Binomial Distribution :-

Binomial distribution occurs whenever

(i) an experiments is repeated n times, n is finite.

(ii) all n trials are independent of each other.

(iii) each trials has only two possibilities, where one is taken as 'success' and other as 'failure'.

Let X denotes the number of successes out of n trials.

Then X takes values 0, 1, 2, ..., n.

Let $p = P\{\text{success in a trial}\}$,

$q = P\{\text{Failure in a trial}\}$,

then $q = 1 - p$

and $P\{X = x\} = {}^{n}C_{x}\, p^{x} q^{n-x}$, $x = 0, 1, \ldots, n$

We say that X follows a Binomial distribution with parameters n and p; denoted by $X \sim B(n, p)$

Mean and Variance of $X \sim B(n, p)$:-

(1) $\mu_X = E(X) = np$

(2) $\sigma_X^2 = Var(X) = npq$

Ex ① A box contains 100 transistors, out of which 20 are defective. If 10 are selected for inspection, find the probability that (i) 5 are defectives (ii) at least one is defective (iii) at most 3 are defective.

Soln Let X: No. of defective transistors out of 10 selected one.

$X \sim B(n, p)$

$n = 10$

$p = P\{\text{A transistor is defective}\}$

$= \frac{20}{100} = 0.2$

$\therefore q = 1 - p = 0.8$

$\therefore P\{X = x\} = {}^{n}C_{x}\, p^{x} q^{n-x}$

$= {}^{10}C_{x} (0.2)^{x} (0.8)^{10-x}$

(i) $P\{\text{5 are defective}\}$

$= P\{X = 5\}$

$= {}^{10}C_{5} (0.2)^{5} (0.8)^{5} = 0.0264$

(ii) $P\{\text{at least one is defective}\}$

$= P\{X \geq 1\} = 1 - P\{X < 1\} = 1 - P\{X = 0\}$

$= 1 - {}^{10}C_{0} (0.2)^{0} (0.8)^{10} = 0.8926$

(iii) $P\{\text{at most 3 are defective}\}$

$= P\{X \leq 3\}$

$= P\{X = 0, 1, 2, 3\}$

$= {}^{10}C_0 (0.2)^0 (0.8)^{10} + {}^{10}C_1 (0.2)^1 (0.8)^9$

$+ {}^{10}C_2 (0.2)^2 (0.8)^8 + {}^{10}C_3 (0.2)^3 (0.8)^7$

$= 0.8791$

Poisson Distribution :-

A r.v. X follows poisson distribution if it assumes values $0, 1, 2, \ldots$

and $P\{X = x\} = e^{-\lambda} \dfrac{\lambda^x}{x!}$, $x = 0, 1, \ldots$

we denote by $X \sim P(\lambda)$ where λ is the parameter of distribution.

Note :-

1. poisson distribution occurs in the events which has large possibility but rare.
e.g. No. of death reported in Mumbai in a day due to heart attack.

Here the parameter λ is taken as an average number of outcomes.

(2) Poisson distribution is applied in queuing problems.

e.g. No. of OLA taxies picking up the passengers at the Airport in a month.

If K is the average value of outcomes in unit time, then for the total time T, the parameter $\lambda = KT$.

(3) Poisson distribution is a good approximation of Binomial distribution.

Let $X \sim B(n, p)$.

If number of trials n is large then $X \sim P(\lambda)$ (approx) with

$\lambda = np$ [Approx. is good for $\underline{n \geq 20}$]

<u>Mean and Variance of Poisson Distribution</u>:-

Let $X \sim P(\lambda)$

then $P\{X = x\} = e^{-\lambda} \dfrac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$

Mgf (moment generating function) of X is

$$M(t) = E(e^{Xt})$$
$$= \sum_{x=0}^{\infty} e^{xt} P\{X = x\}$$

$$\therefore M(t) = \sum_{x=0}^{\infty} e^{xt} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} e^{xt} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$\therefore$ Mean of X is

$$\mu_X = E(X) = \frac{d}{dt} M(t) \Big|_{t=0}$$

$$M'(t) = e^{-\lambda} \lambda e^t e^{\lambda e^t}$$

$$\therefore \mu_X = e^{-\lambda} \lambda e^{\lambda} = \lambda e^0 = \lambda$$

|||^{ly} $E(X^2) = \frac{d^2}{dt^2} M(t) \Big|_{t=0}$

$$M''(t) = \lambda e^{-\lambda} [e^t e^{\lambda e^t} + e^t \lambda e^t e^{\lambda e^t}]$$

$$= \lambda e^{-\lambda} [e^t + \lambda e^{2t}] e^{\lambda e^t}$$

$$\therefore E(X^2) = \lambda e^{-\lambda} [1 + \lambda] e^{\lambda}$$

$$= \lambda [1 + \lambda] = \lambda + \lambda^2$$

$$\therefore Var(X) = E(X^2) - (E(X))^2$$

$$= \lambda + \lambda^2 - (\lambda)^2 = \lambda$$

Thus, for $X \sim P(\lambda)$

$$\boxed{\begin{aligned} \text{Mean}(X) &= E(X) = \lambda \\ \text{Var}(X) &= \lambda \end{aligned}}$$

E.x.

① After correcting 50 pages of the proof of a book, the proof reader finds that there are on the average 2 errors per 3 pages. How many pages would one expect to find with 0, 1, 2 errors in 1000 pages of the first print of the book.

Soln

X: No. of errors per page.

$X : 0, 1, 2, \ldots$

$X \sim P(\lambda)$,

$\lambda = 2/3$

$$\therefore P\{X = x\} = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$= e^{-2/3} \frac{(2/3)^x}{x!}$$

$$P\{X = 0\} = e^{-2/3} \frac{(2/3)^0}{0!} = 0.5134$$

$\therefore$ No. of pages out of 1000 pages with 0 error $= 1000 \times 0.5134 = 513$

$$P\{X=1\} = \frac{e^{-2/3}(2/3)^1}{1!} = 0.3423$$

$\therefore$ No. of pages out of 1000 with 1 error

$= 1000 \times 0.3423 = 342$

$$P\{X=2\} = \frac{e^{-2/3}(2/3)^2}{2!} = 0.1141$$

$\therefore$ No. of pages out of 1000 with 2 errors

$= 1000 \times 0.1141 = 114$

(2) On an average 3 trucks per hour arrive at the certain dock to be unloaded. What is the probability of not more than 10 trucks arriving for unloading in 8 hrs.

Soln X: No. of trucks arriving for unloading in 8 hrs.

$X \sim P(\lambda)$,

$\lambda = 3 \times 8 = 24$

$$\therefore P\{X=x\} = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-24}\,24^x}{x!}, \quad x=0,1,\dots$$

P{Not more than 10 trucks arriving}

$= P\{X \le 10\} = P\{X=0,1,\dots,10\}$

$$= e^{-24}\left[1 + \frac{24}{1!} + \frac{24^2}{2!} + \dots + \frac{24^{10}}{10!}\right] = 0.0011$$

(3) It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to market in packets of 20, find the number of packets containing at least, exactly and at most 2 defective items in a consignment of 1000 packets.

<u>Sol</u>

Soln X: No. of defective items in a packet containing 20 items.

$X: 0, 1, \cdots, 20$

$X \sim B(n, p)$

$n = 20, \quad p = 0.05$

$n = 20$ is large,

therefore Binomial distribution is approximated by poisson distribution.

$\Rightarrow X \sim P(\lambda)$

$\lambda = np = 20 \times 0.05 = 1$

$$\therefore P\{X = x\} = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-1} 1^x}{x!}$$

$$= e^{-1} \frac{1}{x!}, \quad x = 0, 1, \cdots, 20$$

(i) P{ atleast 2 defective in a packet}

$= P\{X \geq 2\}$

$= 1 - P\{X < 2\} = 1 - P\{X = 0, 1\}$

$= 1 - e^{-1}\left[1 + \frac{1}{1!}\right] = 1 - e^{-1} \times 2$

$= 0.2642$

∴ No. of packets out of 1000 with atleast 2 defective items

$= 1000 \times 0.2642 = 264$

(ii) P{ exactly 2 defective in a packet}

$= P\{X = 2\}$

$= e^{-1} \frac{1}{2!} = 0.1839$

∴ No. of packets out of 1000 with exactly 2 defective items

$= 1000 \times 0.1839 = 184$

(iii) P{ at most two defective in a packet}

$= P\{X \leq 2\} = e^{-1}\left[1 + \frac{1}{1!} + \frac{1}{2!}\right] = 0.9197$

∴ No. of packets out of 1000 with at most 2 defective

$= 1000 \times 0.9197 = 920$

③ Suppose X follows Poisson distribution with parameter λ and $P\{X=2\} = \frac{2}{3} P\{X=1\}$.
Find (i) $P\{X=3\}$ (ii) $E(X^2)$.

Soln

$X \sim P(\lambda)$

$\lambda = ?$

$P\{X=x\} = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, \ldots$

$\therefore P\{X=2\} = \frac{2}{3} P\{X=1\}$

$\Rightarrow e^{-\lambda} \frac{\lambda^2}{2!} = \frac{2}{3} e^{-\lambda} \frac{\lambda^1}{1!}$

$\Rightarrow \frac{\lambda^2}{2} = \frac{2}{3} \lambda$

$\Rightarrow \lambda^2 = \frac{4}{3} \lambda \Rightarrow \lambda^2 - \frac{4}{3} \lambda = 0$

$\Rightarrow \lambda = 0, \ \lambda = 4/3$

$\lambda = 0$ (not possible)

$\Rightarrow \lambda = 4/3$

(i) $\therefore P\{X=3\} = e^{-\lambda} \frac{\lambda^3}{3!} = \frac{e^{-4/3} (4/3)^3}{6}$

$= 0.1041$

(ii) $E(X) = \lambda = 4/3$

$Var(X) = \lambda = 4/3$

and $Var(X) = E(X^2) - (E(X))^2$

$\therefore E(X^2) - (E(X))^2 = 4/3$

$E(X^2) - (4/3)^2 = 4/3$

$\Rightarrow E(X^2) = (4/3)^2 + \frac{4}{3} = \frac{28}{9}$

(5) Fit a poisson distribution to the following data.

x:	0	1	2	3	4	5	6
f:	143	90	42	12	9	3	1

Soln

x	f	e	$\ne$ e
0	143	123.2	123
1	90	109.7	110
2	42	48.8	49
3	12	14.5	14
4	9	3.2	3
5	3	0.6	1
6	1	0.1	0
			300

Let $x \sim P(\lambda)$

then $\lambda = \text{mean}(x)$

$$= \frac{\sum xf}{\sum f} = \frac{267}{300} = 0.89$$

$\therefore\ P\{X=x\} = e^{-\lambda}\dfrac{\lambda^{x}}{x!}$

$= e^{-0.89}\dfrac{(0.89)^{x}}{x!}, \quad x=0,1,2,\ldots$

$\therefore$ Expected frequency is

e = Total frequency $\times\ P\{X=x\}$

$= N \times P\{X=x\}$

$= 300 \times e^{-0.89}\dfrac{(0.89)^{x}}{x!}, \quad x=0,1,2,\ldots$

Normal Distribution (Gaussian Distribution):-

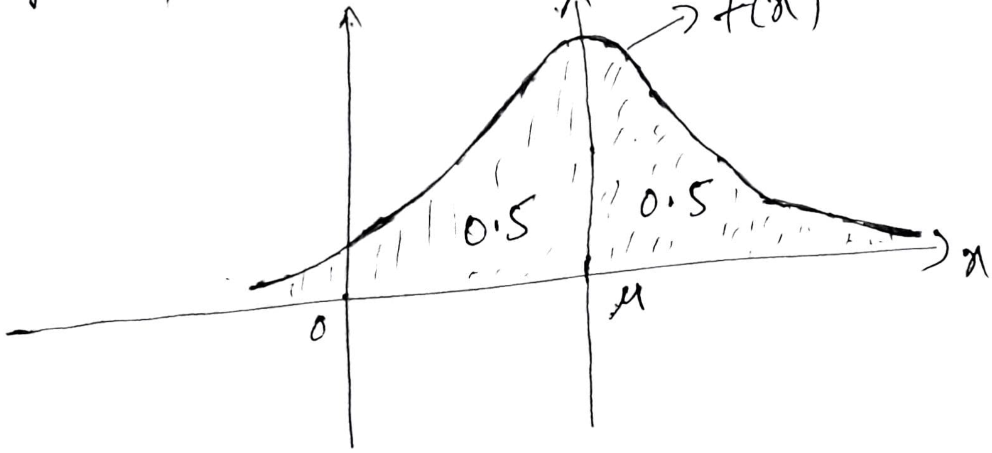
A r.v. X is said to follow Normal distribution if its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

We denote it as $X \sim N(\mu, \sigma^2)$

where μ and σ are parameters of the distribution.

Graph of the distribution is



Mean and Variance of Normal Distribution:-

Let $X \sim N(\mu, \sigma^2)$; then

$$\text{mean}(X) = E(X) = \mu$$

$$\text{Var}(X) = \sigma^2$$

Standard Normal Distribution:-

Let $X \sim N(\mu, \sigma^2)$; then

for the r.v. $Z = \dfrac{X-\mu}{\sigma}$,

$$\text{mean}(Z) = 0 \quad \text{and} \quad \text{Var}(Z) = 1$$

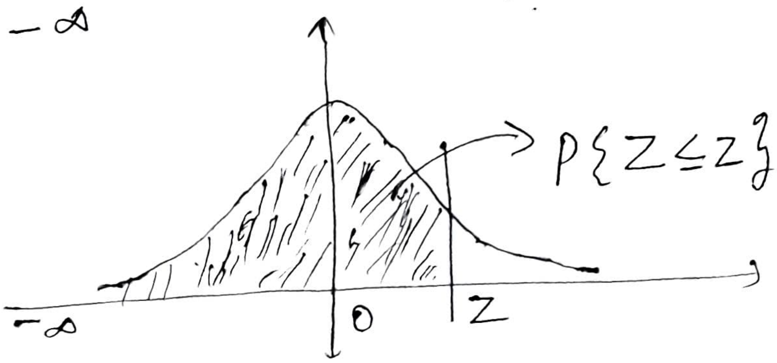
$\therefore$ $Z \sim N(0,1)$, known as standard normal distribution.

The pdf of Z is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

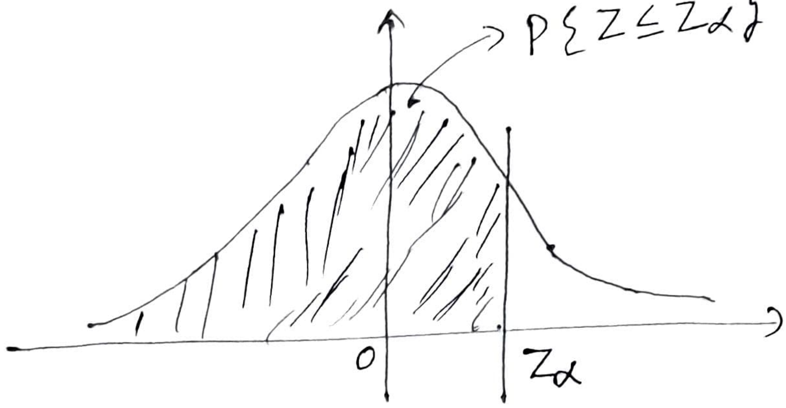
The distribution of Z is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2}\, dz = P\{Z \le z\}$$

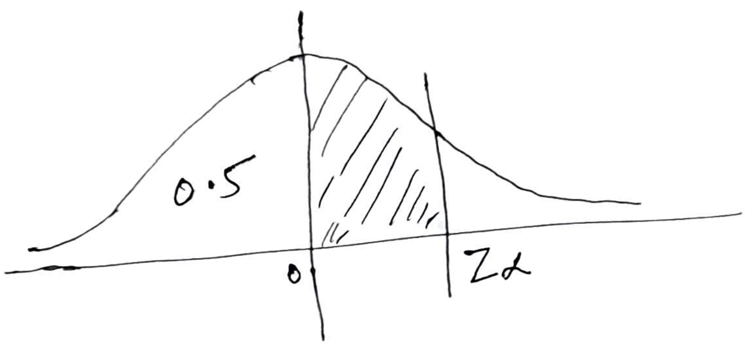


Note:-

① $P\{Z \le Z_\alpha\}$ = Area under the curve from $-\infty$ to Z_α

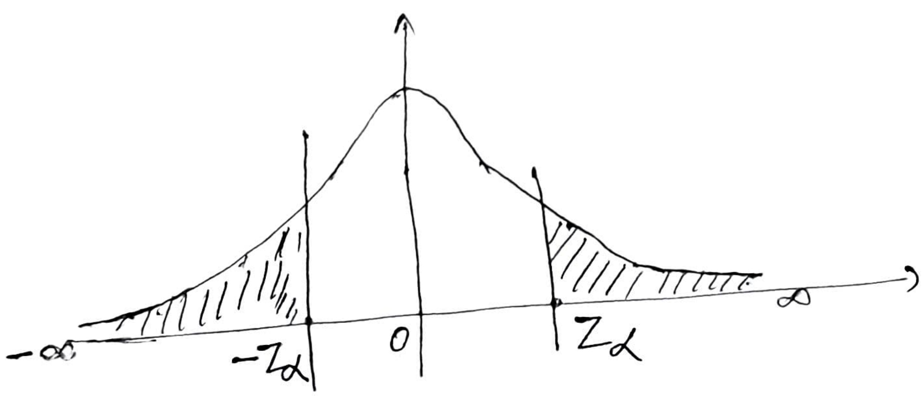


②



$$P\{Z \le Z_\alpha\} = P\{Z \le 0\} + P\{0 \le Z \le Z_\alpha\}$$
$$= 0.5 + P\{0 \le Z \le Z_\alpha\}$$

③



$$P\{Z \le -Z_\alpha\} = P\{Z \ge Z_\alpha\}$$
$$= 1 - P\{Z \le Z_\alpha\}$$

(4) Let $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0,1)$, then

(i) $P\{X \le a\} = P\{Z \le \frac{a-\mu}{\sigma}\}$

(ii) $P\{a \le X \le b\} = P\{\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\}$

Normal Approximation of the Binomial distribution:-

Let $X \sim B(n, p)$

If n is large then the distribution is approximately normal i.e

$X \sim N(\mu, \sigma^2)$

where $\mu = np$ and $\sigma^2 = npq$

Note:-

(1) Approximation is good for $n \ge 20$.

(2) Since Binomial distribution is a discrete distribution therefore probability at a single point is non zero.

(i) $P\{X = a\} = P\{a - 0.5 \le X \le a + 0.5\}$

(ii) $P\{X < a\} = P\{X \le a - 0.5\}$

E.X.

(1) If X is a normal variate with mean 10 and standard deviation 4, find

(i) $P\{X \leq 12\}$ (ii) $P\{5 \leq X \leq 18\}$

(iii) $P\{|X-14| < 1\}$

Solⁿ

$X \sim N(\mu, \sigma^2)$

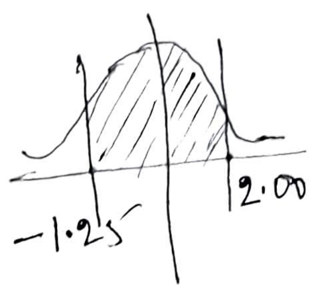
$\mu = 10, \quad \sigma = 4$

(i) $P\{X \leq 12\} = P\{Z \leq \frac{12-10}{4}\}$

$= P\{Z \leq 0.5\}$

$= 0.6915$

(ii) $P\{5 \leq X \leq 18\} = P\{\frac{5-10}{4} \leq Z \leq \frac{18-10}{4}\}$



$= P\{-1.25 \leq Z \leq 2.00\}$

$= P\{Z \leq 2.00\} - P\{Z \leq -1.25\}$

$= P\{Z \leq 2.00\} - [1 - P\{Z \leq 1.25\}]$

$= 0.9772 - [1 - 0.8944]$

$= 0.9772 - 0.1056$

$= 0.8716$

(iii) $P\{|X-14| < 1\}$

$= P\{-1 < X-14 < 1\}$

$$= P\{13 < X < 15\}$$
$$= P\{X < 15\} - P\{X \leq 13\}$$
$$= P\{Z \leq \frac{15-10}{4}\} - P\{Z \leq \frac{13-10}{4}\}$$
$$= P\{Z \leq 1.25\} - P\{Z \leq 0.75\}$$
$$= 0.8944 - 0.7734$$
$$= 0.121$$

(2) The local authorities in a certain city install 10,000 electric lamps in a streets of the city. If these lamps have an average life of 1000 burning hrs with s.d. 200 hrs;
how many lamps are expected to fail
(i) in the first 800 hrs?
(ii) between 800 and 1300 hrs?
after how many hrs would you expect to
(iii) 10% of the lamps to fail?
(iv) 15% of the lamps to be still burning?

Soln X: Life of a lamp.

$$X \sim N(\mu, \sigma^2)$$

$\mu = 1000$ hrs , $\sigma = 200$ hrs

(i) P{A lamp fail in the first 800 hrs}

$= P\{X \leq 800\}$

$= P\{Z \leq \frac{800-1000}{200}\} = P\{Z \leq -1.00\}$

$= 0.1587$

$\therefore$ No. of lamps that fails in first 800 hrs out of 10000 lamps

$= 10000 \times 0.1587 = 1587$

(ii) P{A lamp fail in between 800 and 1300 hrs}

$= P\{800 \leq X \leq 1300\}$

$= P\{X \leq 1300\} - P\{X \leq 800\}$

$= P\{Z \leq \frac{1300-1000}{200}\} - P\{Z \leq \frac{800-1000}{200}\}$

$= P\{Z \leq 1.50\} - P\{Z \leq -1.00\}$

$= 0.9332 - 0.1587 = 0.7745$

$\therefore$ No. of lamps out of 10000 that fails between 800 and 1300 hrs

$= 10,000 \times 0.7745 = 7745$

(iii) Let after K hrs, 10% of lamps has failed.

$P\{\text{Life of a lamp} \leq K\} = 10\%$

$\Rightarrow P\{X \leq K\} = \frac{10}{100} = 0.1$

$P\{Z \leq \frac{K-1000}{200}\} = 0.1$

$\Rightarrow \frac{K-1000}{200} = -1.28$

$\Rightarrow K = 1000 - 1.28 \times 200$

$= 744$ hrs

(iv) Let after K hrs, 15% of the lamps are still burning.

$P\{X \geq K\} = \frac{15}{100} = 0.15$

$\Rightarrow 1 - P\{X \leq K\} = 0.15$

$\Rightarrow P\{X \leq K\} = 1 - 0.15 = 0.85$

$P\{Z \leq \frac{K-1000}{200}\} = 0.85$

$\Rightarrow \frac{K-1000}{200} = 1.04$

$\Rightarrow K = 1000 + 200 \times 1.04 = 1208$ hrs