Linear Algebra: Matrix Theory

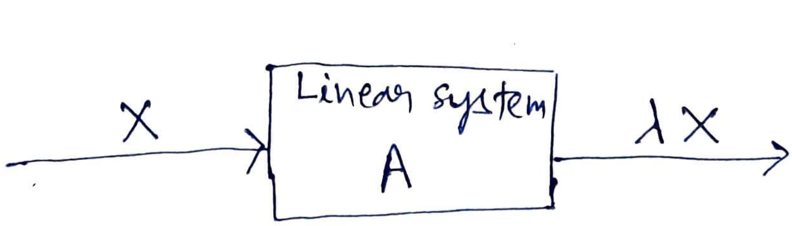
Definition :-

Let $A = (a_{ij})_n$ be a square matrix.
If there is non zero vector $X \neq 0$
such that

$$AX = \lambda X$$

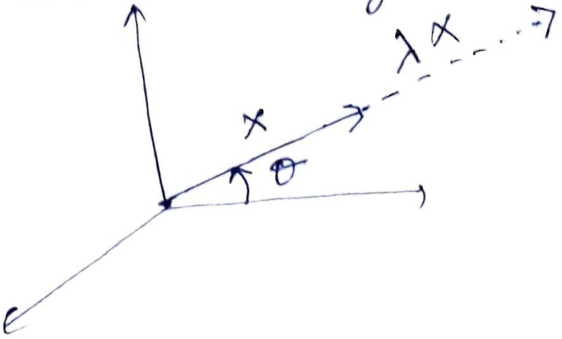
for some scalar λ, then X is called an eigen vector or characteristic vector or latent vector and scalar λ is called an eigen value or characteristic value (root) or latent value of matrix A.

Physical Meaning :-



X is such a vector (input) for the Linear system A that when processed under it, its characteristic does not changes but only get scaled.

Geometrical Meaning:-



For the linear system A, X is such a vector that when transformation A is applied to X, its magnitude changes but its direction remains same.

We have $$AX = \lambda X$$

$$\Leftrightarrow AX - \lambda X = 0$$

$$\Leftrightarrow (A - \lambda I)X = 0 \quad \text{——— (I)}$$

which is a system of homogeneous equation.

The above system of equations has a non-zero solution $(X \neq 0)$

iff $$|A - \lambda I| = 0 \quad \text{——— (II)}$$

The equation (II) is known as characteristic equation of matrix A and solutions of equation (II) are the eigen values of A. Solution of equation (I) for a known value

of λ gives the eigen ~~values~~ vectors of A.
The polynomial $|A - \lambda I|$ is known as the characteristic polynomial of matrix A.

Note :- For a 3X3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

characteristic equation is

$$|A - \lambda I| = 0$$

$$\Leftrightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$\Leftrightarrow \lambda^3 - \left(\begin{array}{l}\text{sum of all minors of order one} \\ \text{along the principal diagonal}\end{array}\right)\lambda^2$

$+ \left(\begin{array}{l}\text{sum of all minors of order two} \\ \text{along the principal diagonal}\end{array}\right)\lambda$

$- |A| = 0$

E.X.

Find the eigen values and corresponding linearly independent eigen vectors.

(1) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Cht. Eq of A is

$$|A - \lambda I| = 0$$

$$\lambda^3 - (2+2+2)\lambda^2 + (5+3+3)\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

For $\lambda = 1$,

$$(A - \lambda I) X = 0$$

$$\Rightarrow (A - I) X = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 - x_2 + x_3 = 0,$

$2x_2 - 2x_3 = 0$

$\Rightarrow \quad x_2 = x_3, \quad x_1 = 0$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$,

$(A - 2I)X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_2 + x_3 = 0, \quad x_1 - x_3 = 0$

$\Rightarrow \quad x_2 = x_3, \; x_1 = x_3$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3$,

$(A - 3I)X = 0$

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} R_2 + R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_2 + x_3 = 0, \quad -2x_2 = 0$

$\Rightarrow x_2 = 0, \quad x_1 = x_3$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(2) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Sol^n Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

cht. Eq is

$$|A - \lambda I| = 0$$

$\Rightarrow \lambda^3 - (2+3+2)\lambda^2 + (4+4+3)\lambda - |A| = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\Rightarrow \lambda = 5, 1, 1$

For $\lambda = 5$,

$$(A - 5I)X = 0$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - R_3 \begin{bmatrix} -3 & 2 & 1 \\ 0 & -4 & 4 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -4x_2 + 4x_3 = 0, \quad x_1 + 2x_2 - 3x_3 = 0$

$\Rightarrow \quad x_2 = x_3, \quad x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 1$,

$(A - I)X = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + 2x_2 + x_3 = 0$

$\Rightarrow \quad x_3 = -x_1 - 2x_2$

$\Rightarrow$ There are two L.I. eigen vectors

$X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

(3) $\quad A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

cht. eq is

$|A - \lambda I| = 0$

$$\Rightarrow \lambda^3-(4+3-2)\lambda^2+(6+4-2)\lambda-|A|=0$$

$$\Rightarrow \lambda^3-5\lambda^2+8\lambda-4=0$$

$$\Rightarrow \lambda=1,2,2$$

For $\lambda=1$,

$$(A-I)X=0$$

$$\begin{bmatrix}3 & 6 & 6\\ 1 & 2 & 2\\ -1 & -5 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$R_3+R_2\begin{bmatrix}3 & 6 & 6\\ 1 & 2 & 2\\ 0 & -3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow x_1+2x_2+2x_3=0, \quad -3x_2-x_3=0$$

$$\Rightarrow x_3=-3x_2, \quad x_1=4x_2$$

$$\therefore X_1=\begin{bmatrix}4\\ 1\\ -3\end{bmatrix}$$

For $\lambda=2$,

$$(A-2I)X=0$$

$$\begin{bmatrix}2 & 6 & 6\\ 1 & 1 & 2\\ -1 & -5 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$R_3+R_2\begin{bmatrix}2 & 6 & 6\\ 1 & 1 & 2\\ 0 & -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow \quad x_1 + x_2 + 2x_3 = 0, \quad -4x_2 - 2x_3 = 0$

$\Rightarrow \quad x_3 = -2x_2, \quad x_1 = 3x_2$

$\Rightarrow$ There is only one L.I. eigen vector

$$X_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

(4) $$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

cht. Eq is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda)(2-\lambda)(2-\lambda) = 0$

$\Rightarrow \quad \lambda = 2, 2, 2$

For $\lambda = 2$,

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_2 = 0, \quad x_3 = 0$

$\Rightarrow$ There is only one L.I. eigen vector

$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(5) $$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigen values of A are

$\lambda = 2, 2, 2$

$(A - 2I)X = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 = 0$

$\Rightarrow$ There are two L.I. eigen vectors

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition:-

Algebraic Multiplicity (AM)

= No. of times an eigen value is repeated

Geometric Multiplicity (GM)

= No. of L.I. eigen vectors possible for an eigen value.

Note that, in the previous example,

λ	AM	G.M
2	3	2

(6) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

then $\lambda = 2, 2, 2$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ All x_1, x_2, x_3 can take independent values.

$\Rightarrow$ There are three L.I. eigen vectors.

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Here for $\lambda = 2$, $AM = 3$, $GM = 3$