Application of Residue to Evaluate the Integral of Real value Functions :-

(I) To evaluate the integral of the form

$$I = \int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$

put $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$\Rightarrow$ $C : |z| = 1$

$dz = i e^{i\theta} d\theta = iz\, d\theta$

$\Rightarrow$ $d\theta = \dfrac{1}{iz} dz$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$$

$\therefore$ $I = \int_C f(z)\, dz$

(II) To evaluate $I = \int_{-\infty}^{\infty} f(x)\, dx$

Let $f(z)$ is not Analytic at finite number of points where $z_1, z_2, \ldots, z_n$ lies on upper half plane and $a_1, a_2, \ldots, a_m$ lies on the real axis, then

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \left[\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) \right]$$
$$+ \pi i \left[\operatorname*{Res}_{z=a_1} f(z) + \cdots + \operatorname*{Res}_{z=a_m} f(z) \right]$$

Application of Residue to evaluate Integration of Real value functions

Ex. 1 Evaluate $\displaystyle\int_0^{\pi} \frac{d\theta}{17-\cos\theta}$

Sol^n Let $\displaystyle I = \int_0^{\pi} \frac{d\theta}{17-\cos\theta}$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{17-\cos\theta}\, d\theta$$

Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$$dz = i e^{i\theta}\, d\theta = iz\, d\theta$$

$$\Rightarrow \quad d\theta = \frac{1}{iz}\, dz$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$$

$$= \frac{z^2+1}{2z}$$

$C: |z| = 1$

$$\therefore \quad I = \frac{1}{2}\int_C \frac{1}{17-\frac{(z^2+1)}{2z}}\, \frac{dz}{iz}$$

$$= \frac{1}{\cancel{2}}\int_C \frac{\cancel{2}z}{34z-z^2-1}\, \frac{dz}{iz}$$

$$I = -\frac{1}{i} \int_C \frac{1}{z^2 - 34z + 1} \, dz$$

Let $f(z) = \dfrac{1}{z^2 - 34z + 1}$

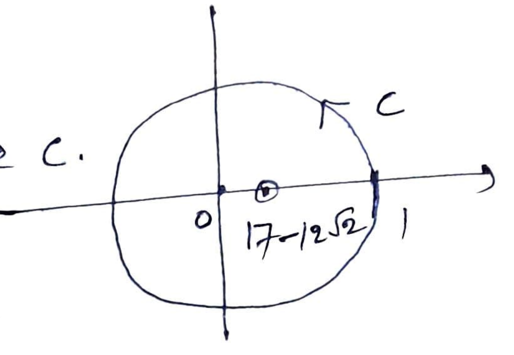
$f(z)$ is not Analytic at

$z^2 - 34z + 1 = 0$

$\Rightarrow z = 17 + 12\sqrt{2}, \; 17 - 12\sqrt{2}$

$C: \; |z| = 1$

$z = 17 - 12\sqrt{2}$ lies inside C.



$z = 17 - 12\sqrt{2}$ is a pole of order one.

$$\therefore \operatorname*{Res}_{z = 17 - 12\sqrt{2}} f(z) = \left.\frac{1}{2z - 34}\right|_{z = 17 - 12\sqrt{2}}$$

$$= -\frac{1}{24\sqrt{2}}$$

$$\therefore I = 2\pi i \left(-\frac{1}{i}\right) \times \left(-\frac{1}{24\sqrt{2}}\right)$$

$$= \frac{\pi}{12\sqrt{2}}$$

Ex 2 Evaluate $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$

Sol^n
$$I = \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$$

$$= 2\int_0^{\pi} \frac{1}{1+\sin^2\theta}\, d\theta$$

$$= 2 \times \frac{1}{2} \int_0^{2\pi} \frac{1}{1+\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{1+\sin^2\theta}\, d\theta$$

Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$\Rightarrow$ $C : |z| = 1$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}$$

$$= \frac{z^2 - 1}{2iz}$$

$$dz = i\, e^{i\theta} = iz\, d\theta$$

$$\Rightarrow \quad d\theta = \frac{1}{iz}\, dz$$

$$\therefore\quad I = \int_C \frac{1}{1+\left(\frac{z^2-1}{2iz}\right)^2}\,\frac{dz}{iz}$$

$$= \int_C \frac{1}{1-\frac{(z^2-1)^2}{4z^2}}\,\frac{dz}{iz}$$

$$= \int_C \frac{4z^2}{4z^2-z^4+2z^2-1}\,\frac{dz}{iz}$$

$$= -\frac{4}{i}\int_C \frac{z}{z^4-6z^2+1}\,dz$$

$f(z) = \cancel{z^4-6z}\ \dfrac{z}{z^4-6z^2+1}$ is

not Analytic at

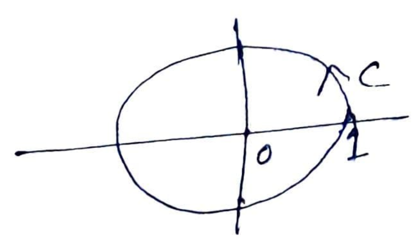
$$z^4-6z^2+1=0$$

$$\Rightarrow (z^2)^2-6z^2+1=0$$

$$\Rightarrow z^2 = 3+2\sqrt{2},\quad z^2 = 3-2\sqrt{2}$$

$$\Rightarrow z = \pm\sqrt{3+2\sqrt{2}},\quad z = \pm\sqrt{3-2\sqrt{2}}$$

$C: |z| = 1$



$z = \sqrt{3-2\sqrt{2}}$ and $z = -\sqrt{3-2\sqrt{2}}$ lies inside. &

Both are poles of order one

$$\therefore \operatorname*{Res}_{z=\sqrt{3-2\sqrt{2}}} f(z) = \left.\frac{z}{4z^3-12z}\right|_{z=\sqrt{3-2\sqrt{2}}}$$

$$= \left.\frac{1}{4z^2-12}\right|_{z=\sqrt{3-2\sqrt{2}}}$$

$$= -\frac{1}{8\sqrt{2}}$$

$$\text{lly} \quad \operatorname*{Res}_{z=-\sqrt{3-2\sqrt{2}}} f(z) = \left.\frac{1}{4z^2-12}\right|_{z=-\sqrt{3-2\sqrt{2}}}$$

$$= -\frac{1}{8\sqrt{2}}$$

$$\therefore I = -\frac{4}{i} \times 2\pi i \left[-\frac{1}{8\sqrt{2}} - \frac{1}{8\sqrt{2}}\right]$$

$$= \pi \frac{2}{\sqrt{2}} = \pi\sqrt{2}$$

Ex. $$I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$$

$$= Re \int_0^{2\pi} \frac{e^{i3\theta}}{5-4\cos\theta}\, d\theta$$

Let $z = e^{i\theta}$; $0 \le \theta \le 2\pi$

$\Rightarrow$ $c : |z| = 1$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$dz = i e^{i\theta}\, d\theta = iz\, d\theta$$

$$\Rightarrow d\theta = \frac{1}{iz}\, dz$$

$$\therefore I = Re\int_c \frac{z^3}{5-4\left(\frac{z^2+1}{2z}\right)} \frac{1}{iz}\, dz$$

$$= Re\,(I_1)$$

$$I_1 = \frac{1}{i}\int_c \frac{z^2}{5-\frac{2(z^2+1)}{z}}\, dz$$

$$= \frac{1}{i}\int_c \frac{z^3}{5z-2z^2-2}\, dz$$

$$I_1 = -\frac{1}{i}\int_C \frac{z^3}{2z^2-5z+2}\,dz$$

$$f(z) = \frac{z^3}{2z^2-5z+2}$$ is not

Analytic at $2z^2-5z+2=0$

$$\Rightarrow z=2,\ z=\frac{1}{2}$$

$C: \ |z|=1$

$z = 1/2$ lies inside C and is a pole of order one.

$$\therefore \operatorname*{Res}_{z=1/2} f(z) = \left.\frac{z^3}{4z-5}\right|_{z=1/2}$$

$$= -\frac{1}{24}$$

$$\therefore I_1 = -\frac{1}{i}\times 2\pi i\left[-\frac{1}{24}\right]$$

$$= \frac{\pi}{12}$$

$$\therefore I = \operatorname{Re}(I_1) = \frac{\pi}{12}$$

Ex. Evaluate $\int_{-\infty}^{\infty} \frac{x}{(x^2-1)(x^2+2x+2)} \, dx$

Soln Let $I = \int_{-\infty}^{\infty} \frac{x}{(x^2-1)(x^2+2x+2)} \, dx$,

$f(z) = \frac{z}{(z^2-1)(z^2+2z+2)}$ is not

Analytic at

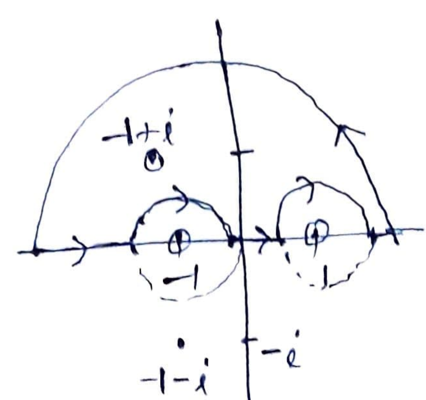
$$(z^2-1)(z^2+2z+2) = 0$$

$$\Rightarrow (z^2-1) = 0, \quad z^2+2z+2 = 0$$

$$\Rightarrow z = 1, -1, -1+i, -1+i$$

$z = 1, -1$ lies on Real axis

and $z = -1+i$ lies on upper half plane.



All $z = 1, -1, -1+i$ are poles of order one.

$$\operatorname*{Res}_{z=1} f(z) = \left. \frac{z}{2z(z^2+2z+2)} \right|_{z=1}$$

$$= \frac{1}{10}$$

$$\operatorname*{Res}_{z=-1} f(z) = \left. \frac{z}{2z\,(z^2+2z+2)} \right|_{z=-1}$$

$$= \frac{1}{2}$$

$$\operatorname*{Res}_{z=-1+i} f(z) = \left. \frac{z}{(z^2-1)\,(2z+2)} \right|_{z=-1+i}$$

$$= \frac{-1+i}{((-1+i)^2-1)\,2i}$$

$$= \frac{-3}{10} + \frac{1}{10} i$$

$$\therefore I = 2\pi i \left[\frac{-3}{10} + \frac{1}{10} i \right]$$

$$+ \pi i \left[\frac{1}{10} + \frac{1}{2} \right]$$

$$= \pi i \left[\frac{-3}{5} + \frac{1}{5} i + \frac{3}{5} \right]$$

$$= -\frac{\pi}{5}$$

Ex. Evaluate $\int_0^\infty \frac{x^2}{x^6+1}\,dx$

Soln

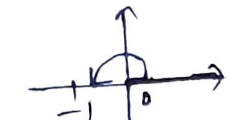
$$I = \int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$$

$f(z) = \frac{z^2}{z^6+1}$ is not Analytic at

$z = -1$

$|z| = 1$

$\theta = \pi$



$$z^6 + 1 = 0$$

$$\Rightarrow z^6 = -1$$

$$\Rightarrow z = (-1)^{1/6}$$

$$= \left(1\, e^{i(\pi + 2n\pi)}\right)^{1/6}$$

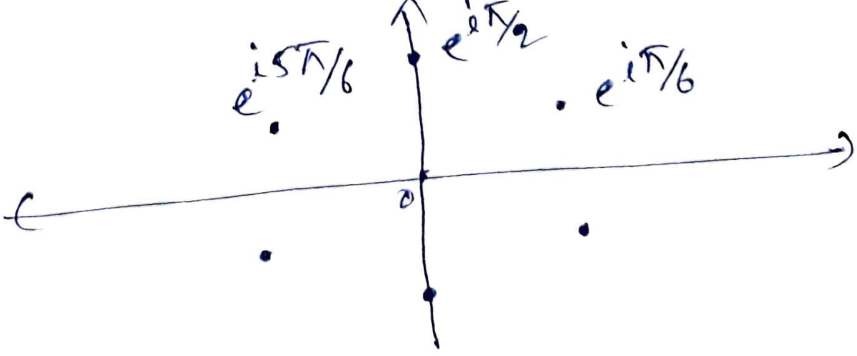
$$= e^{i(2n+1)\frac{\pi}{6}}, \quad n = 0, 1, \ldots, 5$$

i.e. $z = e^{i\pi/6},\ e^{i\frac{3\pi}{6}},\ e^{i\frac{5\pi}{6}},$

$e^{i\frac{7\pi}{6}},\ e^{i\frac{9\pi}{6}},\ e^{i\frac{11\pi}{6}}$

where $z = e^{i\pi/6},\ e^{i\frac{3\pi}{6}},\ e^{i\frac{5\pi}{6}}$

lies on upper half plane.



All $z = e^{i\pi/6}$, $e^{i\pi/2}$, $e^{i5\pi/6}$

are poles of order one.

$$\underset{z=e^{i\pi/6}}{\text{Res}} f(z) = \left.\frac{z^2}{6z^5}\right|_{z=e^{i\pi/6}}$$

$$= \left.\frac{1}{6}\frac{1}{z^3}\right|_{z=e^{i\pi/6}}$$

$$= \frac{1}{6}\frac{1}{e^{i\pi/2}}$$

$e^{i\pi/2}$
$= \cos\pi/2 + i\sin\pi/2$
$= i$

$$= \frac{1}{6i}$$

$$\underset{z=e^{i\pi/2}}{\text{Res}} f(z) = \frac{1}{6\,(e^{i\pi/2})^3} = \frac{1}{6\,(i)^3}$$

$$= -\frac{1}{6i}$$

$$\underset{z=e^{i5\pi/6}}{\text{Res}} f(z) = \frac{1}{6}\frac{1}{(e^{i5\pi/6})^3}$$

$$= \frac{1}{6}\frac{1}{e^{i5\pi/2}} = \frac{1}{6i}$$

$$\therefore\quad I = \frac{1}{2}\,2\pi i\left[\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right]$$

$$= \frac{\pi}{6}$$

Ex. Evaluate $\int_0^\infty \frac{x^2 \cos ax}{x^4+4}\, dx$

Solⁿ

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 \cos ax}{x^4+4}\, dx$$

$$= \frac{1}{2} \operatorname{Re} \underbrace{\int_{-\infty}^{\infty} \frac{x^2 e^{iax}}{x^4+4}}_{I_1}\, dx$$

$$f(z) = \frac{z^2 e^{iaz}}{z^4+4}$$ is not Analytic at

$$z^4+4=0 \Rightarrow z^4=-4$$

$$\Rightarrow z = (-4)^{1/4} = \left(4 e^{i(\pi+2n\pi)}\right)^{1/4}$$

$$= \sqrt{2}\, e^{i(2n+1)\frac{\pi}{4}}, \quad n=0,1,2,3$$

i.e. $z = \sqrt{2}\, e^{i\pi/4}, \ \sqrt{2}\, e^{i\frac{3\pi}{4}},$

$\sqrt{2}\, e^{i\frac{5\pi}{4}}, \ \sqrt{2}\, e^{i\frac{7\pi}{4}}$

where $z = \sqrt{2}\, e^{i\pi/4}, \ \sqrt{2}\, e^{i\frac{3\pi}{4}}$

lies on upper half plane.

All $z = \sqrt{2}\, e^{i\frac{\pi}{4}}$, $\sqrt{2}\, e^{i\frac{3\pi}{4}}$ are poles of order one.

$$\operatorname*{Res}_{z=\sqrt{2}e^{i\pi/4}} f(z) = \left.\frac{z^2 e^{iaz}}{4z^3}\right|_{z=\sqrt{2}e^{i\frac{\pi}{4}}}$$

$$= \left.\frac{e^{iaz}}{4z}\right|_{z=\sqrt{2}e^{i\pi/4}} = \frac{1}{4}\frac{e^{ia\sqrt{2}e^{i\frac{\pi}{4}}}}{\sqrt{2}\, e^{i\pi/4}}$$

$$= \frac{1}{4\sqrt{2}} e^{ia\sqrt{2}e^{i\pi/4}} \cdot e^{-i\frac{\pi}{4}}$$

$$= \frac{1}{4\sqrt{2}} e^{ia(1+i)} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$$

$$= \frac{1}{8} e^{a(-1+i)} (1-i)$$

$$\boxed{e^{i\pi/4} = \cos\pi/4 + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}}$$

||ly $$\operatorname*{Res}_{z=\sqrt{2}e^{i\frac{3\pi}{4}}} f(z) = \frac{1}{4\sqrt{2}} e^{ia\sqrt{2}e^{i\frac{3\pi}{4}}} \cdot e^{-i\frac{3\pi}{4}}$$

$$= \frac{1}{8} e^{a(-1-i)} (-1-i)$$

$$I_1 = 2\pi i \left[\frac{1}{8} e^{a(-1+i)} (1-i) + \frac{1}{8} e^{a(-1-i)} (-1-i)\right]$$

$$I_1 = \frac{\pi}{4} i \left[e^{-a} e^{ia} (1-i) + e^{-a} e^{-ia} (-1-i) \right]$$

$$= \frac{\pi}{4} e^{-a} i \left[(\cos a + i \sin a)(1-i) + (\cos a - i \sin a)(-1-i) \right]$$

$$= \frac{\pi}{4} e^{-a} \left[(\cos a + i \sin a)(i+1) + (\cos a - i \sin a)(-i+1) \right]$$

$$\therefore I = \frac{1}{2} \left[\frac{\pi e^{-a}}{4} \left(\cos a - \sin a + \cos a - \sin a \right) \right]$$

$$= \frac{\pi}{8} e^{-a} (2\cos a - 2\sin a)$$

$$= \frac{\pi}{4} e^{-a} (\cos a - \sin a)$$

<u>Ex.</u> Evaluate $\int_0^\infty \frac{\sin 5x}{x^5}\, dx$

<u>Solⁿ</u>

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin 5x}{x^5}\, dx$$

$$= \frac{1}{2} \operatorname{Im} \underbrace{\int_{-\infty}^{\infty} \frac{e^{i5x}}{x^5}\, dx}_{I_1}$$

$f(z) = \dfrac{e^{i5z}}{z^5}$ is not Analytic at

$z^5 = 0 \Rightarrow z = 0$

$z=0$ lies on real axis.

$z=0$ is a pole of order 5.

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{4!} \frac{d^4}{dz^4}\left[e^{i5z}\right]\Big|_{z=0}$$

$$= \frac{1}{4!}(i5)^4 e^{i5z}\Big|_{z=0}$$

$$= \frac{5^4}{4!} = \frac{625}{24}$$

$$\therefore I_1 = \pi i \times \frac{\cancel{65}\,625}{24}$$

$$\therefore I = \frac{1}{2}\operatorname{Im}(I_1)$$

$$= \frac{625}{48}\pi$$