# Z - Transform

## Sequence:-

An ordered list of real or complex numbers is known as sequence.

e.g. 10, 2, 4, 0, 3, −6, 8

We can associate a name to the members of a sequence as $f(k)$ or $f_k$ for some integer $k$. For instance:

$f(1)=10,\ f(2)=2,\ f(3)=4,\ f(4)=0,\ f(5)=3,$
$f(6)=-6,\ f(7)=8$

or

$f(0)=10,\ f(1)=2,\ f(2)=4,\ f(3)=0,\ f(4)=3,$
$f(5)=-6,\ f(6)=8$

or

$f(-4)=10,\ f(-3)=2,\ f(-2)=4,\ f(-1)=0,$
$f(0)=3,\ f(1)=-6,\ f(2)=8$

Note that: In general $f(k)$ is the name (index) associated with $k^{th}$ member of the given sequence, $k$ being an integer.

Generally the zeroeth position member is indicated by an $\uparrow$.

e.g. $f(K)$; $10, 2, \underset{\uparrow}{4}, 0, 3, -6, 8$

$\phantom{e.g. f(K); } -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$f(-2)=10,\ f(-1)=2,\ f(0)=4,\ f(1)=0,$

$f(2)=3,\ f(3)=-6,\ f(4)=8$

If no arrow is indicated:

$\{f(K)\} = \{10, 2, 4, 0, 3, -6, 8\}$

we name it as:

$f(0)=10,\ f(1)=2,\ f(2)=4,\ f(3)=0,\ f(4)=3,$

$f(5)=-6,\ f(6)=8$

A sequence may be finite or infinite.

e.g. $-6, 3, 6, 7, 10, 11, 9, 10, 4, 87, \ldots$

The value of $f(K)$ may or may not depends on $K$.

e.g. $2, 4, 6, 8, 10, 12, \ldots$

Then $f(K) = 2K,\ K = 1, 2, 3, \ldots$

e.g. $f(K) = K\,2^K,\ K \geq 0$

$\{f(K)\} = \{0, 2, 2\,2^2, 3\,2^3, 4\,2^4, \ldots\}$

e.g. $f(K) = K^2+1, \ -3 \le K < 2$

$\{f(K)\} = \{10, 5, 2, 1, 2\}$

Thus a sequence is a function of set of integers and denoted by

$f(K), \ n_1 \le K \le n_2$ or $\{f(K)\}_{K=n_1}^{n_2}$

Infinite sequence:

$f(K)$ or $\{f(K)\}$ or $\{f(K)\}_{K=-\infty}^{\infty}$

## Z- Transform:-

Z transform of a sequence $\{f(K)\}$ is defined as

$$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K)\, z^{-K}$$

$$= \dots + f(-2)\, z^2 + f(-1)\, z + f(0) + f(1)\frac{1}{z} + f(2)\frac{1}{z^2} + \dots$$

**Note:** $Z\{f(K)\} = F(z)$, some function of variable $z$.

E.X.

Find the Z-transform of the following sequences.

① $15, 10, 7, \underset{\uparrow}{4}, 1, -1, 0, 6$

Sol$^n$ $f(K):\ 15,\ 10,\ 7,\ \underset{\uparrow}{4},\ 1,\ -1,\ 0,\ 6$

$K:\ \ -3\ \ -2\ \ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4$

$$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K)\, Z^{-K}$$

$$= 15Z^3 + 10Z^2 + 7Z + 4 + \frac{1}{Z} - \frac{1}{Z^2} + \frac{6}{Z^4}$$

② $f(K) = 2K^2 - K, \quad -4 \le K < 3$

Sol$^n$ $$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K)\, Z^{-K}$$

$$= \sum_{K=-4}^{2} (2K^2 - K)\, Z^{-K}$$

$$= 36Z^4 + 21Z^3 + 10Z^2 + 3Z + \frac{1}{Z} + \frac{6}{Z^2}$$

(3) $f(K) = 1, \quad K \geq 0$

<u>Sol^n</u> $Z\{f(K)\} = \sum_{K=0}^{\infty} f(K)\, Z^{-K}$

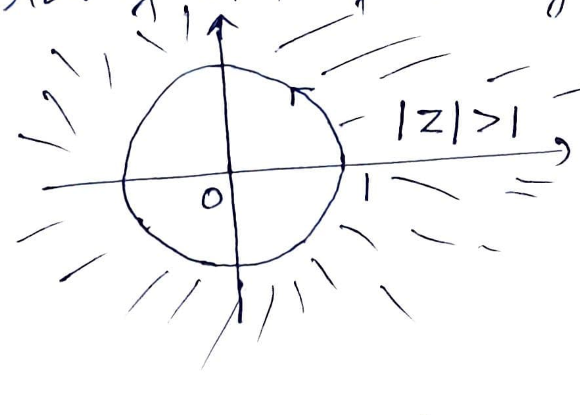
$$= \sum_{K=0}^{\infty} Z^{-K} = \sum_{K=0}^{\infty} \left(\frac{1}{Z}\right)^{K}$$

$$= 1 + \frac{1}{Z} + \left(\frac{1}{Z}\right)^{2} + \left(\frac{1}{Z}\right)^{3} + \cdots$$

$$= \frac{1}{1 - \frac{1}{Z}} = \frac{Z}{Z-1}$$

for $\left|\frac{1}{Z}\right| < 1 \iff 1 < |Z| \iff |Z| > 1$

<u>Note that</u>: $Z\{f(K)\}$ converges to $\frac{Z}{Z-1}$ for all $|Z| > 1$. Thus region $|Z| > 1$ is called the Region of convergence (ROC) of the Z-transform of the given sequence.




<u>Properties of Z-Transform:-</u>

(1) Linearity:

$$Z\{a f(K) + b\, g(K)\} = a Z\{f(K)\} + b Z\{g(K)\}$$

② Change of scale:

$$Z\{a^K f(K)\} = Z\{f(K)\}_{Z \to \frac{Z}{a}}$$

③ Multiplication by K:

$$Z\{K f(K)\} = -Z \frac{d}{dZ} Z\{f(K)\}$$

④ Division by K:

$$Z\left\{\frac{f(K)}{K}\right\} = -\int \frac{1}{Z} Z\{f(K)\}\, dZ$$

⑤ convolution property:

$$\text{conv}\{f(K), g(K)\} = \sum_{n=-\infty}^{\infty} f(n)\, g(K-n)$$

$$Z\{\text{conv}\{f(K), g(K)\}\} = Z\{f(K)\}\, Z\{g(K)\}$$

E.X.

Find the Z-transform of the following and find its region of convergence on Z-plane.

① $U(K)$, Discrete unit step function.

Soln Discrete unit step function is defined as

$$U(K) = \begin{cases} 1 & K = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$= 1, \quad K \geq 0$$

$$Z\{U(K)\} = \sum_{K=0}^{\infty} 1\, z^{-K}$$

$$= \sum_{K=0}^{\infty} \left(\frac{1}{z}\right)^K$$

$$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

for $\left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1$

② $a^K,\ K \geq 0$

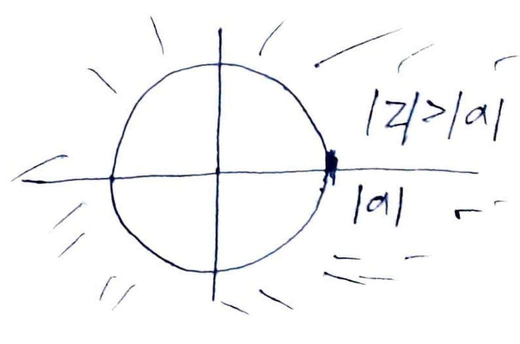
Sol$^n$ $f(K) = a^K,\ K \geq 0$

$$\therefore Z\{f(K)\} = \sum_{K=0}^{\infty} a^K z^{-K}$$

$$= \sum_{K=0}^{\infty} \frac{a^K}{z^K} = \sum_{K=0}^{\infty} \left(\frac{a}{z}\right)^K$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

for $\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$




or

$f(K) = a^K,\ K \geq 0$

$= a^K\, U(K)$

We know that

$$Z(U(K)) = \frac{z}{z-1} \quad \text{for } |z| > 1$$

$$\therefore Z\{f(K)\} = Z\{a^K U(K)\}$$

$$= Z\{U(K)\}_{z \to \frac{z}{a}} \quad \text{(change of scale property)}$$

$$= \frac{z/a}{z/a - 1} = \frac{z}{z-a}$$

$$\text{for } \left|\frac{z}{a}\right| > 1 \Rightarrow |z| > |a|$$

(3) $K, \ K \geq 0$

Soln $f(K) = K, \ K \geq 0$

$$Z\{f(K)\} = \sum_{K=0}^{\infty} K z^{-K}$$

$$= \sum_{K=0}^{\infty} K \left(\frac{1}{z}\right)^K$$

$$= \sum_{K=1}^{\infty} K \left(\frac{1}{z}\right)^K$$

$$= \sum_{K=0}^{\infty} (K+1) \left(\frac{1}{z}\right)^{K+1}$$

$$= \frac{1}{z} \sum_{K=0}^{\infty} (K+1) \left(\frac{1}{z}\right)^K$$

$$= \frac{1}{Z} \frac{1}{(1-\frac{1}{Z})^2} = \frac{1}{Z} \frac{Z^2}{(Z-1)^2}$$

$$= \frac{Z}{(Z-1)^2}$$

for $\left|\frac{1}{Z}\right| < 1 \Rightarrow |Z| > 1$

or

$f(K) = K, \quad K \geq 0$

$= K\, U(K)$

$\therefore Z\{f(K)\} = -Z \frac{d}{dZ} Z\{U(K)\}$

$$= -Z \frac{d}{dZ}\left(\frac{Z}{Z-1}\right)$$

$$= -Z\left[\frac{Z-1-Z}{(Z-1)^2}\right]$$

$$= \frac{Z}{(Z-1)^2}$$

for $|Z| > 1$

(4) $K^2, \quad K \geq 0$

Solⁿ $f(K) = K^2, \quad K \geq 0$

$= K^2 U(K)$

$$Z\{U(K)\} = \frac{Z}{Z-1} \quad \text{for } |Z|>1$$

$$\therefore Z\{K\,U(K)\} = -Z\frac{d}{dZ}\left(\frac{Z}{Z-1}\right)$$

$$= \frac{Z}{(Z-1)^2} \quad \text{for } |Z|>1$$

$$Z\{K^2 U(K)\} = -Z\frac{d}{dZ}\left(\frac{Z}{(Z-1)^2}\right)$$

$$= -Z\left[\frac{(Z-1)^2 - Z\cdot 2(Z-1)}{(Z-1)^4}\right]$$

$$= -Z\left[\frac{Z-1-2Z}{(Z-1)^3}\right]$$

$$= \frac{Z(Z+1)}{(Z-1)^3} \quad \text{for } |Z|>1$$

5) $f(K) = \begin{cases} 5^K & K<0 \\ 3^K & K\geq 0 \end{cases}$

Sol

Soln $Z\{f(K)\}$

$$= \sum_{K=-\infty}^{\infty} f(K)\,Z^{-K}$$

$$= \sum_{K=-\infty}^{-1} 5^K Z^{-K} + \sum_{K=0}^{\infty} 3^K Z^{-K}$$

$$= \sum_{K=1}^{\infty} 5^{-K} Z^K + \sum_{K=0}^{\infty} \left(\frac{3}{Z}\right)^K$$

$$= \sum_{K=1}^{\infty} \left(\frac{Z}{5}\right)^K + \sum_{K=0}^{\infty} \left(\frac{3}{Z}\right)^K$$

$$= \sum_{K=0}^{\infty} \left(\frac{Z}{5}\right)^{K+1} + \frac{1}{1-\frac{3}{Z}}$$

$$= \frac{Z}{5} \sum_{K=0}^{\infty} \left(\frac{Z}{5}\right)^K + \frac{Z}{Z-3}$$

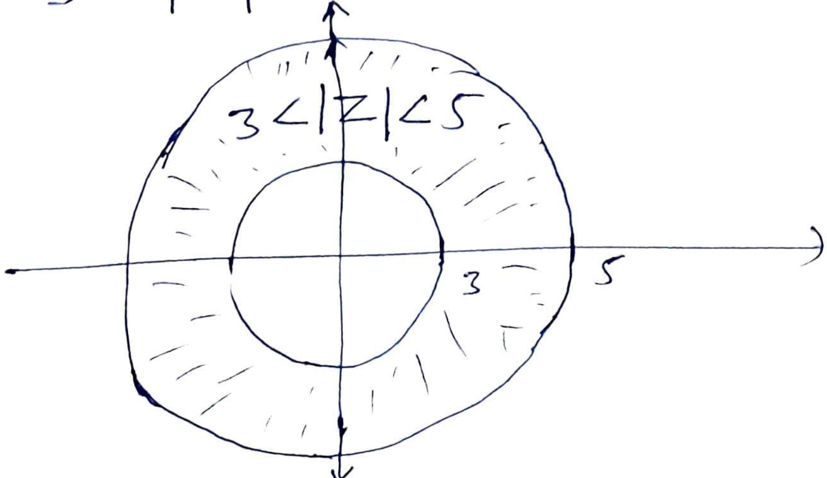
$$= \frac{Z}{5} \frac{1}{1-\frac{Z}{5}} + \frac{Z}{Z-3}$$

$$= \frac{Z}{5-Z} + \frac{Z}{Z-3}$$

for $\left|\frac{Z}{5}\right| < 1$ and $\left|\frac{3}{Z}\right| < 1$

$\Rightarrow$ $|Z| < 5$ and $3 < |Z|$

$\Rightarrow$ $3 < |Z| < 5$



(6) $a^{|k|}$

Sol$^n$ $f(k) = a^{|k|}$

$$|k| = \begin{cases} -k & k < 0 \\ k & k \geq 0 \end{cases}$$

$$\therefore f(k) = \begin{cases} a^{-k} & k < 0 \\ a^{k} & k \geq 0 \end{cases}$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^{k} z^{-k} \quad \text{——— (4)}$$

$$\sum_{k=-\infty}^{-1} a^{-k} z^{-k} = \sum_{k=1}^{\infty} a^{k} z^{k}$$

$$= \sum_{k=1}^{\infty} (az)^{k} = \sum_{k=0}^{\infty} (az)^{k+1}$$

$$= az \sum_{k=0}^{\infty} (az)^{k}$$

$$= az \frac{1}{1-az} = \frac{az}{1-az}$$

for $|az| < 1 \Rightarrow |z| < \frac{1}{|a|}$

$$\sum_{k=0}^{\infty} a^{k} z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^{k} = \frac{1}{1-\frac{a}{z}}$$

$$= \frac{z}{z-a}$$

for $\left|\frac{a}{z}\right| < 1 \Rightarrow |a| < |z|$

$$\therefore Z\{f(K)\} = \frac{aZ}{1-aZ} + \frac{Z}{Z-a}$$

for $|a| < |Z| < \frac{1}{|a|}$

(7) $\sin\alpha K$, $K \geq 0$

Sol$^n$ $f(K) = \sin\alpha K$

$$= \frac{1}{2i}\left(e^{i\alpha K} - e^{-i\alpha K}\right), \quad K \geq 0$$

$$Z\left(e^{i\alpha K}\right) = \sum_{K=0}^{\infty} e^{i\alpha K} z^{-K}$$

$$= \sum_{K=0}^{\infty}\left(\frac{e^{i\alpha}}{z}\right)^K$$

$$= \frac{1}{1-\frac{e^{i\alpha}}{z}} = \frac{z}{z-e^{i\alpha}}$$

for $\left|\frac{e^{i\alpha}}{z}\right| < 1 \Rightarrow |e^{i\alpha}| < |z|$

$\Rightarrow 1 < |z| \Rightarrow |z| > 1$

$|||^{ly}$ $Z\left(e^{-i\alpha K}\right) = \frac{z}{z-e^{-i\alpha}}$

for $|e^{-i\alpha}| < |z| \Rightarrow 1 < |z| \Rightarrow |z| > 1$

$$\therefore Z\{f(K)\} = \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right]$$

$$= \frac{z}{2i}\left[\frac{e^{i\alpha} - e^{-i\alpha}}{z^2 - (e^{i\alpha} + e^{-i\alpha})z + 1}\right]$$

$$\Rightarrow Z\{f(K)\} = \frac{z}{2i}\left[\frac{2i\sin\alpha}{z^2-2\cos\alpha\, z+1}\right]$$

$$= \frac{z\sin\alpha}{z^2-2\cos\alpha\, z+1}$$

for $|z|>1$

(8) $c^K \cosh\alpha K$, $K \geq 0$

Soln $f(K) = c^K \cosh\alpha K$

$$= c^K \frac{1}{2}\left(e^{\alpha K}+e^{-\alpha K}\right)$$

$$= \frac{1}{2}\left(c^K e^{\alpha K}+c^K e^{-\alpha K}\right)$$

$$= \frac{1}{2}\left((c e^{\alpha})^K+(c e^{-\alpha})^K\right) \quad (1)$$

$$Z\{(c e^{\alpha})^K\} = \sum_{K=0}^{\infty}(c e^{\alpha})^K z^{-K}$$

$$= \sum_{K=0}^{\infty}\left(\frac{c e^{\alpha}}{z}\right)^K$$

$$= \frac{1}{1-\frac{c e^{\alpha}}{z}} = \frac{z}{z-c e^{\alpha}}$$

for $\left|\frac{c e^{\alpha}}{z}\right|<1 \Rightarrow |c|e^{\alpha}<|z|$

$$\Rightarrow |z|>|c|e^{\alpha}$$

lly

$$Z\{(c e^{-\alpha})^K\} = \frac{z}{z-c e^{-\alpha}}$$

for $|z|>|c|e^{-\alpha}$

$\therefore Z\{f(K)\}$

$$= \frac{1}{2}\left[\frac{Z}{Z - Ce^{\alpha}} + \frac{Z}{Z - Ce^{-\alpha}}\right]$$

$$= \frac{Z}{2}\left[\frac{2Z - C(e^{\alpha} + e^{-\alpha})}{Z^2 - C(e^{\alpha} + e^{-\alpha})Z + C^2}\right]$$

$$= \frac{Z}{2}\left[\frac{2Z - 2C\cosh\alpha}{Z^2 - 2C\cosh\alpha\, Z + C^2}\right]$$

$$= \frac{Z(Z - C\cosh\alpha)}{Z^2 - 2C\cosh\alpha\, Z + C^2}$$

(9) $\frac{1}{(2K)!}, \quad K \geq 0$

Soln $f(K) = \frac{1}{(2K)!}, \quad K \geq 0$

$$Z\{f(K)\} = \sum_{K=0}^{\infty} \frac{1}{(2K)!} Z^{-K}$$

$$= \sum_{K=0}^{\infty} \frac{1}{(2K)!} \left(\frac{1}{Z}\right)^{K}$$

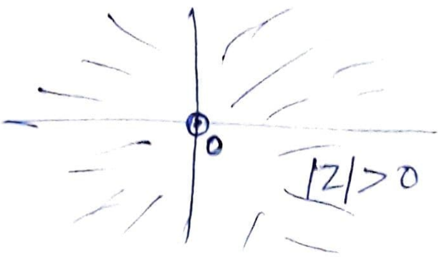
$$= \sum_{K=0}^{\infty} \frac{1}{(2K)!} \left(\left(\frac{1}{Z}\right)^{1/2}\right)^{2K}$$

$$= \cosh\left(\left(\frac{1}{Z}\right)^{1/2}\right) = \cosh\left(\frac{1}{\sqrt{Z}}\right)$$

for $\left|\left(\frac{1}{Z}\right)^{1/2}\right| < \infty \Rightarrow \frac{1}{|Z|^{1/2}} < \infty$

$\Rightarrow \frac{1}{\infty} < |z|^{1/2} \Rightarrow 0 < |z|^{1/2} \Rightarrow 0 < |z|$

$\Rightarrow |z| > 0$




(10) $\frac{1}{K+1}$ , $K \geq 0$

Sol^n $f(K) = \frac{1}{K+1}$ , $K \geq 0$

$$Z\{f(K)\} = \sum_{K=0}^{\infty} \frac{1}{(K+1)} z^{-K}$$

$$= \sum_{K=0}^{\infty} \frac{1}{(K+1)} \left(\frac{1}{z}\right)^{K}$$

$$= \sum_{K=0}^{\infty} \frac{1}{(K+1)} \left(\frac{1}{z}\right)^{K+1-1}$$

$$= \left(\frac{1}{z}\right)^{-1} \sum_{K=0}^{\infty} \frac{1}{(K+1)} \left(\frac{1}{z}\right)^{K+1}$$

$$= -z \sum_{K=0}^{\infty} -\frac{1}{(K+1)} \left(\frac{1}{z}\right)^{K+1}$$

$$= -z \log\left(1 - \frac{1}{z}\right) = -z \log\left(\frac{z-1}{z}\right)$$

for $\left|\frac{1}{z}\right| < 1 \Rightarrow 1 < |z|$

$\Rightarrow |z| > 1$