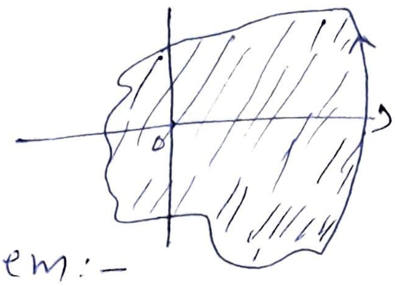
Cauchy's Integral Theorem:-

If $f(z)$ is Analytic and $f'(z)$ is continuous on and inside a closed contour C, then

$$\int_C f(z)\,dz = 0$$



Cauchy-Goursat Theorem:-

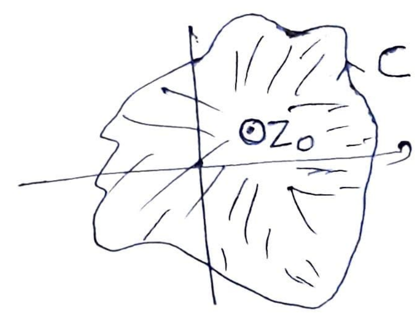
If $f(z)$ is Analytic on and inside a closed contour C, then

$$\int_C f(z)\,dz = 0$$

Cauchy's Integral Formula:-

If $f(z)$ is Analytic on and inside a closed contour C except at z_0, which lies completely inside 'C' and

$$f(z) = \frac{g(z)}{(z-z_0)^n}, \text{ then}$$



$$\int_C f(z)\,dz = \int_C \frac{g(z)}{(z-z_0)^n}\,dz$$

$$= \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}} g(z)\right]_{z=z_0}$$

Note: For $n=1$,

$$\int_C f(z)\,dz = \int_C \frac{g(z)}{(z-z_0)}\,dz = 2\pi i\left[g(z)\right]_{z=z_0}$$

E.X.

① Evaluate $\int_C \frac{z^3-1}{(z+1)^2(z-2)}\,dz$

where C is (i) $|z| = \frac{3}{2}$ (ii) $|z-2| = 2$

(iii) $|z| = 3$ (iv) $|z+2i| = 1$

Soln Let $I = \int_C \frac{z^3-1}{(z+1)^2(z-2)}\,dz$

$$f(z) = \frac{z^3-1}{(z+1)^2(z-2)}$$

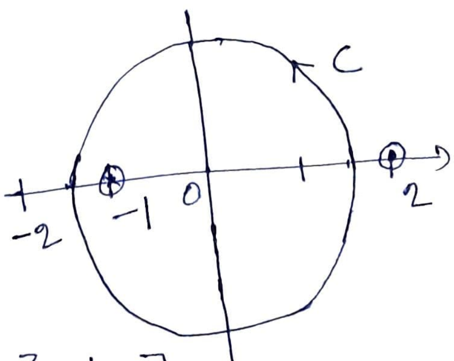
$f(z)$ is not Analytic at

$$(z+1)^2(z-2) = 0$$

$$\Rightarrow z=-1, \quad z=2$$

(i) $C: |z| = \frac{3}{2}$

$z=-1$ lies inside C.



$$\therefore I = \int_C f(z)\,dz$$

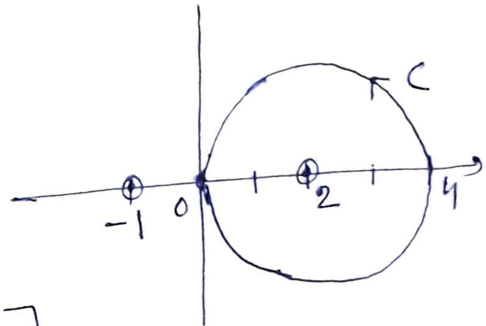
$$= \frac{2\pi i}{1!} \frac{d}{dz}\left[\frac{z^3-1}{z-2}\right]_{z=-1}$$

$$= 2\pi i \left[\frac{3z^2(z-2)-(z^3-1)}{(z-2)^2}\right]_{z=-1}$$

$$= 2\pi i \left[\frac{3\times(-3)-(-2)}{(-3)^2}\right] = -\frac{14}{9}\pi i$$

(ii) C: $|z-2| = 2$

$z = 2$ lies inside C.



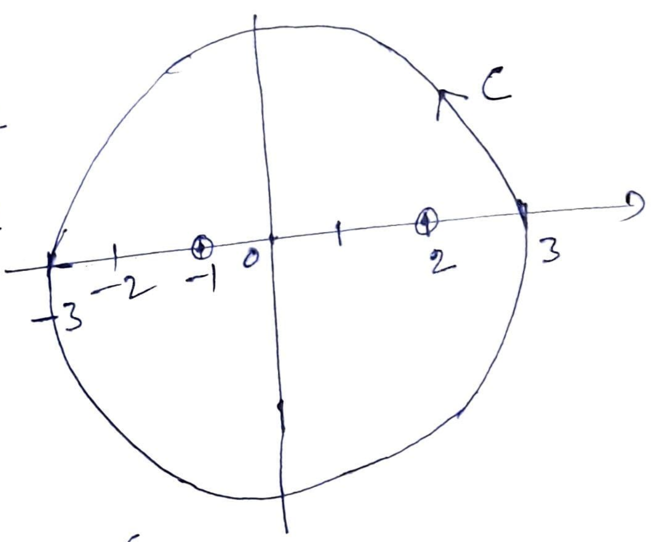
$\therefore I = \int_C f(z)\, dz$

$= 2\pi i \left[\frac{z^3-1}{(z+1)^2} \right]_{z=2}$

$= 2\pi i \left[\frac{8-1}{(3)^2} \right] = \frac{14}{9} \pi i$

(iii) C: $|z|=3$

$z=-1$ and $z=2$ both lies inside C.



$\frac{1}{(z+1)^2 (z-2)}$

$= \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-2)}$

$\Rightarrow A(z+1)(z-2) + B(z-2) + C(z+1)^2 = 1$

$z=-1; \quad -3B = 1 \Rightarrow B = -1/3$

$z = 2, \quad 9C = 1 \Rightarrow C = \frac{1}{9}$

coeff z^2, $\quad A + C = 0 \Rightarrow A = -C = -\frac{1}{9}$

$\therefore f(z) = -\frac{1}{9} \frac{(z^3-1)}{(z+1)} - \frac{1}{3} \frac{(z^3-1)}{(z+1)^2} + \frac{1}{9} \frac{(z^3-1)}{(z-2)}$

$$\therefore I = \int_C f(z)\,dz$$

$$= -\frac{1}{9}\int_C \frac{z^3-1}{(z+1)}\,dz - \frac{1}{3}\int_C \frac{z^3-1}{(z+1)^2}\,dz + \frac{1}{9}\int_C \frac{z^3-1}{z-2}\,dz$$

$$= -\frac{1}{9}\,2\pi i\left[z^3-1\right]_{z=-1} - \frac{1}{3}\,\frac{2\pi i}{1!}\,\frac{d}{dz}(z^3-1)\Big|_{z=-1}$$
$$+ \frac{1}{9}\,2\pi i\left[z^3-1\right]_{z=2}$$

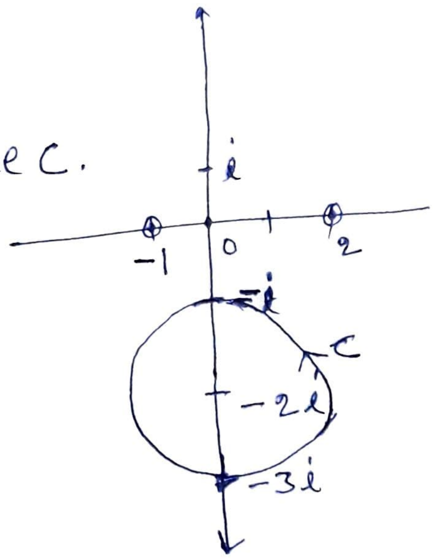
$$= -\frac{2\pi i}{9}\left[-2\right] - \frac{2\pi i}{3}\left[3z^2\right]_{z=-1}$$
$$+ \frac{2\pi i}{9}\left[7\right]$$

$$= \frac{4\pi i}{9} - 2\pi i + \frac{14\pi i}{9}$$

$$= 0$$

(iv) $C : |z+2i| = 1$

Both $z=-1, 2$ lies outside C.



$\Rightarrow f(z)$ is Analytic on and inside the circle C

$\Rightarrow$ By C.I.T.

$$\int_C f(z)\,dz = 0$$

2. Evaluate $\int_C \frac{z^2}{(z-i)(z+2)^2}\,dz$ where

(i) $C: |z-i|=1$ (ii) $C: |z+3|=2$

Sol^n

$$I = \int_C \frac{z^2}{(z-i)(z+2)^2}\,dz$$

$$f(z) = \frac{z^2}{(z-i)(z+2)^2}$$

$f(z)$ is not Analytic at

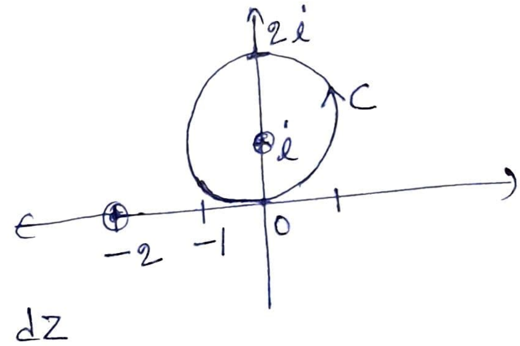
$$(z-i)(z+2)^2 = 0$$

$$\Rightarrow z = i, -2$$

(i) $C: |z-i|=1$

$z=i$ lies inside C

$\therefore$ by C.I.F.



$$I = \int_C \frac{z^2}{(z-i)(z+2)^2}\,dz$$

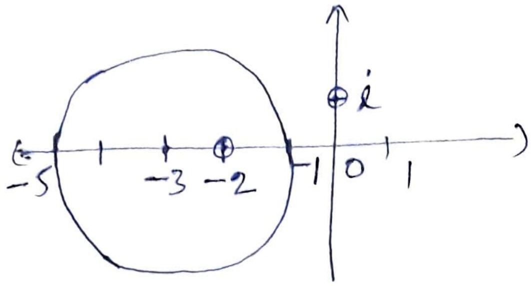
$$= 2\pi i \left[\frac{z^2}{(z+2)^2}\right]_{z=i}$$

$$= 2\pi i \frac{(i)^2}{(i+2)^2}$$

$$= \left(-\frac{8}{25} - \frac{6}{25} i\right)\pi$$

(ii) $C: |z+3| = 2$

$z=-2$ lies inside C



$\therefore$ by C.I.F.

$$\int_C f(z)\, dz = \int_C \frac{z^2}{(z-i)(z+2)^2}\, dz$$

$$= \frac{2\pi i}{1!} \frac{d}{dz}\left[\frac{z^2}{z-i}\right]_{z=-2}$$

$$= 2\pi i \left[\frac{2z(z-i)-z^2}{(z-i)^2}\right]_{z=-2}$$

$$= 2\pi i \left[\frac{-4(-2-i)-4}{(-2-i)^2}\right]$$

$$= \left(\frac{8}{25} + \frac{56}{25} i\right)\pi$$

(3) Evaluate $\displaystyle\int_C \frac{2z-1}{z^4-2z^3-3z^2}\, dz$

where C is (i) $|z-\frac{1}{2}| = 1$ (ii) $|z+1| = 2$

(iii) $|z-2| = \frac{5}{2}$ (iv) $|z-2i| = 1$

Sol^n

$$I = \int_C \frac{2z-1}{z^4-2z^3-3z^2}\, dz$$

$$f(z) = \frac{2z-1}{z^4-2z^3-3z^2}$$

$f(z)$ is not Analytic at

$z^4 - 2z^3 - 3z^2 = 0$

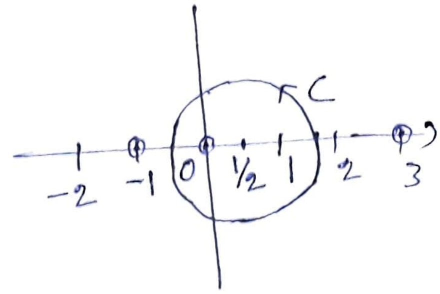
$\Rightarrow z^2 (z^2 - 2z - 3) = 0$

$\Rightarrow z^2 = 0 \quad , \quad z^2 - 2z - 3 = 0$

$\Rightarrow z = 0 \quad , \quad z = -1, 3$

(i) $C: |z - \frac{1}{2}| = 1$

$z = 0$ lies inside C.



$$\therefore \int_C f(z)\, dz = \int_C \frac{2z-1}{z^2 (z+1)(z-3)}\, dz$$

$$= \frac{2\pi i}{1!} \frac{d}{dz} \left[\frac{2z-1}{(z+1)(z-3)} \right]_{z=0}$$

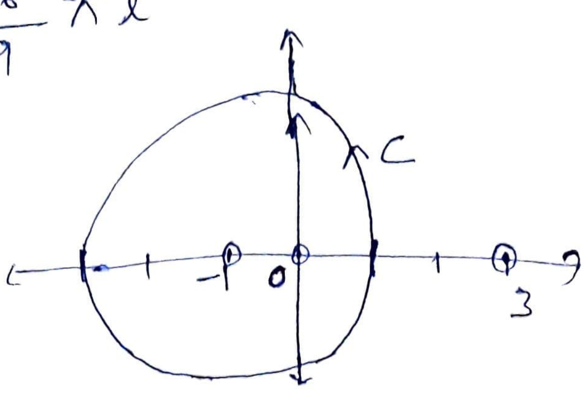
$$= 2\pi i \left[\frac{2(z+1)(z-3) - (2z-1)\left[(z-3) + (z+1)\right]}{(z+1)^2 (z-3)^2} \right]_{z=0}$$

$$= 2\pi i \left[\frac{2(-3) + (-2)}{(-3)^2} \right]$$

$$= -\frac{16}{9} \pi i$$

(ii) $C: |z+1| = 2$

$z = -1, 0$ lies inside C.



$\therefore \quad f(z) = \dfrac{2z-1}{z^2(z+1)(z-3)}$

$= \dfrac{A}{z} + \dfrac{B}{z^2} + \dfrac{C}{(z+1)} + \dfrac{D}{(z-3)}$

$\therefore \quad Az(z+1)(z-3) + B(z+1)(z-3) + Cz^2(z-3)$
$+ Dz^2(z+1) = 2z-1$

$z=0, \quad -3B = -1 \Rightarrow B = \dfrac{1}{3}$

$z=-1, \quad -4C = -3 \Rightarrow C = \dfrac{3}{4}$

$z=3, \quad 36D = 5 \Rightarrow D = \dfrac{5}{36}$

coeff z^3, $\quad A+C+D=0 \Rightarrow A = -C-D = -\dfrac{8}{9}$

$\therefore \int_C f(z)\,dz = -\dfrac{8}{9}\int_C \dfrac{1}{z}dz + \dfrac{1}{3}\int_C \dfrac{1}{z^2}dz$

$+ \dfrac{3}{4}\int_C \dfrac{1}{z+1}dz + \dfrac{5}{36}\int_C \dfrac{1}{z-3}dz$

$= -\dfrac{8}{9}2\pi i\left[1\right]_{z=0} + \dfrac{1}{3}2\pi i \dfrac{d}{dz}\left[1\right]_{z=0}$

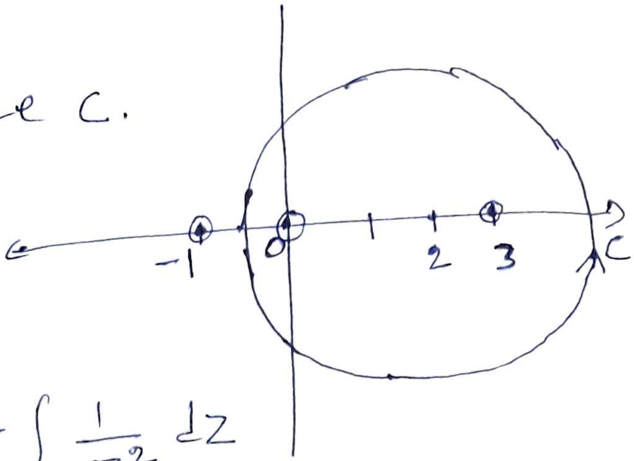
$+ \dfrac{3}{4}2\pi i\left[1\right]_{z=-1} + \dfrac{5}{36}\times 0$

$= -\dfrac{16}{9}\pi i + 0 + \dfrac{3}{2}\pi i$

$= -\dfrac{5}{18}\pi i$

(iii) $C: |z-2| = \frac{5}{2}$

$z = 0, 3$ lies inside C.



$\therefore \int_C f(z)\, dz$

$= -\frac{8}{9} \int_C \frac{1}{z} dz + \frac{1}{3} \int_C \frac{1}{z^2} dz$

$+ \frac{3}{4} \int_C \frac{1}{z+1} dz + \frac{5}{36} \int_C \frac{1}{z-3} dz$

$= -\frac{8}{9} 2\pi i \left[1\right]_{z=0} + \frac{1}{3} 2\pi i \frac{d}{dz}\left[1\right]_{z=0}$

$+ 0 + \frac{5}{36} 2\pi i \left[1\right]_{z=3}$

$= -\frac{16}{9}\pi i + 0 + 0 + \frac{5}{18}\pi i$

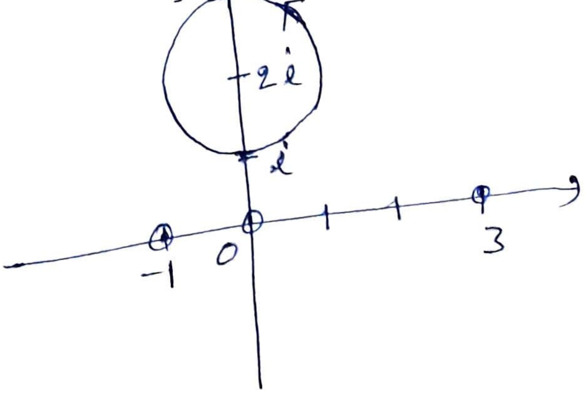
$= -\frac{3}{2}\pi i$

(iv) $C: |z-2i| = 1$

All $z = 0, -1, 3$ lies outside C.

$\Rightarrow f(z)$ is Analytic on and inside C.

$\therefore$ by C. I. T.



$\int_C f(z)\, dz = 0$

(4) Evaluate $\int_C \frac{\cos^2 z}{z^6}\, dz$ along the circle $|z|=1$.

<u>Soln</u>

$$I = \int_C \frac{\cos^2 z}{z^6}\, dz$$

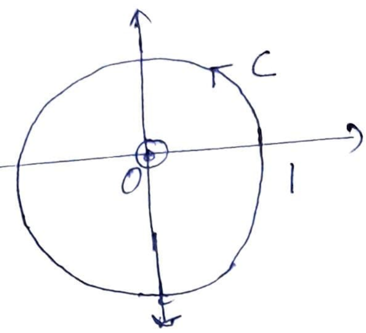
$$f(z) = \frac{\cos^2 z}{z^6}$$

$f(z)$ is not Analytic at

$z^6 = 0 \Rightarrow z = 0$

$C: |z| = 1$

$z=0$ lies inside C.



$$I = \int_C \frac{\cos^2 z}{z^6}\, dz$$

$$= \frac{2\pi i}{5!} \frac{d^5}{dz^5}\left[\cos^2 z\right]_{z=0}$$

$$= \frac{2\pi i}{5!} \frac{d^5}{dz^5}\left[\frac{1+\cos 2z}{2}\right]_{z=0}$$

$$= \frac{\pi i}{5!}\left[-32 \sin 2z\right]_{z=0}$$

$$= \frac{\pi i}{5!}\left[0\right]$$

$$= 0$$

$1+\cos 2z$

$\Rightarrow -2 \sin 2z$

$-4 \cos 2z$

$8 \sin 2z$

$16 \cos 2z$

$-32 \sin 2z$

(5) Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^n}\,dz$,

$c: |z|=1$ for $n=1,3$.

Soln

$$I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^n}\,dz$$

$$f(z) = \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^n}$$

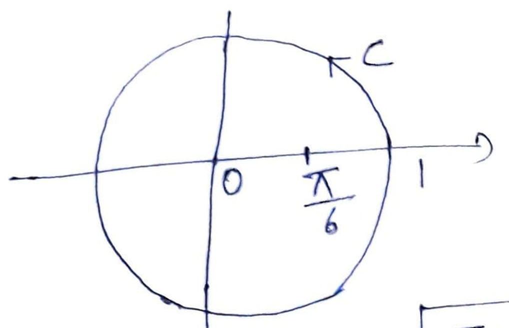
$f(z)$ is not Analytic at

$$\left(z-\frac{\pi}{6}\right)^n = 0$$

$$\Rightarrow z = \frac{\pi}{6}$$

$C: |z|=1$

$z=\frac{\pi}{6}$ lies inside C.



$\pi/6 = \frac{3.14}{6}$

(i) For $n=1$

$$I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)}\,dz$$

$$= 2\pi i \left[\sin^6 z\right]_{z=\frac{\pi}{6}}$$

$$= 2\pi i \left(\sin \pi/6\right)^6$$

$$= 2\pi i \left(\frac{1}{2}\right)^6$$

$$= \frac{\pi i}{32}$$

(ii) For $n=3$,

$$I = \int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}\, dz$$

$$= \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left[\sin^6 z\right]_{z=\frac{\pi}{6}}$$

$$= \pi i \frac{d}{dz}\left[6 \sin^5 z \cos z\right]_{z=\frac{\pi}{6}}$$

$$= \pi i \left[30 \sin^4 z \cos^2 z - 6 \sin^6 z\right]_{z=\frac{\pi}{6}}$$

$$= \pi i \left[30 \left(\sin \tfrac{\pi}{6}\right)^4 \left(\cos \pi/6\right)^2 - 6 \left(\sin \pi/6\right)^6\right]$$

$$= \pi i \left[30 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{\sqrt{3}}{2}\right)^2 - 6 \left(\tfrac{1}{2}\right)^6\right]$$

$$= \pi i \left[\frac{45}{32} - \frac{3}{32}\right]$$

$$= \frac{21}{16} \pi i$$

(6) If $f(a) = \int_C \frac{4z^2+z+5}{(z-a)}\, dz$

where C is $4x^2+9y^2=36$, find the values of $f(1)$, $f(i)$, $f'(-i)$, $f'(-3-2i)$, $f''(2+i)$.

Sol^n

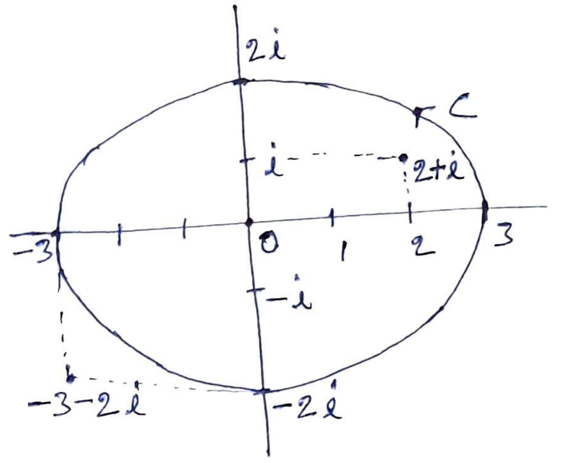
$$f(a) = \int_C \frac{4z^2+z+5}{(z-a)}\, dz$$

Let $g(z) = \dfrac{4z^2+z+5}{(z-a)}$

$g(z)$ is not Analytic at

$z - a = 0$

$\Rightarrow z = a$



$C: 4x^2 + 9y^2 = 36$

$\Rightarrow \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

$\Rightarrow \dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$

$$f(1) = \int_C \frac{4z^2+z+5}{(z-1)} dz$$

$$= 2\pi i \left[4z^2+z+5\right]_{z=1}$$

$$= 2\pi i [10] = 20\pi i$$

$$f(i) = \int_C \frac{4z^2+z+5}{(z-i)} dz$$

$$= 2\pi i \left[4z^2+z+5\right]_{z=i}$$

$$= 2\pi i [-4+i+5]$$

$$= 2\pi i [1+i]$$

$$= 2\pi(-1+i)$$

$$f'(a) = \frac{d}{da} \int_C \frac{4z^2+z+5}{(z-a)} dz$$

$$\Rightarrow f'(a) = \int_C \frac{4z^2+z+5}{(z-a)^2}\, dz$$

$$\frac{1}{(z-a)}$$

$$\Rightarrow \frac{-1 \times (-1)}{(z-a)^2}$$

$$= \frac{1}{(z-a)^2}$$

$$\therefore f'(-i) = \frac{2\pi i}{1!} \frac{d}{dz}\left[4z^2+z+5\right]_{z=-i}$$

$$= 2\pi i \left[8z+1\right]_{z=-i}$$

$$= 2\pi i \left[-8i+1\right]$$

$$= 2\pi (8+i)$$

$$f'(-3-2i) = \int_C \frac{4z^2+z+5}{(z+3+2i)^2}\, dz$$

$$= 0$$

$$f''(a) = \frac{d}{da} \int_C \frac{4z^2+z+5}{(z-a)^2}\, dz$$

$$= \int_C \frac{2(4z^2+z+5)}{(z-a)^3}\, dz$$

$$f''(2+i) = 2 \cdot \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left[4z^2+z+5\right]_{z=2+i}$$

$$= 2\pi i \frac{d}{dz}\left[8z+1\right]_{z=2+i}$$

$$= 2\pi i \left[8\right]_{z=2+i}$$

$$= 16\pi i$$

(7) Evaluate $\int_C \frac{e^{3z}}{(z-\pi i)^2}\, dz$ where

$C: |z-2|+|z+2|=8$

<u>Soln</u>

$$I = \int_C \frac{e^{3z}}{(z-\pi i)^2}\, dz$$

$$f(z) = \frac{e^{3z}}{(z-\pi i)^2}$$

$f(z)$ is not Analytic at $z-\pi i = 0$

$\Rightarrow z = \pi i$

$C: |z-2|+|z+2|=8$

For $z=\pi i$

$$|z-2|+|z+2| = |\pi i-2|+|\pi i+2|$$

$$= \sqrt{\pi^2+4}+\sqrt{\pi^2+4}$$

$$= 2\sqrt{\pi^2+4} < 8$$

$\Rightarrow z=\pi i$ lies inside C.

$\therefore$ By CIF

$$I = \int_C \frac{e^{3z}}{(z-\pi i)^2}\, dz$$

$$= \frac{2\pi i}{1!}\frac{d}{dz}\left[e^{3z}\right]_{z=\pi i}$$

$$= 2\pi i\left[3e^{3z}\right]_{z=\pi i} = 6\pi i\, e^{3\pi i}$$

$$= -6\pi i$$

$z = x+iy$

$|z-2|+|z+2|=8$

$|z-2| = 8-|z+2|$

$\sqrt{(x-2)^2+y^2}$

$= 8-\sqrt{(x+2)^2+y^2}$

$\Rightarrow$ squaring both side & simplifying

$3x^2+4y^2=48$

$\Leftrightarrow$

$$\frac{x^2}{4^2}+\frac{y^2}{(\sqrt{12})^2}=1$$

$e^{3\pi i}$

$= \cos 3\pi + i\sin 3\pi$

$= -1$

⑧ Evaluate $\int_{C:|z|=1} \frac{Re(z)}{(z-a)} dz, \quad 0<|a|<1$

Sol^n

$$I = \int_{C:|z|=1} \frac{Re(z)}{(z-a)} dz$$

We know that $z = x+iy$

then $\bar{z} = x-iy$

$\therefore \quad z+\bar{z} = 2x \Rightarrow x = \frac{1}{2}(z+\bar{z})$

i.e. $Re(z) = \frac{1}{2}(z+\bar{z})$

Also $z\cdot\bar{z} = x^2+y^2 = |z|^2$

$\therefore$ on $C: |z|=1$

$$|z|^2 = 1 \Rightarrow z\cdot\bar{z} = 1$$

$$\Rightarrow \bar{z} = \frac{1}{z}$$

$$\therefore \quad I = \int_C \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{(z-a)} dz$$

$$= \frac{1}{2}\int_C \frac{(z^2+1)}{z(z-a)} dz$$

$f(z) = \frac{z^2+1}{z(z-a)}$ is not

Analytic at

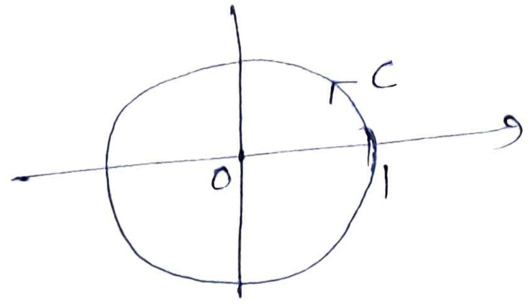
$$z(z-a) = 0$$

$$\Rightarrow z=0, \ z=a$$

$C: |z|=1$

& $0<|a|<1$

$\Rightarrow z=0,\ a$ lies inside C.



$$\frac{1}{z(z-a)} = \frac{1}{a}\,\frac{z-(z-a)}{z(z-a)}$$

$$= \frac{1}{a}\left[\frac{1}{(z-a)} - \frac{1}{z}\right]$$

$$\therefore\ f(z) = \frac{1}{a}\left[\frac{z^2+1}{(z-a)} - \frac{z^2+1}{z}\right]$$

$$\therefore\ I = \frac{1}{2a}\left[\int_C \frac{z^2+1}{(z-a)}\,dz - \int_C \frac{z^2+1}{z}\,dz\right]$$

$$= \frac{1}{2a}\left[2\pi i(a^2+1) - 2\pi i(1)\right]$$

$$= \frac{1}{2a}\,2\pi i a^2$$

$$= \pi i a$$

(9) Evaluate $\int_C \frac{z^2}{(z-z_0)^3}\,dz$

where (i) C is any closed contour containing z_0 in its interior.

(ii) C is any closed contour not containing z_0.

Soln $I = \int_C \frac{z^2}{(z-z_0)^3} dz$

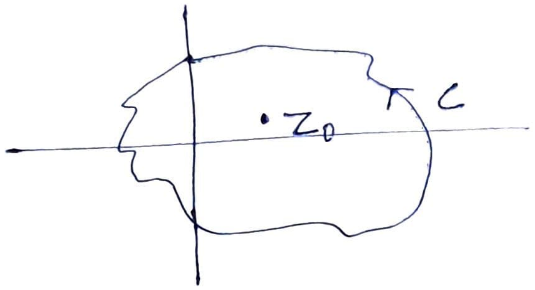
$$f(z) = \frac{z^2}{(z-z_0)^3}$$

$f(z)$ is not Analytic at

$$(z-z_0)^3 = 0$$

$$\Rightarrow z = z_0$$

(i) C:

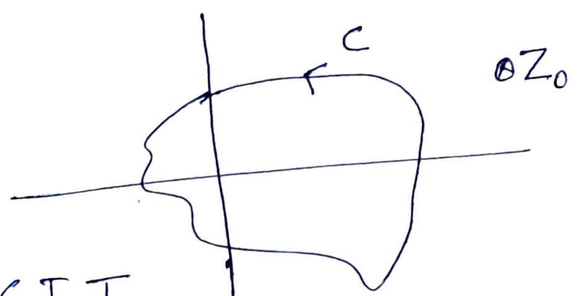


z^2
$\downarrow$
$2z$
2

$$I = \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left[z^2\right]_{z=z_0}$$

$$= \pi i \left[2\right]_{z=z_0} = 2\pi i$$

(ii) C:



$\therefore$ By CIT

$\therefore$ $I = \int_C f(z)\, dz = 0$