Non Linear Programming Problems

Non Linear Problem without any Constraint:

Optimize (Maximize or Minimize) $Z = f(x_1, x_2, ..., x_n)$

The stationary point at which the objective function optimizes are given by

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \; ...., \; \frac{\partial f}{\partial x_n} = 0$$

Let X be a stationary point for the objective function obtain from the above equations

At the stationary point X

Consider the Matrix
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & .... & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & .... & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ .... & .... & .... & .... \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & .... & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

The matrix H is known as Hessian matrix.

Find the principal Minors $\Delta_1, \Delta_2, ..., \Delta_n$ of H

If the **signs** of $\Delta_1, \Delta_2, ..., \Delta_n$ is alternate starting from $(-)$ve sign then the stationary point **maximizes** the objective function

If the **signs** of $\Delta_1, \Delta_2, ..., \Delta_n$ are all $(+)$ve then the stationary point **minimizes** the objective function

Non Linear Programming Problem

Ex. Determine the relative maximum and minimum (if any) of the following function.

① $x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$

Solⁿ Let $f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$

stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_3} = 0$$

$\Rightarrow 1 - 2x_1 = 0 \qquad \Rightarrow 2x_1 = 1$

$x_3 - 2x_2 = 0 \qquad -2x_2 + x_3 = 0$

$2 + x_2 - 2x_3 = 0 \qquad x_2 - 2x_3 = -2$

$\Rightarrow x_1 = \frac{1}{2}, \; x_2 = 2/3, \; x_3 = 4/3$

At the stationary point, we have the Hessian matrix

$$H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\Delta_1 = |-2| = -2$$

$$\Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$$\Delta_3 = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = -6$$

$\Rightarrow$ $x_1 = 1/2$, $x_2 = 2/3$, $x_3 = 4/3$ is point of maxima. Maximum value of f is

$$f_{max} = f(1/2, 2/3, 4/3) = \frac{57}{36}$$

(2) $f(x_1, x_2, x_3) = 4x_1^2 + 3x_2^2 + x_3^2 - 6x_1 x_2 + x_1 x_3 - \frac{1}{2}x_1 - 2x_2 + 15$

Sol^n we have

$$\frac{\partial f}{\partial x_1} = 8x_1 - 6x_2 + x_3 - 1/2$$

$$\frac{\partial f}{\partial x_2} = 6x_2 - 6x_1 - 2$$

$$\frac{\partial f}{\partial x_3} = 2x_3 + x_1$$

Stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_3} = 0$$

$$\Rightarrow \quad 8x_1 - 6x_2 + x_3 = 1/3$$
$$-6x_1 + 6x_2 = 2$$
$$x_1 + 2x_3 = 0$$

$\Rightarrow \quad x_1 = 5/3, \quad x_2 = 2, \quad x_3 = -5/6$

we have Hessian matrix

$$H = \begin{bmatrix} 8 & -6 & 1 \\ -6 & 6 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\Delta_1 = |8| = 8$$

$$\Delta_2 = \begin{vmatrix} 8 & -6 \\ -6 & 6 \end{vmatrix} = 12$$

$$\Delta = \begin{vmatrix} 8 & -6 & 1 \\ -6 & 6 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 18$$

$\Rightarrow \quad x_1 = 5/3, \; x_2 = 2, \; x_3 = -5/6$ minimizes the objective function

$$\Rightarrow \quad f_{min} = f(5/3, 2, -5/6)$$
$$= 12.59$$

Lagrange's Method for Non Linear Problem with Equality Constraints:

Consider a Non Linear Problem in n variables and m equality constraints $(m<n)$

Optimize (Maximize or Minimize) $Z=f(x_1,x_2,\ldots,x_n)$

Subject to

$$\begin{aligned}
&h_1(x_1,x_2,\ldots,x_n)=0\\
&h_2(x_1,x_2,\ldots,x_n)=0\\
&\ldots\\
&h_m(x_1,x_2,\ldots,x_n)=0\\
&x_1,x_2,\ldots,x_n\geq 0
\end{aligned}$$

We define a Lagrange's function

$$L=f(x_1,x_2,\ldots,x_n)-\lambda_1 h_1(x_1,x_2,\ldots,x_n)-\lambda_2 h_2(x_1,x_2,\ldots,x_n)\ldots\ldots-\lambda_m h_m(x_1,x_2,\ldots,x_n)$$

The stationary point at which the objective function optimizes are given by

$$\frac{\partial L}{\partial x_1}=0,\ \frac{\partial L}{\partial x_2}=0,\ \ldots,\ \frac{\partial L}{\partial x_n}=0$$

And

$$\begin{aligned}
&h_1(x_1,x_2,\ldots,x_n)=0\\
&h_2(x_1,x_2,\ldots,x_n)=0\\
&\ldots\\
&h_m(x_1,x_2,\ldots,x_n)=0
\end{aligned}$$

Let X be a stationary point for the objective function obtain from the above equations

At the stationary point X

Consider the Matrix $H^B=\begin{bmatrix} O & P\\ P^T & Q\end{bmatrix}_{(m+n)\times(m+n)}$

Where $O=\begin{bmatrix} 0 & \ldots & 0\\ \ldots & \ldots & \ldots\\ 0 & \ldots & 0\end{bmatrix}_{m\times m}$, $P=\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \ldots & \frac{\partial h_1}{\partial x_n}\\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \ldots & \frac{\partial h_2}{\partial x_n}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \ldots & \frac{\partial h_m}{\partial x_n}\end{bmatrix}$, $P^T=$ Transpose of P,

And $Q=\begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1\partial x_2} & \ldots & \frac{\partial^2 L}{\partial x_1\partial x_n}\\ \frac{\partial^2 L}{\partial x_2\partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \ldots & \frac{\partial^2 L}{\partial x_2\partial x_n}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{\partial^2 L}{\partial x_n\partial x_1} & \frac{\partial^2 L}{\partial x_n\partial x_2} & \ldots & \frac{\partial^2 L}{\partial x_n^2}\end{bmatrix}$

The matrix H^B is known as Bordered Hessian matrix .

Particular case

I. For a NLPP with n variables and **one** constrains , we have

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}_{(n+1)\times(n+1)}$$

Find the Last $(n-1)$ Principal Minors of H^B starting from $\Delta_3, \Delta_4, \ldots, \Delta_{n+1}$

If the **signs** of $\Delta_3, \Delta_4, \ldots, \Delta_{n+1}$ is alternate starting from $(+)$ve sign then the stationary point **maximizes** the objective function
If the **signs** of $\Delta_3, \Delta_4, \ldots, \Delta_{n+1}$ are all $(-)$ve then the stationary point **minimizes** the objective function

II. For a NLPP with n variables and **two** constrains , we have

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}_{(n+2)\times(n+2)}$$

Find the Last $(n-2)$ Principal Minors of H^B starting from $\Delta_5, \Delta_6, \ldots, \Delta_{n+2}$

If the **signs** of $\Delta_5, \Delta_6, \ldots, \Delta_{n+2}$ is alternate starting from $(-)$ve sign then the stationary point **maximizes** the objective function
If the **signs** of $\Delta_5, \Delta_6, \ldots, \Delta_{n+2}$ are all $(+)$ve then the stationary point **minimizes** the objective function

Karush Kuhn Tucker (KKT) Conditions for a Non Linear Programming Problem with Inequality Constraints:

Consider a Non Linear Problem in n variables and m Inequality constraints $(m < n)$

Optimize (Maximize or Minimize) $Z = f(x_1, x_2, ..., x_n)$

Subject to

$$h_1(x_1, x_2, \dots, x_n) \le 0$$
$$h_2(x_1, x_2, \dots, x_n) \le 0$$
$$....$$
$$h_m(x_1, x_2, \dots, x_n) \le 0$$
$$x_1, x_2, ..., x_n \ge 0$$

We define a Lagrange's function

$$L = f(x_1, x_2, ..., x_n) - \lambda_1 h_1(x_1, x_2, \dots, x_n) - \lambda_2 h_2(x_1, x_2, \dots, x_n) ..... - \lambda_m h_m(x_1, x_2, \dots, x_n)$$

The stationary point at which the objective function optimizes must satisfies the conditions given below known as **Karush Kuhn Tucker conditions**

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad .... , \quad \frac{\partial L}{\partial x_n} = 0$$
$$\lambda_1 \, h_1(x_1, x_2, \dots, x_n) = 0$$
$$\lambda_2 \, h_2(x_1, x_2, \dots, x_n) = 0$$
$$....$$
$$\lambda_m \, h_m(x_1, x_2, \dots, x_n) = 0$$

$$h_1(x_1, x_2, \dots, x_n) \le 0$$
$$h_2(x_1, x_2, \dots, x_n) \le 0$$
$$....$$
$$h_m(x_1, x_2, \dots, x_n) \le 0$$

And $\lambda_1, \lambda_2, ..., \lambda_n \ge 0$

The stationary point obtained using above condition will optimizes the objective function.

The stationary point will maximize or minimize the objective function will depends on the Principal Minors of the H^B described above.

Ex. optimize

$$Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

sub to $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$

Soln We have the Lagrange function

$$L = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda(x_1 + x_2 + x_3 - 20)$$

$\therefore$ Point of maxima or minima are given by

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0 \quad \text{—— (1)}$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda = 0 \quad \text{—— (2)}$$

$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 - \lambda = 0 \quad \text{—— (3)}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 20) = 0 \quad \text{—— (4)}$$

From (1), $\lambda = 4x_1 + 10$

From (2), $-4x_1 + 2x_2 = 2$

From (3), $-4x_1 + 6x_3 = 4$

From (4), $x_1 + x_2 + x_3 = 20$

$\Rightarrow x_1 = 5, \ x_2 = 11, \ x_3 = 4$

To test whether this point maximizes or minimizes the objective function; we have Bordered Hessian matrix

$$H^B = \left[\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{array}\right]$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix}$$

$$= -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -44$$

$\Rightarrow x_1 = 5, \ x_2 = 11, \ x_3 = 4$ is a point of minima

& $Z_{min} = 281$

E.x. Solve the NLPP

optimize $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1 x_2$

Sub to $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20,$

$x_1, x_2, x_3 \geq 0$

Sol^n Define the Lagrangian function

$$L = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1 x_2 - \lambda_1 (x_1 + x_2 + x_3 - 15)$$
$$- \lambda_2 (2x_1 - x_2 + 2x_3 - 20)$$

The stationary points are given by

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0 \quad \text{--- (4)}$$

$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0 \quad \text{--- (5)}$$

From (1) $\lambda_1 + 2\lambda_2 = 8x_1 - 4x_2$

From (2) $\lambda_1 - \lambda_2 = 4x_2 - 4x_1$

From (3), $\lambda_1 + 2\lambda_2 = 2x_3$

$\Rightarrow \quad 2x_3 = 8x_1 - 4x_2$

$\Rightarrow \quad -8x_1 + 4x_2 + 2x_3 = 0$

$\Rightarrow \quad -4x_1 + 2x_2 + x_3 = 0$

From (4) $\quad x_1 + x_2 + x_3 = 15$

From (5) $\quad 2x_1 - x_2 + 2x_3 = 20$

$\Rightarrow \quad x_1 = \frac{11}{3}, \; x_2 = \frac{10}{3}, \; x_3 = 8$

At the stationary point, we have the Bordered Hessian matrix;

$$H^B = \left[\begin{array}{cc:ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hdashline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & -2 \end{array}\right]$$

$$\Delta_5 = |H^B|$$

$$= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= -3 \times (-6) + 0 + 3 \times 24 = 90$$

$\Rightarrow x_1 = \frac{11}{3}, \; x_2 = \frac{10}{3}, \; x_3 = 8$

minimizes the objective function.

$\therefore \; Z_{min} = \frac{820}{9}$

Karush Kuhn Tucker (KKT) method for NLPP with inequality constraints:-

(1) $\text{Max } Z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

sub to $\quad 2x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0$

Soln Define the Lagrange function

$$L = 10x_1 + 4x_2 - 2x_1^2 - x_2^2 - \lambda(2x_1 + x_2 - 5)$$

The stationary point satisfies the following KKT conditions

$$\frac{\partial L}{\partial x_1} = 10 - 4x_1 - 2\lambda = 0 \quad \text{——— (1)}$$

$$\frac{\partial L}{\partial x_2} = 4 - 2x_2 - \lambda = 0 \quad \text{——— (2)}$$

$$\lambda(2x_1 + x_2 - 5) = 0 \quad \text{——— (3)}$$

$$2x_1 + x_2 - 5 \leq 0 \quad \text{——— (4)}$$

$$\lambda \geq 0 \quad \text{——— (5)}$$

case (1)

For $\lambda = 0$;

$10 - 4x_1 = 0 \Rightarrow x_1 = 5/2$

$4 - 2x_2 = 0 \Rightarrow x_2 = 2$

$2x_1 + x_2 - 5 = 5 + 2 - 5 = 2 > 0$

$\Rightarrow$ (4) condition is not satisfied

$\Rightarrow$ solution is not feasible, therefore rejected.

case (2)

$\lambda \neq 0$

$\Rightarrow \quad -4x_1 - 2\lambda = -10$

$-2x_2 - \lambda = -4$

$2x_1 + x_2 = 5$

$\Rightarrow \quad x_1 = \frac{11}{6}, \; x_2 = 4/3, \; \lambda = 4/3$

and all conditions (1) to (5) are satisfied.

$\Rightarrow$ solution is feasible and since it is unique; it is also optimal.

$\Rightarrow \quad Z_{max} = \frac{91}{6}$

(2) Max $Z = 7x_1^2 + 6x_1 + 5x_2^2$

Sub to $x_1 + 2x_2 \leq 10$

$x_1 - 3x_2 \leq 9$

$x_1, x_2 \geq 0$

Sol^n^ Lagrange function is defined as

$$L = 7x_1^2 + 6x_1 + 5x_2^2 - \lambda_1 (x_1 + 2x_2 - 10) - \lambda_2 (x_1 - 3x_2 - 9)$$

Stationary point must satisfies following KKT conditions:-

$$\frac{\partial L}{\partial x_1} = 14x_1 + 6 - \lambda_1 - \lambda_2 = 0 \quad \text{——— (1)}$$

$$\frac{\partial L}{\partial x_2} = 10x_2 - 2\lambda_1 + 3\lambda_2 = 0 \quad \text{——— (2)}$$

$$\lambda_1 (x_1 + 2x_2 - 10) = 0 \quad \text{——— (3)}$$

$$\lambda_2 (x_1 - 3x_2 - 9) = 0 \quad \text{——— (4)}$$

$$x_1 + 2x_2 - 10 \leq 0 \quad \text{——— (5)}$$

$$x_1 - 3x_2 - 9 \leq 0 \quad \text{——— (6)}$$

$$\lambda_1, \lambda_2 \geq 0 \quad \text{——— (7)}$$

case (1) $\lambda_1 = 0, \lambda_2 = 0$

$$x_1 = -\frac{6}{14}, \quad x_2 = 0$$

$\Rightarrow$ solution is infeasible

case (2) $\lambda_1 = 0, \lambda_2 \neq 0$; $\because$ from (1), (2) & (4)

$$14x_1 - \lambda_2 = -6$$

$$10x_2 + 3\lambda_2 = 0$$

$$x_1 - 3x_2 = 9$$

$\Rightarrow$ $\cancel{x_1 = \frac{19}{119}}$, $\cancel{x_2 = -1050}$

$\cancel{x_1 = -\frac{36}{29}}$, $\cancel{x_2 =}$

$$x_1 = \frac{9}{34}, \quad x_2 = -\frac{99}{34}, \quad \lambda_2 = \frac{165}{17}$$

$\Rightarrow$ solution is infeasible.

case (3) $\lambda_1 \neq 0, \lambda_2 = 0$; from (1), (2), (3)

$$14x_1 - \lambda_1 = -6$$

$$10x_2 - 2\lambda_1 = 0$$

$$x_1 + 2x_2 = 10$$

$$\Rightarrow x_1 = \frac{38}{33}, \quad x_2 = \frac{146}{33}, \quad \lambda_1 = \frac{730}{33}$$

(4), (5), (7) are satisfied.

$x_1 - 3x_2 - 9 < 0 \Rightarrow$ (6) is satisfied.

$\Rightarrow$ solution is feasible.

$$Z = 7\left(\frac{38}{33}\right)^2 + 6 \times \frac{38}{33} + 5\left(\frac{146}{33}\right)^2$$

$$= \frac{3764}{33} = 114.1$$

case (4) $\lambda_1 \neq 0$, $\lambda_2 \neq 0$; from (1) to (4)

$$14x_1 + 6 - \lambda_1 - \lambda_2 = 0$$

$$10x_2 - 2\lambda_1 + 3\lambda_2 = 0$$

$$x_1 + 2x_2 = 10$$

$$x_1 - 3x_2 = 9$$

$$\Rightarrow x_1 = \frac{48}{5}, \; x_2 = \frac{1}{5}$$

$$\Rightarrow \lambda_1 + \lambda_2 = \frac{702}{5}$$

$$2\lambda_1 - 3\lambda_2 = 2$$

$$\lambda_1 = \frac{2116}{25}, \quad \lambda_2 = \frac{1394}{25}$$

$\Rightarrow$ (5), (6), (7) are satisfied

$\Rightarrow$ solution is feasible

$$Z = 7\left(\frac{48}{5}\right)^2 + 6 \times \frac{48}{5} + 5 \times \left(\frac{1}{5}\right)^2$$

$$= \frac{17573}{25} = 702.9$$

Hence optimal solution is

$x_1 = \frac{48}{5}$, $x_2 = \frac{1}{5}$

& $Z_{max} = \frac{17573}{25}$

(3) optimize $Z = 2x_1 + 3x_2 - x_1^2 - x_2^2 - x_3^2$

sub to $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 6$

$x_1, x_2 \geq 0$

$\underline{\text{sol}^n}$ Defining the Lagrange function

$L = 2x_1 + 3x_2 - x_1^2 - x_2^2 - x_3^2 - \lambda_1(x_1 + x_2 - 1)$
$- \lambda_2(2x_1 + 3x_2 - 6)$

The stationary point must satisfies following KKT conditions.

$\frac{\partial L}{\partial x_1} = 2 - 2x_1 - \lambda_1 - 2\lambda_2 = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 3 - 2x_2 - \lambda_1 - 3\lambda_2 = 0$ — (2)

$\frac{\partial L}{\partial x_3} = -2x_3 = 0$ — (3)

$\lambda_1(x_1 + x_2 - 1) = 0$ — (4)

$\lambda_2(2x_1 + 3x_2 - 6) = 0$ — (5)

$x_1 + x_2 - 1 \leq 0$ — (6)

$2x_1 + 3x_2 - 6 \leq 0$ — (7)

$\lambda_1, \lambda_2 \geq 0$ — (8)

case ① $\lambda_1 = 0, \lambda_2 = 0$; from ①, ②, ⑤

$x_1 = 1, \; x_2 = 3/2, \; x_3 = 0$

③, ④ are satisfied.

$x_1 + x_2 - 1 = 1 + 3/2 - 1 > 0$

$\Rightarrow$ ⑥ is not satisfied

$\Rightarrow$ solution is infeasible

case ② $\lambda_1 = 0, \lambda_2 \neq 0$; from ① ②, ③ & ⑤

$-2x_1 - 2\lambda_2 = -2$

$-2x_2 - 3\lambda_2 = -3$

$x_3 = 0$

$2x_1 + 3x_2 = 6$

$\Rightarrow x_1 = \frac{12}{13}, \; x_2 = \frac{18}{13}, \; x_3 = 0, \; \lambda_2 = \frac{1}{13}$

④ & ⑦, ⑧ are satisfied

$x_1 + x_2 - 1 = \frac{12}{13} + \frac{18}{13} - 1 > 0$

$\Rightarrow$ ⑥ is not satisfied

$\Rightarrow$ solution is infeasible.

case ③ $\lambda_1 \neq 0, \lambda_2 = 0$; from ① to ④

$2x_1 + \lambda_1 = 2$

$2x_2 + \lambda_2 = 3$

$x_3 = 0$

$x_1 + x_2 = 1$

$\Rightarrow x_1 = 1/4,\ x_2 = 3/4,\ x_3 = 0,\ \lambda_1 = 3/2$

(5), (6), (8) are satisfied

$2x_1 + 3x_2 - 6 = \frac{1}{2} + \frac{9}{4} - 6 = -\frac{13}{4} < 0$

$\Rightarrow$ (7) is satisfied

$\Rightarrow$ solution is feasible.

$$\Rightarrow \quad Z = 2\times\frac{1}{4} + \frac{9}{4} - \left(\frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - 0$$

$$= \frac{17}{8}$$

case (4) $\lambda_1 \neq 0,\ \lambda_2 \neq 0$; From (1) to (5)

$$2x_1 + \lambda_1 + 2\lambda_2 = 2$$

$$2x_2 + \lambda_1 + 3\lambda_2 = 3$$

$$x_3 = 0$$

$$x_1 + x_2 = 1$$

$$2x_1 + 3x_2 = 6$$

$\Rightarrow x_1 = -3,\ x_2 = 4,\ x_3 = 0$

$\Rightarrow$ solution is infeasible.

Hence optimal solution is

$$x_1 = \frac{1}{4},\ x_2 = \frac{3}{4},\ x_3 = 0$$

Now whether this optimal solution is going to maximize or minimize the objective function is not clear.

In this case, we apply the 2nd derivative test.

We have the Bordered Hessian matrix;

$$H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ \hline 1 & 2 & -2 & 0 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$$\Delta_5 = |H^B|$$

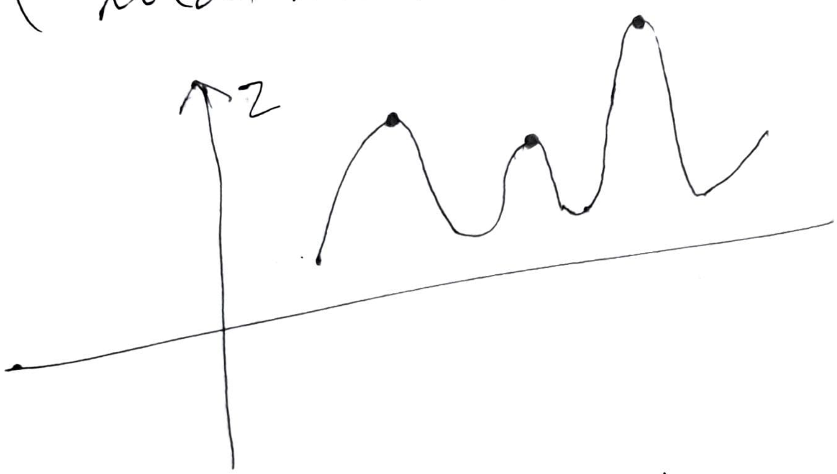
$$= \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

$$= 1 \times [-6+4] = -2$$

$\Rightarrow$ The solution is a maximizer of objective function.

$\therefore$ $x_1 = 1/4$, $x_2 = 3/4$, $x_3 = 0$ is a point of maxima & $Z_{max} = \frac{17}{8}$.

Note:- suppose, there were two ~~feasi~~ or more feasible solution, then all ~~points~~ solutions are maxima of objective function (local maximum point)



The ~~B S~~ feasible solution at which the objective function is maximum is called the Globle maxima or maximum point & gives the Zmax & therefore will be the optimal solution.