Solutions and Commentary

Collected Mathematical Problems for First Year Students

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This document outlines the solutions to the problem set and the methods used to find them. I have added comments for each problem to identify important learning opportunities and interesting results, some of which use language and notation which may be unfamiliar to a student approaching them – you have been warned (the internet is a very useful thing!).

1. Determine the value of x which satisfies the following equation:

$$\frac{9^{-1}}{81^x} = 9^{x^2}$$

$$\frac{9^{-1}}{81^x} = 9^{x^2}$$
$$\implies 9^{-1} = 9^{x^2} 81^x$$
$$\implies 9^{-1} = 9^{x^2} 9^{2x}$$
$$\therefore 1 = 9^{x^2} 9^{2x} 9$$
$$\implies 1 = 9^{x^2+2x+1}$$
$$\implies 0 = x^2 + 2x + 1$$

$$\boxed{x = -1}$$

This is a relatively simple problem, with the key step being to take the logarithm base 9 in the final stage to form a quadratic equation in terms of x to solve. There are alternative methods with differing rearrangement orders, but they are trivial.

2. Suppose I had a basket of green apples and red apples. Of the 100 total apples in the basket, 99% of them were red. Assuming that there are only red and green apples in the basket, how many red apples would I have to remove to change the percentage of red apples in the basket to 98%?

We can set up an equation to represent the situation:

$$\frac{100}{100-x} = 2 \tag{1}$$

This is derived from the standard percentage formula to find the percentage for 1 apple within a basket of $(100 - x)$, using x as the number of apples removed, and equating the result to 2%.

Then, solving (1);

$$\implies 100 = 2\,(100 - x)$$

$$\implies 100 = 200 - 2x$$

$$\therefore 2x = 100$$

$$\boxed{x = 50 \text{ apples}}$$

This result can be counter-intuitive, and this problem has featured in alternative forms on certain UK television programmes. The most common answer is simply to remove one red apple from the basket, which is clearly wrong under a mathematical treatment. Formal equations are not required to obtain the correct result, but they do help!

3. Find x, where:

$$\frac{9^{2x+1}3^{4-3x}}{27^{2-x}} = 9$$

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$$\implies 9^{2x+1}3^{4-3x} = 27^{2-x}9$$

$$\implies 3^{2(2x+1)}3^{4-3x} = 3^{3(2-x)}3^2$$

$$\therefore 3^{2(2x+1)+(4-3x)} = 3^{3(2-x)+2}$$

$$\implies 2(2x+1)+(4-3x)=3(2-x)+2$$
$$\implies (4x+2)+(4-3x)=(6-3x)+2$$
$$\therefore x+6=8-3x$$
$$\implies 4x=2$$

$$\boxed{x=\frac{1}{2}}$$

This problem is solved using similar techniques to number [1.], with a more complicated equation to solve. A key step not highlighted in the first problem's comments is the manipulation used to get all the terms to the same base to enable the logarithm to be taken (concisely, at least) by using 'power rules'.

An added difficulty for those who read the initial instructions is the non-integer result, which was designed to teach that reading instructions really is important!

4. On a hot summer's day, mathematicians at a gala are queuing at an ice cream van. The mathematician at the front of the queue requests that his cone be filled with two scoops of ice cream such that they perfectly fill the cone without exceeding the rim of the cone. They ask for one scoop to be 6.0 cm in diameter and the other to be 3.0 cm in diameter. Assuming that the scoops form uniform spheres and the cone has no thickness, find the height of the ice cream cone.

There are two ways to approach this problem: one uses an infinite series sum (a) and the other uses trigonometry (b). I will approach using the infinite series method first;

(a) By examining the problem, we can constrain it to a limiting case of filling in a line segment using incrementally smaller segments. This is not necessarily a logical step, but leads to the conclusion that the original line segment can be completely filled using an infinite series of line segments which incrementally halve in length. The following set represents the line segements of the system:

$$\left\{\sigma \;\middle|\; 6, 3, \frac{3}{2}, \frac{3}{4}, \cdots, \sigma_n = 6\left(\frac{1}{2}\right)^{n-1}, \cdots\right\}$$

Which results in this series sum:

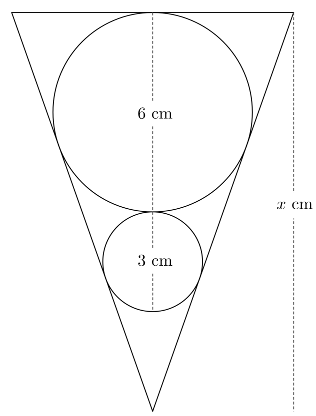
$$\sum_{n=1}^{\infty} \sigma_n = \sum_{n=1}^{\infty} 6\left(\frac{1}{2}\right)^{n-1} = 6\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$$

This is a multiple of a standard series sum and can be expressed as follows:

$$6\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} = 6(2)$$

$$\boxed{\text{height} = 12\text{ cm}}$$

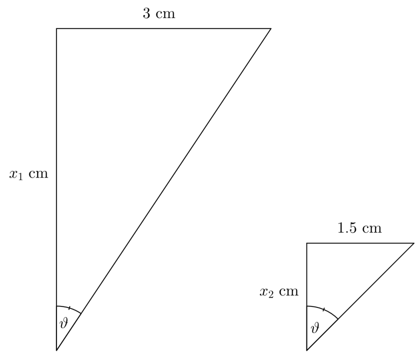
(b) A diagram for the situation looks like this:



To determine the height, x, we first need to find the height from the centre of the first and second circles, which I will call x_1 and x_2 (where x_1 represents the distance $(x-3)$ and x_2 the distance $(x-7.5)$). We can also state that half of the triangle's bottom angle, ϑ, will be a common base angle for the two right-angle triangles formed by x_1 and x_2 with their associated radii (see diagram overleaf).

As both triangles are formed from sections of the same triangle, they are congruent, and we can use this property to determine the lengths x_1 and x_2. To do this, we can write the following system of simultaneous equations:

$$\begin{cases} x_1 \tan\vartheta = 3 \\ x_2 \tan\vartheta = 1.5 \end{cases} \tag{2}$$



Which becomes:

$$\begin{cases} (x-3)\tan\vartheta = 3 \\ (x-7.5)\tan\vartheta = 1.5 \end{cases} \tag{3}$$

Rearranging system (3) then gives:

$$\frac{3}{x-3} = \frac{1.5}{x-7.5}$$
$$\implies 3\,(x-7.5) = 1.5\,(x-3)$$
$$\implies 3x - 22.5 = 1.5x - 4.5$$
$$\therefore 1.5x = 18$$

$$\boxed{x = 12 \text{ cm}}$$

This demonstrates the two methods for solving this problem and that they result in identical solutions. When approaching problems, there are generally multiple different methods that can be used; in this case, one is conceptually easier, whilst the other is computationally easier.

I was originally inspired to write this problem by my favourite mathematical joke, which runs something like this:

"An infinite line of mathematicians walks into a bar. The first asks for a pint of lager, the next a half pint, and the third for a quarter pint. Before the fourth mathematician has a chance to place his order, the bartender places two pints of lager on the bar. He says: 'That's the problem with you mathematicians, you just don't know your limits!'"

The problem is a direct application of this in three dimensions (the series, not the pints).

I would also add that there are other trigonometric approaches to this problem, such as a ratio method. I have only included the simplest trigonometric method here.

5. Solve for x:

$$2\log_2 x - \frac{1}{2}\log_2 x = 3$$

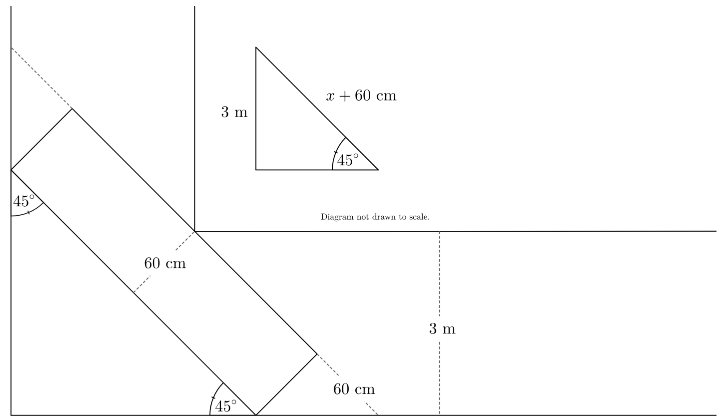
$$\begin{aligned} 2\log_2 x - \frac{1}{2}\log_2 x &= 3 \\ \implies \log_2 x^2 - \log_2 \sqrt{x} &= 3 \\ \implies \log_2 \frac{x^2}{\sqrt{x}} &= 3 \\ \implies \log_2 x^{\frac{3}{2}} &= 3 \\ \therefore x^{\frac{3}{2}} &= 8 \end{aligned}$$

$$\boxed{x = 4}$$

This is a marginally harder version of problems [1.] and [3.], which makes use of (binary) logarithms directly, thus requiring the inverse thought process to raise all of the terms to a common base.

6. A chaise longue is a poncy kind of French chair most likely owned by Louis XVI just before he got 'the chop'. In the Palace of Versailles, Louis XVI wants to put a chaise longue in a long gallery such that it can be wheeled around a corner. Given that the corner angle is exactly 90 degrees, that the gallery has a uniform width of 3 m, and that all the chaise longues in France are 60 cm wide and a whole multiple of 10 cm long; how long is the longest chaise longue that Louis XVI can use in the long gallery?

We need to consider the extreme case, which is when the chaise longue is 'stuck' in the long gallery at a 45° angle to the inside wall - this has been drawn in the diagram below:



From this diagram of the long gallery, we can draw a 45° right-angle triangle (top right) with dimensions 3 m by $(x + 60)$ cm [$(x + 0.6)$ m]. Then, solving for x:

$$x + 0.6 = \frac{3}{\sin 45^\circ}$$

$$\implies x + 0.6 = 3\sqrt{2}$$

$$\therefore x = \left(3\sqrt{2} - 0.6\right)$$

Since x is half of the total length of the maximal chaise longue, we need to double it to get our final answer. This also needs to be rounded down to the nearest multiple of 10 cm due to the requirements of the problem:

$$\boxed{\text{maximal length} = 720 \text{ cm}}$$

This is a challenging problem and was originally inspired by the 'optimal sofa' problem - a problem regarding the optimum shape to make the largest sofa able to fit around a corner. I probably do not have to say that the inspiration was a lot harder than this problem.

I like this problem as it is the kind of real-world challenge that the recreational mathematician might encounter, making it very accessible for students. It also highlights the need for good diagrams – you would be lost without them!

7. Determine the non-zero value of x which satisfies:

$$\sqrt{x} + \sqrt{x} = x$$

$$\sqrt{x} + \sqrt{x} = x$$

$$\implies 2\sqrt{x} = x$$

$$\therefore 2 = \sqrt{x}$$

$$\boxed{x = 4}$$

This is a neat little problem which requires a minuscule effort for a fantastic proof. You see, this problem can be arranged to work with any integer square number x, so long as the number of square roots in the equation equals the square root of x ($\forall x \in \mathbb{N}^2, \sqrt{x}\sqrt{x} = x \implies \forall n \in \mathbb{N}, n\sqrt{n^2} = n^2$) - which sounds trivial, but goes a little way to explaining the process of squaring numbers.

8. A 10 pence coin is made to roll along the edge of a 1 pound coin. Given that a 10 pence coin has a diameter of 28.5 mm and that a 1 pound coin has twelve sides, each measuring 6.00 mm; calculate how many revolutions the 10 pence coin passes through as it makes one full revolution of the 1 pound coin. (Give your answer to the nearest fifth of a whole rotation.)

For this question, we first need to calculate the perimeters of the two coins, labelling the perimeter of the 10 pence coin as P_{10} and the 1 pound coin as P_{100}, these are:

$$\begin{cases} P_{10} = \pi D = 28.5\pi \text{ mm} \\ P_{100} = 12l = 72 \text{ mm} \end{cases} \tag{4}$$

Then, we need to find an equation to calculate the number of rotations, ρ, which is equal to:

$$\rho = \frac{P_{100} + P_{10}}{P_{10}} = \frac{P_{100}}{P_{10}} + 1$$

Substituting (4) into this formula then gives the correct result.

$$\boxed{\rho = 2.2 \text{ rotations}}$$

This is a hard problem and, famously, a basic version of this was marked incorrectly in the 1982 SAT (US college admissions test). The key insight is that the revolving coin not only travels the perimeter of the fixed coin, but also its own perimeter. This leads to the additional term required in the equation.

The problem I chose based on this principle applies it to a system of two different sized objects, one of which is non-uniform in shape. Whilst this sounds as though it complicates the problem, the underlying mathematics is identical and it requires a very similar thought process.

9. Solve for x:

$$\frac{x\left(x\left(x\left(x\left(x-20\right)+160\right)-640\right)+1280\right)-1024}{\left(x-4\right)^{4}}=5$$

$$\frac{x\left(x\left(x\left(x\left(x-20\right)+160\right)-640\right)+1280\right)-1024}{\left(x-4\right)^{4}}=5$$

$$\implies \frac{x^5-20x^4+160x^3-640x^2+1280x-1024}{\left(x-4\right)^{4}}=5$$

$$\implies \frac{\left(x-4\right)^{5}}{\left(x-4\right)^{4}}=5$$

$$\therefore x-4=5$$

$$\boxed{x=9}$$

The factor that makes this problem hard is figuring out that the numerator is equal to $(x-4)^5$ – although polynomial long division could have also been used. Working backwards from a polynomial to a factorised bracket is a difficult thing to do and requires rather a lot of practice.

To explain the mental reasoning in more depth, the first step after expanding the numerator would be to find the fifth root of the final term ($\sqrt[5]{-1024} = -4$), substituting this into a binomial system (e.g. $(x-4)^5$) and calculating a few terms to verify that the system is correct.

10. One day, Leonardo Fibonacci is playing poker with a standard deck of playing cards and asks himself how many arrangements of the 52 cards leads to all of the cards of the 4 suits being separated – regardless of their individual order. Given that there are 13 cards in each suit and no jokers, what is the solution to Fibonacci's problem? (A modern calculator will give the answer in standard form to nine significant figures, Fibonacci's abacus probably would not. I suggest you use the former.)

This is a combinatorics problem, first consider the system involved:

We have 4 suits, each of 13 cards, and want to find how many ways there are to separate the four suits – regardless of order. So, we will evaluate it in 2 steps, starting with a system of 13 cards. How many ways are there to arrange 13 cards? In this case, the order of the 13 cards is important, so we can use a permutation calculation to find the total number of possible arrangements of cards, α_c;

$$\alpha_c = {}^{13}P_{13} = \frac{13!}{(13-13)!} = 13!\,(= 6227020800)$$

Then we need to consider this across the 4 suits, which is simply α_c raised to the power of 4, as each of the suits has α_c arrangements, giving α_s total arrangements across the suits;

$$\alpha_s = {\alpha_c}^4 = \left({}^{13}P_{13}\right)^4 = (13!)^4$$

$$\boxed{\alpha_s = 1.503561738 \times 10^{39} \text{ arrangements}}$$

This is a thought-provoking problem which demonstrates not only the use for combinatorics, but also how large answers can be from a relatively simple setup. The principles involved are the same for many other problems within combinatorics and probability theory, such as dice rolls and even some mathematics with statistical distributions.

11. Solve for x:

$$2 = \sqrt{x\sqrt{x\sqrt{x\cdots}}}$$

$$2 = \sqrt{x\sqrt{x\sqrt{x\cdots}}}$$

$$\implies 4 = x\sqrt{x\sqrt{x\cdots}}$$

$$\implies 4 = 2x \because 2 = \sqrt{x\sqrt{x\cdots}}$$

$$\boxed{x = 2}$$

This is a challenging problem as it involves using the initial series definition in the rearrangement, however, once that has been established, the solution appears trivial.

In reality, we can state that $\forall n \in \mathbb{N}, n = \sqrt{n\sqrt{n\sqrt{n\cdots}}}$ which provides the interesting result an infinite series of square roots of this form will provide an approximation for n – although I am not convinced it will ever be a useful substitution to make.