frac{d\mathbf{u}}{dx} \\&= \frac{d(\cos(x))^3}{d\cos(x)} \cdot \frac{d\cos(x)}{x} \\&= 3(\cos(x))^2 \cdot -\sin(x) \\&= -3(\cos(x))^2 \sin(x)\end{aligned}$$

## 15.22. Implicit Differentiation

When a function is not explicitly solved for one variable in terms of another. E.g.:

$$x^2 + y^2 = 1$$

Instead of solving for  $y$  explicitly in terms of  $x$ , implicit differentiation allows you to differentiate both sides of an equation directly, treating  $y$  as an implicit function of  $x$ .

### Steps for Implicit Differentiation

1. Differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a function of  $x$
2. Apply the chain rule whenever differentiating  $y$ , since  $y = y(x)$
2. Solve for  $\frac{dy}{dx}$

$$x^2 + y^2 = 1$$

1. Differentiate both sides

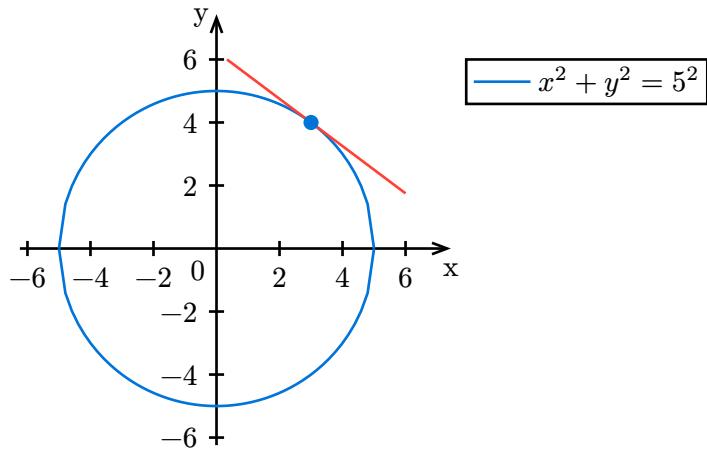
$$\begin{aligned}\frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}[1] \\ \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] &= \frac{d}{dx}[1] \\ \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] &= 0\end{aligned}$$

2. Apply the chain rule to  $y^2$

$$2x + 2y \frac{dy}{dx} = 0$$

3. Slove for  $\frac{dy}{dx}$

$$\begin{aligned}2y \cdot \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= -\frac{x}{y}\end{aligned}$$



$$x^2 + y^2 = 5^2$$

$$2xdx + 2ydy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

### 15.23. Derivatives of Inverse Functions

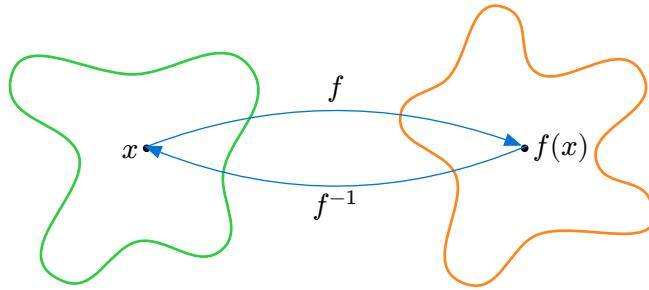
Finding the derivative of the inverse function  $f^{-1}(x)$  at a given point directly from the function  $f(x)$ . Instead of explicitly computing the inverse function  $f^{-1}(x)$ , we use the inverse function derivative formula:

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

This approach allows us to determine the derivative of the inverse function without needing to express  $f^{-1}(x)$  explicitly. Instead, we find the value of  $x$  that satisfies  $f(x) = a$  (where  $a$  is the given point), evaluate  $f'(x)$ , and apply the formula.

#### 1. Definition of Inverse Function

$$\begin{aligned} f(f^{-1}(x)) &= x \\ f^{-1}(f(x)) &= x \end{aligned}$$



#### 2. Differentiate Both Sides

Differentiate both sides with respect to  $x$

$$\frac{d}{dx}[f(f^{-1}(x))] = \frac{d}{dx}[x]$$

The right-hand side simplifies to:

$$\frac{d}{dx}[f(f^{-1}(x))] = 1$$

Using chain-rule:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

The left side expands as:

$$\frac{d}{dx}[f(f^{-1}(x))] = f'(f^{-1}(x)) \cdot \frac{d}{dx}[f^{-1}(x)]$$

This we get:

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}[f^{-1}(x)] = 1$$

### 3. Solve for $\frac{d}{dx}[f^{-1}(x)]$

Rearrange to isolate  $\frac{d}{dx}[f^{-1}(x)]$ :

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

Given the function:

$$f(x) = x^3$$

We want to find  $\frac{d}{dx}f^{-1}(x)$  at  $x = 0.5$  using the inverse function derivative formula:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

#### 1. Compute $f'(x)$

Differentiate  $f(x)$ :

$$f'(x) = 3x^2$$

#### 2. Solve for $x$ such that $f(x) = 0.5$

We need to find  $x$  such that:

$$x^3 = 0.5$$

Solving for  $x$ :

$$x = \sqrt[3]{0.5}$$

#### 3. Compute $f'(\sqrt[3]{0.5})$

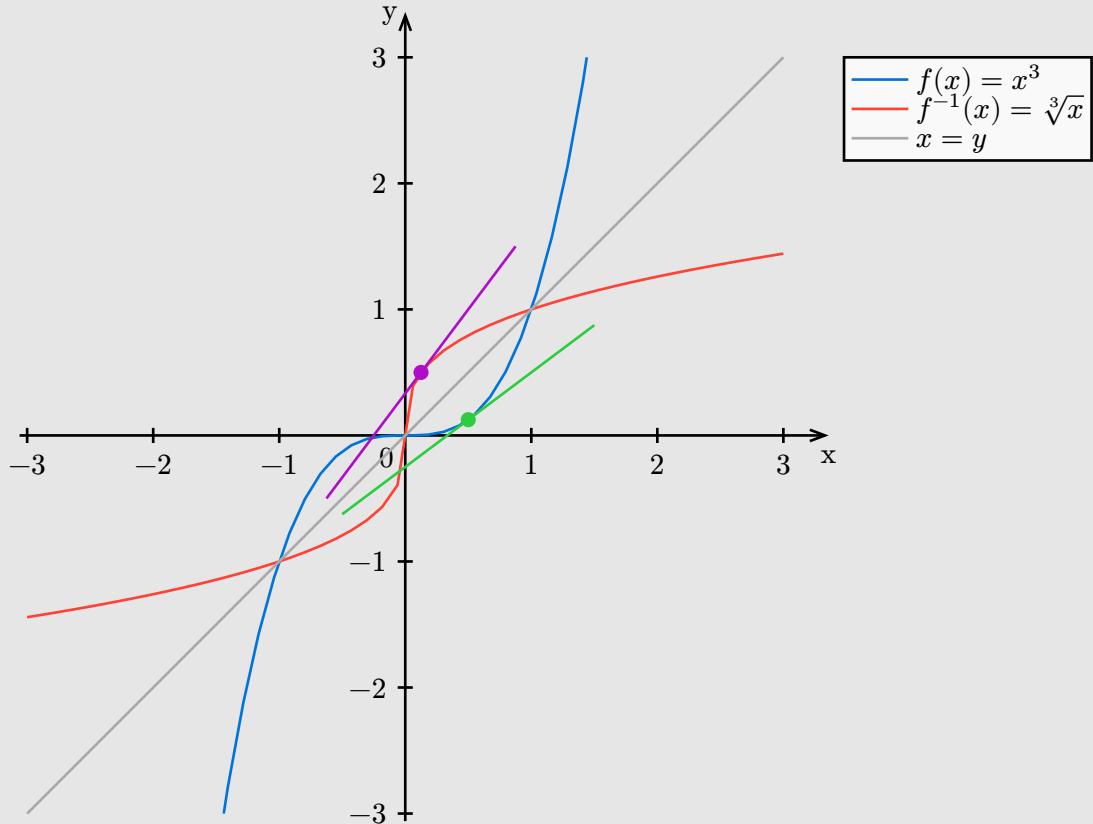
Evaluate the derivative at  $x = \sqrt[3]{0.5}$ :

$$f'(\sqrt[3]{0.5}) = 3(\sqrt[3]{0.5})^2$$

#### 4. Use the formula

$$\frac{d}{dx} f^{-1}(0.5) = \frac{1}{3(\sqrt[3]{0.5})^2}$$

#### 5. Interpretation



The expression:

$$\frac{d}{dx} f^{-1}(0.5) = \frac{1}{3(\sqrt[3]{0.5})^2}$$

represents the derivative of the inverse function  $f^{-1}(x)$  evaluated at  $x = 0.5$ . This means it gives the slope of the tangent line to the inverse function at  $x = 0.5$ .

```
from sympy import symbols, solve

# y = x**3 + x

x, y = symbols('x y')

f = x**3 + x - y

inverse = solve(f, x)

print(inverse)
```

### 15.23.1. Derivative Inverse Sin

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

### 15.23.2. Derivative Inverse Cos

$$\frac{d}{dx}[\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$$

### 15.23.3. Derivative Inverse Tan

$$\frac{d}{dx}[\tan^{-1}(x)] = -\frac{1}{1+x^2}$$

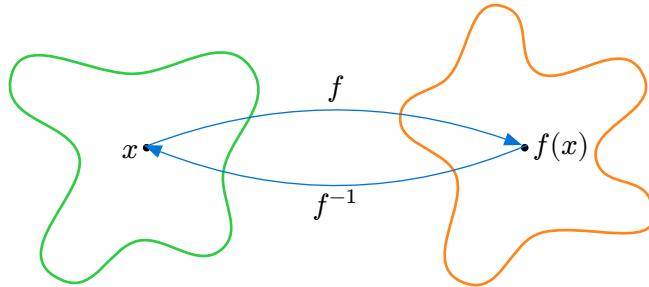
$$\frac{d}{dx}[\arctan(x)] = -\frac{1}{1+x^2}$$

## 15.24. Inverse Functions

### 1. Definition

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x$$



A function  $f : A \rightarrow B$  has an inverse function  $f^{-1}$  if and only if  $f$  is **bijective** (i.e., both one-to-one and onto):

- **Injective (One-to-One):**

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2$$

No two inputs map to the same output

- **Surjective (Onto):**

Every element in  $B$  is mapped from some element in  $A$

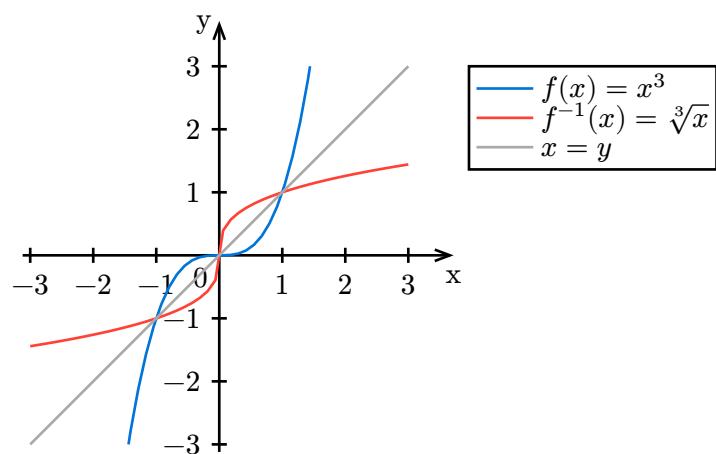
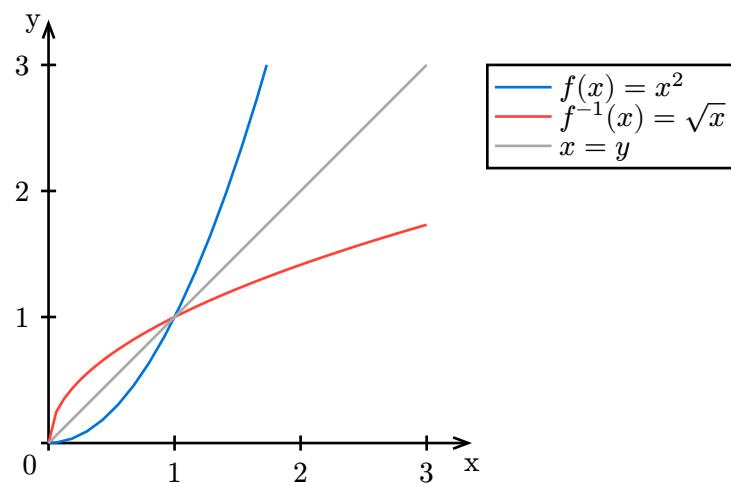
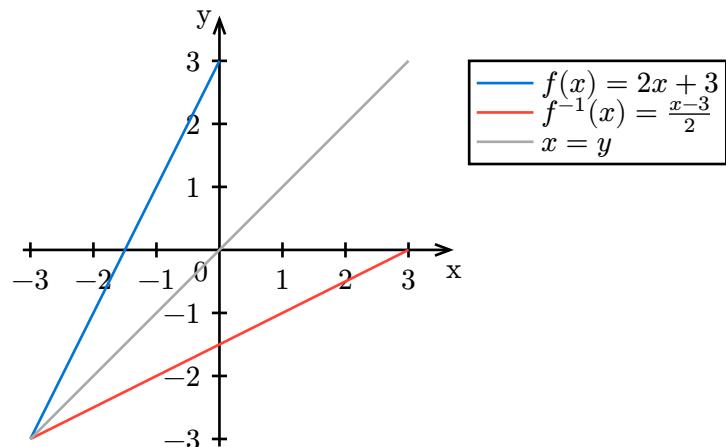
### 2. Finding Inverse Function

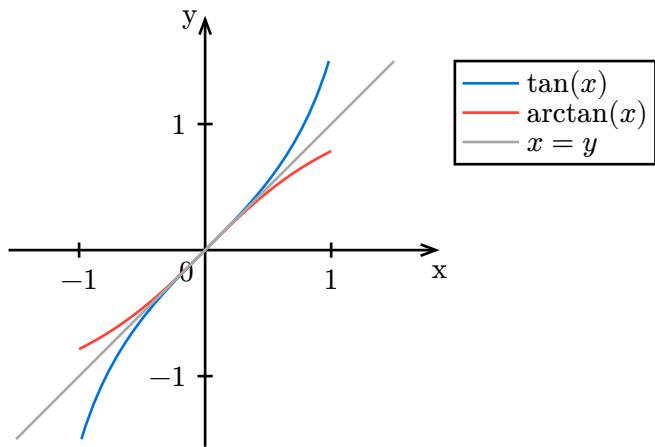
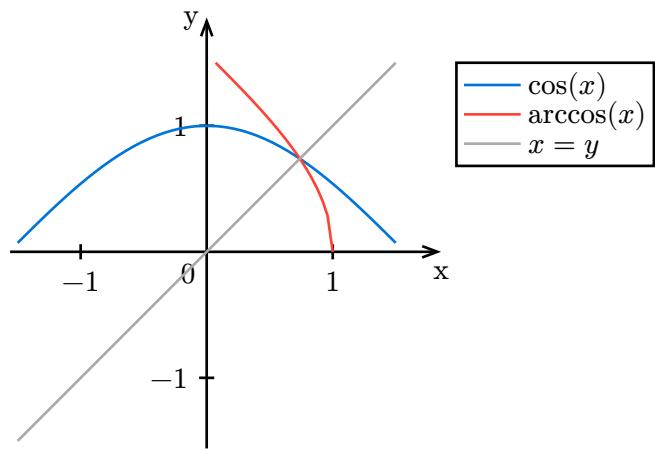
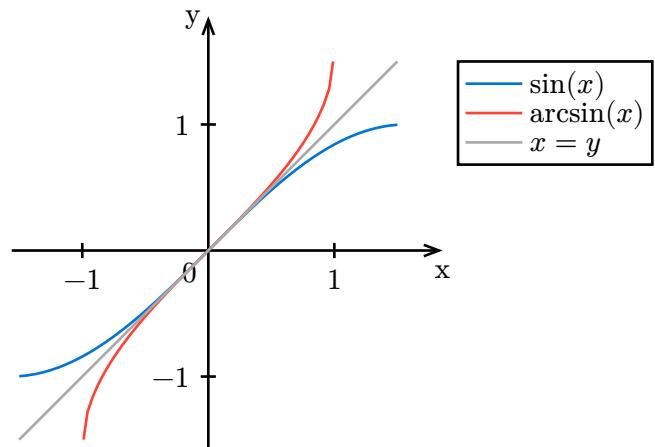
To determine  $f^{-1}$ :

- Express  $y$  in terms of  $x$ :  $y = f(x)$
- Solve for  $x$  in terms of  $y$
- Swap  $x$  and  $y$ , remaining  $y$  as  $f^{-1}(x)$

### 3. Graphical Representation

The graph of  $f^{-1}$  is a reflection of the graph of  $f$  across the line  $x = y$





### 15.25. L'Hôpital's Rule

Evaluating limits that result in an indeterminate form like  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$

If  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$  (or both go to  $\pm\infty$ ), and  $f(x)$  and  $g(x)$  are differentiable near  $a$ , then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Consider:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2}$$

**Step 1:** Direct Substitution

Substituting  $x = 0$ :

$$\frac{1 - \cos(0)}{0^2} = \frac{1 - 1}{0} = \frac{0}{0}$$

Since this is an indeterminate form, we apply L'Hôpital's Rule.

**Step 2:** First Application of L'Hôpital's Rule

Differentiate the numerator and denominator:

- Numerator:  $f(x) = 1 - \cos(x) \Rightarrow f'(x) = \sin(x)$
- Denominator:  $g(x) = x^2 \Rightarrow g'(x) = 2x$

Thus, applying L'Hôpital's Rule:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \rightarrow 0} \frac{\sin(x)}{2x}$$

**Step 3:** Second Check for Indeterminate Form

Substituting  $x = 0$ :

$$\frac{\sin(0)}{2(0)} = \frac{0}{0}$$

Since this is still an indeterminate form, we apply L'Hôpital's Rule again.

**Step 4:** Second Application of L'Hôpital's Rule

Differentiate again:

- Numerator:  $f'(x) = \sin(x) \Rightarrow f''(x) = \cos(x)$
- Denominator:  $g'(x) = 2x \Rightarrow g''(x) = 2$

Applying L'Hôpital's Rule again:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{2x} = \lim_{x \rightarrow 0} \frac{\cos(x)}{2}$$

**Step 5:** Evaluate the Limit

Now, substituting  $x = 0$ :

$$\frac{\cos(0)}{2} = \frac{1}{2}$$

**Final Answer:**

$$\boxed{\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}}$$

### 15.26. Mean Value Theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function that satisfies the following conditions:

1.  $f$  is **continuous** on the closed interval  $[a, b]$
2.  $f$  is **differentiable** on the open interval  $(a, b)$

Then there exists at least one point  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This means that the instantaneous rate of change (derivative) at some point  $c$  is equal to the average rate of change over the entire interval

Consider  $f(x) = x^2$  on  $[1, 3]$

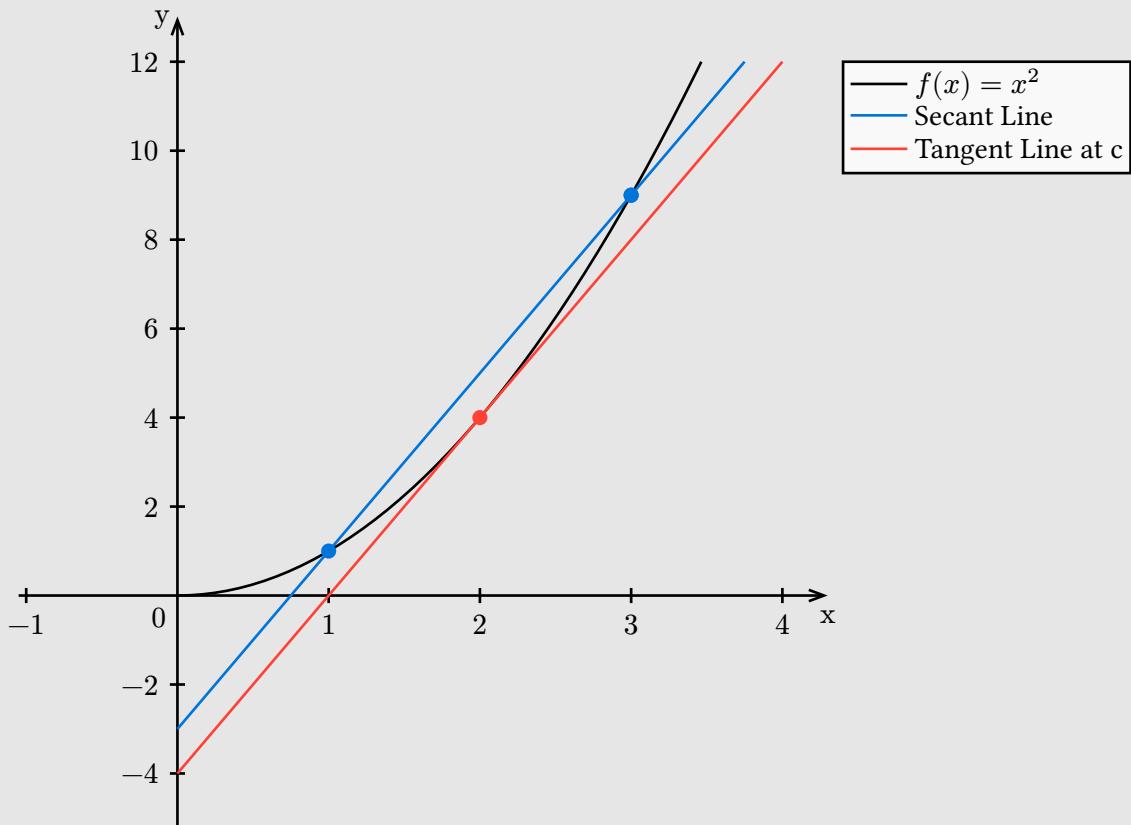
- The average rate of change is:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4$$

- The derivative for  $f(x)$  is  $f'(x) = 2x$
- Setting  $f'(c) = 4$ , we solve:

$$2c = 4 \Rightarrow c = 2$$

Thus, at  $c = 2$ , the instantaneous rate of change matches the average rate of change

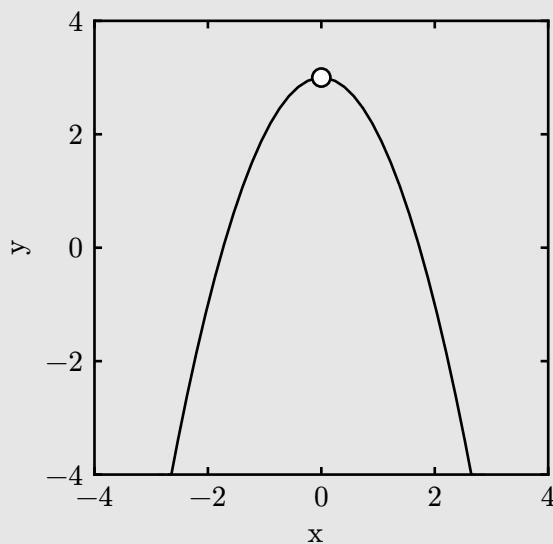


## 15.27. Extreme Value Theorem

The Extreme Value Theorem states that if a function  $f(x)$  is continuous on a closed interval  $[a, b]$ , then  $f(x)$  must attain both a maximum and a minimum value within that interval. This means there exist points  $c, d \in [a, b]$  such that:

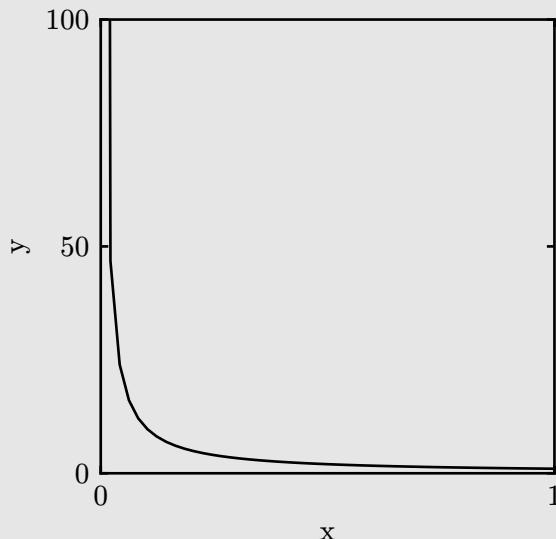
$$f(c) \geq f(x) \quad \text{and} \quad f(d) \leq f(x) \quad \text{for all } x \in [a, b]$$

1. **Continuity:** The function must be continuous on  $[a, b]$ . Discontinuities (jumps, asymptotes, holes) can prevent the function from attaining an extreme value



2. **Closed Interval:** If the function is defined on an open interval  $(a, b)$ , an extremum may not exist

$f(x) = \frac{1}{x}$  on  $(0, 1]$  has no maximum because it keeps increasing as  $x \rightarrow 0$ .

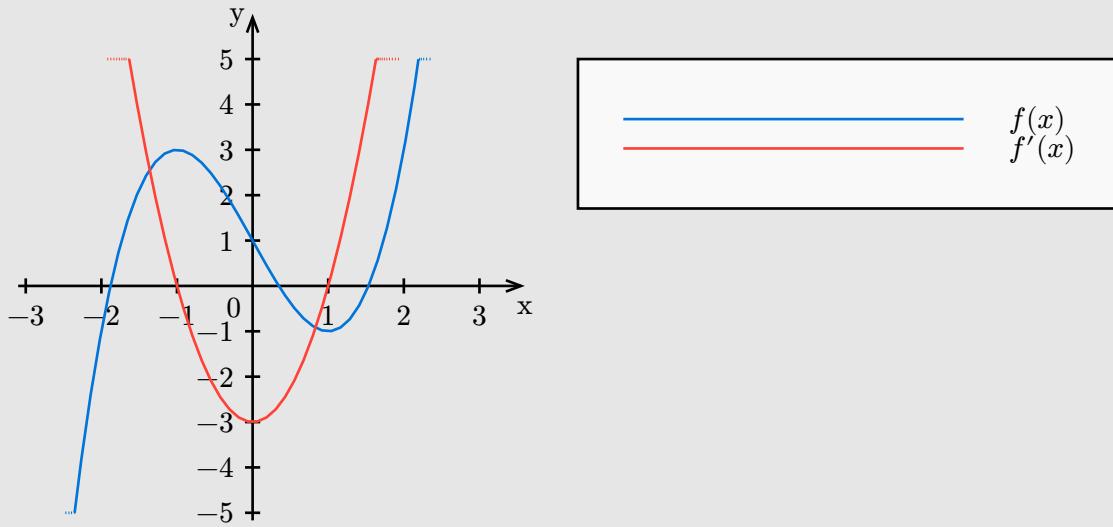


### 15.27.1. Critical points

A critical point of a function  $f(x)$  is a point in the domain where either:

- $f'(x) = 0$
- $f'(x)$  is undefined

$f(x) = \frac{1}{x}$  on  $(0, 1]$  has no maximum because it keeps increasing as  $x \rightarrow 0$ .



### 15.27.2. Global vs. Local Extrema

- $f(c)$  is a **relative maximum** if  $f(c) \geq f(x)$  for all  $x \in (c - h, c + h)$  for  $h > 0$
- $f(d)$  is a **relative minimum** if  $f(d) \leq f(x)$  for all  $x \in (d - h, d + h)$  for  $h > 0$

### 15.27.3. First and Second Derivative Tests

First Derivative Test and the Second Derivative Test are used to classify critical points, and both aim to determine whether the point is a local maximum, local minimum, or neither

#### 15.27.3.1. First Derivative Test

If  $f'(x)$  changes sign around a critical point  $c$ , we can determine if  $f(c)$  is a local maximum or minimum:

- If  $f'(x)$  changes from **positive to negative** at  $c$ , then  $f(c)$  is a **local maximum**
- If  $f'(x)$  changes from **negative to positive** at  $c$ , then  $f(c)$  is a **local minimum**
- If  $f'(x)$  does **not** change sign,  $f(c)$  is **not** a local extremum

Function:

$$f(x) = x^3 - 3x$$

Derivative:

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ &= 3(x - 1)(x + 1) \end{aligned}$$

Critical points ( $f'(x) = 0$ ):

- $x = -1$
- $x = 1$

Analysis of  $f'(x)$

- Pick point left of  $-1$ , say  $x = -2$ :

$$f'(-2) = 3(4 - 1) = 9 > 0 \rightarrow \text{increasing}$$

- Pick a point between  $-1$  and  $1$ , say  $x = 0$ :

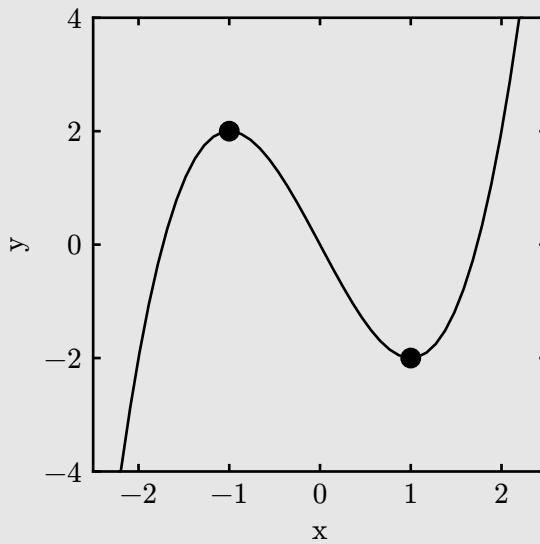
$$f'(0) = -3 < 0 \rightarrow \text{decreasing}$$

- Pick a point right of  $1$ , say  $x = 2$ :

$$f'(2) = 3(4 - 1) = 9 > 0 \rightarrow \text{increasing}$$

Conclusion

- At  $x = -1$ ,  $f'(x)$  changes from **positive to negative**  $\rightarrow$  **local maximum**
- At  $x = 1$ ,  $f'(x)$  changes from **negative to positive**  $\rightarrow$  **local minimum**



#### 15.27.3.2. Second Derivative Test

If  $f''(x)$  is continuous near a critical point  $c$ , and  $f'(c) = 0$ , then:

- If  $f''(c) > 0$ , then  $f(c)$  is a **local minimum** (concave up)
- If  $f''(c) < 0$ , then  $f(c)$  is a **local maximum** (concave down)
- If  $f''(c) = 0$ , the test is **inconclusive** – use the first derivative test or other methods

Function:

$$f(x) = x^4 - 4x^2$$

First Derivative:

$$f'(x) = 4x^3 - 8x$$

Second Derivative:

$$f''(x) = 12x^2 - 8$$

Critical points ( $f'(x) = 0$ ):

- $x = 0$
- $x = \sqrt{2}$

- $x = -\sqrt{2}$

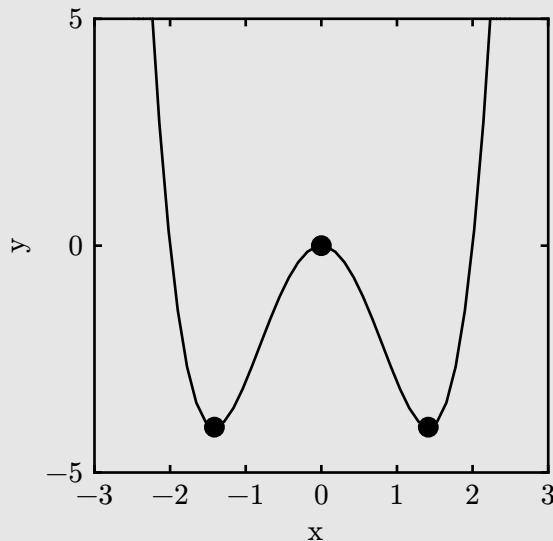
Evaluate  $f''(x)$  at each point

- At  $x = 0$

$$f''(0) = 12(0)^2 - 8 = -8 \rightarrow \text{concave down} \rightarrow \text{local maximum}$$

- At  $x = \sqrt{2}$  or  $x = -\sqrt{2}$

$$f''(\sqrt{2}) = f''(-\sqrt{2}) = 12(2)^2 - 8 = 16 \rightarrow \text{concave up} \rightarrow \text{local minimum}$$



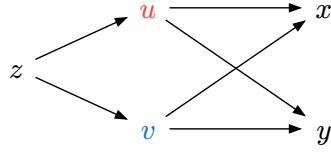
## 15.28. Differentiation Rules

Power	$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$
Sum	$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
Product	$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
Chain	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

# Calculus II

$$z = f(\textcolor{red}{u}, \textcolor{blue}{v})$$

$$\textcolor{red}{u} = f(x, y) \quad \textcolor{blue}{v} = f(x, y)$$



The derivative of  $z$  with respect to  $x$  is the sum of all possible paths from  $z$  to  $x$ :

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

Let:

- $u = x + y$
- $v = xy$
- $z = u^2 + \sin(v)$

We want to compute:

$$\frac{\partial z}{\partial x}$$

**Step 1:** Apply the multivariable chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

**Step 2:** Compute partial derivatives

- $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x + y) = 1 + 0 = \boxed{1}$
- $\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(xy) = 1y = \boxed{y}$
- $\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(u^2 + \sin(v)) = 2u + 0 = \boxed{2u}$
- $\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}(u^2 + \sin(v)) = 0 + \cos(v) = \boxed{\cos(v)}$

**Step 3:** Plug in:

$$\begin{aligned} \frac{\partial z}{\partial x} &= 2u \cdot 1 + \cos(v) \cdot y \\ &= 2(x + y) + y \cos(xy) \end{aligned}$$

# Calculus III



# Statistics

# 16. Probability Theory

## 16.1. Probability Axioms

### 16.1.1. Non-Negativity

Probability of any event cannot be negative

$$P(A) \geq 0$$

### 16.1.2. Normalization

The probability of the sample space is always 1

$$P(S) = 1$$

Where:

- $S$  is the sample space (the set of all possible outcomes)

### 16.1.3. Additivity

If two events are mutually exclusive (cannot happen at the same time), the probability of either occurring is the sum of their individual probabilities

$$P(A \cup B) = P(A) + P(B)$$

## 16.2. Rules

### 16.2.1. Complement Rule

The probability of the complement of an event  $A$  is  $P(A^c) = 1 - P(A)$

How likely it is that the event does not occur.

Consider a fair die:

- Let  $A$  be the event “rolling a 4”. Then  $P(A) = \frac{1}{6}$
- The complement of  $A$ , denoted  $P(A^c)$ , is “not rolling 4”
- $P(A^c) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$

### 16.2.2. Multiplication Rule

- Independent Events

If  $A$  and  $B$  are independent, then  $P(A \cap B) = P(A) \times P(B)$ . The probability of both events occurring is the product of their individual probabilities.

Consider flipping two fair coins. Let  $A$  be the event “the first coin is heads” and  $B$  be the event “the second coin is heads”

- Since the flips are independent,  $P(A \cap B) = P(A) \times P(B)$
- $P(A) = \frac{1}{2}$  and  $P(B) = \frac{1}{2}$
- Thus,  $P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

- Dependent Events

If  $A$  and  $B$  are dependent,  $P(A \cap B) = P(A) \times P(B | A)$ , where  $P(B | A)$  is the conditional probability of  $B$  given  $A$ .

Draw two cards from a standard deck without replacement. Let  $A$  be the event “drawing an Ace on the first draw” and  $B$  be the event “drawing an Ace on the second draw”

- $P(A) = \frac{4}{52} = \frac{1}{13}$
- If  $A$  occurs (i.e., an Ace is drawn first), there are 3 Aces left out of 51 cards. So,  $P(B | A) = \frac{3}{51} = \frac{1}{17}$
- Thus,  $P(A \cap B) = P(A) \times P(B | A) = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$

#### 16.2.3. Addition Rule

For Any Two Events:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ . This accounts for the overlap between the two events to avoid double counting.

Suppose you roll a die, and you want to find the probability of rolling a 2 or a 4.

- Let  $A$  be the event “rolling a 2,” and  $B$  be the event “rolling a 4”
- $P(A) = \frac{1}{6}$  and  $P(B) = \frac{1}{6}$
- Since  $A$  and  $B$  are mutually exclusive,  $P(A \cap B) = 0$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{6} + \frac{1}{6} - 0 = \frac{2}{6} = \frac{1}{3}$

#### 16.2.4. Conditional Probability

The probability of an event  $A$  given that  $B$  has occurred is  $P(A | B) = \frac{P(A \cap B)}{P(B)}$ , provided  $P(B) > 0$ .

In a deck of 52 cards, if you know a card is a spade, what is the probability that it is an Ace?

- Let  $A$  be the event “drawing an Ace,” and  $B$  be the event “drawing a spade”
- $P(B) = \frac{12}{52} = \frac{1}{4}$
- There is 1 Ace of Spades out of 13 spades, so  $P(A \cap B) = \frac{1}{32}$
- Thus,  $P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{32}}{\frac{1}{4}} = \frac{1}{13}$

#### 16.2.5. Law of Total Probability

If  $\{B_i\}$  is a partition of the sample space, then for any event  $A$ :

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A | B_i) \times P(B_i)$$

Suppose you want to calculate the probability of raining on a given day. You know that it’s either sunny or cloudy, and the probability of rain is different in each condition.

- Let  $B_1$  be the event “sunny” and  $B_2$  be the event “cloudy”
- $P(B_1) = \frac{3}{4}$  and  $P(B_2) = \frac{1}{4}$

- The probability of rain given sunny is  $P(A | B_1) = \frac{1}{10}$ , and given cloudy is  $P(A | B_2) = \frac{2}{5}$
- The total probability of rain is:

$$\begin{aligned}
 P(A) &= P(A | B_1) \times P(B_1) + P(A | B_2) \times P(B_2) \\
 &= \frac{1}{10} \times \frac{3}{4} + \frac{2}{5} \times \frac{1}{4} \\
 &= \frac{3}{40} + \frac{2}{20} \\
 &= \frac{3}{40} + \frac{4}{40} \\
 &= \frac{7}{40}
 \end{aligned}$$

### 16.2.6. Law of Large Numbers

As the number of trials of a random experiment increases, the sample mean will converge to the expected value (mean) of the random variable

You flip a fair coin 1000 times. As you increase the number of flips, the proportion of heads should get closer to the probability of heads (which is 0.5).

### 16.2.7. Central Limit Theorem

For a sufficiently large number of trials, the distribution of the sample mean approaches a normal distribution, regardless of the distribution of the population

Suppose you are rolling a fair die repeatedly and calculating the average of the results. If you take 30 samples, each being the average of 10 die rolls, the distribution of those sample means will be approximately normal with a mean of 3.5 (the mean of a fair die) and a standard deviation that can be computed based on the original die's distribution.

## 16.3. Bayes Theorem

$$P(A | B) = \frac{P(B | A) \cdot P(A)}{P(B)}$$

- $P(A)$ : prior probability (initial belief about event  $A$ )
- $P(B | A)$ : likelihood (probability of observing  $B$  given that  $A$  is true)
- $P(B)$ : marginal likelihood (overall probability of observing  $B$  under all possible conditions)
- $P(A | B)$ : posterior probability (updated belief after considering the evidence  $B$ )

## 17. Descriptive Statistics

### 17.1. Central Tendency

#### 17.1.1. Mean

Sum of all the values divided by the number of values

$$\mu = \frac{\sum_{i=1}^n x_i}{n}$$

[1, 2, 3]

$$\bar{x} = \frac{1+2+3}{3} = 150$$

### 17.1.2. Median

Middle value in a set of values when they are arranged in ascending or descending order

#### 1. Odd Number of Values

[1, 2, 3]

- **Step 1:** Arrange the Data in Ascending Order

[1, 2, 3]

- **Step 2:** Identify the Median

Median = 2

#### 2. Even Number of Values

[1, 2, 3, 4]

- **Step 1:** Arrange the Data in Ascending Order

[1, 2, 3, 4]

- **Step 2:** Identify the Median

$$\text{Median} = \frac{2+3}{2} = \frac{5}{2} = 2.5$$

### 17.1.3. Mode

Value that appears most frequently

[1, 1, 2, 3]

- **Step 1:** Identify the Most Frequent Number

1 : 2

2 : 1

3 : 1

- **Step 2:** Determine the Mode

Mode = 1

## 17.2. Dispersion

### 17.2.1. Range

Range = max - min

[1, 2, 3, 4, 5]

- **Step 1:** Identify the Maximum and Minimum Values

$$\begin{aligned}\max &= 5 \\ \min &= 1\end{aligned}$$

- **Step 2:** Calculate the Range

$$\text{Range} = 5 - 1 = 4$$

### 17.2.2. Variance

Quantifies the spread or dispersion of a set of data points in a dataset

- Population

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^n (x_i - \mu)^2$$

- Sample

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

[70, 75, 80, 85, 90]

**Step 1:** Find mean

$$\bar{x} = \frac{70 + 75 + 80 + 85 + 90}{5} = \frac{400}{5} = 80$$

**Step 2:** Subtract the Mean and Square the result

$$\begin{aligned}(70 - 80)^2 &= (-10)^2 = 100 \\ (75 - 80)^2 &= (-5)^2 = 25 \\ (80 - 80)^2 &= 0^2 = 0 \\ (85 - 80)^2 &= 5^2 = 25 \\ (90 - 80)^2 &= 10^2 = 100\end{aligned}$$

**Step 3:** Calculate variance

$$s^2 = \frac{100 + 25 + 0 + 25 + 100}{5-1} = \frac{250}{4} = 62.5$$

### 17.2.3. Standard deviation

Quantifies the amount of variation or dispersion in a set of data points

- Population

$$\sigma = \sqrt{\sigma^2}$$

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2}$$

- Sample

$$s = \sqrt{s^2}$$

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^N (x_i - \bar{x})^2}$$

#### 17.2.4. Interquartile Range (IQR)

$$\text{IQR} = \text{Q3} - \text{Q1}$$

$$[1, 2, 3, 4, 5, 6, 7]$$

Step 1: Arrange the Data in Ascending Order

$$[1, 2, 3, 4, 5, 6, 7]$$

Step 2: Find the Quartiles

1. Calculate the Median (Q2)

$$\text{Median (Q2)} = 4$$

2. Find First Quartile (Q1)

Q1 is the median of the first half of the dataset

$$\text{Q1} = 2$$

3. Find Third Quartile (Q3)

Q3 is the median of the second half of the dataset

$$\text{Q3} = 5$$

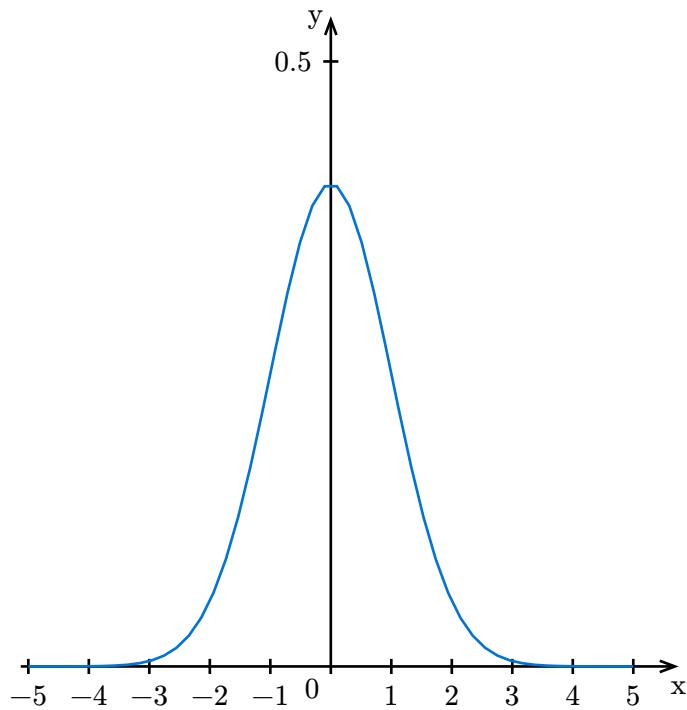
Step 3: Calculate the Interquartile Range (IQR)

$$\text{IQR} = 5 - 2 = 3$$

## 18. Probability Distributions

### 18.1. Gaussian (Normal) distribution

$$f(x | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

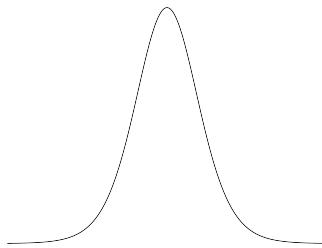


## 18.2. t-Distribution

$$f(t \mid \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Where:

- $t$ : t-statistic
- $\nu$  (or  $df$ ): degrees of freedom
- $\Gamma$ : Gamma function (generalizes the factorial function)



Continuous probability distribution for estimating the mean of a normally distributed population in situations where:

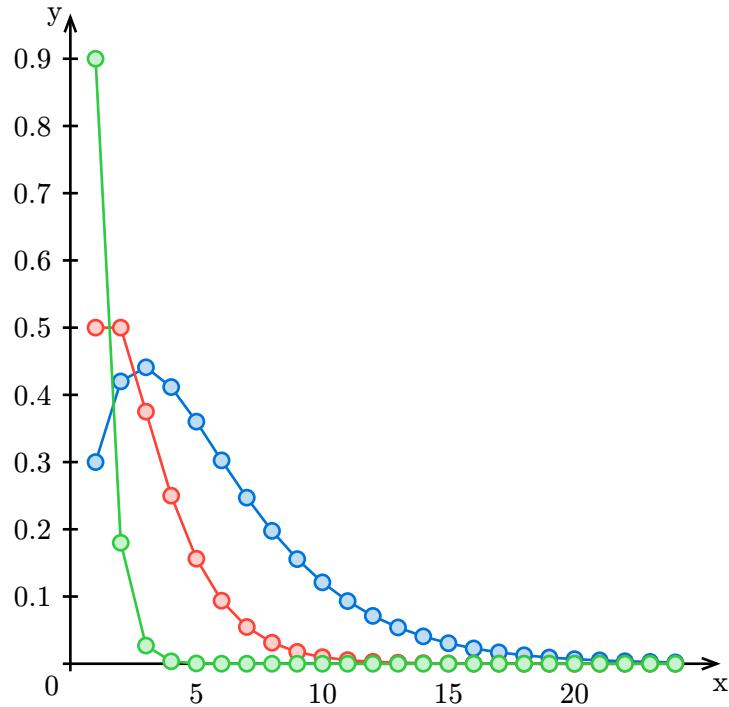
- Sample size is small
- Population standard deviation is unknown

Similar in shape to the normal distribution but has heavier tails, which means it gives more probability to values further from the mean

## 18.3. Binomial distribution

Discrete probability distribution that describes the number of successes in a fixed number of independent Bernoulli trials, each with the same probability of success

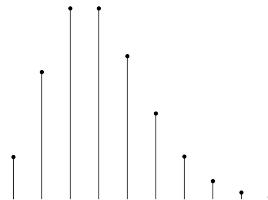
$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$



#### 18.4. Poisson distribution

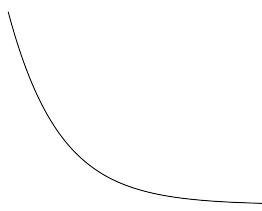
Used to model the number of events that occur within a fixed interval of time or space, given a constant mean rate and assuming that these events occur independently of each other

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$



#### 18.5. Exponential distribution

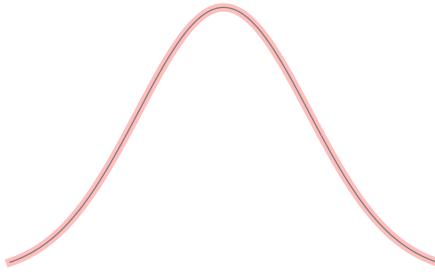
$$f(x \mid \lambda) = \lambda e^{-\lambda x}$$



## 19. Functions

### 19.1. PDF (Probability Density Function)

Function that describes the likelihood of a continuous random variable taking on a particular value



Properties:

- The area under the curve of a PDF over the entire range of possible values equals 1
- The PDF itself is non-negative everywhere
- The probability that the variable falls within a certain range is given by the integral of the PDF over that range

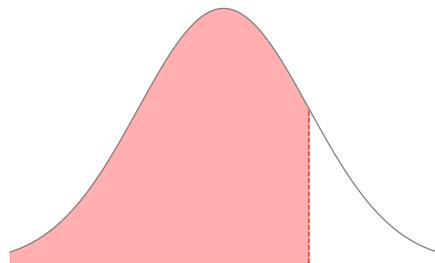
### 19.2. PMF (Probability Mass Function)

Frequency distribution that provides the probability that a categorical variable takes on each of its possible values

### 19.3. CDF (Cumulative Distribution Function)

Gives the probability that  $X$  will take a value less than or equal to  $x$

$$F(x) = P(X \leq x)$$



1. Categorical

$$F(x) = \int_{-\infty}^x f(t)dt$$

2. Continuous

$$F(x) = \sum_{t \leq x} P(X = t)$$

```
from scipy.stats import norm
```

```

x = 1
mu = 0
sigma = 1

norm.cdf(x, loc=mu, scale=sigma)

```

## 19.4. PPF (Percent-Point Function)

Gives the value  $x$  such that the probability of a random variable being less than or equal to  $x$  is equal to a given probability  $p$

```

from scipy.stats import norm

p = 0.95
mu = 0
sigma = 1

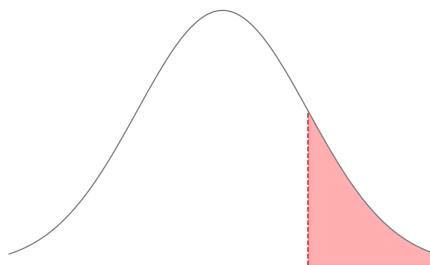
x_value = norm.ppf(p, loc=mu, scale=sigma)

```

## 19.5. SF (Survival Function)

Probability that a certain event has not occurred by a certain time

$$S(t) = P(T > t)$$



Relationship to PDF:

$$S(t) = 1 - F(t)$$

```

from scipy.stats import norm

z = 3.4
mu = 0
sigma = 1

norm.sf(z, loc=mu, scale=sigma)

```

# 20. Error Metrics

## 20.1. MAE (Mean Absolute Error)

Average of squared differences between predicted ( $\hat{y}_i$ ) and actual values ( $y_i$ )

$$\text{MAE} = \frac{1}{n} \sum_{i=1}^n |y_i - \hat{y}_i|$$

## 20.2. MSE (Mean Squared Error)

Average of squared differences between predicted ( $\hat{y}_i$ ) and actual ( $y_i$ ) values

$$\text{MSE} = \frac{1}{2} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

## 20.3. RMSE (Root Mean Squared Error)

square root of the average squared differences between predicted ( $\hat{y}_i$ ) and actual ( $y_i$ ) values

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2}$$

## 20.4. MAPE (Mean Absolute Percentage Error)

Average percentage difference between predicted ( $\hat{y}_i$ ) and actual ( $y_i$ ) values

$$\text{MAPE} = \frac{100}{n} \sum_{i=1}^n \left| \frac{y_i - \hat{y}_i}{y_i} \right|$$

## 20.5. R-squared

Proportion of variance explained by the model

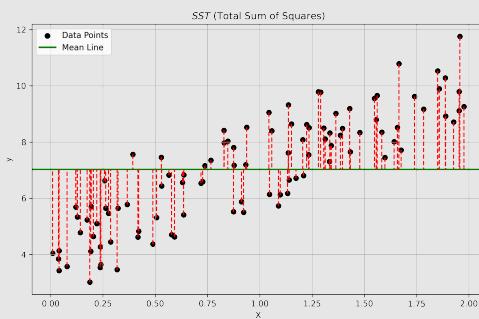
$$R^2 = 1 - \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{\sum_{i=1}^n (y_i - \bar{y})^2}$$

- 1: model explains all the variance in the dependent variable
- 0: model explains none of the variance in the dependent variable

**Step 1:** Fit the Regression Model

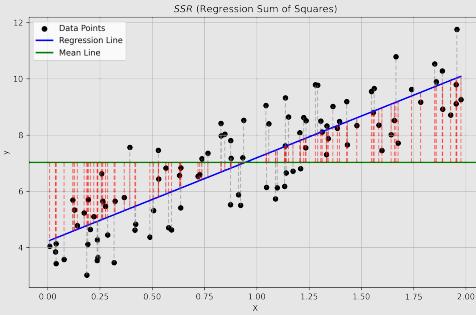
**Step 2:** Compute **Total Sum of Squares (SST)**

$$\text{SST} = \sum_{i=1}^n (y_i - \bar{y})^2$$



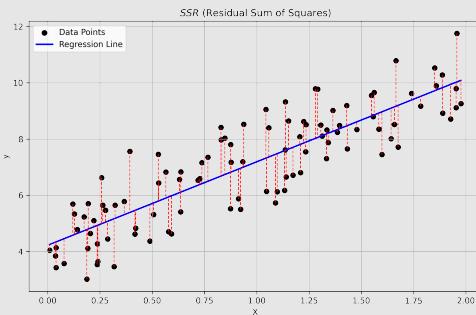
**Step 3:** Compute **Regression Sum of Squares (SSR)**

$$SSR = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2$$



#### Step 4: Compute Residual Sum of Squares (SSE)

$$SSE = \sum_{i=1}^n (y_i - \hat{y}_i)^2$$



#### Step 5: Calculate $R^2$

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

### 20.6. Adj R-squared

Adjusts the  $R^2$  value based on the number of predictors (penalty for adding non-informative variables)

$$\text{Adj } R^2 = 1 - (1 - R^2) \frac{n - 1}{n - p - 1}$$

### 20.7. MSLE (Mean Squared Logarithmic Error)

When predictions and actual values span several orders of magnitude

$$\text{MSLE} = \frac{1}{n} \sum_{i=1}^n (\log(1 + y_i) - \log(1 + \hat{y}_i))^2$$

### 20.8. Cross-Entropy Loss (Log Loss)

Binary and multi-class classification

$$\text{Log Loss} = -\frac{1}{n} \sum_{i=1}^n [y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i)]$$

## 21. Hypothesis Testing

### 21.1. Hypotheses

#### 21.1.1. Null ( $H_0$ )

No effect or difference (observed effect is due to sampling variability)

$$H_0 : \mu = \mu_0$$

#### 21.1.2. Alternative ( $H_1$ or $H_a$ )

Presence of an effect or a difference

- Two-Tailed

$$H_1 : \mu \neq \mu_0$$

- Right-Tailed

$$H_1 : \mu > \mu_0$$

- Left-Tailed

$$H_1 : \mu < \mu_0$$

### 21.2. Error Types

#### 21.2.1. Type I ( $\alpha$ )

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

#### 21.2.2. Type II ( $\beta$ )

$$\beta = P(\text{Fail to Reject } H_0 \mid H_1 \text{ is true})$$

### 21.3. t-Tests

#### 21.3.1. One-sample

Tests if the mean of a single sample differs from a known or hypothesized population mean.

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

- $\bar{x}$ : sample mean
- $\mu_0$ : hypothesized population mean
- $s$ : sample standard deviation
- $n$ : sample size

$[78, 82, 89]$

Step 1: State Hypotheses

$H_0 : \mu = 85 \text{ cm}$

$$H_1 : \mu \neq 85 \text{ cm}$$

Step 2: Summarize Data

- Sample values:

$$x_1 = 78, x_2 = 82, x_3 = 89$$

- Sample size:

$$n = 3$$

Step 3: Calculate Sample Mean ( $\bar{x}$ )

$$\bar{x} = \frac{x_1 + x_2 + x_3}{n} = \frac{78 + 82 + 89}{3} = 83$$

Step 4: Calculate Sample Standard Deviation:

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}$$

- Find the deviations from the mean and square them

$$(x_1 - \bar{x})^2 = (78 - 83)^2 = (-5)^2 = 25$$

$$(x_2 - \bar{x})^2 = (82 - 83)^2 = (-1)^2 = 1$$

$$(x_3 - \bar{x})^2 = (89 - 83)^2 = (6)^2 = 36$$

- Sum of squared deviations

$$\text{SSD} = 25 + 1 + 36$$

- Calculate variance of sample

$$s^2 = \frac{\text{SSD}}{n-1} = \frac{62}{3-1} = \frac{62}{2} = 31$$

- Calculate Sample Standard Deviation

$$s = \sqrt{s^2} = \sqrt{32} = 5.57$$

Step 5: Calculate the Test Statistic

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{83 - 85}{\frac{5.57}{\sqrt{3}}} = -\frac{3}{3.22} = -0.62$$

Step 6: Determine the Degrees of Freedom

$$df = n - 1 = 15 - 1 = 14$$

Step 7: Find Critical t-value

- For a two-tailed test at a significance level ( $\alpha$ ) of 0.05 and 2 degrees of freedom ( $df$ )

$$4.303$$

Step 8: Compare the t-Value to the Critical t-Value

- If the absolute value of the test statistic is greater than the critical t-value, reject the null hypothesis.
- If the absolute value of the test statistic is less than the critical t-value, fail to reject the null hypothesis.

Step 8: Find the p-Value

$$t = 4.303 \text{ and } df = 3$$

$$\text{p-value} = 0.58$$

```
from scipy import stats
rvs = stats.uniform.rvs(size=50)
stats.ttest_1samp(rvs, popmean=0.5)
```

### 21.3.2. Independent

Compares the means of two independent samples.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s_2^2}{n_2}\right)}}$$

- $\bar{x}_1$  and  $\bar{x}_2$ : sample means
- $s_1^2$  and  $s_2^2$ : sample variances
- $n_1$  and  $n_2$ : sample sizes

```
from scipy import stats
rvs1 = stats.norm.rvs(loc=5, scale=10, size=500)
rvs2 = stats.norm.rvs(loc=5, scale=10, size=500)
stats.ttest_ind(rvs1, rvs2)
```

### 21.3.3. Paired

$$t = \frac{\bar{D}}{\frac{s_D}{\sqrt{n}}}$$

- $\bar{D}$ : mean of the differences between paired observations
- $s_D$ : standard deviation of the differences
- $n$ : number of pairs

## 21.4. Chi-square tests

### 21.4.1. Goodness of Fit Test

Compares an **observed** categorical distribution to a **theoretical** categorical distribution.

$$\chi^2 = \sum_{i=1}^k \frac{(o_i - e_i)^2}{e_i}$$

- $o_i$ : observed frequency in category  $i$
- $e_i$ : expected frequency in category  $i$

```
from scipy import stats
f_obs = np.array([43, 52, 54, 40])
f_exp = np.array([47, 47, 47, 47])
stats.chisquare(f_obs=f_obs, f_exp=f_exp)
```

### 21.4.2. Test of independence

Compares two **observed** categorical distributions.

$$\chi^2 = \sum_{i=1}^k \frac{(o_{ij} - e_{ij})^2}{e_{ij}}$$

- $o_{ij}$ : observed frequency in cell  $(i, j)$
- $e_{ij}$ : expected frequency in category  $(i, j)$  calculated as:

$$E_{ij} = \frac{R_i \times C_j}{N}$$

- $R_i$ : Row total for row  $i$
- $C_j$ : Column total for column  $j$
- $N$ : Total number of observations

## 21.5. ANOVA (Analysis of Variance)

### 21.5.1. One-way

Compares the means of three or more groups based on one independent variable

- $H_0: \mu_1 = \mu_2 = \dots = \mu_k$
- $H_1$ : At least one  $\mu_i$  differs from the others

**Step 1:** Calculate Between-Group Variation ( $SS_{\text{between}}$ )

$$SS_{\text{between}} = \sum_{i=1}^k n_i (\bar{X}_i - \bar{X}_{\text{overall}})^2$$

- $n_i$ : Number of observations in group  $i$
- $\bar{X}_i$ : Mean of group  $i$
- $\bar{X}_{\text{overall}}$ : Overall mean of all groups

**Step 2:** Calculate Within-Group Variation ( $SS_{\text{within}}$ )

$$SS_{\text{within}} = \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$$

- $X_{ij}$ : Observation  $j$  in group  $i$

**Step 3:** Calculate Total Variation ( $SS_{\text{total}}$ )

$$SS_{\text{total}} = SS_{\text{between}} + SS_{\text{within}}$$

**Step 4:** Calculate Mean Squares

- Mean Square Between ( $MS_{\text{between}}$ )

$$MS_{\text{between}} = \frac{SS_{\text{between}}}{k - 1}$$

- Mean Square Within ( $MS_{\text{within}}$ )

$$MS_{\text{within}} = \frac{SS_{\text{within}}}{N - k}$$

- $N$ : total number of observations
- $k$ : number of groups

**Step 5:** Calculate the F-statistic

$$F = \frac{MS_{\text{between}}}{MS_{\text{within}}}$$

**Step 6:** Decision Rule

- Compare the F-statistic to the critical value from the F-distribution table (based on chosen significance level  $\alpha$ ).
- Alternatively, compare the p-value to the significance level  $\alpha$ .
- Reject  $H_0$  if the F-statistic is greater than the critical value or if the p-value is less than  $\alpha$ , indicating that at least one group mean is significantly different.

#### 21.5.2. Two-way

## 22. Regression Analysis:

### 22.1. Simple linear regression

$$y = \beta_0 + \beta_1 x + \varepsilon$$

- $y$ : dependent variable
- $x$ : independent variables
- $\beta_0$ : intercept (value of  $y$  when  $x = 0$ )
- $\beta_1$ : slope (change in  $y$  for a one-unit change in  $x$ )
- $\varepsilon$ : error term (difference between the actual data points and the predicted values)

Data

$x$ (Hours Studied)	$y$ (Test Score)
1	2
2	3
3	5
4	4
5	6

**Step 1:** Calculate Means

$$\bar{x} = 3$$

$$\bar{y} = 4$$

**Step 2:** Calculating Slope  $\beta_1$ 

$$\beta_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

- The numerator

$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) = 9$$

- The denominator

$$\sum_{i=1}^n (x_i - \bar{x})^2 = 10$$

- The slope  $\beta_1$  is

$$\beta_1 = \frac{9}{10} = 0.9$$

**Step 3:** Calculate Intercept  $\beta_0$ 

$$\beta_0 = \bar{y} - \beta_1 \bar{x} = 1.3$$

**Step 4:** Calculate p-Value

- Calculate Standard Error of the Slope ( $SE_{\beta_1}$ )

$$SE_{\beta_1} = \sqrt{\frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{(n-2) \cdot \sum_{i=1}^n (x_i - \bar{x})^2}}$$

- Calculate the Residual Sum of Squares (RSS)

$$RSS = \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

- Calculate the t-statistic for the Slope

$$t = \frac{\beta_1}{SE_{\beta_1}}$$

- Determine Degrees of Freedom

$$df = n - 2$$

- Look up the p-value corresponding to  $t$  with 3 degrees of freedom

**Step 5:** Calculate  $R^2$  ( $R_{\text{adj}}^2$ )

$$R^2 = \frac{SS_{\text{reg}}}{SS_{\text{total}}}$$

$SS_{\text{reg}}$  (Regression Sum of Squares): sum of the squared differences between the predicted  $\hat{y}$  values and the mean of the observed  $y$  values.

$$SS_{\text{reg}} = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2$$

$SS_{\text{total}}$  (Total Sum of Squares): sum of the squared differences between the observed  $y$  values and the mean of the observed  $y$  values.

$$SS_{\text{total}} = \sum_{i=1}^n (y_i - \bar{y})^2$$

Adjusting for the number of independent variables

$$R_{\text{adj}}^2 = 1 - \left( \frac{(1 - R^2)(n - 1)}{n - k - 1} \right)$$

- $n$ : number of observations
- $k$ : number of independent variables

## 22.2. Multiple regression

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \varepsilon$$

Matrix form:

$$Y = X\beta + \varepsilon$$

Estimating Coefficients (OLS):

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

Where:

- $\hat{\beta}$ : vector of estimated coefficients

## 22.3. Logistic Regression

Binary classification (1 or 0, true or false, yes or no) based on one or more predictor variables

Sigmoid function:

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Where:

$$z = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n$$

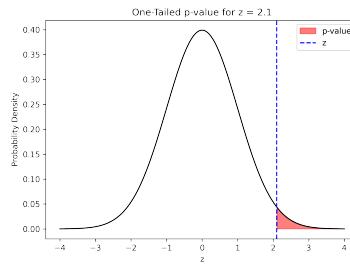
## 22.4. Model diagnostics

### 22.4.1. p-Values

Probability of obtaining results at least as extreme as the observed results

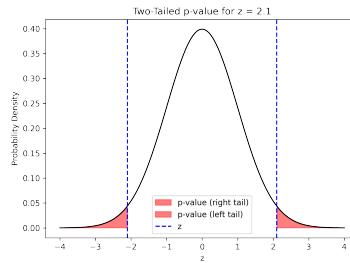
1. One-Tailed

$$p = P(Z > z_{\text{observed}})$$



## 2. Two-Tailed

$$p = 2 \cdot P(Z > |z_{\text{observed}}|)$$



```

z = 2.1
df = 3
scipy.stats.norm.sf(abs(z), df=df)

```

### 22.4.2. F-Statistic

$$F = \frac{\text{MSR}}{\text{MSE}}$$

Where:

- Mean Square Regression (MSR):

$$\text{MSR} = \frac{\text{SSR}}{\text{df}_{\text{regression}}}$$

- $\text{df}_{\text{regression}} = p$

Where:

- $p$ : Number of independent
- Mean Square Error (MSE):

$$\text{MSE} = \frac{\text{SSE}}{\text{df}_{\text{error}}}$$

- $\text{df}_{\text{error}} = n - p - 1$

Where:

- $p$ : Number of independent
- $n$ : Number of observations
- 1: Constant (for intercept)

#### 22.4.3. Confidence Intervals (CI)

Range within which we can be confident that the true value (population parameter) lies, based on the sample data

1. Known population standard deviation ( $\sigma$ ):

$$CI = \bar{x} \pm z \frac{\sigma}{\sqrt{n}}$$

Where:

- $\bar{x}$ : sample mean
- $z$ : z-score corresponding to the desired confidence level
- $\sigma$ : population standard deviation
- $n$ : sample size

2. Unknown population standard deviation ( $\sigma$ ):

$$CI = \bar{x} \pm t \frac{s}{\sqrt{n}}$$

Where:

- $\bar{x}$ : sample mean
- $t$ : critical value from t-distribution
- $s$ : sample standard deviation
- $n$ : sample size

## 23. Correlation

### 23.1. Pearson

Linear relationships

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \times (y_i - \bar{y})^2}}$$

Interpretation:

- 1: perfect positive linear relationship
- -1: perfect negative linear relationship
- 0: no linear relationship

$$X : [1, 2, 3, 4, 5]$$

$$Y : [2, 4, 6, 8, 10]$$

**Step 1:** Calculate the Means of X and Y

$$\bar{X} = 3$$

$$\bar{Y} = 6$$

**Step 2:** Calculate the Differences from the Mean

### 23.2. Spearman's Rank

Non-linear relationships

$$\rho = 1 - \frac{6 \sum_{i=1}^n d_i^2}{n(n^2 - 1)}$$

Where

- $\rho$ : Spearman rank correlation coefficient
- $d_i$ : difference between the ranks of corresponding values
- $n$ : number of observations

Rank: position of a value within a data set when the values are ordered in ascending order

Observation $i$	$X$	Rank $X$	$Y$	Rank $Y$	$d_i = \text{Rank } X - \text{Rank } Y$	$d_i^2$
1	3	3	8	5	-2	4
2	1	1	6	3	-2	4
3	4	4	7	4	0	0
4	2	2	4	1	1	1
5	5	5	5	2	3	9

Interpretation

- $\rho = 1$ : Perfect positive correlation
- $\rho = -1$ : Perfect negative correlation
- $\rho = 0$ : No correlation

## 24. Non-Parametric Statistics

### 24.1. Mann-Whitney U

Determine whether there is a significant difference between the distributions of two independent samples (used as an alternative to the independent samples t-test when the assumptions of normality are not met)

### 24.2. Wilcoxon Signed-Rank

Compare two paired samples or to assess whether the median of a single sample is different from a specified value (used as a non-parametric alternative to the paired t-test when the data does not meet the assumptions of normality)

### 24.3. Kolmogorov-Smirnov

Determine if a sample is drawn from a population with a specific distribution.

It compares the empirical distribution function (EDF) of the sample with the cumulative distribution function (CDF) of the reference distribution.

### 24.4. Kruskal-Wallis

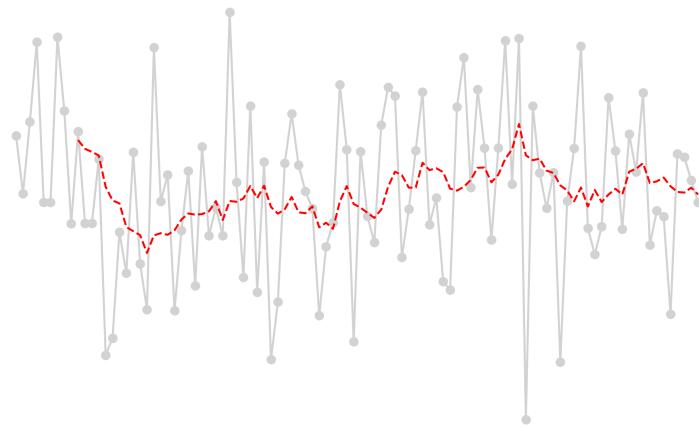
Determine if there are significant differences between the medians of three or more independent groups (extension of the Mann-Whitney U test, which is used for comparing two groups)

## 25. Time Series

### 25.1. SMA (Simple Moving Averages)

Creating a series of averages of different subsets (i.e., window) of the full data set

Minimize impact of short-term fluctuations



#### 3-Day Simple Moving Average for the 7 Day Time Series

Data

- Day 1: \$10
- Day 2: \$12
- Day 3: \$14
- Day 4: \$16
- Day 5: \$18
- Day 6: \$20
- Day 7: \$22

$$\text{SMA for Day 3} = \frac{\text{Day 1} + \text{Day 2} + \text{Day 3}}{3}$$

$$\text{SMA for Day 4} = \frac{\text{Day 2} + \text{Day 3} + \text{Day 4}}{3}$$

$$\text{SMA for Day 5} = \frac{\text{Day 3} + \text{Day 4} + \text{Day 5}}{3}$$

$$\text{SMA for Day 6} = \frac{\text{Day 4} + \text{Day 5} + \text{Day 6}}{3}$$

$$\text{SMA for Day 7} = \frac{\text{Day 5} + \text{Day 6} + \text{Day 7}}{3}$$

```
pl.col('X').rolling_mean(window_size)
```

## 25.2. WMA (Weighted Moving Average)

Each data point in the window is assigned a specific weight (usually decrease linearly)

$$\text{WMA} = \frac{\sum_{i=1}^n (x_i w_i)}{\sum_{i=1}^n w_i}$$

### Data

- Day 1: \$10
- Day 2: \$12
- Day 3: \$14
- Day 4: \$13
- Day 5: \$15

### Weights

- 1st most recent: 3
- 2nd most recent: 2
- 3rd most recent: 1

$$\text{WMA for Day 3} = \frac{(14 \times 3) + (12 \times 2) + (10 \times 1)}{3 + 2 + 1} = 12.67$$

$$\text{WMA for Day 4} = \frac{(13 \times 3) + (14 \times 2) + (12 \times 1)}{3 + 2 + 1} = 13.17$$

$$\text{WMA for Day 5} = \frac{(15 \times 3) + (13 \times 2) + (14 \times 1)}{3 + 2 + 1} = 14.7$$

## 25.3. Exponential Smoothing

Gives more weight to recent data points (more responsive to new information)

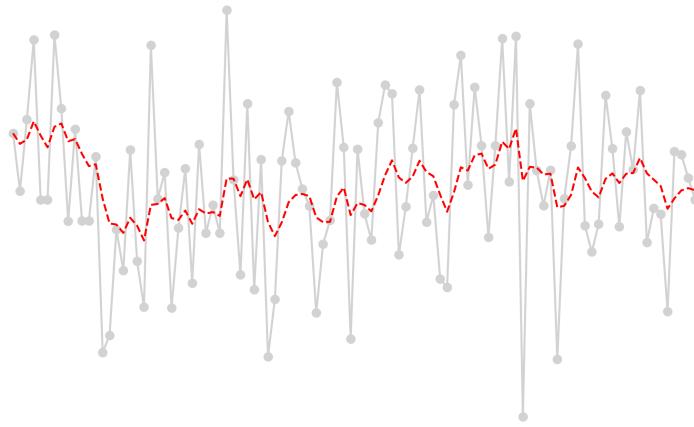
$$\text{EMA}_t = \alpha X_t + (1 - \alpha) \text{EMA}_{t-1}$$

Where

- $\alpha$ : smoothing factor

$$\alpha = \frac{2}{N + 1}$$

- $N$ : Period (i.e., window size) of the EMA



### Data

- Day 1: \$10
- Day 2: \$12
- Day 3: \$14
- Day 4: \$13
- Day 5: \$15

### Calculate Smoothing Factor

$$\alpha = \frac{2}{3+1} = 0.5$$

### Calculate SMA for First Value

$$\text{SMA for Day 3} = \frac{10 + 12 + 14}{3}$$

### Calculate EMA

$$\text{EMA for Day 4} = (0.5 \times 13) + (12 \times 0.5) = 12.5$$

$$\text{EMA for Day 5} = (0.5 \times 15) + (12.5 \times 0.5) = 13.75$$

```
pl.col("X").ewm_mean(span=window_size, adjust=False)
```

## 25.4. Seasonal Decomposition

- Level: baseline value around which the time series fluctuates
- Trend: long-term progression or direction of the time series
- Seasonality: regular, repeating patterns or cycles in the time series
- Noise: random fluctuations or irregular variations (cannot be explained by the trend, seasonality, or other components)

### Additive Decomposition

When the magnitude of seasonal fluctuations and trend does not change over time

$$Y_t = L_t + T_t + S_t + N_t$$

Where

- $L_t$ : Level at time  $t$
- $T_t$ : Trend at time  $t$
- $S_t$ : Seasonal component at time  $t$
- $N_t$ : Noise (residuals) at time  $t$

### **Multiplicative Decomposition**

When the magnitude of seasonal fluctuations and trend change proportionally over time

$$Y_t = L_t \times T_t \times S_t \times N_t$$

Where

- $L_t$ : Level at time  $t$
- $T_t$ : Trend at time  $t$
- $S_t$ : Seasonal component at time  $t$
- $N_t$ : Noise (residuals) at time  $t$

### **25.5. ARMA (AutoRegressive Moving Average)**

### **25.6. ARIMA (AutoRegressive Integrated Moving Average)**

# Industrial Engineering & Operations Research

## 26. Causes of variation

### 26.1. Common

Common Cause Variation: Normal, expected variation inherent to the process; predictable and stable.

- Temperature
- Humidity
- Material properties

### 26.2. Special

Special Cause Variation: Abnormal, unexpected variation due to specific causes; unpredictable and requires immediate action.

External disruptions

- Machine malfunction
- Faulty raw materials
- Staff shortages
- Weather events

## 27. Design of Experiments (DOE):

## 28. Control Charts:

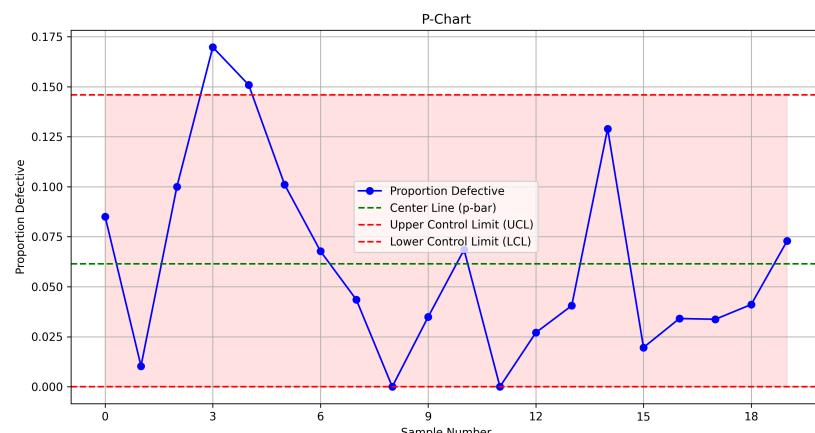
### 28.1. P-charts (Proportion)

Proportion of defective items

$$\hat{p} = \frac{D}{n}$$

Where:

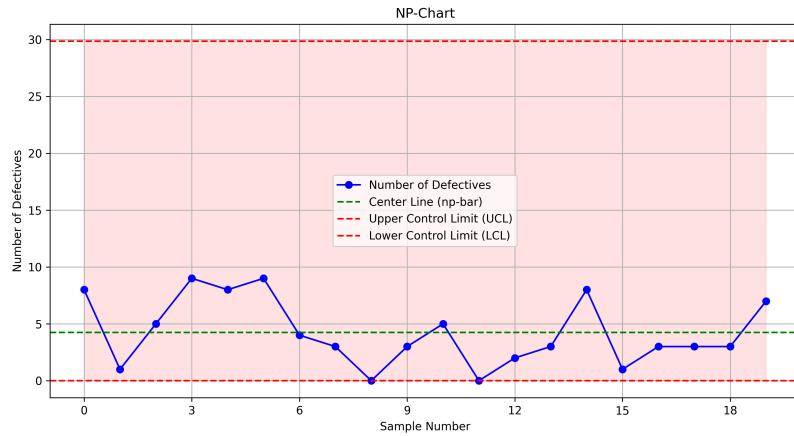
- $D$ : number of defective items
- $n$ : sample size



## 28.2. NP-charts (Number Proportion)

Number of defective items (constant sample size)

- $N$ : count of defective items in each sample



## 28.3. C-charts (Count)

Count of defects (fixed unit size)

Number of defects observed in each sample or inspection unit

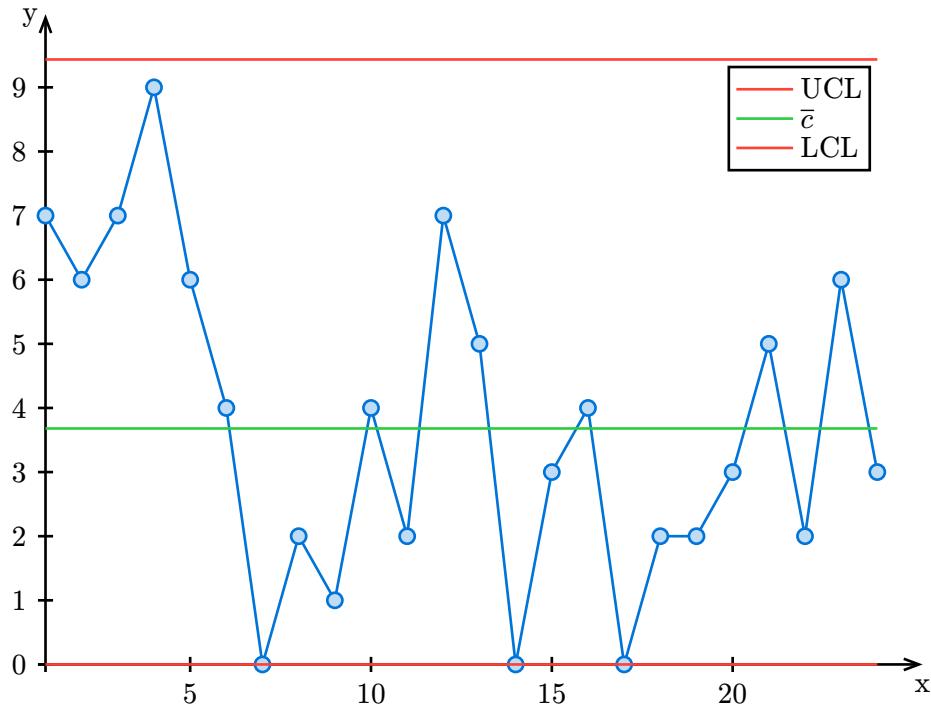
$$\bar{c} = \frac{\sum_{i=1}^k c_i}{k}$$

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}}$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}}$$

Where:

- $c$ : number of defects
- $k$ : number of samples



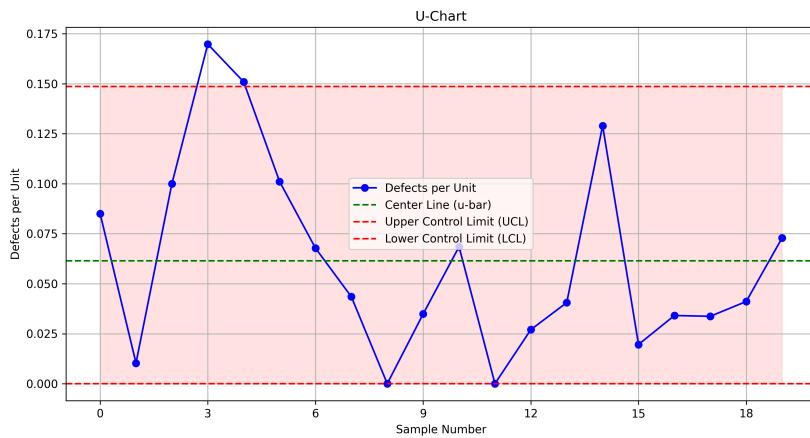
## 28.4. U-charts (Unit)

Defects per unit (variable unit size)

$$\hat{u} = \frac{C}{n}$$

Where:

- $C$ : number of defects
- $n$ : size of unit



## 28.5. $\bar{X}$ -chart

## 28.6. R-chart

# 29. Process Capability Analysis

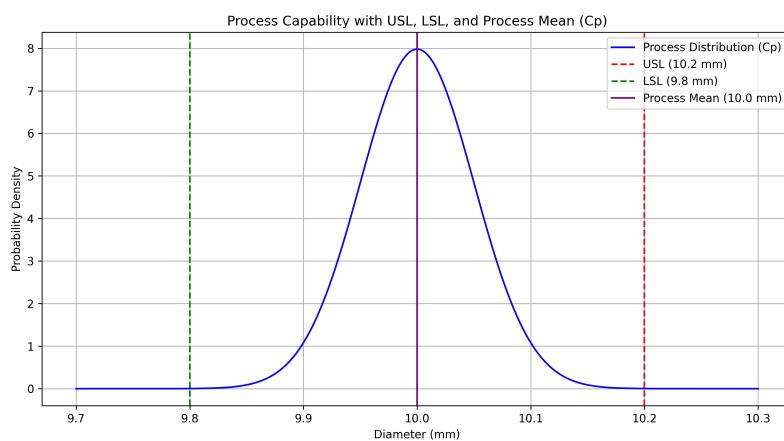
## 29.1. $C_p$ (Process Capability Index)

Measure how well a process can produce outputs within specified limits

$$C_p = \frac{USL - LSL}{6\sigma}$$

- $C_p > 1$ : The process variation is smaller than the specification range (good capability).
- $C_p = 1$ : The process variation matches the specification range (barely acceptable).
- $C_p < 1$ : The process variation exceeds the specification range (poor capability).

**Assumption:** Process is **centered** within the specification limits



Suppose a company manufactures metal rods, and the specification limits for the diameter of the rods are:

- Upper Specification Limit (USL): 10.2 mm
- Lower Specification Limit (LSL): 9.8 mm

The process has a standard deviation **0.05** of 0.05 mm.

**Step 1:** Determine the Specification Width

The specification width is the difference between the USL and LSL.

$$\text{Specification Width} = \text{USL} - \text{LSL} = 10.2 \text{ mm} - 9.8 \text{ mm}$$

**Step 2:** Calculate the Process Capability Index  $C_p$

The formula for  $C_p$  is:

$$C_p = \frac{\text{Specification Width}}{6\sigma} = \frac{\text{USL} - \text{LSL}}{6\sigma}$$

Substitute the values:

$$C_p = \frac{0.4 \text{ mm}}{6 \times 0.05 \text{ mm}} = \frac{0.4 \text{ mm}}{0.3 \text{ mm}} = 1.33$$

Interpretation:

$C_p = 1.33$  means the process spread (6 0.05) fits 1.33 times within the tolerance range (the distance between the Upper Specification Limit and Lower Specification Limit).

- $C_p = 1.00$ : Process variation fits exactly within the specification limits. 99.73% of the output will be within specifications **if the process is centered** (3 sigma process).
- $C_p > 1.00$ : Process variation is narrower than the specification limits. The higher the  $C_p$ , the more capable the process is, meaning it can produce parts within the tolerance more consistently.
- $C_p < 1$ : Process variation is wider than the specification limits. Significant portion of the output will fall outside the specification limits.

Limitations:

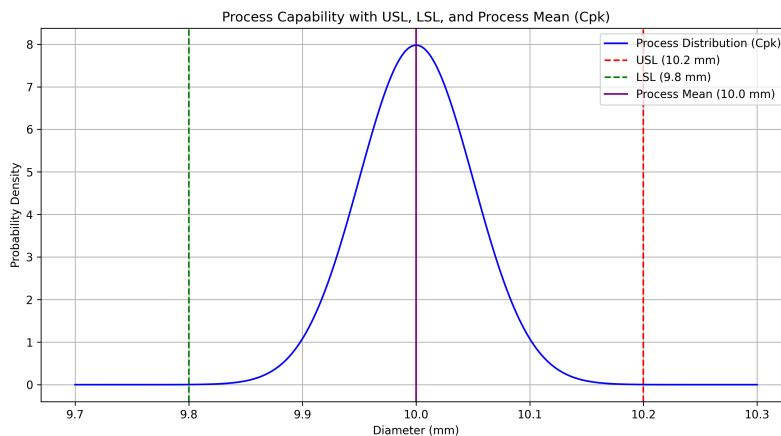
Since  $C_p$  **does not account for the centering of the process**, it may give a false sense of security if the process mean is off-center (see  $C_{pk}$ ).

Use:

When you are interested in understanding the **potential capability** of a process under ideal conditions, typically in a short-term study where the process is stable and controlled.

## 29.2. $C_{pk}$ (Process Capability Index with Centering)

$$C_{pk} = \text{Min} \left( \frac{\text{USL} - \bar{x}}{3\sigma}, \frac{\bar{x} - \text{LSL}}{3\sigma} \right)$$



Suppose a company manufactures metal rods, and the specification limits for the diameter of the rods are:

- Upper Specification Limit (**USL**): 10.2 mm
- Lower Specification Limit (**LSL**): 9.8 mm

The process has:

- A standard deviation  $\sigma$  of 0.05 mm.

- A process mean  $\mu$  of 10.1 mm.

**Step 1:** Calculate the distance from the mean to the USL:

$$\frac{USL - \mu}{3\sigma} = \frac{10.2 \text{ mm} - 10.1 \text{ mm}}{3 \times 0.05 \text{ mm}} = \frac{0.1 \text{ mm}}{0.15 \text{ mm}} = 0.67$$

**Step 2:** Calculate the distance from the mean to the LSL:

$$\frac{\mu - LSL}{3\sigma} = \frac{10.1 \text{ mm} - 9.8 \text{ mm}}{3 \times 0.05 \text{ mm}} = \frac{0.3 \text{ mm}}{0.15 \text{ mm}} = 2.00$$

**Step 3:** Determine  $C_{pk}$ :

$$C_{pk} = \min(0.67, 2.00) = 0.67$$

Interpretation:

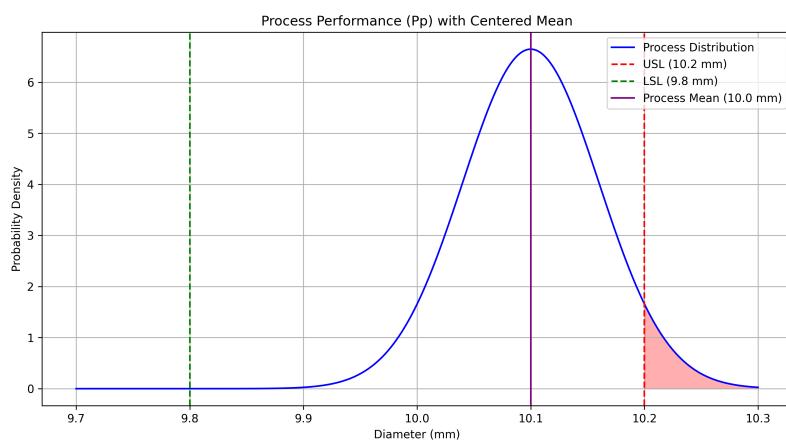
$C_p = 1.33$  means the process spread ( $6\sigma$ ) fits 1.33 times within the tolerance range (the distance between the Upper Specification Limit and Lower Specification Limit).

- $C_p = 1.00$ : Process mean is exactly at the midpoint of the specification limits, and the process variation fits exactly within these limits. 99.73% of the output will be within specifications, indicating a capable process (3 sigma process).
- $C_p > 1.00$ : The higher the  $C_{pk}$ , the more capable and stable the process is, meaning it can consistently produce parts within tolerance with minimal risk of defects.
- $C_p < 1$ : The process mean is off-center or the variation is wider than the specification limits, or both. A significant portion of the output may fall outside the specification limits.

### 29.3. $C_{pm}$ (Taguchi Capability Index)

### 29.4. $P_p$ (Process Performance Index)

$$P_p = \frac{USL - LSL}{6\sigma_{\text{overall}}}$$



Suppose a company manufactures metal rods, and the specification limits for the diameter of the rods are:

- Upper Specification Limit (**USL**): 10.2 mm
- Lower Specification Limit (**LSL**): 9.8 mm
- Overall standard deviation ( $\sigma_{\text{overall}}$ ): 0.06 mm

### Step 1: Determine the Specification Width

The specification width is the difference between the USL and LSL.

$$\text{Specification Width} = \text{USL} - \text{LSL} = 10.2 \text{ mm} - 9.8 \text{ mm} = 0.4$$

### Step 2: Calculate the Process Performance Index $P_p$

The formula for  $P_p$  is:

$$P_p = \frac{\text{Specification Width}}{6\sigma_{\text{overall}}} = \frac{\text{USL} - \text{LSL}}{6\sigma_{\text{overall}}}$$

Substitute the values:

$$P_p = \frac{0.4 \text{ mm}}{6 \times 0.06 \text{ mm}} = \frac{0.4 \text{ mm}}{0.36 \text{ mm}} = 1.11$$

Interpretation:

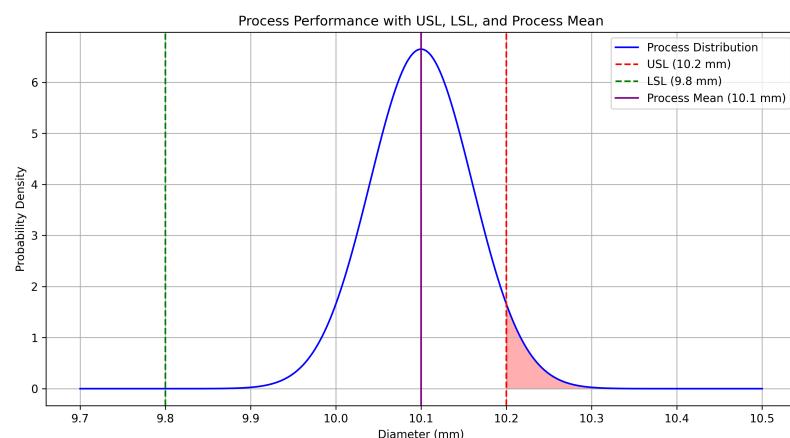
- A  $P_p$  of 1.11 indicates that the process performance, considering all sources of variation over time, is capable but less so than the potential capability indicated by  $C_p$ . The value being slightly above 1 suggests that the process can generally produce rods within specifications, but there might be more variability in the process compared to the short-term capability measured by  $C_p$
- Decrease from the  $C_p$  value (1.33 to 1.11) reflects the impact of additional variability when evaluating the process over a longer time or under different conditions.

Use:

When you need to evaluate the **actual performance** of a process over a longer period, considering all sources of variation, including shifts, drifts, and other long-term factors.

### 29.5. $P_{pk}$ (Process Performance Index with Centering)

$$P_{pk} = \text{Min} \left( \frac{\text{USL} - \mu_{\text{overall}}}{3\sigma}, \frac{\mu_{\text{overall}} - \text{LSL}}{3\sigma} \right)$$



Suppose a company manufactures metal rods, and the specification limits for the diameter of the rods are:

- Upper Specification Limit (**USL**): 10.2 mm
- Lower Specification Limit (**LSL**): 9.8 mm
- Overall standard deviation ( $\sigma_{\text{overall}}$ ): 0.06 mm
- Overall process mean ( $\mu_{\text{overall}}$ ): 10.1 mm

**Step 1:** Calculate the Distance from the Process Mean to the Specification Limits

Calculate the distance from the overall process mean to both the USL and LSL:

$$\text{USL} - \mu_{\text{overall}} = 10.2 \text{ mm} - 10.1 \text{ mm} = 0.1 \text{ mm}$$

$$\mu_{\text{overall}} - \text{LSL} = 10.1 \text{ mm} - 9.8 \text{ mm} = 0.3 \text{ mm}$$

**Step 2:** Calculate the Process Performance Index  $P_k$

The formula for  $P_k$  is:

$$P_{pk} = \text{Min} \left( \frac{\text{USL} - \mu_{\text{overall}}}{3\sigma}, \frac{\mu_{\text{overall}} - \text{LSL}}{3\sigma} \right)$$

Substitute the values:

$$P_{pk} = \text{Min} \left( \frac{0.1 \text{ mm}}{3 \times 0.06 \text{ mm}}, \frac{0.3 \text{ mm}}{3 \times 0.06 \text{ mm}} \right)$$

$$P_{pk} = \text{Min} \left( \frac{0.1 \text{ mm}}{0.18 \text{ mm}}, \frac{0.3 \text{ mm}}{0.18 \text{ mm}} \right)$$

$$P_{pk} = \min(0.56, 1.67) = 0.56$$

Interpretation:

- A  $P_p$  of 1.11 indicates that the process performance, considering all sources of variation over time, is capable but less so than the potential capability indicated by  $C_p$ . The value being slightly above 1 suggests that the process can generally produce rods within specifications, but there might be more variability in the process compared to the short-term capability measured by  $C_p$
- Decrease from the  $C_p$  value (1.33 to 1.11) reflects the impact of additional variability when evaluating the process over a longer time or under different conditions.

Use:

When you need to evaluate the **actual performance** of a process over a longer period, considering all sources of variation, including shifts, drifts, and other long-term factors.

## 30. Inventory Management

### 30.1. Newsvendor

determine the optimal order quantity  $Q^*$  that minimizes the total expected cost or maximizes the expected profit, based on the trade-off between the overage and underage costs

Assumptions

- Products are separable

- Planning is done for a single period
- Demand is random
- Deliveries are made in advance of demand
- Costs of overage or underage are linear

1. Parameters

- $P$ : Sale Price
- $C$ : Purchase Cost
- $S$ : Unsold Value
- $\mu$ : Mean Demand
- $\sigma$ : Standard Deviation Demand

2. Calculate Underage and Overage Costs:

- Underage Cost ( $C_u$ ): Profit lost for each unit of demand not met

$$C_u = P - C$$

- Overage Cost ( $C_o$ ): This is the cost of holding an unsold newspaper.

$$C_o = C - S$$

3. Calculate Critical Ratio ( $CR$ )

$$CR = \frac{C_u}{C_u - C_o}$$

4. Find z-score: Find the number of standard deviations away from the mean corresponding to the critical ratio:

$$z^* = \Phi^{-1}(CR)$$

Where:

- $\Phi^{-1}$ : Inverse of the CDF of the standard normal distribution (PPF)

5. Calculate Optimal Order Quantity ( $Q^*$ )

$$Q^* = \mu + z^* \sigma$$

Where:

- $z^*$ : z-score corresponding to the critical ratio  $CR$  from the standard normal distribution

Consider a newsvendor selling newspapers:

1. Parameters

- Sale Price ( $P$ ): \$3 (per unit)
- Purchase Cost ( $C$ ): \$1 (per unit)
- Unsold Value ( $S$ ): \$0 (per unit)
- Mean Demand ( $\mu$ ): 100 (units)
- Standard Deviation Demand ( $\sigma$ ): 20 (units)

2. Calculate Underage and Overage Costs:

- Underage Cost ( $C_u$ ): Profit lost for each unit of demand not met

$$C_u = P - C = 3 - 1 = 2$$

- Overage Cost ( $C_o$ ): This is the cost of holding an unsold newspaper.

$$C_o = C - S = 1 - 0 = 1$$

### 3. Calculate Critical Ratio ( $CR$ )

$$CR = \frac{C_u}{C_u - C_o} = \frac{2}{2 + 1} = \frac{2}{3} = 0.67$$

### 4. Find z-score: Find the number of standard deviations away from the mean corresponding to the critical ratio:

$$z^* = \Phi^{-1}(CR) = 0.44$$

### 5. Calculate Optimal Order Quantity ( $Q^*$ )

$$Q^* = \mu + z^* \sigma = 100 + 0.44 \cdot 20 = 108.8$$

```

import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt

# Parameters for the newsvendor example
selling_price = 3 # Selling price per newspaper
purchase_cost = 1 # Purchase cost per newspaper
unsold_value = 0 # Value of unsold newspapers
mu = 100 # Mean of demand
sigma = 20 # Standard deviation of demand

# Calculate underage and overage costs
C_u = selling_price - purchase_cost # Underage cost
C_o = purchase_cost - unsold_value # Overage cost

# Calculate the critical ratio
CR = C_u / (C_u + C_o)

# Assume a normal distribution for demand
mu = 50 # Mean demand
sigma = 10 # Standard deviation of demand

# Calculate the optimal order quantity (Q*)
z_star = stats.norm.ppf(CR) # z-score corresponding to the critical ratio
Q_star = mu + z_star * sigma # Optimal order quantity

```

## 30.2. ABC Analysis

categorizes inventory items into three groups (A, B, and C) based on their importance

- A: Top 70-80% of the total annual consumption value
- B: Next 15-25% of the total annual consumption value
- C: Remaining 5-10% of the total annual consumption value

Item	Usage Quantity (Annual)	Unit Cost	Consumption Value (Annual)
$I_1$	50	\$100	\$5000
$I_2$	150	\$20	\$3000
$I_3$	300	\$10	\$3000
$I_4$	400	\$5	\$2000
$I_5$	500	\$1	\$500

**Step 1:** Calculate Annual Consumption Values

**Step 2:** Sort Items by Annual Consumption Value (Descending)

**Step 3:** Calculate Total Annual Consumption Value

$$\text{Total} = \$5,000(I_1) + \$3,000(I_2) + \$3,000(I_3) + \$2,000(I_4) + \$500(I_5) = \$13500$$

**Step 4:** Calculate Cumulative Consumption Value Percentages

- $I_1 : \frac{5000}{13500} \times 100\% = 37.04\%$
- $I_2 : \frac{3000}{13500} \times 100\% = 22.22\%$
- $I_3 : \frac{3000}{13500} \times 100\% = 22.22\%$
- $I_4 : \frac{2000}{13500} \times 100\% = 14.81\%$
- $I_5 : \frac{500}{13500} \times 100\% = 3.7\%$

**Step 5:** Cumulative Percentages

- $I_1 : 37.04\%$
- $I_1 + I_2 : 59.26\%$
- $I_1 + I_2 + I_3 : 81.48\%$
- $I_1 + I_2 + I_3 + I_4 : 96.3\%$
- $I_1 + I_2 + I_3 + I_4 + I_5 : 100\%$

**Step 6:** Categorize Items

- A:  $I_1, I_2, I_3$
- B:  $I_4$
- C:  $I_5$

### 30.3. Fill Rate

Percentage of customer demand that is satisfied from available inventory

$$F = \frac{U_f}{U_o} \times 100\%$$

Where:

- $F$ : Fill Rate
- $U_f$ : Number of units fulfilled
- $U_o$ : Total number of units ordered

Customers order 100 units of a product and 90 units are fulfilled from stock:

$$F = \frac{90}{100} \times 100\% = 90\%$$

### 30.4. (OCT) Order Cycle Time

Measures the total time taken from when a customer places an order to when the order is delivered

$$\text{OCT} = T_{\text{order}} + T_{\text{processing}} + T_{\text{production}} + T_{\text{shipping}}$$

Where:

- $T_{\text{order}}$ : Order Entry Time (time it takes to receive and log the order)
- $T_{\text{processing}}$ : Order Processing Time (time to check inventory, verify details, and prepare for production or shipment)
- $T_{\text{production}}$ : Production Time (time to manufacture or prepare the product)
- $T_{\text{shipping}}$ : Shipping Time (time it takes to deliver the product from the warehouse to the customer)

### 30.5. ROP (Reorder Point)

The inventory level at which a new order should be placed to avoid stockouts

$$\text{ROP} = (\text{Average Demand per Period} \times \text{Lead Time})$$

Suppose your business sells 50 units per week, and the lead time for a new order is 2 weeks.

Using the formula:

$$\text{ROP} = 50(\text{units per week}) \times 2(\text{weeks}) = 100 \text{ units}$$

This means that when your inventory level drops to 100 units, you should place a new order to avoid running out of stock.

### 30.6. XYZ Analysis

Categorization based on variability

- X: Low variability
- Y: Moderate variability
- Z: High variability

Coefficient of Variation:

$$CV = \frac{\sigma}{\mu}$$

- X Items: Low CV
  - ▶  $CV < k_1$ , where  $k_1$  is the threshold value indicating low variability
- Y Items: Moderate CV

- $CV < k_2$ , where  $k_2$  is the moderate value indicating low variability
- Z Items: High CV
  - $CV < k_3$ , where  $k_3$  is the threshold value indicating high variability

**Step 1:** Collect Historical D./ata (12 months)

- Product A: [100, 105, 98, 102, 101, 104, 103, 100, 99, 100, 101, 102]
- Product B: [150, 155, 145, 160, 140, 150, 155, 150, 165, 155, 150, 140]
- Product C: [200, 180, 220, 190, 210, 240, 180, 230, 220, 210, 250, 190]

**Step 2:** Calculate the Mean and Standard Deviation

- Product A:
  - $\mu = 101$
  - $\sigma = 2$
- Product B:
  - $\mu = 150$
  - $\sigma = 8$
- Product C:
  - $\mu = 210$
  - $\sigma = 25$

**Step 3:** Calculate the Coefficient of Variation (CV)

- $CV = \frac{2}{101} = 0.0198$
- $CV = \frac{8}{150} = 0.0533$
- $CV = \frac{25}{210} = 0.1190$

**Step 4:** Categorization

- X: Product A (Low variability)
- Y: Product B (Moderate variability)
- Z: Product C (High variability)

### 30.7. EOQ (Economic Order Quantity)

Optimal order quantity that **minimizes the total cost**, which includes both **holding costs** and **ordering costs**

$$EOQ = \sqrt{\frac{2DS}{H}}$$

Where:

- $D$ : Demand per time period
- $S$ : Ordering cost per order
- $H$ : Holding cost per unit, per time period

A company sells widgets and wants to determine the optimal order quantity for inventory.

- The annual demand for widgets ( $D$ ) is 12,000 units.
- The cost to place an order ( $S$ ) is \$50.

The holding cost per unit per year ( $H$ ) is \$2.

$$EOQ = \sqrt{\frac{2 \times 12000 \times 50}{2}} = 775$$

The company should order 775 widgets each time they place an order to minimize the total cost, which includes both ordering and holding costs

### 30.7.1. Perfect Order Rate

Measures the percentage of orders delivered to customers in full, on time, and without any damage

$$\text{Perfect Order Rate} = \frac{\text{Number of Perfect Orders}}{\text{Total Number of Orders}} \times 100\%$$

Where:

- **Number of Perfect Orders:** The number of orders that are delivered on time, complete, and undamaged.
- **Total Number of Orders:** The total number of orders fulfilled within a specific period.

Suppose you received 1,000 orders over a quarter, and 900 of those orders were delivered on time, complete, and without damage.

$$\text{Perfect Order Rate} = \frac{900}{1000} \times 100\% = 90\%$$

## 30.8. Safety Stock

Additional quantity of inventory kept on hand to protect against uncertainties in demand or supply. buffer to prevent stockouts due to unexpected variations in demand or delays in supply.

### 1. Constant Demand & Constant Lead Time

$$SS = Z \times \sigma_D$$

Where:

- $Z$ : Z-score corresponding to the desired service level
- $\sigma_D$ : standard deviation of demand

### 2. Variable Demand & Constant Lead Time

$$SS = Z \times \sigma_D \times \sqrt{L}$$

Where:

- $Z$ : Z-score corresponding to the desired service level
- $\sigma_D$ : Standard deviation of demand per unit of time
- $L$ : Lead time

### 3. Variable Lead Time & Constant Demand

$$SS = Z \times \bar{D} \times \sigma_L$$

Where:

- $Z$ : Z-score corresponding to the desired service level
- $\bar{D}$ : Average demand
- $\sigma_L$ : Standard deviation of lead time

#### 4. Variable Demand & Variable Lead Time

$$SS = Z \times \sqrt{(\bar{D}^2 \times \sigma_D^2) + (L \times \sigma_L^2)}$$

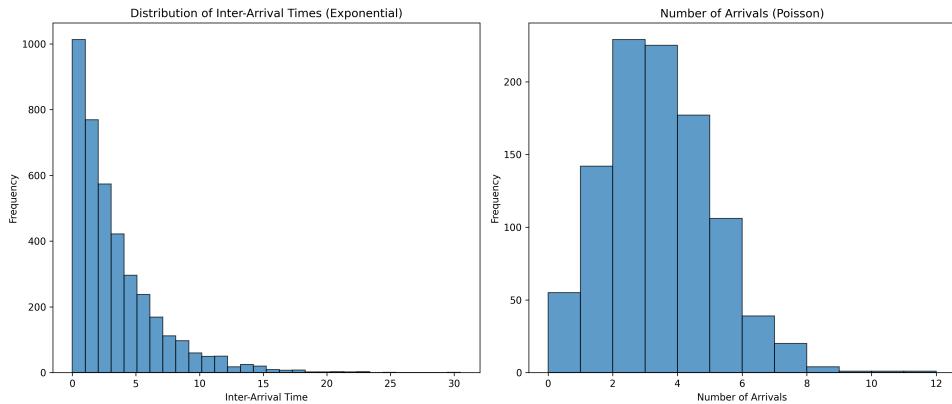
Where:

- $\bar{L}$ : Average Lead Time (average time it takes to receive inventory after placing an order)
- $\sigma_D^2$ : Demand Variance (variability in demand during the lead time)
- $\bar{D}$ : Average Demand (mean quantity of demand per time period)
- $\sigma_L^2$ : Lead Time Variance (variability in lead time)

### 31. Queuing Theory

#### 31.1. M/M/1

- Arrival rate ( $\lambda$ ): Average number of customers arriving per unit of time
- Service rate ( $\mu$ ): Average number of customers served per unit of time



##### 1. Utilization ( $\rho$ )

Fraction of time the server is busy

$$\rho = \frac{\lambda}{\mu}$$

##### 2. Average Number of Customers in the System ( $L$ )

Average number of customers (both waiting and being served) in the system

$$L = \frac{\rho}{1 - \rho}$$

##### 3. Average Number of Customers in the Queue ( $L_q$ )

Average number of customers waiting in the queue

$$L_q = \frac{\rho^2}{1 - \rho}$$

##### 4. Average Time a Customer Spends in the System ( $W$ )

Average time a customer spends in the system (from arrival until they are done being served)

$$W = \frac{1}{\mu - \lambda}$$

5. Average Waiting Time in the Queue ( $W_q$ )

Average time a customer spends just waiting in line before being served

$$W_q = \frac{\rho}{\mu - \lambda}$$

6. Probability that the System is Empty ( $P_0$ )

Probability that there are zero customers in the system (no one is being served and no one is waiting)

$$P_0 = 1 - \rho$$

7. Probability that n Customers are in the System ( $P_n$ )

Probability that there are n customers in the system (either waiting or being served)

$$P_n = (1 - \rho) \cdot \rho^n$$

8. Probability the Queue is Full (if the Queue has Limited Capacity) ( $P_{n_{\max}}$ )

Probability that the system is at full capacity

$$P_{n_{\max}} = (1 - \rho) \cdot \rho^{n_{\max}}$$

9. System Throughput

Rate at which customers are served and leave the system

$$\text{Throughput} = \lambda$$

10. Expected Time in Service ( $W_s$ )

Average time a customer spends actually being served (not including waiting time)

$$W_s = \frac{1}{\mu}$$

11. Idle Time ( $1 - \rho$ )

Fraction of time that the server is idle (i.e., not serving any customers)

$$\text{Idle Time} = 1 - \rho$$

12. Probability of Having to Wait in the Queue ( $P_w$ )

Probability that an arriving customer will have to wait before being served, i.e., that the server is busy when the customer arrives

$$P_w = \rho$$

13. Variance of the Number of Customers in the System ( $\text{Var}(L)$ )

Variance of the number of customers in the system

$$\text{Var}(L) = \frac{\rho}{(1 - \rho)^2}$$

A bank with a single teller

- **Arrival rate ( $\lambda$ ):** On average, 4 customers arrive every 10 minutes ( $\lambda = 4$  customers per 10 minutes)
- **Service Rate ( $\mu$ ):** The teller can serve 6 customers every 10 minutes ( $\mu = 6$  customers per 10 minutes)

### 1. Utilization ( $\rho$ )

$$\rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.67$$

The teller is busy 67% of the time. The remaining 33% of the time, the teller is idle, waiting for the next customer

### 2. Average Number of Customers in the System ( $L$ )

$$L = \frac{\rho}{1 - \rho} = \frac{0.67}{1 - 0.67} = 2$$

On average, there are 2 customers in the coffee shop at any given time, either being served or waiting in line

### 3. Average Number of Customers in the Queue ( $L_q$ )

$$L_q = \frac{\rho^2}{1 - \rho} = \frac{0.67^2}{1 - 0.67} = 1.33$$

On average, about 1.33 customers are waiting in line at any time

### 4. Average Time a Customer Spends in the System ( $W$ )

$$W = \frac{1}{\mu - \lambda} = \frac{1}{6 - 4} = 0.5$$

On average, a customer spends 5 minutes ( $0.5 \times 10$  minutes) in the shop (including both waiting in line and getting served)

### 5. Average Waiting Time in the Queue ( $W_q$ )

$$W_q = \frac{\rho}{\mu - \lambda} = \frac{0.67}{6 - 4} = 0.33$$

On average, a customer waits 3.3 minutes ( $0.33 \times 10$  minutes) in line before being served by the teller

### 6. Probability that the System is Empty ( $P_0$ )

$$P_0 = 1 - \rho = 1 - 0.67 = 0.33$$

There is a 33% chance that the coffee shop is empty, meaning there is no customer in the queue or being served

### 7. Probability that $n$ Customers are in the System ( $P_n$ )

$$P_n = (1 - \rho) \cdot \rho^n = (1 - 0.67) \cdot 0.67^n = 0.148$$

There is a 14.8% chance that exactly 2 customers are either in line or being served

8. Probability the Queue is Full (if the Queue has Limited Capacity) ( $P_{n_{\max}}$ )

$$P_{n_{\max}} = (1 - \rho) \cdot p^{n_{\max}} = (1 - 0.67) \cdot 0.67^5 = 0.028$$

There is a 2.8% chance that the system is full, and no new customers can enter

9. System Throughput

$$\text{Throughput} = \lambda = 4$$

The coffee shop serves 4 customers every 10 minutes, on average

10. Expected Time in Service ( $W_s$ )

$$W_s = \frac{1}{\mu} = \frac{1}{6} = 0.167$$

On average, a customer spends 1.67 minutes being served by the teller

11. Idle Time ( $1 - \rho$ )

$$\text{Idle Time} = 1 - \rho = 1 - 0.67 = 0.33$$

The teller is idle 33% of the time

12. Probability of Having to Wait in the Queue ( $P_w$ )

$$P_w = \rho = 0.67$$

There is a 67% chance that a customer will have to wait when they arrive

13. Variance of the Number of Customers in the System ( $\text{Var}(L)$ )

$$\text{Var}(L) = \frac{\rho}{(1 - \rho)^2} = \frac{0.67}{(1 - 0.67)^2} = 6.12^2$$

The queue length varies significantly, with a variance of 6.12 customers

Costs

- **Cost per Waiting Customer per Unit Time ( $C_w$ )**: cost incurred per customer for each unit of time they spend waiting in the queue

$$\text{Total Waiting Cost} = L_q \times C_w \times \text{Unit Time}$$

- **Cost per Idle Server per Unit Time ( $C_s$ )**: cost incurred per unit time when the server is not serving customers

$$\text{Total Idle Cost} = (1 - \rho) \times C_s \times \text{Unit Time}$$

Total Cost

$$\text{Total Cost} = \text{Total Waiting Cost} + \text{Total Idle Cost}$$

- Arrival Rate ( $\lambda$ ): 4 customers per 10 minutes
- Service Rate ( $\mu$ ): 6 customers per 10 minutes
- Cost per Waiting Customer per Hour ( $C_w$ ): \$10
- Cost per Idle Server per Hour ( $C_s$ ): \$20

- Operational Time: 1 hour

### 1. Utilization

$$\rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.67$$

### 2. Average Number of Customers in Queue ( $L_q$ )

$$L_q = \frac{\rho^2}{1 - \rho} = \frac{0.67^2}{1 - 0.67} = 1.33 \text{ cusommers}$$

### 3. Total Waiting Cost

$$\text{Total Waiting Cost} = L_q \times C_w \times \text{Unit Time} = 1.33 \times 10 \times 1 = \$13.33$$

### 4. Idle Time

$$\text{Idle Time} = 1 - \rho = 1 - 0.67 = 0.33$$

### 5. Total Idle Cost

$$\text{Total Idle Cost} = (1 - \rho) \times C_s \times \text{Unit Time} = 0.33 \times 20 \times 1 = \$6.67$$

### 6. Total Cost

$$\text{Total Cost} = \text{Total Waiting Cost} + \text{Total Idle Cost} = 13.33 + 6.67 = \$20$$

## 32. Network Optimization

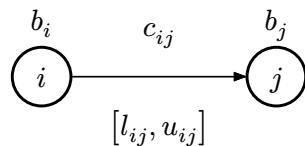
### 32.1. Shortest Path

### 32.2. Maximum Flow

### 32.3. Netwrok Flow Optimization

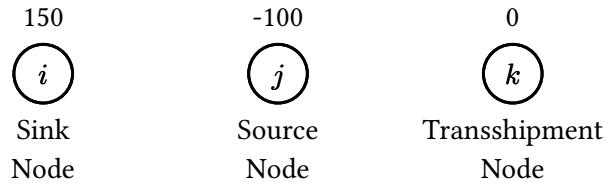
Types of Problems:

- Maximum Flow
- Minimum Cost Flow
- Multi-Commodity Flow



Node  $i$

- Sink node:  $b_i > 0$  Has demand of  $b_i$  units
- Source node:  $b_i < 0$  Has supply of  $-b_i$  units
- Transsipation node:  $b_i = 0$  Neither supply or demand



Units shipped from node  $i$  to node  $j$ :

$$x_{ij}$$

Minimize:

$$Z = \sum_{i,j} c_{ij} x_{ij}$$

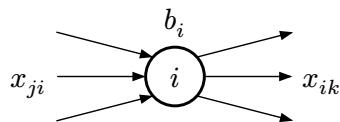
s.t.

$$\sum_k x_{ki} - \sum_l x_{il} = \text{or } \leq \text{ or } \geq b_1$$

$$l_{ij} \leq x_{ij} \leq u_{ij}$$

Where:

- $c_{ij}$ : Unit cost of flow from node  $i$  to node  $j$
- $b_i$ : Demand on node  $i$
- $l_{ij}$ : flow lower bound from  $i$  to  $j$
- $u_{ij}$ : Flow upper bound from  $i$  to  $j$
- $\sum_k x_{ki}$ : Inflow to  $i$
- $\sum_l x_{il}$ : Outflow from  $j$



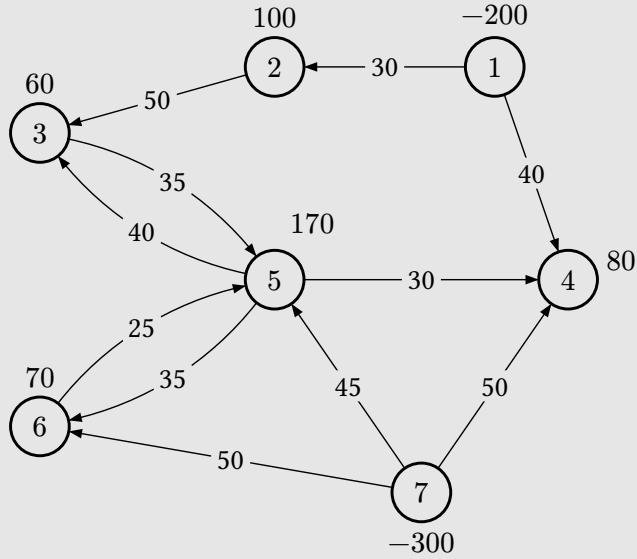
If:

- Total Supply = Total Demand  $\quad [Inflow\ to\ i] - [Outflow\ from\ i] = b_i$
- Total Supply > Total Demand  $\quad [Inflow\ to\ i] - [Outflow\ from\ i] \geq b_i$
- Total Supply < Total Demand  $\quad [Inflow\ to\ i] - [Outflow\ from\ i] \leq b_i$

Important:

- One decision variable  $x_{ij}$  for each edge  $(i, j)$

- One flow balancing constraint for each node  $i$



Minimize

$$\begin{aligned}
 Z = & 30x_{12} + 40x_{14} + 50x_{23} + 35x_{35} + 40x_{53} \\
 & + 30x_{54} + 35x_{56} + 25x_{65} + 50x_{74} + 45x_{75} + 50x_{76}
 \end{aligned}$$

s.t.

$$x_{12} + x_{14} \leq 200 \quad (\text{Node 1})$$

$$x_{12} + x_{23} \geq 100 \quad (\text{Node 2})$$

$$x_{23} + x_{53} - x_{35} \geq 60 \quad (\text{Node 3})$$

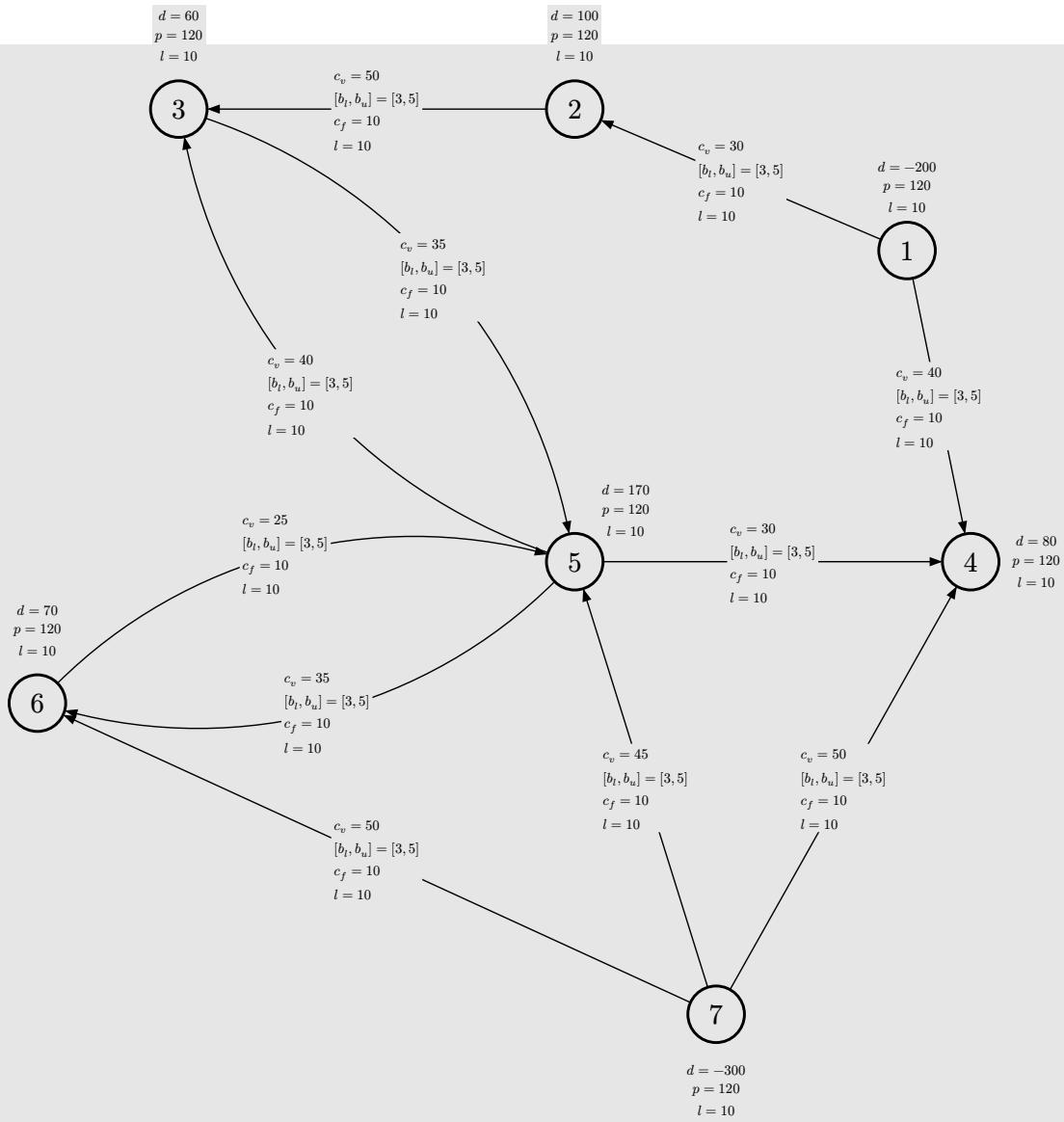
$$x_{14} + x_{54} + x_{74} \geq 80 \quad (\text{Node 4})$$

$$x_{35} + x_{65} + x_{75} - x_{53} - x_{54} - x_{56} \geq 170 \quad (\text{Node 5})$$

$$x_{56} + x_{76} - x_{65} \geq 70 \quad (\text{Node 6})$$

$$x_{76} + x_{75} + x_{74} \leq 300 \quad (\text{Node 7})$$

$$x_{ij} \geq 0 \quad \forall (i, j) \in E$$



Minimize:

$$\sum_{ij} (c_{ij}^f \cdot y_{ij}) + \sum_{ij} (c_{ij}^v \cdot x_{ij}) + \sum_i (c_i^f \cdot y_i) + \sum_i (p_i \cdot s_i) +$$

Where:

- $\sum_{ij} (c_{ij}^f \cdot y_{ij})$ : Edge fixed cost contribution
  - $\sum_{ij} (c_{ij}^v \cdot x_{ij})$ : Variable cost contribution
  - $\sum_i (c_i^f \cdot y_i)$ : Node fixed cost contribution
  - $\sum_i (p_i \cdot s_i)$ : Penalty contribution
  - $\sum_{ij} (l_{ij} \cdot x_{ij})$ : Edge lead time weighted by flow
  - $\sum_i l_i \cdot \sum_j x_{ij}$ : Node service time weighted by flow

s.t.

$\sum_j x_{ji} - \sum_j x_{ij} = \text{or } \leq \text{ or } \geq d_i$	Flow Conservation
$b_{ij}^l \leq x_{ij} \leq b_{ij}^u$	Lower & Upper Flow Bound
$x_{ij} \leq M \cdot y_{ij}$	Fixed Cost Route
$\sum_i (x_{ji} + x_{ij}) \leq M \cdot y_i$	Fixed Cost Node
$\sum_i x_{ij} + s_j \geq d_j$	Unmet Demand Penalty
$x_{ij} \geq 0 \quad \forall (i, j) \in E$	Non Negative Flow

---

Node Properties:

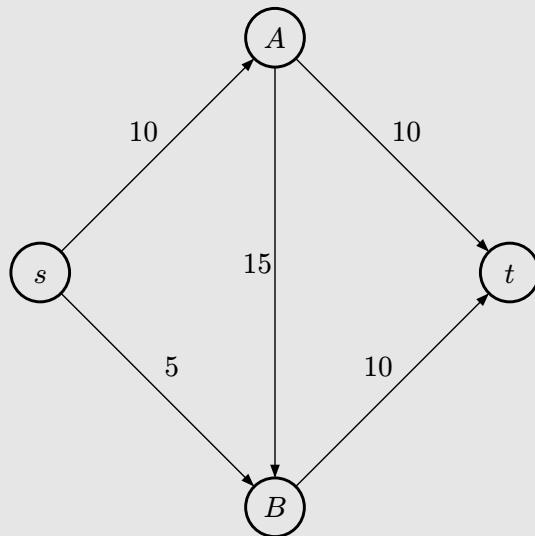
- **Node Type:** Source, sink, or intermediary.
- **Supply (Source):** Flow capacity of source nodes.
- **Holding Cost (Intermediary):** Inventory cost for stored goods.
- **Service Time:** Processing time at the node.
- **Demand:** Required flow at sink nodes.
- **Storage Capacity (Intermediary):** Maximum amount of goods that can be held at a node
- **Penalty for Unfulfilled Demand (Sink):** Cost for unmet demand.
- **Disruption Risk (All Nodes):** Probability of a node being unavailable due to unforeseen circumstances

Edge Properties:

- **Edge Type:** Transport mode (air, water, road, rail).
- **Fixed Cost:** Cost incurred for using the edge, regardless of flow.
- **Reliability:** Probability of edge availability.
- **Flow Bounds:** Minimum and maximum allowable flow.
- **Unit Cost:** Cost per unit of flow.
- **Lead Time:** Time it takes for flow to travel along the edge.
- **Environmental Impact:** Account for the carbon footprint of using certain transport modes.

### 32.4. Ford-Fulkerson

Find augmenting paths in the network and increase the flow until no more augmenting paths can be found



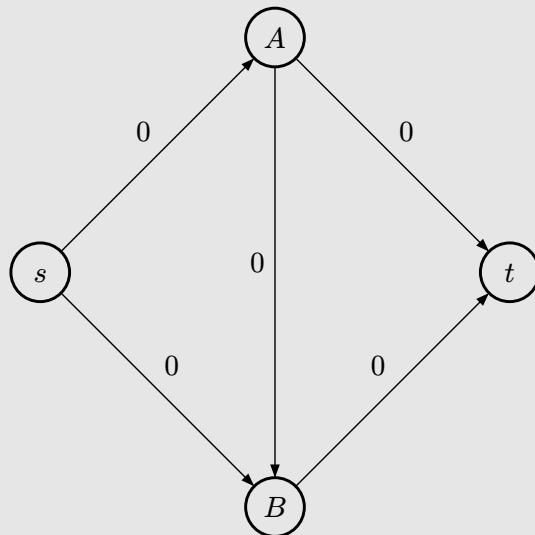
- $s$ : Source
- $A$  &  $B$ : Intermediate nodes
- $t$ : Sink

Capacities:

- $s \rightarrow t$ : 10
- $s \rightarrow B$ : 5
- $A \rightarrow B$ : 15
- $A \rightarrow t$ : 10
- $B \rightarrow t$ : 10

**Step 1:** Initialize flow to 0

All flows through the edges are initially set to 0.



**Step 2:** Find an augmenting path

Find an augmenting path using Depth-First Search (DFS). Start from the source  $s$

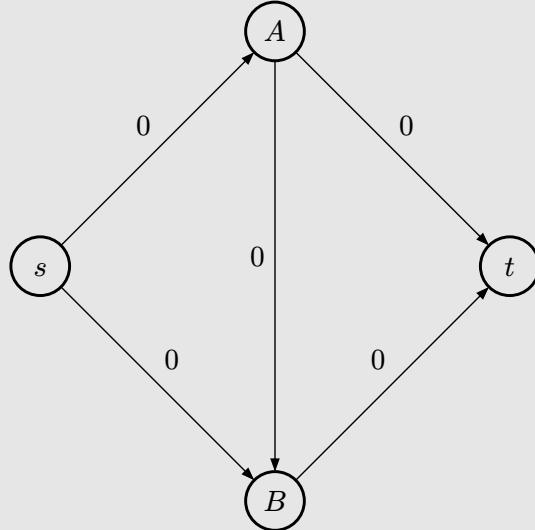
$s \rightarrow A \rightarrow t$

The minimum capacity along this path is 10 (bottleneck on edge  $A \rightarrow t$ )

We can push a flow of 10 units along this path.

**Step 3:** Update the residual graph

- $s \rightarrow A$ : Capacity becomes  $10 - 10 = 0$  (no residual capacity)
- $A \rightarrow t$ : Capacity becomes  $10 - 10 = 0$  (no residual capacity)



**Step 4:** Find another augmenting path

Find another augmenting path.

We cannot use  $s \rightarrow A$  or  $A \rightarrow t$  because their capacities are 0.

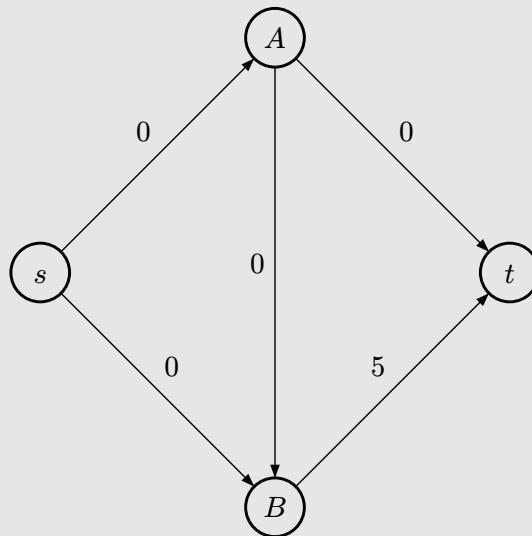
$s \rightarrow B \rightarrow t$

The minimum capacity along this path is 5 (bottleneck on edge  $s \rightarrow B$ )

We can push a flow of 5 units along this path

**Step 5:** Update the residual graph

- $s \rightarrow B$ : Capacity becomes  $5 - 5 = 0$
- $B \rightarrow t$ : Capacity becomes  $10 - 5 = 5$



#### Step 6: Termination

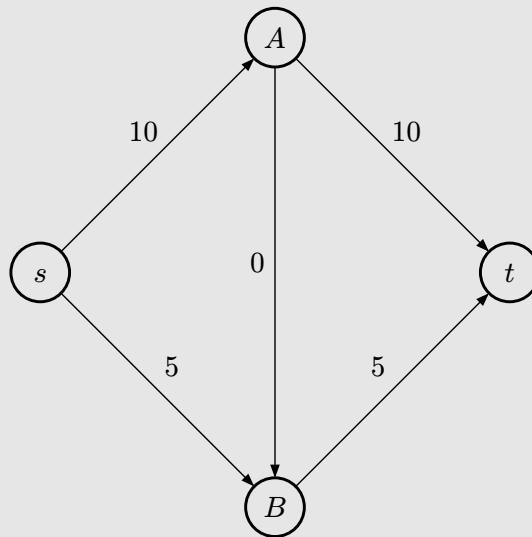
No more augmenting paths from  $s$  to  $t$  can be found, as all edges from  $s$  are fully saturated

Maximum flow:

- $s \rightarrow A \rightarrow t$ : 10 units
- $s \rightarrow B \rightarrow t$ : 5 units

Thus, the maximum flow from source  $s$  to sink  $t$  is 15 units

#### Step 7: Final Flow Distribution



## 33. Optimization

### 33.1. Mathematical Programs

#### 33.1.1. Formulation

$$\begin{array}{lll} \min & f(x_1, x_2, \dots, x_n) & \text{(objective function)} \\ \text{s.t.} & g_i(x_1, x_2, \dots, x_n) \leq b_i \quad \forall i \in 1, \dots, m & \text{(constraints)} \\ & x_j \in \mathbb{R} \quad \forall j \in 1, \dots, n & \text{(decision variables)} \end{array}$$

- $m$ : Number of constraints
- $n$ : Number of variables
- $x_1, x_2, \dots, x_n$ : decision variables
- $f : \mathbb{R}^n \rightarrow \mathbb{R}$
- $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\begin{array}{ll} \min & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \leq b_i \quad \forall i = 1, \dots, m \end{array}$$

- $A_{ij}$ : constraint coefficients
- $b_i$ : right-hand-side values (RHS)
- $c_j$ : objective coefficients

#### Vector

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \leq b_i \quad \forall i = 1, \dots, m \end{array}$$

- $a_i \in \mathbb{R}^n$
- $b_i \in \mathbb{R}$
- $c \in \mathbb{R}^n$
- $x \in \mathbb{R}^n$

#### Matrix

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

- $A \in \mathbb{R}^{m \times n}$
- $b \in \mathbb{R}^m$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \dots & a_{mn}x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$c^T x = [c_1, c_2, \dots, c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^n c_j x_j$$

### 33.1.2. Transformation

- **Min and Max**

$$\max f(x) \Leftrightarrow \min -f(x)$$

- **= and  $\geq$**

$$g_i(x) \geq b_i \Leftrightarrow -g_i(x) \leq -b_i$$

$$g_i(x) = b_i \Leftrightarrow g_i(x) \geq b_i \wedge g_i(x) \leq b_i$$

- **E.g.,**

$$\begin{array}{ll} \max & x_1 - x_2 \\ \text{s.t.} & -2x_1 + x_2 \geq -3 \\ & x_1 + 4x_2 = 5 \end{array} \Leftrightarrow \begin{array}{ll} \min & -x_1 + x_2 \\ \text{s.t.} & 2x_1 - x_2 \leq 3 \\ & x_1 + 4x_2 \leq 5 \\ & -x_1 + 4x_2 \leq -5 \end{array}$$

### 33.1.3. Constraints

- **Sign Constraints**

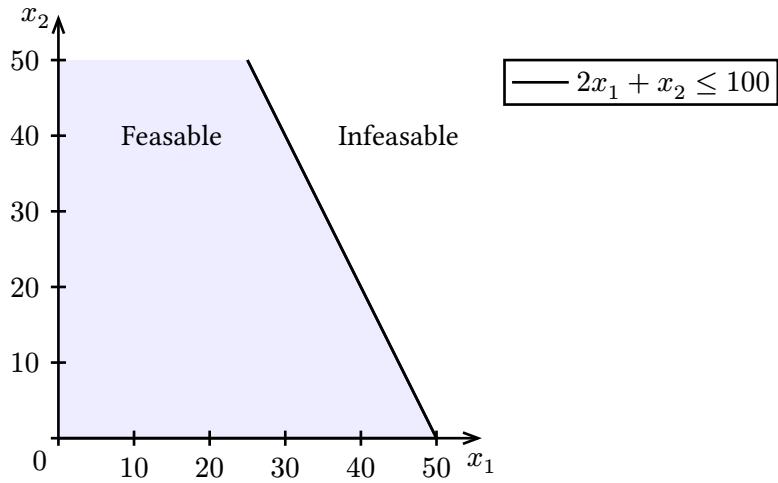
$$x_i \geq 0 \text{ or } x_i \leq 0$$

- **Functional Constraints**

$$\sum_{i=1}^n c_i x_i \leq b_i$$

### 33.1.4. Feasible Solutions

- Feasible solution: Satisfies all constraints
- Infeasible solution: violates at least one constraint



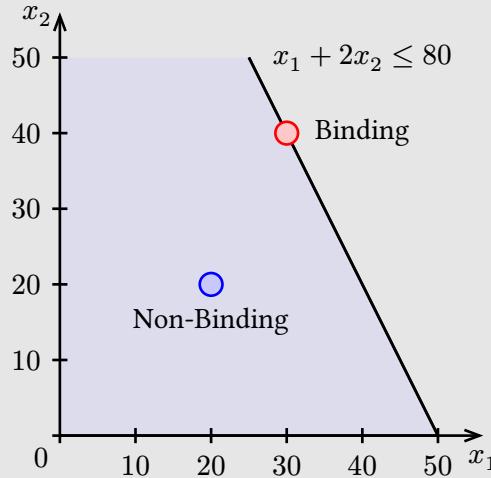
### 33.1.5. Feasible Region and Optimal Solution

- Feasible Region: Set of feasible solutions
- Optimal Solution: Attains largest (maximization) or smallest (minimization) objective value

### 33.1.6. Binding Constraint

Let  $g(\cdot) \leq b$  be an inequality constraint and  $|(x)$  be a solution.  $g(\cdot) \leq b$  is binding at  $|(x)$  if  $g(|(x)) = b$

$2x_1 + x_2 \leq 100$	is binding (active) at the point	$(x_1, x_2) = (30, 40)$
$2x_1 + x_2 \leq 100$	is non-binding (inactive) at the point	$(x_1, x_2) = (20, 20)$

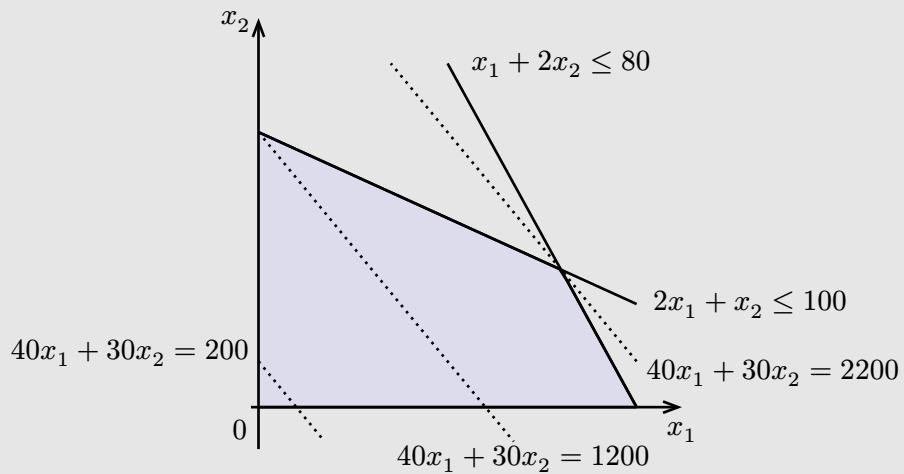


### 33.1.7. Strict and Weak Inequalities

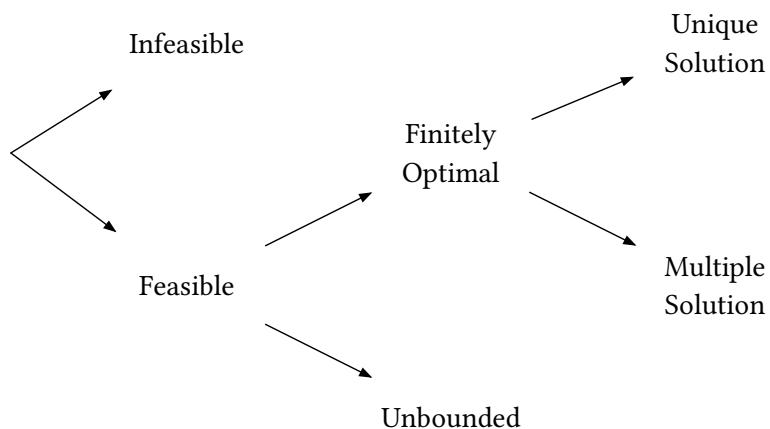
- Strict: The two sides **cannot be equal** (e.g.,  $x_1 + x_2 > 5$ )
- Weak: The two sides **can be equal** (e.g.,  $x_1 + x_2 \geq 5$ )

### 33.1.8. Graphical Approach

$\max$	$40x_1$	$+$	$30x_2$
$s.t.$	$2x_1$	$+$	$x_2 \leq 100$
	$x_1$	$+$	$2x_2 \leq 80$



### 33.1.9. Types of LP



### 33.1.10. LP Formulation

#### Production

We produce desks and tables

- Producing a desk requires 3 units of wood, 1 hour of labor and 50 minutes of machine time
- Producing a table requires 5 units of wood, 2 hours of labor and 20 minutes of machine time

For each day, we have

- 200 workers that each work 8 hours
- 50 machines that each run for 16 hours
- A supply of 3600 units of wood

Desks and tables are sold \$700 and \$900 per unit, respectively

Everything that is produced is sold

#### 1. Define Variables

$x_1$ : number of desks produced per day

$x_2$ : number of tables produced per day

## 2. Formulate Objective Function

$$\max 700x_1 + 900x_2$$

## 3. Formulate Constraints

Resource	Consumption per		Total supply
	Desk	Table	
Wood	3 units	5 units	3600 units
Labor	1 hours	2 hours	200 workers $\times$ 8 hours = 1600 hours
Machine time	50 minutes	20 minutes	50 machines $\times$ 16 hours = 800 hours

$$\begin{aligned} 3x_1 + 5x_2 &\leq 3600 \\ x_1 + 2x_2 &\leq 1600 \\ 50x_1 + 20x_2 &\leq 48000 \end{aligned}$$

## 4. Complete Formulation

$$\begin{aligned} \max \quad & 700x_1 + 900x_2 \\ \text{s.t.} \quad & 3x_1 + 5x_2 \leq 3600 \quad (\text{wood}) \\ & x_1 + 2x_2 \leq 1600 \quad (\text{labor}) \\ & 50x_1 + 20x_2 \leq 48000 \quad (\text{machine}) \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

## Compact Formulation

$$\begin{aligned} \max \quad & \sum_{j=1}^n P_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n A_{ij} x_j \leq R_i \quad \forall i = 1, \dots, m \\ & x_j \geq 0 \quad \forall j = 1, \dots, n \end{aligned}$$

Where:

- $n$ : number of products
- $m$ : number of resources
- $j$ : indices of products
- $i$ : indices of resources

- $P_j$ : unit sale price of product  $j$
- $R_i$ : supply limit of resource  $i$
- $A_{ij}$ : unit of resource  $i$  to produce one unit of product  $j$
- $i = 1, \dots, m$
- $j = 1, \dots, n$
- $x_j$ : production quantity for product  $j$ ,  $j = 1, \dots, n$

---

## model.py

```

from pyomo.environ import *
from pyomo.dataportal import DataPortal

model = AbstractModel()

# Sets
model.Products = Set()
model.Resources = Set()

# Parameters
model.Profit = Param(model.Products)
model.Supply = Param(model.Resources)
model.Consumption = Param(model.Resources, model.Products)

# Variables
model.x = Var(model.Products, domain=NonNegativeReals)

# Objective: Maximize profit
def objective_rule(model):
    return sum(model.Profit[j] * model.x[j] for j in model.Products)
model.OBJ = Objective(rule=objective_rule, sense=maximize)

# Constraints: Do not exceed resource supply
def constraint_rule(model, i):
    return sum(model.Consumption[i, j] * model.x[j] for j in model.Products) <=
model.Supply[i]
model.ResourceConstraint = Constraint(model.Resources, rule=constraint_rule)

# Load data from .dat file
data = DataPortal()
data.load(filename='data.dat', model=model)

# Create an instance of the model
instance = model.create_instance(data)

# Create solver
solver = SolverFactory('glpk')
solver.options['tmlim'] = 60

# Solve with solver timeout (optional)
results = solver.solve(instance, tee=True)

# Display results
instance.display()

```

### data.dat

```
set Products := Desk Table ;
set Resources := Wood Labor Machine ;

param Profit :=
Desk 700
Table 900 ;

param Supply :=
Wood 3600
Labor 1600
Machine 48000 ;

param Consumption:
    Desk Table :=
Wood      3      5
Labor      1      2
Machine   50     20 ;
```

### Output:

```
Variables:
x : Size=2, Index=Products
    Key : Lower : Value          : Upper : Fixed : Stale : Domain
    Desk : 0 : 884.210526315789 : None : False : False :
NonNegativeReals
    Table : 0 : 189.473684210526 : None : False : False :
NonNegativeReals

Objectives:
OBJ : Size=1, Index=None, Active=True
    Key : Active : Value
    None : True : 789473.6842105257

Constraints:
ResourceConstraint : Size=3
    Key : Lower : Body          : Upper
    Labor : None : 1263.157894736841 : 1600.0
    Machine : None : 47999.9999999997 : 48000.0
    Wood : None : 3599.99999999997 : 3600.0
```

### Multi Period (Inventory)

We produce and sell products

For the coming 4 days, the demand will be

- Days 1: 100
- Days 2: 150
- Days 3: 200
- Days 4: 170

The unit production costs are different for different days

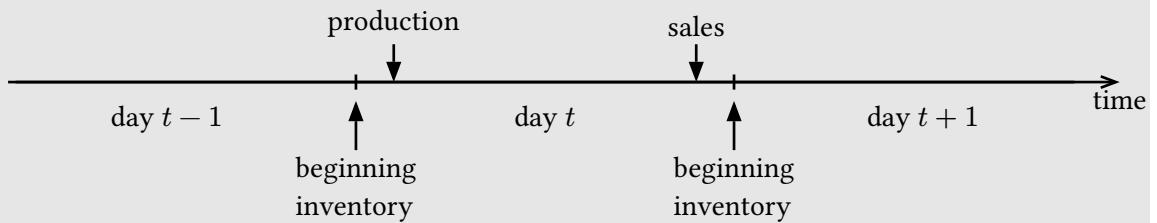
- Days 1: \$9
- Days 2: \$12
- Days 3: \$10
- Days 4: \$12

Prices are all fixed. So maximizing profit is equivalent to minimizing costs.

We may store products and sell them later. Inventory cost is \$1 per unit per day.

E.g., producing 620 units on day 1 to fulfill all demands:

$$\$9 \times 620 + \$1 \times 150 + \$2 \times 200 + \$3 \times 170 = \$6,640$$



$$I_{t-1} + x_t - d_t = I_t$$

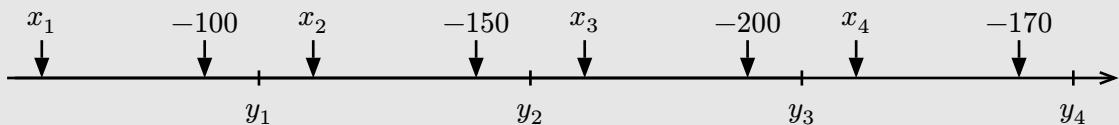
- $I_{t-1}$ : inventory at the end of day  $t-1$  (i.e., beginning inventory for day  $t$ )
- $x_t$ : units produced on day  $t$
- $d_t$ : demand (or sales) on day  $t$
- $I_t$ : inventory at the end of day  $t$

Inventory costs are calculated according to **ending inventory**

- $x_t$ : production quantity of day  $t$ ,  $t = 1, \dots, 4$
- $y_t$ : **ending inventory** of day  $t$ ,  $t = 1, \dots, 4$

Objective function:

$$\min \quad 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4$$



Inventory balancing constraints:

- Day 1:  $x_1 - 100 = y_1$
- Day 2:  $y_1 + x_2 - 150 = y_2$
- Day 3:  $y_2 + x_3 - 200 = y_3$
- Day 4:  $y_3 + x_4 - 170 = y_4$

Demand fulfillment constraints:

- $x_1 \geq 100$
- $y_1 + x_2 \geq 150$
- $y_2 + x_3 \geq 200$
- $y_3 + x_4 \geq 170$

Non-negativity constraints:

- $y_t \geq 0$
- $x_t \geq 0$

Complete Formulation

$$\begin{aligned}
 \min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4 \\
 \text{s.t.} \quad & x_1 - 100 = y_1 \\
 & y_1 + x_2 - 150 = y_2 \\
 & y_2 + x_3 - 200 = y_3 \\
 & y_3 + x_4 - 170 = y_4 \\
 & x_1 \geq 100 \\
 & y_1 + x_2 \geq 150 \\
 & y_2 + x_3 \geq 200 \\
 & y_3 + x_4 \geq 170 \\
 & x_t, y_t \geq 0 \quad \forall t = 1, \dots, 4
 \end{aligned}$$

Simplification (Redundant Constraints)

- Inventory balancing & non-negativity imply fulfillment
  - E.g., in day 1,  $x_1 - 100 = y_1$  and  $y_1 \geq 0$  means  $x_1 \geq 100$

$$\begin{aligned}
 \min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4 \\
 \text{s.t.} \quad & x_1 - 100 = y_1 \\
 & y_1 + x_2 - 150 = y_2 \\
 & y_2 + x_3 - 200 = y_3 \\
 & y_3 + x_4 - 170 = y_4 \\
 & x_t, y_t \geq 0 \quad \forall t = 1, \dots, 4
 \end{aligned}$$

- No need to have ending inventory in period 4 (costly but useless)

$$\begin{aligned}
 \min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4 \\
 \text{s.t.} \quad & x_1 - 100 = y_1 \\
 & y_1 + x_2 - 150 = y_2 \\
 & y_2 + x_3 - 200 = y_3 \\
 & y_3 + x_4 - 170 = 0 \\
 & x_t \geq 0 \quad \forall t = 1, \dots, 4 \\
 & y_t \geq 0 \quad \forall t = 1, \dots, 3
 \end{aligned}$$

## Compact Formulation

$$\begin{aligned}
 \min \quad & \sum_{t=1}^4 (C_t x_t + y_t) \\
 \text{s.t.} \quad & y_t - 1 + x_t - D_t = y_t \quad \forall t = 1, \dots, 4 \\
 & y_0 = 0 \\
 & x_t, y_t \geq 0 \quad \forall t = 1, \dots, 4
 \end{aligned}$$

Where:

- $D_t$ : demand on day  $t$
- $y_t$ : ending inventory of day  $t$
- $C_t$ : unit production cost on day  $t$

## Personnel Scheduling

Scheduling employees

Each employee must work for 5 consecutive days and then take 2 consecutive rest days

Number of employees required for each day

Mon	Tue	Wed	Thu	Fri	Sat	Sun
110	80	150	30	70	160	120

Seven shifts

- Mon to Fri
- Tue to Sat
- ...

Minimize number of employees hired

Let  $x_i$  be the number of employees assigned to work from day  $i$  for 5 consecutive days

Objective function:

$$\min x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

Demand Fulfilment Constraints

- 110 employees needed Monday

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 110$$

An employee that works on Tues ( $t_2$ ) or Wed ( $t_3$ ) does not work on Mon ( $t_1$ )

- 80 employees needed Tuesday

$$x_1 + x_2 + x_5 + x_6 + x_7 \geq 80$$

An employee that works on Wed ( $t_3$ ) or Thu ( $t_4$ ) does not work on Tue ( $t_2$ )

- 120 employees needed Sunday

$$x_3 + x_4 + x_5 + x_6 + x_7 \geq 120$$

An employee that works on Mon ( $t_1$ ) or Tue ( $t_2$ ) does not work on Sun ( $t_7$ )

Non-Negativity Constraints

$$x_i \geq 0 \quad \forall i = 1, \dots, 7$$

Complete Formulation

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{s.t.} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq 110 \\ & x_1 + x_2 + x_5 + x_6 + x_7 \geq 80 \\ & x_1 + x_2 + x_3 + x_6 + x_7 \geq 150 \\ & x_1 + x_2 + x_3 + x_4 + x_7 \geq 30 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \geq 70 \\ & x_2 + x_3 + x_4 + x_5 + x_6 \geq 160 \\ & x_3 + x_4 + x_5 + x_6 + x_7 \geq 120 \\ & x_i \geq 0 \quad \forall i = 1, \dots, 7 \end{aligned}$$

**model.py**

```
from pyomo.environ import *
from pyomo.dataportal import DataPortal

model = AbstractModel()

# Sets
model.Days = Set(ordered=True) # Days 1..7 (Mon..Sun)
model.Shifts = Set(ordered=True) # Shifts starting on days 1..7

# Parameters
model.Demand = Param(model.Days) # Daily staffing requirements
model.Cover = Param(model.Days, model.Shifts, within=Binary) # Coverage matrix
# (1 if shift j covers day i)

# Variables
model.x = Var(model.Shifts, domain=NonNegativeReals)

# Objective: Minimize total employees hired
def obj_rule(model):
    return sum(model.x[s] for s in model.Shifts)
model.OBJ = Objective(rule=obj_rule, sense=minimize)

# Constraints: Cover daily demand
def demand_rule(model, d):
    return sum(model.Cover[d, s] * model.x[s] for s in model.Shifts) >=
model.Demand[d]
model.DemandConstraint = Constraint(model.Days, rule=demand_rule)

# Load data from .dat file
data = DataPortal()
data.load(filename='data.dat', model=model)
```

```

# Create an instance of the model
instance = model.create_instance(data)

# Create solver
solver = SolverFactory('glpk')
solver.options['tmlim'] = 60

# Solve with solver timeout (optional)
results = solver.solve(instance, tee=True)

# Display results
instance.display()

```

### data.dat

```

set Days := Mon Tue Wed Thu Fri Sat Sun ;
set Shifts := s1 s2 s3 s4 s5 s6 s7 ;

param Demand :=
Mon 110
Tue 80
Wed 150
Thu 30
Fri 70
Sat 160
Sun 120 ;

# Each shift covers 5 consecutive days starting on the shift's day
# 1 if shift s_j covers day d_i, 0 otherwise

param Cover:
      s1 s2 s3 s4 s5 s6 s7 :=
Mon  1  0  0  1  1  1  1
Tue  1  1  0  0  1  1  1
Wed  1  1  1  0  0  1  1
Thu  1  1  1  1  0  0  1
Fri  1  1  1  1  1  0  0
Sat  0  1  1  1  1  1  0
Sun  0  0  1  1  1  1  1 ;

```

### Output:

```

Variables:
x : Size=7, Index=Shifts
      Key : Lower : Value          : Upper : Fixed : Stale : Domain
            s1 : 0 : 3.33333333333333 : None : False : False : NonNegativeReals
            s2 : 0 : 40.0 : None : False : False : NonNegativeReals
            s3 : 0 : 13.3333333333333 : None : False : False : NonNegativeReals
            s4 : 0 : 0.0 : None : False : False : NonNegativeReals
            s5 : 0 : 13.3333333333333 : None : False : False : NonNegativeReals
            s6 : 0 : 93.3333333333333 : None : False : False : NonNegativeReals
            s7 : 0 : 0.0 : None : False : False : NonNegativeReals

```

```

Objectives:
  OBJ : Size=1, Index=None, Active=True
    Key : Active : Value
    None :  True : 163.3333333333323

Constraints:
  DemandConstraint : Size=7
    Key : Lower : Body           : Upper
    Fri : 70.0 : 69.9999999999993 : None
    Mon : 110.0 : 109.9999999999993 : None
    Sat : 160.0 : 159.999999999999 : None
    Sun : 120.0 : 119.999999999999 : None
    Thu : 30.0 : 56.66666666666663 : None
    Tue : 80.0 : 149.999999999994 : None
    Wed : 150.0 : 149.999999999994 : None

```

### 33.2. IP (Integer Programming)

#### 33.3. Selection and Logical Relations on Binary Variables $x_i \in \{0, 1\}$

##### 33.3.1. Cardinality Constraints

Given a subset  $S \subseteq \{1, \dots, n\}$ :

- At least  $k$  of the items in  $S$

$$\sum_{i \in S} x_i \geq k$$

We want to select at least 2 of items 1, 2, 3

$$x_1 + x_2 + x_3 \geq 2$$

- At most  $k$  of the items in  $S$

$$\sum_{i \in S} x_i \leq k$$

We want to select at most 2 of items 1, 2, 3

$$x_1 + x_2 + x_3 \leq 2$$

- Exactly  $k$  of the items in  $S$

$$\sum_{i \in S} x_i = k$$

We want to select exactly 2 of items 1, 2, 3

$$x_1 + x_2 + x_3 = 2$$

##### 33.3.2. Logical OR

Given a subset  $S \subseteq \{1, \dots, n\}$ :

- At least 1 item in  $S$

$$\sum_{i \in S} x_i \geq 1$$

Select item 1 or item 2, or both

$$x_1 + x_2 \geq 1$$

### 33.3.3. Conditional Selection: “If X, then Y”

(Implication Constraints)

- If  $x_a = 1$ , then  $x_b = 1$

$$x_a \leq x_b$$

If item 1 is selected, then item 2 must also be selected

$$x_1 \leq x_2$$

$x_a$	$x_b$	Valid?
0	0	✓
0	1	✓
1	0	✓
1	1	✗

- If  $x_a = 1$ , then  $x_b = 0$

$$x_a + x_b \leq 1$$

If you choose project A, then you cannot choose project B (mutually exclusive).

$x_a$	$x_b$	Valid?
0	0	✓
0	1	✓
1	0	✓
1	1	✗

- If  $x_a = 1$ , then  $x_b + x_c \leq 0$  (generalized)

$$x_b + x_c \leq M(1 - x_a)$$

For binary variables  $M = 2$  works

If you choose project A, then you must not choose either project B or C.

$x_a$	$x_b$	$x_c$	Valid?
0	0	0	✓

1	0	0	✓
1	1	0	✗
1	1	1	✗
0	1	1	✓
0	0	1	✓

### 33.3.4. Conditional OR: “If not X, then select Y and Z”

- If  $x_j = 0$ , then at least  $k$  items in  $S$  must be selected (generalized)

$$\sum_{i \in S} x_i \geq k(1 - x_j)$$

If you do not choose item 1, then you must choose both item 2 and item

If  $x_1 = 0$ , then  $x_2 = x_3 = 1$

$$x_2 + x_3 \geq 2(1 - x_1)$$

$x_1$	$x_2$	$x_3$	LHS	RHS	Valid?
0	1	1	2	2	✓
0	1	0	1	2	✗
0	0	0	0	2	✗
1	0	0	0	0	✓
1	1	0	1	0	✓
1	1	1	2	0	✓

### 33.3.5. XOR / Exclusive Conditions

- Exactly one item in a set  $S$

$$\sum_{i \in S} x_i = 1$$

- Select one or the other but not both (i.e.,  $x_i \oplus x_k$ )

$$x_a + x_b = 1$$

You can select item 1 or item 2, but not both

$x_1$	$x_2$	Sum	Valid?
0	1	1	✓
1	0	1	✓
1	1	2	✗

$$\overline{0} \quad 0 \quad 0 \quad \text{X}$$

### 33.4. Constraint Activation and Relaxation via Binary Indicators $z_i \in \{0, 1\}$

#### 33.4.1. Base Case: At Least One of Two Constraints Satisfied

You are given 2 constraints

$$\begin{aligned} g_1(x) &\leq b_1 \\ g_2(x) &\leq b_2 \end{aligned}$$

We want at least one of these two constraints to be satisfied.

Binary Indicator Variables

Define:

$$z_i = \begin{cases} 1 & \text{if } g_{i(x)} \leq b_i \text{ is enforced} \\ 0 & \text{if } g_{i(x)} \leq b_i \text{ may be violated} \end{cases} \quad \text{for } i = 1, 2$$

Let  $M_1$  and  $M_2$  be upper bounds on how much each constraint's LHS  $g_{i(x)}$  can exceed  $b_i$ :

$$M_i \geq \max_x (g_{i(x)} - b_i)$$

Big-M Reformulation

We relax each constraint as follows:

$$\begin{aligned} g_1(x) - b_1 &\leq M_1(1 - z_1) \\ g_2(x) - b_2 &\leq M_2(1 - z_2) \end{aligned}$$

When  $z_i = 1$ , the original constraint is enforced

When  $z_i = 0$ , the original constraint can be violated by up to  $M_i$

At Least One Constraint Enforced

We require:

$$z_1 + z_2 \geq 1$$

#### 33.4.2. General Formulation for $m$ Constraints with At Least $k$ Satisfied

1. Introduce a binary variable  $z_i \in \{0, 1\}$  for each constraint  $g_{i(x)} \leq b_i$ , where:

- $z_i = 1$ : the constraint is enforced (i.e.,  $g_{i(x)} \leq b_i$ )
- $z_i = 0$ : the constraint may be violated

2. Use “big-M” constraint to relax  $g_{i(x)} \leq b_i$  when  $z_i = 0$ , like:

$$g_{i(x)} - b_i \leq M_i(1 - z_i), \quad \forall i = 1, \dots, m$$

Where  $M_i \geq \max_x (g_{i(x)} - b_i)$

3. Impose a condition that at least  $k$  of the  $m$  constraints must be satisfied:

$$\sum_{i=1}^m z_i \geq k$$

### 33.5. Fixed Charge Constraints (Setup Costs)

- $n$ : Number of factories
- $K_i$ : Capacity of factory  $i$
- $C_i$ : Unit production cost at factory  $i$
- $S_i$ : Fixed setup cost for factory  $i$
- $D$ : Total Demand

Decision Variables

- $x_i$ : production quantity at factory  $i$ ,  $i = 1, \dots, n$
- $y_i$ : Binary variable indicating whether factory  $i$  is used

$$y_i = \begin{cases} 1 & \text{if factory } i \text{ is used (i.e., } x_i > 0) \\ 0 & \text{if factory } i \text{ is not used (i.e., } x_i = 0) \end{cases}$$

Objective Function

$$\min \underbrace{\sum_{i=1}^n C_i x_i}_{\text{Production Cost}} + \underbrace{\sum_{i=1}^n S_i y_i}_{\text{Setup Cost}}$$

Capacity Constraints

$$x_i \leq K_i \quad \forall i = 1, \dots, n$$

Demand Fullfilment

$$\sum_{i=1}^n x_i \geq D$$

Logical Linking of  $x_i$  and  $y_i$

To ensure  $y_i = 1$  if any production occurs at factory  $i$ :

$$x_i \leq K_i y_i \quad \forall i = 1, \dots, n$$

- If  $x_i > 0$ , this forces  $y_i = 1$
- If  $x_i = 0$ ,  $y_i$  can be 0 or 1 (but to minimize cost,  $y_i$  will be 0)

Binary & Non-negative Constraints

$$y_i \in \{0, 1\}, \quad x_i \geq 0 \quad \forall i = 1, \dots, n$$

General Form (Big-M Formulation)

$$x \leq M y$$

Where  $M$  must be set to the upper bound of  $x$

### 33.6. Facility Location

Problem	Definition
Set Covering	Select the minimum number of facilities such that every demand point is within a specified coverage distance of at least one facility
Maximum Covering	Select a fixed number of facilities to maximize the number of demand points covered within a specified distance

Problem	Definition
Fixed Charge Location	Determine which facilities to open and how to assign demand to them to minimize the total cost

### 33.6.1. Set Covering Problem

**Question:** How to allocate as few facilities as possible to cover all demand nodes?

- $I$ : Set of demand nodes
- $J$ : Set of location nodes
- $d_{ij} > 0$ : Distance between demand node  $i \in I$  and location node  $j \in J$
- $s$ : Maximum allowable distance (a facility at  $j$  can cover demand node  $i$  if  $d_{ij} < s$ )
- Demand  $i$  is “covered” by location  $j$  if  $d_{ij} < s$
- $w_j > 0$ : Cost of building facility at location  $j$  (usually set to  $w_j = 1$  for minimizing number of facilities, but can reflect other costs)
- $a_{ij} = \begin{cases} 1 & \text{if } d_{ij} < s \\ 0 & \text{otherwise} \end{cases}$  : Indicator of whether location  $j$  can cover demand  $i$
- $x_j = \begin{cases} 1 & \text{if facility is built at location } j \\ 0 & \text{otherwise} \end{cases}$

Complete Formulation

$$\begin{aligned} \min \quad & \sum_{j \in J} w_j x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq 1 \quad \forall i \in I \\ & x_j \in \{0, 1\} \quad \forall j \in J \end{aligned}$$

### 33.6.2. Maximum Covering Problem

**Question:** How to allocate at most  $p$  facilities to cover as many demand nodes as possible?

- $I$ : Set of demand nodes
- $J$ : Set of location nodes
- $d_{ij} > 0$ : Distance between demand node  $i \in I$  and location node  $j \in J$
- $p \in \mathbb{N}$ : maximum number of facilities
- $w_j > 0$ : Cost of building facility at location  $j$  (usually set to  $w_j = 1$  for minimizing number of facilities, but can reflect other costs)
- $x_j = \begin{cases} 1 & \text{if facility is built at location } j \\ 0 & \text{otherwise} \end{cases}$
- $a_{ij} = \begin{cases} 1 & \text{if } d_{ij} < s \\ 0 & \text{otherwise} \end{cases}$  : Indicator of whether location  $j$  can cover demand  $i$
- $y_i = \begin{cases} 1 & \text{if demand } i \in I \text{ is covered by any facility} \\ 0 & \text{otherwise} \end{cases}$

Complete Formulation

$$\begin{aligned}
\min \quad & \sum_{i \in I} w_i y_i \\
s.t. \quad & \sum_{j \in J} a_{ij} x_j \geq y_i \quad \forall i \in I \\
& \sum_{j \in J} x_j \leq p \quad \forall j \in J \\
& x_j \in \{0, 1\} \quad \forall j \in J \\
& y_i \in \{0, 1\} \quad \forall i \in I
\end{aligned}$$

### Fixed Charge Location Problems

**Question:** How to allocate some facilities to minimize the total shipping and construction costs?

Uncapacitated Facility Location Problem (UFL)

- $I$ : Set of demand nodes
- $J$ : Set of location nodes
- $h_i > 0$ : Demand size at demand node  $i$
- $d_{ij}$ : Unit shipping cost from location node  $j$  to demand node  $i$
- $f_j > 0$ : Fixed construction cost at location  $j$
- $x_j = \begin{cases} 1 & \text{if facility is built at location } j \\ 0 & \text{otherwise} \end{cases}$
- $y_{ij} = \begin{cases} 1 & \text{if demand } i \in I \text{ is served by facility at location } j \in J \\ 0 & \text{otherwise} \end{cases}$

Complete Formulation

$$\begin{aligned}
\min \quad & \underbrace{\sum_{i \in I} \sum_{j \in J} h_i d_{ij} y_{ij}}_{\text{Shipping Costs}} + \underbrace{\sum_{j \in J} f_j x_j}_{\text{Fixed Costs}} \\
s.t. \quad & y_{ij} \leq x_j \quad \forall i \in I, j \in J \\
& \sum_{j \in J} y_{ij} = 1 \quad \forall i \in I \\
& x_j \in \{0, 1\} \quad \forall j \in J \\
& y_{ij} \in \{0, 1\} \quad \forall i \in I
\end{aligned}$$

Capacitated Facility Location Problem (CFL)

If locations have a capacity, we may add constraint

$$\sum_{i \in I} h_i y_{ij} \leq K_j \quad \forall j \in J$$

Where:

- $K_j > 0$ : Capacity of location  $j$

$$\begin{aligned}
\min \quad & \underbrace{\sum_{j=1}^5 f_j x_j}_{\text{Fixed Costs}} + \underbrace{\sum_{i=1}^5 \sum_{j=1}^5 c_{ij} y_{ij}}_{\text{Shipping Costs}} \\
s.t. \quad & \sum_{i=1}^5 y_{ij} \leq K_j x_j \quad \forall j = 1, \dots, 5 \quad (\text{Capacity Constraint}) \\
& \sum_{j=1}^5 y_{ij} \geq D_i \quad \forall i = 1, \dots, 5 \quad (\text{Demand Constraint}) \\
& x_j \in \{0, 1\} \quad \forall j = 1, \dots, 5 \quad (\text{Binary Constraint}) \\
& y_{ij} \geq 0 \quad \forall i = 1, \dots, 5, j = 1, \dots, 5 \quad (\text{Positivity Constraint})
\end{aligned}$$

## Parameters

- $f_j$ : weekly operating cost of distribution center  $j$
- $c_{ij}$ : shipping cost per book from distribution center  $j$  to region  $i$
- $K_j$ : capacity of distribution center  $j$
- $D_i$ : book demand of region  $i$

## Decision variable

- $x_{ij}$ : binary variable

$$x_{ij} = \begin{cases} 1 & \text{if a distribution center is build at location } j \\ 0 & \text{otherwise} \end{cases}$$

- $y_{ij}$ : number of books shipped from distribution center  $j$  to region  $i$

## Supply

	WA	NV	NE	PA	FL	Demand ( $D_i$ )
NorthWest	$y_{11}$	$y_{12}$	$y_{13}$	$y_{14}$	$y_{15}$	8000
SouthWest	$y_{21}$	$y_{22}$	$y_{23}$	$y_{24}$	$y_{25}$	12000
MidWest	$y_{31}$	$y_{32}$	$y_{33}$	$y_{34}$	$y_{35}$	9000
SouthEast	$y_{41}$	$y_{42}$	$y_{43}$	$y_{44}$	$y_{45}$	14000
NorthEast	$y_{51}$	$y_{52}$	$y_{53}$	$y_{54}$	$y_{55}$	17000

## Fixed Costs ( $f_j$ ) and Capacity ( $K_j$ )

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
Operation Cost ( $f_i$ )	40000	30000	25000	40000	30000
Capacity ( $K_i$ )	20000	20000	15000	25000	15000

## Shipping Cost ( $c_j$ )

	WA	NV	NE	PA	FL
NorthWest	2.4	3.25	4.05	5.25	6.95
SouthWest	3.5	2.3	3.25	6.05	5.85

	WA	NV	NE	PA	FL
MidWest	4.8	3.4	2.85	4.3	4.8
SouthEast	6.8	5.25	4.3	3.25	2.1
NorthEast	5.75	6	4.75	2.75	3.5

### Binary Decision Variable

WA	$x_1$
NV	$x_2$
NE	$x_3$
PA	$x_4$
FL	$x_5$

### model.py

```

from pyomo.environ import *
from pyomo.dataportal import DataPortal

model = AbstractModel()

# Sets
model.I = Set(doc='Regions (demand points)')
model.J = Set(doc='Candidate distribution centers')

# Parameters
model.D = Param(model.I, within=NonNegativeReals, doc='Demand at region i')
model.K = Param(model.J, within=NonNegativeReals, doc='Capacity at distribution center j')
model.f = Param(model.J, within=NonNegativeReals, doc='Fixed cost to open center j')
model.c = Param(model.I, model.J, within=NonNegativeReals, doc='Shipping cost from j to i')

# Decision Variables
model.x = Var(model.J, within=Binary, doc='1 if center j is opened')
model.y = Var(model.I, model.J, within=NonNegativeReals, doc='Units shipped from j to i')

# Objective Function: Minimize total cost
def total_cost_rule(model):
    fixed = sum(model.f[j] * model.x[j] for j in model.J)
    shipping = sum(model.c[i, j] * model.y[i, j] for i in model.I for j in model.J)
    return fixed + shipping
model.TotalCost = Objective(rule=total_cost_rule, sense=minimize)

# Constraints

# Capacity Constraint: Total shipped from center ≤ capacity if opened
def capacity_rule(model, j):
    return sum(model.y[i, j] for i in model.I) <= model.K[j] * model.x[j]

```

```

model.CapacityConstraint = Constraint(model.J, rule=capacity_rule)

# Demand Constraint: Total received by region  $\geq$  its demand
def demand_rule(model, i):
    return sum(model.y[i, j] for j in model.J)  $\geq$  model.D[i]
model.DemandConstraint = Constraint(model.I, rule=demand_rule)

# Load data from .dat file
data = DataPortal()
data.load(filename='data.dat', model=model)

# Create an instance of the model
instance = model.create_instance(data)

# Create solver
solver = SolverFactory('glpk')
solver.options['tmlim'] = 60

# Solve with solver timeout (optional)
results = solver.solve(instance, tee=True)

# Display results
instance.display()

```

### data.dat

```

set I := NorthWest SouthWest MidWest SouthEast NorthEast;
set J := WA NV NE PA FL;

param: D :=
NorthWest 8000
SouthWest 12000
MidWest 9000
SouthEast 14000
NorthEast 17000;

param: f :=
WA 40000
NV 30000
NE 25000
PA 40000
FL 30000;

param: K :=
WA 20000
NV 20000
NE 15000
PA 25000
FL 15000;

param c:
      WA      NV      NE      PA      FL :=
NorthWest 2.4    3.25   4.05   5.25   6.95

```

SouthWest	3.5	2.3	3.25	6.05	5.85
MidWest	4.8	3.4	2.85	4.3	4.8
SouthEast	6.8	5.25	4.3	3.25	2.1
NorthEast	5.75	6	4.75	2.75	3.5

### Output:

```

Variables:
x : 1 if center j is opened
Size=5, Index=J
Key : Lower : Value : Upper : Fixed : Stale : Domain
    FL :    0 :  1.0 :    1 : False : False : Binary
    NE :    0 :  0.0 :    1 : False : False : Binary
    NV :    0 :  1.0 :    1 : False : False : Binary
    PA :    0 :  1.0 :    1 : False : False : Binary
    WA :    0 :  0.0 :    1 : False : False : Binary
y : Units shipped from j to i
Size=25, Index=I*J
Key : Lower : Value : Upper : Fixed : Stale : Domain
    ('MidWest', 'FL') :    0 : 1000.0 : None : False : False :
NonNegativeReals
    ('MidWest', 'NE') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('MidWest', 'NV') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('MidWest', 'PA') :    0 : 8000.0 : None : False : False :
NonNegativeReals
    ('MidWest', 'WA') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthEast', 'FL') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthEast', 'NE') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthEast', 'NV') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthEast', 'PA') :    0 : 17000.0 : None : False : False :
NonNegativeReals
    ('NorthEast', 'WA') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthWest', 'FL') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthWest', 'NE') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthWest', 'NV') :    0 : 8000.0 : None : False : False :
NonNegativeReals
    ('NorthWest', 'PA') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('NorthWest', 'WA') :    0 :  0.0 : None : False : False :
NonNegativeReals
    ('SouthEast', 'FL') :    0 : 14000.0 : None : False : False :
NonNegativeReals
    ('SouthEast', 'NE') :    0 :  0.0 : None : False : False :
NonNegativeReals

```

```

        ('SouthEast', 'NV') : 0 : 0.0 : None : False : False :
NonNegativeReals
        ('SouthEast', 'PA') : 0 : 0.0 : None : False : False :
NonNegativeReals
        ('SouthEast', 'WA') : 0 : 0.0 : None : False : False :
NonNegativeReals
        ('SouthWest', 'FL') : 0 : 0.0 : None : False : False :
NonNegativeReals
        ('SouthWest', 'NE') : 0 : 0.0 : None : False : False :
NonNegativeReals
        ('SouthWest', 'NV') : 0 : 12000.0 : None : False : False :
NonNegativeReals
        ('SouthWest', 'PA') : 0 : 0.0 : None : False : False :
NonNegativeReals
        ('SouthWest', 'WA') : 0 : 0.0 : None : False : False :
NonNegativeReals

Objectives:
  TotalCost : Size=1, Index=None, Active=True
    Key : Active : Value
      None : True : 268950.0

Constraints:
  CapacityConstraint : Size=5
    Key : Lower : Body : Upper
      FL : None : 0.0 : 0.0
      NE : None : 0.0 : 0.0
      NV : None : 0.0 : 0.0
      PA : None : 0.0 : 0.0
      WA : None : 0.0 : 0.0
  DemandConstraint : Size=5
    Key : Lower : Body : Upper
      MidWest : 9000.0 : 9000.0 : None
      NorthEast : 17000.0 : 17000.0 : None
      NorthWest : 8000.0 : 8000.0 : None
      SouthEast : 14000.0 : 14000.0 : None
      SouthWest : 12000.0 : 12000.0 : None

```

### 33.7. Machine Scheduling

	Problem	Definition
<b>One Machine</b>	Single Machine Serial Production	Scheduling jobs one after another on a single machine
<b>Multiple Machines</b>	Multiple Parallel Machines	Scheduling jobs on two or more identical machines working in parallel; each job can be assigned to any machine
	Flow Shop	All jobs follow the same sequence of machines; each job visits every machine in the same order
	Job Shop	Each job has its own specific sequence of machines to follow

- Job Splitting
  - Non-preemptive problems

Once a job starts processing on a machine, it must continue without interruption until completion



- ▶ Preemptive problems

A job can be interrupted and resumed later, possibly on a different machine, without losing progress



- Performance Metrics

- ▶ **Makespan:** The sum of completion times of all jobs, possibly weighted by job importance
- ▶ **(Weighted) total completion time:** The sum of completion times of all jobs, possibly weighted by job importance (measures overall throughput)
- ▶ **(Weighted) number of delayed jobs:** The total number of jobs completed after they are due, possibly weighted by job priority
- ▶ **(Weighted) total lateness:** The sum of differences between each job's completion time and due date (can be negative or positive), possibly weighted
- ▶ **(Weighted) total tardiness:** The sum of positive delays (i.e., lateness when a job finishes after its due date), possibly weighted. Negative values (early completions) are treated as zero.
- ▶ ...

### 33.7.1. Single Machine Serial Production



#### Notation

- $n$ : Total number of jobs
- $J = \{1, 2, \dots, n\}$ : Set of jobs
- $j \in J$ : A job
- $p_j$ : processing time of job  $j$

- $x_j$ : completion time of job  $j$

- $z_{ij} \in \{0, 1\}$ : Binary variable

$$z_{ij} = \begin{cases} 1 & \text{if job } j \text{ is before job } i \\ 0 & \text{otherwise} \end{cases}$$

### Job Order Constraints

We must ensure that either job  $i$  comes before job  $j$  or vice versa, but not both

If job  $i$  is before job  $j$  ( $z_{ij} = 0$ )

$$x_j \geq x_i + p_j$$

If job  $j$  is before job  $i$  ( $z_{ij} = 1$ ):

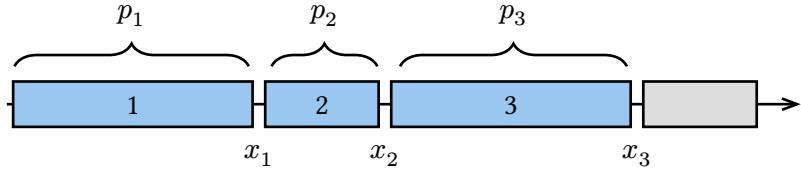
$$x_i \geq x_j + p_i$$

These two cases can be rewritten using a big- $M$  formulation:

$$\begin{aligned} x_i + p_j - x_j &\leq Mz_{ij} \\ x_j + p_i - x_i &\leq M(1 - z_{ij}) \end{aligned}$$

Where:

- $M = \sum_{j \in J} p_j$



### Complete Formulation

$$\begin{aligned} \min \quad & \sum_{j \in J} x_j \\ \text{s.t.} \quad & x_i + p_j - x_j \leq Mz_{ij} \quad \forall i \in J, j \in J, i < j \\ & x_j + p_i - x_i \leq M(1 - z_{ij}) \quad \forall i \in J, j \in J, i < j \\ & x_j \geq p_j \quad \forall j \in J \\ & x_j \geq 0 \quad \forall j \in J \\ & z_{ij} \in \{0, 1\} \quad \forall i \in J, j \in J, i < j \end{aligned}$$

### Interpretation

If jobs are run in order  $1, 2, \dots, n$ , the completion times are:

$$x_1 = p_1$$

$$x_2 = p_1 + p_2$$

$$x_3 = p_1 + p_2 + p_3$$

⋮

$$x_n = \sum_{i=1}^n p_i$$

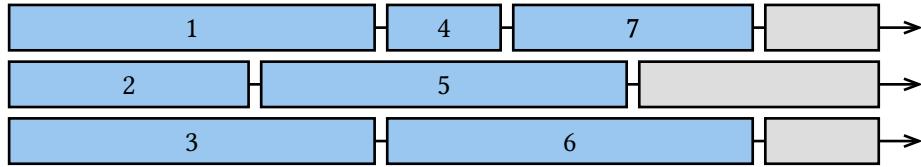
So, the completion time of a job depends on the sum of processing times of all jobs before it in the schedule

### Optional: Release Times

If jobs have release times  $R_j$ , meaning each job  $j$  cannot start before  $R_j$ , then we add:

$$x_j \geq R_j + p_j \quad \forall j \in J$$

#### 33.7.2. Multiple Parallel Machines



We aim to assign jobs to machines such that:

- Each job is assigned to exactly one machine.
- A job can be processed on any machine.
- The objective is to minimize the makespan, which is the time when the last machine finishes processing.

### Notation

- $J = \{1, 2, \dots, n\}$ : set of jobs
- $I = \{1, 2, \dots, m\}$ : set of machines
- $p_j$ : processing time of job  $j \in J$
- $x_{ij}$ : a binary variable that indicates whether job  $j$  is assigned to machine  $i$ :

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ is assigned to machine } i \\ 0 & \text{otherwise} \end{cases}$$

- The completion time of machine  $i$  is the total processing time of all jobs assigned to it

$$C_i = \sum_{j \in J} p_j x_{ij}$$

- The makespan  $w$  is the maximum completion time across all machines:

$$w \geq \sum_{j \in J} p_j x_{ij} \quad \forall i \in I$$

### Objective

Minimize makespan  $w$

### Complete Formulation

$$\begin{aligned}
 \min \quad & w \\
 \text{s.t.} \quad & \sum_{j \in J} p_j x_{ij} \leq w \quad \forall i \in I \\
 & \sum_{i \in I} x_{ij} = 1 \quad \forall j \in J \\
 & x_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J
 \end{aligned}$$

- The first constraint ensures the makespan is at least as large as each machine's workload
- The second ensures each job is assigned to one and only one machine

```

from pyomo.environ import *

model = AbstractModel()

model.MACHINES = Set()
model.JOBS = Set()

model.p = Param(model.JOBS, within=PositiveReals)

model.x = Var(model.MACHINES, model.JOBS, domain=Binary)
model.w = Var(domain=NonNegativeReals)

def obj_rule(m):
    return m.w
model.Obj = Objective(rule=obj_rule, sense=minimize)

def job_assignment_rule(m, j):
    return sum(m.x[i, j] for i in m.MACHINES) == 1
model.JobAssignment = Constraint(model.JOBS, rule=job_assignment_rule)

def machine_completion_rule(m, i):
    return sum(m.p[j] * m.x[i, j] for j in m.JOBS) <= m.w
model.MachineCompletion = Constraint(model.MACHINES, rule=machine_completion_rule)

```

### 33.7.3. Flow Shop



### 33.7.4. Job Shop

## 33.8. Vehicle Routing

### 33.8.1. Traveling Salesperson

- Let  $G = (V, E)$  be a directed complete graph
- Let  $d_{ij}$  be the distance (cost) from node  $i$  to  $j$
- Let  $x_{ij}$  be a binary decision variable:

$$x_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

## Objective

Minimize the total travel cost

$$\min \sum_{(i,j) \in E} d_{ij} x_{ij}$$

## Constraints

### 1. Flow Balancing

For node  $k \in V$

- One **incoming** edge:

$$\sum_{i \in V, i \neq k} = 1$$

- One **outgoing** edge:

$$\sum_{j \in V, j \neq k} = 1$$

### 2. Subtours

#### a. Miller-Tucker-Zemlin (MTZ)

- Let  $u_i$ : be the position of node  $i$  in the tour (e.g.,  $u_i = k$  if node  $i$  is the  $k$ th node to be passed on the tour)

$$\begin{aligned} u_1 &= 1 \\ 2 \leq u_i &\leq n & \forall i \in V \setminus \{1\} \\ u_i - u_j + 1 &\leq (n-1)(1 - x_{ij}) & \forall (i, j) \in E, i \neq 1, j \neq 1 \end{aligned}$$

#### b. Subtour Elimination Constraint (SEC)

For every subset  $S \subset V$  with  $|S| \geq 2$ :

$$\sum_{\substack{i \in S, j \in S \\ i \neq j}} x_{ij} \leq |S| - 1 \quad \forall S \subset V, |S| \geq 2$$

This ensures that no subset of nodes forms a closed tour independent of the main tour.

## Complete Formulation

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} d_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i \in V, i \neq k} = 1 \quad \forall k \in V \\ & \sum_{j \in V, j \neq k} = 1 \quad \forall k \in V \\ & x_{ij} \in \{0, 1\} \quad \forall (i, j) \in E \end{aligned}$$

MTZ or SEC

### 33.9. Gradient Descent

Find the minimum of a function

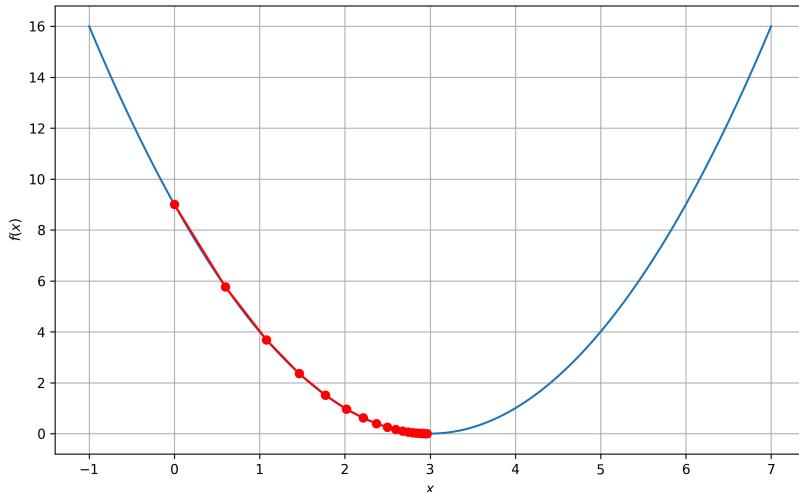
1. **Initialize:** Start with an initial guess for the parameters
2. **Compute Gradient:** Find the gradient of the function at the current parameters
3. **Update Parameters:** Adjust the parameters by moving in the opposite direction of the gradient, scaled by the learning rate
4. **Repeat:** Continue the process until the parameters converge to a minimum or the changes are minimal

**Update Rule:**

$$\theta \leftarrow \theta - \alpha \nabla f(\theta)$$

Where:

- $\theta$ : parameter being optimized
- $\alpha$ : learning rate
- $\nabla f(\theta)$ : gradient of the function  $f$  with respect to  $\theta$



1. Function and Gradient:

- Function:  $\theta_1^2 + \theta_2^2$
- Gradient:  $\nabla f(\theta) = \left( \frac{\partial f}{\partial \theta_1}, \frac{\partial f}{\partial \theta_2} \right) = (2\theta_1, 2\theta_2)$

2. Initial Values:

$$\begin{aligned}\theta_1 &= 1 \\ \theta_2 &= 2\end{aligned}$$

3. Learning Rate:

$$\alpha = 0.1$$

4. Gradient Calculation

- For  $\theta_1 = 1$  and  $\theta_2 = 2$ :

$$\nabla f(\theta) = (2 \cdot 1, 2 \cdot 2) = (2, 4)$$

5. Parameter Update:

- Update  $\theta_1$  and  $\theta_2$  using the rule  $\theta \leftarrow \theta - \alpha \nabla f(\theta)$ :

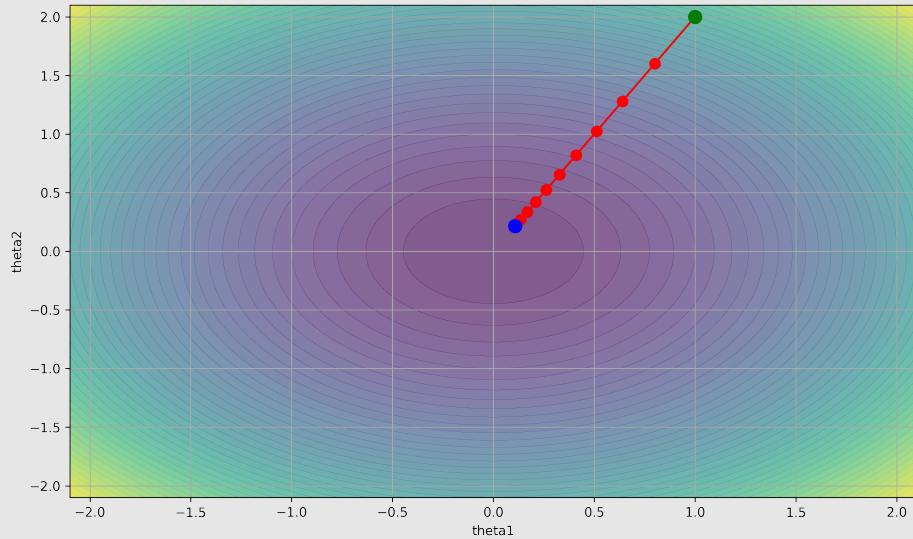
$$\theta_1 \leftarrow 1 - 0.1 \cdot 2 = 1 - 0.2 = 0.8$$

$$\theta_2 \leftarrow 1 - 0.1 \cdot 4 = 2 - 0.4 = 1.6$$

6. New Values:

$$\theta_1 = 0.8$$

$$\theta_2 = 1.6$$



### 33.10. Monte Carlo