Q.1]

$$f(x) = \begin{cases} -1+x & -\pi < x < 0 \\ 1+x & 0 < x < \pi \end{cases}$$

$$\therefore f(x) = \begin{cases} -1-x & -\pi < -x < 0 \\ 1-x & 0 < -x < \pi \end{cases}$$

$$= \begin{cases} -1-x & \pi > x > 0 \\ 1-x & 0 > x > -\pi \end{cases}$$

$$= \begin{cases} 1-x & -\pi < x < 0 \\ -1-x & 0 < x < \pi \end{cases}$$

$$f(x) = -f(x)$$

$\therefore$ Given function is odd.

$\therefore a_0 = 0, \quad a_n = 0$

$$\therefore fx = \sum_{n=1}^{\infty} b_n \sin n\pi$$

$$\therefore b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$$

$$= 2/\pi \int_0^{\pi} (1-x) \sin nx \, dx$$

$$= 2/\pi \left[\int_0^{\pi} \sin nx \, dx - \int_0^{\pi} x \sin nx \, dx \right]$$

$$= 2/\pi \left[\left(-\frac{\cos n\pi}{n} \right)_0^{\pi} - \int_0^{\pi} \left(x_0 - \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right) dx \right.$$

$$\therefore b_n = \frac{2}{\pi} \left\{ \left[\left(\frac{-\cos n\pi}{\pi} \right) - \left(\frac{-\cos na}{a} \right) - \left[x \left(\frac{-\cos n\pi}{\pi} \right) \right] \right] \right\}$$

$$\therefore b_n = \frac{-2(-1)^n}{n}.$$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n \sin(nx)}{n}.$$

2] Given:- $f(x) = 2x-1$ in $0 < x \leq 1$

To Find:- Find the half range sine series for the function

Solution:- sine series $(L=1)$

$$\Rightarrow b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{l}x\,dx = \int_0^1 (2x-x^2)\sin\frac{\pi}{l}x\,dx.$$

$$\Rightarrow \int_0^1 x\sin x\frac{n\pi}{2} + dx$$

$$\Rightarrow \frac{-\cos\frac{n\pi}{2}x.}{\frac{n\pi}{2}}\int_0^1 x\frac{2}{n\pi}\int_0^1 \cos\frac{n\pi}{2}x\,dx$$

$$\Rightarrow \frac{-4}{n\pi}\cos(n\pi) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}x\right)\Big|_0^1$$

$$\Rightarrow \frac{-4}{n\pi}(-1)^n$$

$$(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2}\sin\left(\frac{n\pi}{2}x\right)$$

$$f(1) = 1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots$$ (Hence the answer)

3] Obtain the half range cosine series of

$$f(x) = \begin{cases} 2x, & 0<x<1 \\ 2(2-x), & 1<x<2 \end{cases}$$

$(0,2) = (0,l)$

$\therefore \; l = 2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$$

$$= \frac{2}{2}\left[\int_0^1 2x\,dx + \int_1^2 2(2-x)\,dx\right]$$

$$= \left[[x^2]_0^1 + [4x - x^2]_1^2\right]$$

$$= \left[(1-0) + ((8-4)-(4-1))\right]$$

$$= [1 + (4-3)]$$

$$= 1+1$$

$$= 2$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= 1\left[\int_0^1 2x\cos\left(\frac{n\pi}{2}\right)x\,dx + \int_1^2 (4-2x)\cos\left(\frac{n\pi}{2}\right)x\,dx\right]$$

$2x$	$+$	$\cos\left(\frac{n\pi}{2}\right)x$
2	$-$	$\frac{\sin(n\pi/2)x}{(n\pi/2)}$
0		$\frac{-\cos(n\pi/2)x}{(n\pi/2)}$

$4-2x$	$+$	$\cos\left(\frac{n\pi}{2}\right)x$
-2	$-$	$\frac{\sin(n\pi/2)x}{\left(\frac{n\pi}{2}\right)}$
0		$\frac{-\cos\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)^2}$

$$a_n = \left[\frac{2x\sin(n\pi/2)x}{(n\pi/2)} - \frac{2\cos(n\pi/2)}{(n\pi/2)^2}\right]_0^1$$

$$+\left[\frac{(4-2x)\sin(n\pi/2)x}{(n\pi/2)} + \frac{2\cos(n\pi/2)x}{(n\pi/2)^2}\right]_1^2$$

$$a_n = \left[\frac{2\times 2}{n\pi} - \frac{2\times 4}{n^2\pi^2}\right] + \left[\frac{2\times 4}{n^2\pi^2} - \frac{4}{\pi}\right]$$

4) Let $f(x) = \sum b_n \sin nx$ $\quad [\because l = \pi]$

$$\therefore b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \cdot \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)(1)\right\}_0^{\pi/2} \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)(-1)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{-\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} - 0 - 0\right\} + \left\{0 - 0 + \frac{\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}\right\}\right]$$

$$= \frac{4}{\pi} \cdot \frac{\sin(n\pi/2)}{n^2}$$

$$\therefore b_1 = \frac{4}{\pi} \cdot \frac{1}{1^2},\ b_2 = 0,\ b_3 = \frac{-4}{\pi} \cdot \frac{1}{3^2},\ b_4 = 0 \ldots$$

$$\therefore f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \ldots\right]$$

By Parseval's identity

$$\frac{1}{\pi}\int_0^{\pi} [f(x)]^2 dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots \infty\right]$$

$$\therefore \frac{1}{\pi}\left[\int_0^{\pi/2} x^2 \cdot dx + \int_{\pi/2}^{\pi} (\pi - x)^2 dx\right] = \frac{1}{2}\left[b_1^2 + b_2^2 + \ldots + \infty\right]$$

Now,

$$\frac{1}{\pi}\left[\int_0^{\pi/2} x^2 \cdot dx + \int_{\pi/2}^{\pi} (\pi^2 - 2\pi x + x^2) dx\right] = \frac{1}{\pi}\left[\left(\frac{x^3}{3}\right)_0^{\pi/2} + \left\{\pi^2 x - \pi x^2 + \frac{x^3}{3}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi^3}{24} - 0\right\} + \left\{\left(\pi^3 - \pi^3 + \frac{\pi^3}{3}\right) - \left(\frac{\pi^3}{2} - \frac{\pi^3}{4} + \frac{\pi^3}{24}\right)\right\}\right]$$

$$= \frac{1}{\pi} \cdot \frac{\pi^3}{12} = \frac{\pi^2}{12}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{2}\cdot\left[\frac{16}{\pi^2}\cdot\frac{1}{1^4} + \frac{16}{\pi^3}\cdot\frac{1}{3^4} + \frac{16}{\pi^2}\cdot\frac{1}{5^4} + \cdots \right]$$

$$\therefore \frac{\pi}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

6) Find the series of $F(x) = 4 - x^2$, $0 < x < 2$ of periodicity 2 and hence find the value of series at $x = 1$ & 2

→ The fourier series at interval (0, 2) is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_0^{l} 4 - x^2 \, dx$$

$$\therefore a_0 = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}$$

$$a_n = \frac{1}{l}\int_0^2 (4-x^2)\cos(n\pi x)\cdot dx$$

$$= \frac{1}{l}\left[(4-x^2)\cdot\sin\frac{n\pi x}{n\pi} - \frac{2x\cos n\pi x}{(n\pi)^2} + \frac{2\sin n\pi x}{(n\pi)^3}\right]_0^2$$

$$= \left[\frac{-4}{(n\pi)^2} + 0\right]$$

Now,

$$b_n = \frac{1}{1}\int_0^2 (4-x^2)\sin(n\pi x)dx$$

$$= \left[-(4-x^2)\cdot\frac{\cos n\pi x}{n\pi} - \frac{2x\sin n\pi x}{(n\pi)^2} - \frac{2\cos n\pi x}{(n\pi)^3}\right]$$

$$= \frac{-2\cos(2n\pi)}{(n\pi)^3} + \frac{4\cos 0}{n\pi} + \frac{2\cos 0}{(n\pi)^3}$$

$$\therefore b_n = \frac{4}{n\pi}$$

Now,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$= \frac{16}{3} \cdot \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2} \cos(n\pi x) + \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin(n\pi x)$$

$$= \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \cdots\right]$$

$$+ \frac{4}{\pi}\left[\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \cdots\right]$$

Now,

$F(x) = 4 - x^2 =$ series

$\therefore$ Value of series at $x=1 = F(1)$. ($\because$ function is continuous)

$F(1) = 4 - 1 = 3$

Now,

$F(x)$ is discontinuous at $x=2$

$$\therefore F(2) = \left[\lim_{x\to 2^-} F(x) + \lim_{x\to 2^+} F(x)\right] / 2$$

$$= \frac{0+4}{2} = 2$$