Property Related to Eigen Values and Eigen vectors :-

(1) Eigen vectors corresponding to distinct eigen values of a matrix are Linearly independent (L.I.).

(2) A and A' have the same eigen values.

(3) If λ is an eigen value of matrix A and X is a corresponding eigen vector

then, (i) $K\lambda$ is an eigen value of KA and X is a corresponding eigen vector.

(ii) λ^n is an eigen value of A^n and X is a corresponding eigen vector

(iii) λ^{-1} is an eigen value of A^{-1} and X is a corresponding eigen vector if A^{-1} exist (i.e. A is non-singular $\Leftrightarrow |A| \neq 0$)

(iv) $K_0 + K_1\lambda + K_2\lambda^2 + \cdots + K_n\lambda^n$ is an eigen value of $K_0I + K_1A + K_2A^2 + \cdots + K_nA^n$ and X is a corresponding eigen vector.

(v) $\frac{|A|}{\lambda}$ is an eigen value of adj A and X a corresponding eigen vector if A is a non-singular matrix ($|A| \neq 0$)

(4) $\lambda = 0$ is an eigen value of A iff matrix A is singular (i.e. $|A| = 0$)

(5) The eigen values of a Triangular matrix are all the principal diagonal elements of the matrix.

(6) The eigen values of a symmetric matrix with real numbers are all real numbers.

(7) If A is a symmetric matrix then eigen vectors corresponding to two distinct eigen values are orthogonal i.e. If $\lambda_1 \neq \lambda_2$ and X_1 and X_2 are corresponding eigen vectors then $X_1' X_2 = 0$

(8) Sum of all the eigen values of A is the <u>trace of A</u> (Tr(A)) i.e. sum of all the principal diagonal elements of A.

(9) Product of all eigen values of A is the determinant of A ($|A|$).

E.x.

(1) Let $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(i) Find the eigen values and eigen vectors of matrices A^2, A^{-1}.

(ii) Find the eigen values and eigen vectors of $A^3 + 4A - 3I$

(iii) Find the eigen values and eigen vectors of $3A^{-1} + 2A^2$

(iv) Find the eigen values and eigen vectors of adj A.

(v) Find the eigen values and eigen vectors of adj adj A.

(vi) Verify that eigen vectors corresponding to distinct eigen values of A are orthogonal.

(vii) Find the orthogonal set of eigen vectors of matrix A

(viii) Find the trace and determinant of matrix $(A^{-1})^2$.

Sol^n

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

cht. Eq of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$$

$$\lambda = 5, 2, 2$$

For $\lambda = 5$,

$$(A - 5I)X = 0$$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 + R_3 \quad \begin{bmatrix} -2 & -1 & 1 \\ 0 & -3 & -3 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_2 - 3x_3 = 0$$

$$x_1 - x_2 - 2x_3 = 0$$

$$\Rightarrow x_3 = -x_2, \quad x_1 = -x_2$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$,

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 + R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 + x_3 = 0$$

$$\Rightarrow x_3 = -x_1 + x_2$$

$\Rightarrow$ There are two L.I. eigen vectors

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(i) Eigen values and eigen vectors of A^2 are

$$\lambda_1 = 5^2 = 25, \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2^2 = 4, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 4$$

Eigen values and eigen vectors of A^{-1} are

$$\lambda_1 = 5^{-1} = \frac{1}{5}, \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = \frac{1}{2}$$

(ii) Eigen values and eigen vectors of $A^3 + 4A - 3I$ are

$$\lambda_1 = 5^3 + 4 \times 5 - 3 = 142, \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2^3 + 4 \times 2 - 3 = 13, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 13$$

(iii) Eigen values and eigen vectors of $3A^{-1} + 2A^2$ are

$$\lambda_1 = \frac{3}{5} + 2 \times 5^2 = \frac{253}{5}, \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{3}{2} + 2 \times 4 = \frac{19}{2}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda_3 = \frac{19}{2}, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(iv) We have $|A| = 5\times2\times2 = 20$;

$\therefore$ Eigen values and eigen vectors of adj A are

$$\lambda_1 = \frac{|A|}{\lambda} = \frac{20}{5} = 4, \quad X_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\lambda_2 = \frac{20}{2} = 10, \quad X_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad X_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\lambda_3 = \frac{20}{2} = 10,$$

(v) We have $|\text{adj } A| = 4\times10\times10 = 400$

$\therefore$ Eigen values and eigen vectors of adj adj A are

$$\lambda_1 = \frac{|\text{adj } A|}{\lambda} = \frac{400}{4} = 100, \quad X_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\lambda_2 = \frac{400}{10} = 40, \quad X_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad X_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\lambda_3 = 40$$

(vi) For matrix A, we have

$$\lambda_1 = 5, \quad X_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\lambda_2 = 2, \quad X_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad X_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\lambda_3 = 2$$

For $\lambda_1 \neq \lambda_2$, $X_1' X_2 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$= 1 + 0 - 1 = 0$$

$\Rightarrow$ X_1 and X_2 are orthogonal (perpendicular)

$\lambda_1 \neq \lambda_3$, $X_1' X_3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$= 0$$

$\Rightarrow$ X_1 and X_3 are orthogonal

(vii) Note that, since A is symmetric matrix, eigen vectors corresponding to distinct eigen values are orthogonal.

$\Rightarrow$ X_1 and X_2 are orthogonal and X_1 and X_3 are orthogonal

But for $\lambda_2 = \lambda_3 = 2$,

$X_2' X_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$= 0 + 0 - 1 = -1 \neq 0$$

$\Rightarrow$ X_2 and X_3 are not orthogonal

But given X_1 and X_2, it is possible to find an eigen vector X_3

of matrix A such that X_2 and X_3 are orthogonal.

We have $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$X_1' X_3 = 0$ and $X_2' X_3 = 0$

$\Rightarrow \quad x_1 - x_2 + x_3 = 0$

$x_1 - x_3 = 0$

$\Rightarrow \quad x_3 = x_1, \quad 2x_1 - x_2 = 0$

$\Rightarrow \quad x_2 = 2x_1$

$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow$ X_1, X_2, X_3 is the orthogonal set of eigen vectors of A.

(viii) Eigen values of $(A^{-1})^2$ are

$\lambda_1 = \left(\frac{1}{5}\right)^2 = \frac{1}{25}, \quad \lambda_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad \lambda_3 = \frac{1}{4}$

$\therefore$ Trace $\left((A^{-1})^2\right) = \frac{1}{25} + \frac{1}{4} + \frac{1}{4} = \frac{27}{50}$

$|(A^{-1})^2| = \frac{1}{25} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{400}$

(2) Find the eigen values and eigen vectors of A^{10}, adj A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & -2 & 2 & 0 \\ 5 & 2 & 3 & 3 \end{bmatrix}$$

Sol^n Since A is an lower triangular matrix,

eigen values of A are

$\lambda = 1, 2, 2, 3$

For $\lambda = 1$,

$(A - I) X = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 \\ 5 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + x_2 = 0$

$4x_1 - 2x_2 + x_3 = 0$

$5x_1 + 2x_2 + 3x_3 + 2x_4 = 0$

$\Rightarrow \quad x_2 = -3x_1, \quad x_3 = -10x_1,$

$x_4 = \frac{31}{2} x_1$

$\therefore \quad X_1 = \begin{bmatrix} 2 \\ -6 \\ -20 \\ 31 \end{bmatrix}$

For $\lambda = 2$,

$(A - 2I)X = 0$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 5 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 = 0$

$4x_1 - 2x_2 = 0 \quad \Rightarrow x_2 = 0$

$5x_1 + 2x_2 + 3x_3 + x_4 = 0$

$\Rightarrow x_4 = -3x_3$

$\Rightarrow$ There is only one L.I. eigen vector

$$X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

For $\lambda = 3$

$(A - 3I)X = 0$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 4 & -2 & -1 & 0 \\ 5 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 = 0, \quad 3x_1 - x_2 = 0 \Rightarrow x_2 = 0$

$4x_1 - 2x_2 - x_3 = 0 \Rightarrow x_3 = 0$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(i) $\therefore$ Eigen values and eigen vectors of A^{10} are

$\lambda_1 = 1^{10} = 1$, $\quad X_1 = \begin{bmatrix} 2 \\ -6 \\ -20 \\ 31 \end{bmatrix}$

$\lambda_2 = 2^{10}$, $\quad X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$

$\lambda_3 = 2^{10}$

$\lambda_4 = 3^{10}$, $\quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(ii) $|A| = 1 \times 2 \times 2 \times 3 = 12$

$\therefore$ Eigen values and eigen vectors of adj A are

$\lambda_1 = \frac{|A|}{\lambda} = \frac{12}{1} = 12$, $\quad X_1 = \begin{bmatrix} 2 \\ -6 \\ -20 \\ 31 \end{bmatrix}$

$\lambda_2 = \frac{12}{2} = 6$, $\quad X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$

$\lambda_3 = \frac{12}{2} = 6$

$\lambda_4 = \frac{12}{3} = 4$, $\quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(3) Find the sum and product of the eigen values of A.

$$A = \begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & -2 \end{bmatrix}$$

Soln Sum of all eigen values of A

$= \lambda_1 + \lambda_2 + \lambda_3$

$= \text{Trace}(A) = -17+19-2 = 0$

Product of all eigen values of A

$= \lambda_1 \lambda_2 \lambda_3$

$$= |A| = \begin{vmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & -2 \end{vmatrix}$$

$= -2$

(4) Let A be a 3×3 symmetric matrix with an eigen value 4 of multiplicity one and an eigen value 1 of multiplicity two and corresponding eigen vectors $X_1 = (1,-1,1)$, $X_2 = (1,0,-1)$ and X_3.

Find AX_1, $A^{10}X_3$ and matrix A.

Soln We have A is a symmetric matrix.

$\lambda_1 = 4,\ X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1,\quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_3 = 1,\quad X_3 = ?$

$$A X_1 = \lambda_1 X_1 = 4\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$$

$$A^{10} X_3 = \lambda_3^{10} X_3 = (1)^{10} X_3 = X_3$$

Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

since matrix A is symmetric,

$X_1' X_3 = 0 \Rightarrow x_1 - x_2 + x_3 = 0$

$X_2' X_3 = 0 \Rightarrow x_1 - x_3 = 0$

$\Rightarrow x_1 = x_3$

and $x_2 = 2x_3$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore \quad A^{10} X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Since A is symmetric,

$$a_{12} = a_{21}, \quad a_{13} = a_{31}, \quad a_{23} = a_{32}$$

$$A X_1 = \lambda_1 X_1$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$$

$$A X_2 = \lambda_2 X_2$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

& $\quad A X_3 = \lambda_3 X_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad a_{11} - a_{12} + a_{13} = 4 \quad \text{———} (i)$$

$$a_{11} + 0\,a_{12} - a_{13} = 1 \quad \text{———} (ii)$$

$$a_{11} + 2\,a_{12} + a_{13} = 1 \quad \text{———} (iii)$$

$\Rightarrow \quad a_{11} = 2, \quad a_{12} = -1, \quad a_{13} = 1$

$\therefore \quad a_{21} = -1, \quad a_{31} = 1$

$a_{21} - a_{22} + a_{23} = -4$

$\Rightarrow \quad -a_{22} + a_{23} = -3$

$a_{21} - a_{23} = 0$

$\Rightarrow \quad -a_{23} = 1 \quad \Rightarrow a_{23} = -1$

$\Rightarrow \quad a_{22} = -1 + 3 = 2$

$\therefore \quad a_{32} = -1$

$a_{31} - a_{33} = -1$

$\Rightarrow \quad a_{33} = 1 + 1 = 2$

$$\therefore \quad A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$