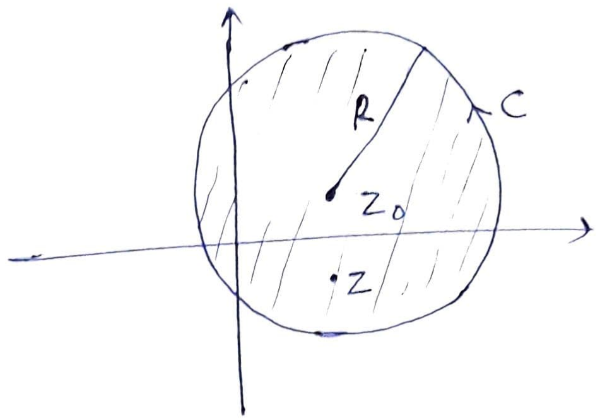
Taylor's Theorem:-

If $f(z)$ is Analytic at $z=z_0$ and Analytic throughout an open disk $|z-z_0|<R$, then $f(z)$ has the series expansion about $z=z_0$ given as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for all } |z-z_0|<R$$

where $a_n = \dfrac{f^n(z_0)}{n!}$



Note:-

(1) The series above is referred as Taylor series expansion of $f(z)$ about the point $z=z_0$.

(2) $f(z)$ admitts the Taylor series expansion about $z=z_0$ in some open disk around z_0 iff $f(z)$ is Analytic at $z=z_0$.

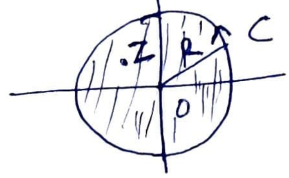
(3) For $z_0=0$ i.e.

If $f(z)$ is Analytic at $z=0$ and in some open disk $|z|<R$ around $z=0$

then,

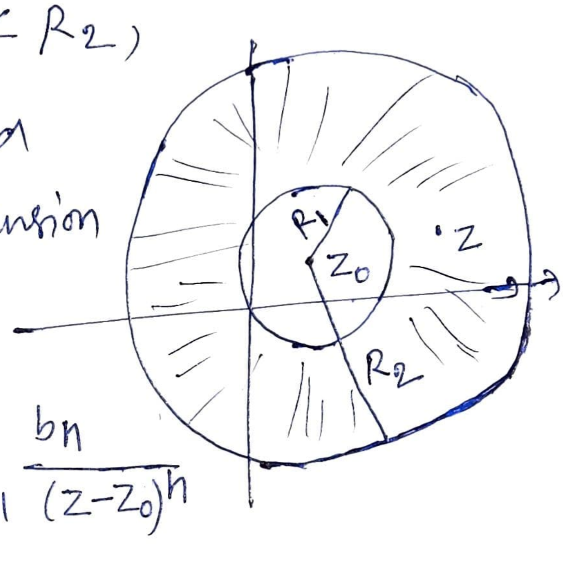
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } |z| < R$$

also known as Maclaurin's series



Laurent's Theorem:-

If $f(z)$ is not Analytic at $z=z_0$ but Analytic through out an Annular domain $R_1 < |z-z_0| < R_2$, then $f(z)$ admitts a Laurent series expansion about $z=z_0$,



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

for all $R_1 < |z-z_0| < R_2$.

Note:- The term $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is known as Taylor part and the term $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is known as Principal part of the Laurent series.

Basic Series and their Region of convergence (ROC):-

① Geometric series:-

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots, \quad |z|<1$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1-z+z^2-\cdots, \quad |z|<1$$

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n = 1+2z+3z^2+\cdots, \quad |z|<1$$

$$\frac{1}{(1+z)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n = 1-2z+3z^2-\cdots,$$

for $|z|<1$

$$\frac{1}{(1-z)^3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2!} z^n, \quad |z|<1$$

$$\frac{1}{(1+z)^3} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2!} z^n, \quad |z|<1$$

② $$\log(1-z) = \sum_{n=0}^{\infty} -\frac{z^{n+1}}{(n+1)} = -z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots$$

for $|z|<1$

$$\log(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{(n+1)}$$

$$= z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots, \quad |z|<1$$

(3) $$e^{z}=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\cdots$$

for $|z|<\infty$

$$e^{-z}=\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}=1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\cdots\cdots$$

for $|z|<\infty$

(4) $$\cos z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\cdots$$

for $|z|<\infty$

(5) $$\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\cdots$$

for $|z|<\infty$

(6) $$\cosh z=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\cdots$$

for $|z|<\infty$

(7) $$\sinh z=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\cdots$$

for $|z|<\infty$

Taylor's and Laurent series Expansion

Ex. Find all possible Laurent's series expansion and specify the domain of convergence.

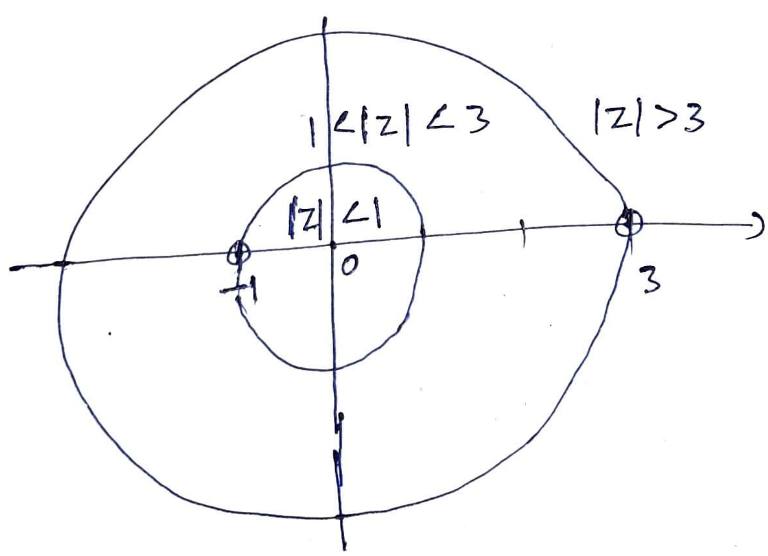
(1) $f(z) = \dfrac{z-1}{z^2-2z-3}$

Sol^n $f(z)$ is not Analytic at

$z^2 - 2z - 3 = 0$

$\Rightarrow \quad z = 3, -1$

$z - 0$
$= z$



$$f(z) = \frac{z-1}{(z-3)(z+1)}$$

$$= \frac{A}{(z-3)} + \frac{B}{(z+1)}$$

$\Rightarrow \quad A(z+1) + B(z-3) = z-1$

For $z=3$; $\quad 4A = 2 \Rightarrow A = \frac{1}{2}$

$z=-1$; $\quad -4B = -2 \Rightarrow B = \frac{1}{2}$

$\therefore \quad f(z) = \frac{1}{2}\frac{1}{(z-3)} + \frac{1}{2}\frac{1}{(z+1)} \quad \text{———} \quad ①$

$$\frac{1}{(z-3)} = -\frac{1}{3}\frac{1}{\left(1-\frac{z}{3}\right)}$$

$$= -\frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right] \quad \text{———(i)}$$

for $\left|\frac{z}{3}\right| < 1 \Rightarrow \frac{|z|}{3} < 1 \Rightarrow |z| < 3$

$$\frac{1}{(z-3)} = \frac{1}{z}\frac{1}{\left(1-\frac{3}{z}\right)}$$

$$= \frac{1}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right]$$

$$= \frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \cdots \quad \text{———(ii)}$$

for $\left|\frac{3}{z}\right| < 1 \Rightarrow 3 < |z| \Rightarrow |z| > 3$

$$\frac{1}{z+1} = \frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

for $|z| < 1$ $\quad$ ——— (iii)

$$\frac{1}{z+1} = \frac{1}{z}\,\frac{1}{\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots \qquad \text{(iv)}$$

for $\left|\frac{1}{z}\right| < 1 \Rightarrow 1 < |z| \Rightarrow |z| > 1$

case 1: For $|z| < 1$; from (i) & (iii)

$$f(z) = -\frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right] + \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right]$$

case 2: For $1 < |z| < 3$; (i) & (iv)

$$f(z) = -\frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right] + \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right]$$

case 3: For $|z| > 3$; (ii) & (iv)

$$f(z) = \frac{1}{2}\left[\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \cdots\right] + \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right]$$

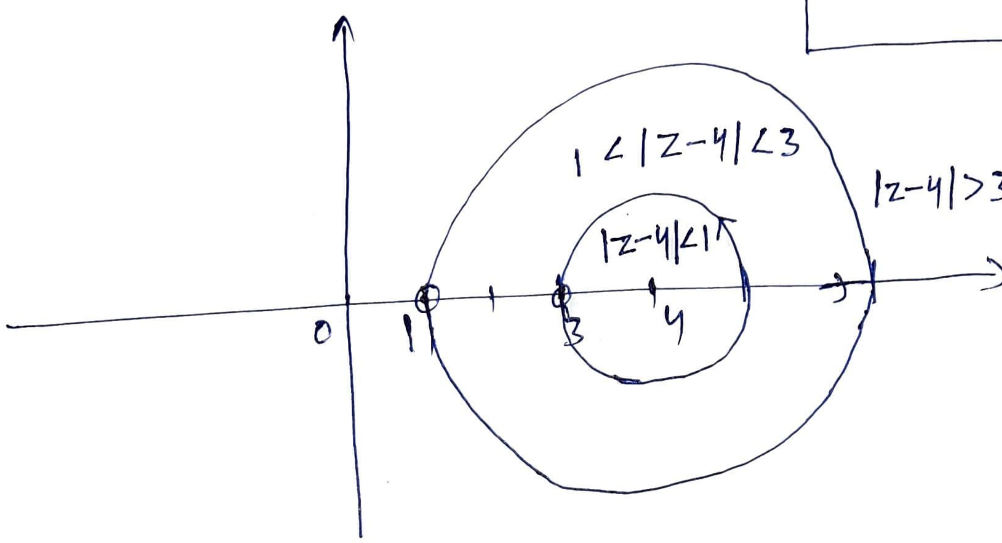
(2) $f(z) = \dfrac{2z-3}{z^2-4z+3}$ about $z=4$

Soln $f(z)$ is not Analytic at

$z^2-4z+3=0$

$\Rightarrow z = 1, 3$

$\boxed{(z-4)}$



$$f(z) = \frac{2z-3}{z^2-4z+3} = \frac{A}{(z-1)} + \frac{B}{(z-3)}$$

$\Rightarrow A(z-3) + B(z-1) = 2z-3$

For $z=1$; $-2A = -1 \Rightarrow A = \frac{1}{2}$

$z=3$; $2B = 3 \Rightarrow B = 3/2$

$\therefore f(z) = \frac{1}{2}\frac{1}{(z-1)} + \frac{3}{2}\frac{1}{(z-3)}$ — (I)

$$\frac{1}{z-1} = \frac{1}{z-4+3} = \frac{1}{3}\frac{1}{\left(1+\frac{(z-4)}{3}\right)}$$

$$= \frac{1}{3}\left[1-\frac{(z-4)}{3}+\frac{(z-4)^2}{3^2}-\cdots\cdots\right]$$

for $\left|\frac{z-4}{3}\right|<1 \Rightarrow |z-4|<3$ ——— (i)

$$\cancel{\frac{1}{z-4}}\ \frac{1}{(z-1)} = \frac{1}{(z-4)+3} = \frac{1}{(z-4)}\frac{1}{\left(1+\frac{3}{(z-4)}\right)}$$

$$= \frac{1}{(z-4)}\left[1-\frac{3}{(z-4)}+\frac{3^2}{(z-4)^2}-\cdots\cdots\right]$$

$$= \frac{1}{(z-4)}-\frac{3}{(z-4)^2}+\frac{3^2}{(z-4)^3}-\cdots\cdots$$

for $\left|\frac{3}{z-4}\right|<1 \Rightarrow 3<|z-4| \Rightarrow |z-4|>3$ ——— (ii)

$$\frac{1}{z-3} = \frac{1}{(z-4)+1} = \frac{1}{1+(z-4)}$$

$$= 1-(z-4)+(z-4)^2-\cdots\cdots$$ ——— (iii)

for $|z-4|<1$

$$\frac{1}{(z-3)} = \frac{1}{(z-4)+1} = \frac{1}{(z-4)}\frac{1}{\left(1+\frac{1}{z-4}\right)}$$

$$= \frac{1}{(z-4)}\left[1-\frac{1}{(z-4)}+\frac{1}{(z-4)^2}-\cdots\right]$$

$$\Rightarrow \frac{1}{z-3} = \frac{1}{(z-4)} - \frac{1}{(z-4)^2} + \frac{1}{(z-4)^3} - \cdots$$

for $\left|\frac{1}{z-4}\right| < 1 \Rightarrow 1 < |z-4| \Rightarrow \underline{|z-4| > 1}$ — (iv)

Case 1: For $|z-4| < 1$; from (i) & (iii)

$$\Rightarrow f(z) = \frac{1}{6}\left[1 - \frac{(z-4)}{3} + \frac{(z-4)^2}{3^2} - \cdots\right]$$
$$+ \frac{3}{2}\left[1 - (z-4) + (z-4)^2 - \cdots\right]$$

Case 2: For $1 < |z-4| < 3$; from (i) & (iv)

$$\Rightarrow f(z) = \frac{1}{6}\left[1 - \frac{(z-4)}{3} + \frac{(z-4)^2}{3^2} - \cdots\right]$$
$$+ \frac{3}{2}\left[\frac{1}{(z-4)} - \frac{1}{(z-4)^2} + \frac{1}{(z-4)^3} - \cdots\right]$$

Case 3: For $|z-4| > 3$; from (ii) & (iv)

$$\Rightarrow f(z) = \frac{1}{2}\left[\frac{1}{(z-4)} - \frac{3}{(z-4)^2} + \frac{3^2}{(z-4)^3} - \cdots\right]$$
$$+ \frac{3}{2}\left[\frac{1}{(z-4)} - \frac{1}{(z-4)^2} + \frac{1}{(z-4)^3} - \cdots\right]$$

E.x. Expand the series of $f(z) = \frac{1}{z(z-1)(z+2)}$ convergent in the domain

(i) $0 < |z| < 1$ (ii) $|z-1| > 3$ (iii) $2 < |z+2| < 3$
(iv) $1 < |z+1| < 2$

Soln

$f(z) = \frac{1}{z(z-1)(z+2)}$ is not Analytic

at $z(z-1)(z+2) = 0$

$\Rightarrow z = 0, 1, -2$

$$f(z) = \frac{1}{z(z-1)(z+2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2}$$

$$\Rightarrow A(z-1)(z+2) + Bz(z+2) + Cz(z-1) = 1$$

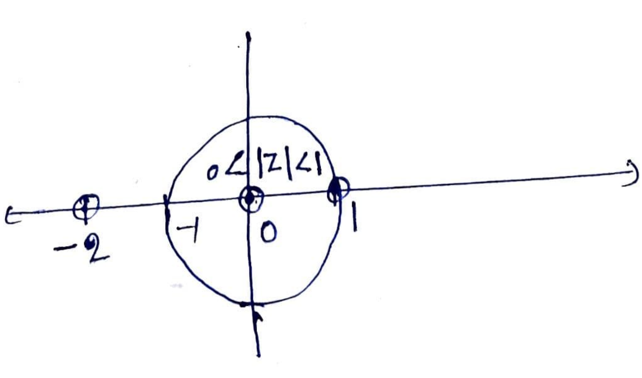
For $z=0$, $-2A = 1 \Rightarrow A = -\frac{1}{2}$

$z=1$, $3B = 1 \Rightarrow B = \frac{1}{3}$

$z=-2$, $6C = 1 \Rightarrow C = \frac{1}{6}$

$$\therefore f(z) = -\frac{1}{2}\frac{1}{z} + \frac{1}{3}\frac{1}{(z-1)} + \frac{1}{6}\frac{1}{(z+2)} \quad \text{(I)}$$

(i) For $0 < |z| < 1$



$$\frac{1}{z-1} = -\frac{1}{(1-z)} = -[1+z+z^2+\cdots\cdots]$$

for $|z|<1$

$$\frac{1}{z+2} = \frac{1}{2}\frac{1}{(1+\frac{z}{2})}$$

$$= \frac{1}{2}\left[1-\frac{z}{2}+\frac{z^2}{2^3}-\cdots\cdots\right]$$

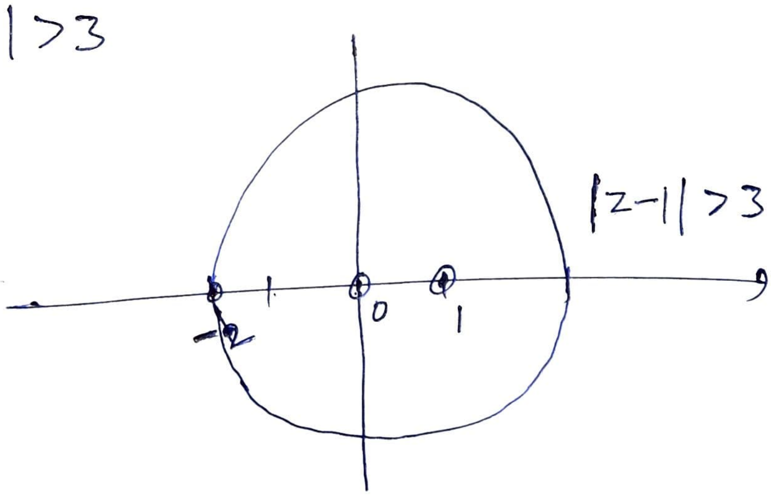
for $\left|\frac{z}{2}\right|<1 \Rightarrow |z|<2$

$$\therefore f(z) = -\frac{1}{2}\frac{1}{z} - \frac{1}{3}[1+z+z^2+\cdots\cdots]$$

$$+\frac{1}{12}\left[1-\frac{z}{2}+\frac{z^2}{2^3}-\cdots\cdots\right]$$

for $0<|z|<1$

(ii) For $|z-1|>3$



$$\frac{1}{z} = \frac{1}{z-1+1} = \frac{1}{(z-1)}\frac{1}{\left(1+\frac{1}{(z-1)}\right)}$$

$$= \frac{1}{(z-1)}\left[1-\frac{1}{(z-1)}+\frac{1}{(z-1)^2}-\cdots\cdots\right]$$

$$= \frac{1}{(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \cdots\cdots$$

for $\left|\frac{1}{z-1}\right|<1 \Rightarrow |z-1|>1$

$$\frac{1}{z+2} = \frac{1}{z-1+3} = \frac{1}{(z-1)}\cdot\frac{1}{\left(1+\frac{3}{(z-1)}\right)}$$

$$= \frac{1}{(z-1)}\left[1-\frac{3}{(z-1)}+\frac{3^2}{(z-1)^2}-\cdots\cdots\right]$$

$$= \frac{1}{(z-1)}-\frac{3}{(z-1)^2}+\frac{3^2}{(z-1)^3}-\cdots\cdots$$

for $\left|\frac{3}{z-1}\right|<1 \Rightarrow 3<|z-1| \Rightarrow |z-1|>3$

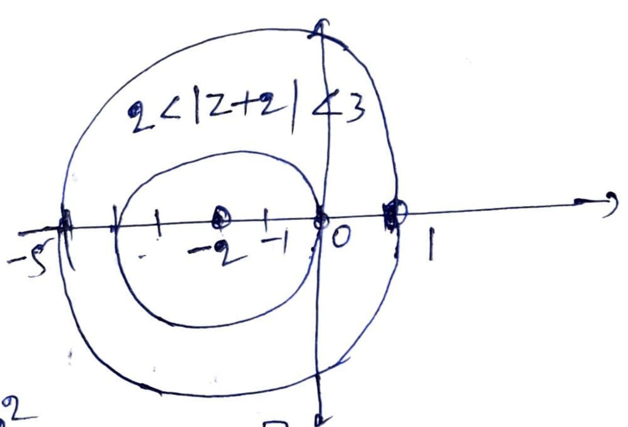
$$\therefore f(z) = -\frac{1}{2}\left[\frac{1}{(z-1)}-\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}-\cdots\cdots\right]$$
$$+\frac{1}{3}\,\frac{1}{(z-1)}+\frac{1}{6}\left[\frac{1}{(z-1)}-\frac{3}{(z-1)^2}+\frac{3^2}{(z-1)^3}-\cdots\cdots\right]$$

for $|z-1|>3$

(iii) For $2<|z+2|<3$

$$\frac{1}{z} = \frac{1}{(z+2)-2}$$

$$= \frac{1}{(z+2)}\,\frac{1}{\left(1-\frac{2}{(z+2)}\right)}$$



$$= \frac{1}{(z+2)}\left[1+\frac{2}{z+2}+\frac{2^2}{(z+2)^2}+\cdots\right]$$

$$= \frac{1}{(z+2)}+\frac{2}{(z+2)^2}+\frac{2^2}{(z+2)^3}+\cdots\cdots$$

for $\left|\frac{2}{z+2}\right|<1 \Rightarrow 2<|z+2|$

$$\frac{1}{z-1} = \frac{1}{z+2-3} = -\frac{1}{3}\frac{1}{\left(1-\frac{(z+2)}{3}\right)}$$

$$= -\frac{1}{3}\left[1+\frac{(z+2)}{3}+\frac{(z+2)^2}{3^2}+\cdots\right]$$

for $\left|\frac{z+2}{3}\right|<1 \Rightarrow |z+2|<3$

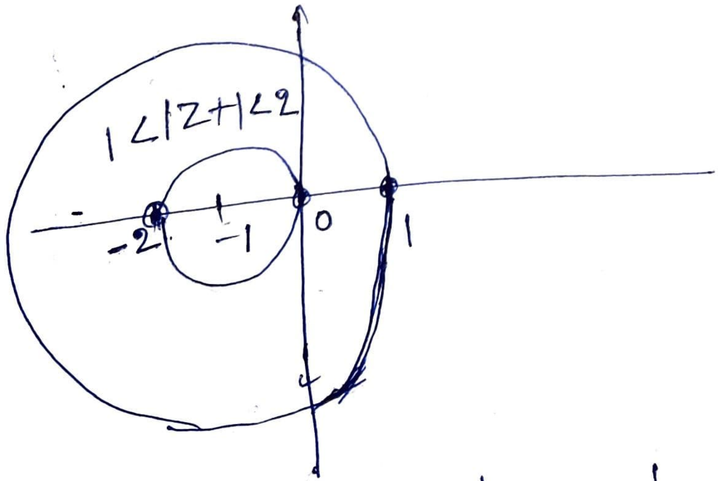
$$\therefore f(z) = -\frac{1}{2}\left[\frac{1}{(z+2)}+\frac{2}{(z+2)^2}+\frac{2^2}{(z+2)^3}+\cdots\right]$$

$$-\frac{1}{9}\left[1+\frac{(z+2)}{3}+\frac{(z+2)^2}{3^2}+\cdots\right]$$

$$+\frac{1}{6}\frac{1}{(z+2)}$$

for $2<|z+2|<3$

(iv) For $1<|z+1|<2$



$$\frac{1}{z} = \frac{1}{(z+1)-1} = \frac{1}{(z+1)}\frac{1}{\left(1-\frac{1}{z+1}\right)}$$

$$= \frac{1}{(z+1)}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\cdots\right]$$

$$= \frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots$$

for $\left|\frac{1}{z+1}\right| < 1 \Rightarrow 1 < |z+1|$

$$\frac{1}{z-1} = \frac{1}{z+1-2} = -\frac{1}{2}\,\frac{1}{\left(1-\frac{(z+1)}{2}\right)}$$

$$= -\frac{1}{2}\left[1 + \frac{(z+1)}{2} + \frac{(z+1)^2}{2^2} + \cdots\right]$$

for $\left|\frac{z+1}{2}\right| < 1 \Rightarrow |z+1| < 2$

$$\frac{1}{z+2} = \frac{1}{(z+1)+1} = \frac{1}{(z+1)}\,\frac{1}{\left(1+\frac{1}{z+1}\right)}$$

$$= \frac{1}{(z+1)}\left[1 - \frac{1}{(z+1)} + \frac{1}{(z+1)^2} - \cdots\right]$$

$$= \frac{1}{(z+1)} - \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} - \cdots$$

for $\left|\frac{1}{z+1}\right| < 1 \Rightarrow 1 < |z+1|$

$$\therefore f(z) = -\frac{1}{2}\left[\frac{1}{(z+1)} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\right]$$

$$-\frac{1}{6}\left[1 + \frac{(z+1)}{2} + \frac{(z+1)^2}{2^2} + \cdots\right]$$

$$+\frac{1}{6}\left[\frac{1}{(z+1)} - \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} - \cdots\right]$$

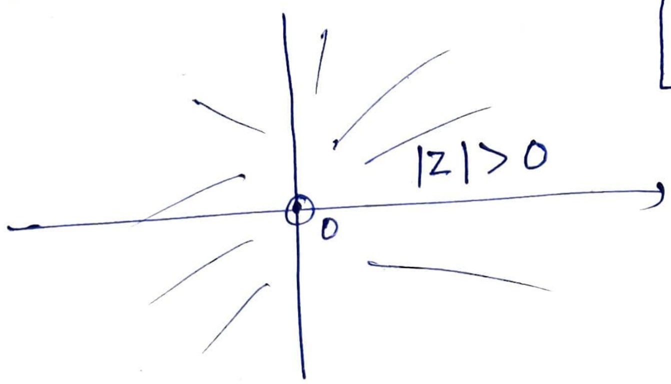
for $1 < |z+1| < 2$

Ex. Find the series expansion of $f(z) = \frac{\sin^2 z}{z}$

Sol^n $f(z) = \frac{\sin^2 z}{z}$ is not Analytic

at $z = 0$.

$\boxed{\begin{array}{l} z-0 \\ = z \end{array}}$



$$\sin^2 z = \frac{1 - \cos 2z}{2}$$

$$= \frac{1}{2}\left[1 - \left[1 - \frac{2^2 z^2}{2!} + \frac{2^4 z^4}{4!} - \frac{2^6 z^6}{6!} + \cdots\right]\right]$$

$$= \frac{1}{2}\left[\frac{2^2 z^2}{2!} - \frac{2^4 z^4}{4!} + \frac{2^6 z^6}{6!} - \cdots\right]$$

for $|2z| < \infty \Rightarrow |z| < \infty$

$$\therefore f(z) = \frac{1}{2z}\left[\frac{2^2 z^2}{2!} - \frac{2^4 z^4}{4!} + \frac{2^6 z^6}{6!} - \cdots\right]$$

$$= \frac{2z}{2!} - \frac{2^3 z^3}{4!} + \frac{2^5 z^5}{6!} - \cdots$$

for $0 < |z| < \infty$

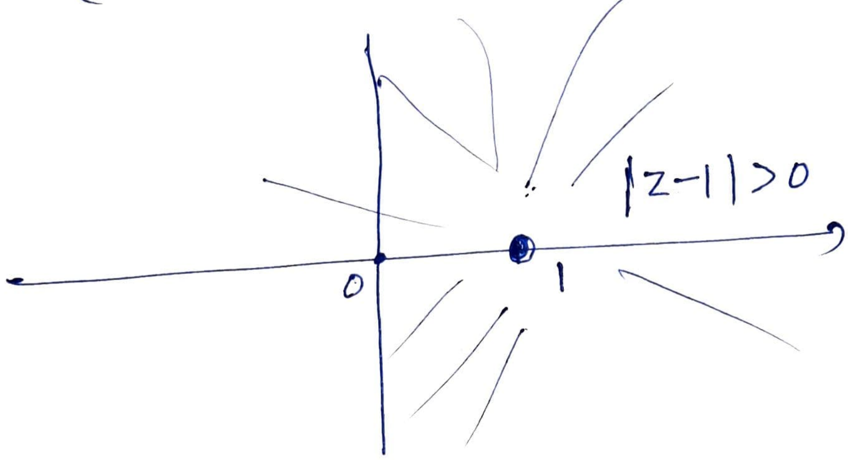
E.x. Find all Laurent series expansion of

$$f(z) = \frac{e^{2z}}{(z-1)^3} \quad \text{about } z=1$$

<u>sol^n</u> $f(z)$ is not Analytic at

$(z-1)^3 = 0 \Rightarrow z=1$

$\boxed{(z-1)}$



$$e^{2z} = e^{2(z-1+1)} = e^{2(z-1)} \cdot e^2$$

$$= e^2\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right]$$

for $|2(z-1)| < \infty \Rightarrow |z-1| < \infty$

$$\therefore f(z) = \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2^2}{3!}\frac{1}{(z-1)} + \frac{2^3}{3!} + \frac{2^4}{4!}(z-1) + \cdots\right]$$