Duality in Linear Programming:

Given a LPP in canonical form

Maximize $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i, \ i = 1, 2, ..., m$ ———— (I)

$x_j \ge 0, \ j = 1, 2, ..., n$

We can write its corresponding dual problem as

Minimize $W = \sum_{i=1}^{m} b_i y_i$

Subject to $\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \ j = 1, 2, ..., n$ ———— (II)

$y_i \ge 0, \ i = 1, 2, ..., n$

The given problem (I) is called Primal problem
And problem (II) is called its Dual problem
Thus problem (I) and (II) are dual of each other.

Note:
1. If given primal problem contains large numbers of constraints than numbers of variables then computational steps can be reduced by converting it into dual and then solving it.

2. It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients formulation of the problem known as sensitivity analysis.

3. An unrestricted variable in sign in primal problem will result in an equality constrain in dual problem and vice versa.

Correspondence between Primal and Dual Optimal solution:

If the dual problem has optimal solution then primal problem also has optimal solution.
The value under the slack variables (if any neglecting the negative sign) and under the artificial variables (if any and neglecting the negative sign and deleting constant M) in the net evaluation row $(c_j - z_j)$ of the optimal table of the dual problem gives the values of basic variables of the primal problem and vice versa.
The optimal value of dual is equal to optimal value of primal and vice versa i.e
$Z_{max} = W_{min}$

Dual LPP :-

Ex. Write the dual of the following problems.

(1) Max $Z = 5x_1 - 4x_2 + 4x_3$

sub to $6x_1 + 5x_2 + 10x_3 \leq 7$

$2x_1 + x_2 - 6x_3 = 20$

$8x_1 - 3x_2 + 6x_3 \geq 50$

$x_1, x_2, x_3 \geq 0$

Soln

cannonical form is

Max $Z = 5x_1 - 4x_2 + 4x_3$

sub to $6x_1 + 5x_2 + 10x_3 \leq 7$

$2x_1 + x_2 - 6x_3 \leq 20$

$-2x_1 - x_2 + 6x_3 \leq -20$

$-8x_1 + 3x_2 - 6x_3 \leq -50$

$x_1, x_2, x_3 \geq 0$

$\therefore$ Dual problem is

Min $W = 7y_1 + 20y_2' - 20y_2'' - 50y_3$

sub to $6y_1 + 2y_2' - 2y_2'' - 8y_3 \geq 5$

$5y_1 + y_2' - y_2'' + 3y_3 \geq -4$

$10y_1 - 6y_2' + 6y_2'' - 6y_3 \geq 4$

$y_1, y_2', y_2'', y_3 \geq 0$

put $y_2' - y_2'' = y_2$;

$$\text{Min } W = 7y_1 + 20y_2 - 50y_3$$

sub to $6y_1 + 2y_2 - 8y_3 \geq 5$

$5y_1 + y_2 + 3y_3 \geq -4$

$10y_1 - 6y_2 - 6y_3 \geq 4$

$y_1, y_3 \geq 0$

and y_2 is unrestricted in sign.

Ex. Solve the LPP using Dual problem.

(1) $\text{Max } Z = 5x_1 - 2x_2 + 3x_3$

sub to $2x_1 + 2x_2 - x_3 \geq 2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

Soln Cannonical form:-

$\text{Max } Z = 5x_1 - 2x_2 + 3x_3$

sub to $-2x_1 - 2x_2 + x_3 \leq -2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

$\therefore$ The corresponding dual problem is

$$\text{Min } W = -2y_1 + 3y_2 + 5y_3$$

sub to $\quad -2y_1 + 3y_2 \geq 5$

$$-2y_1 - 4y_2 + y_3 \geq -2$$

$$y_1 + 3y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

Introducing the slacks and Artificial variables; standard form is

$$\text{Min } W = -2y_1 + 3y_2 + 5y_3 + MA_1 + MA_2$$

sub to $\quad -2y_1 + 3y_2 - S_1 + A_1 = 5$

$$2y_1 + 4y_2 - y_3 + S_2 = 2$$

$$y_1 + 3y_3 - S_3 + A_2 = 3$$

$$y_1, y_2, y_3, S_1, S_2, S_3, A_1, A_2 \geq 0$$

put $y_1 = 0,\ y_2 = 0,\ y_3 = 0,\ S_1 = 0,\ S_3 = 0$

$\Rightarrow \quad A_1 = 5,\ S_2 = 2,\ A_2 = 3$

C_B	Basis	y_1	y_2	y_3	s_1	s_2	s_3	A_1	A_2	Soln	Min Ratio
	C_j	-2	3	5	0	0	0	M	M		
M	A_1	-2	3	0	-1	0	0	1	0	5	$5/3$
0	S_2	2	(4)	-1	0	1	0	0	0	2	$1/2 \rightarrow$
M	A_2	1	0	3	0	0	-1	0	1	3	∞
	Z_j	$-M$	$3M$	$3M$	$-M$	0	$-M$	M	M		
	C_j-Z_j	$2+M$	$3-3M$	$5-3M$	M	0	M	0	0		
			$\uparrow$								
M	A_1	$-7/2$	0	$3/4$	-1	$-\frac{3}{4}$	0	1	0	$7/2$	$14/3$
3	y_2	$1/2$	1	$-1/4$	0	$1/4$	0	0	0	$1/2$	-2
M	A_2	1	0	(3)	0	0	-1	0	1	3	$1 \rightarrow$
	Z_j	$\frac{3}{2}-\frac{5}{2}M$	3	$-\frac{3}{4}+\frac{15}{4}M$	$-M$	$\frac{3}{4}-\frac{3}{4}M$	$-M$	M	M		
	C_j-Z_j	$-\frac{7}{2}+\frac{5}{2}M$	0	$\frac{23}{4}-\frac{15}{4}M$	M	$-\frac{3}{4}+\frac{3}{4}M$	M	0	0		
				$\uparrow$							
M	A_1	-2	0	0	-1	$-\frac{3}{4}$	$(\frac{1}{4})$	1	$-\frac{1}{4}$	$\frac{11}{4}$	$11 \rightarrow$
3	y_2	$7/12$	1	0	0	$1/4$	$-1/12$	0	$1/12$	$3/4$	-9
5	y_3	$1/3$	0	1	0	0	$-1/3$	0	$1/3$	1	-3
	Z_j	$\frac{41}{12}-2M$	3	5	$-M$	$\frac{3}{4}-\frac{3}{4}M$	$-\frac{23}{12}+\frac{M}{4}$	M	$\frac{23}{12}-\frac{M}{4}$		
	C_j-Z_j	$\frac{65}{12}+2M$	0	0	M	$-\frac{3}{4}+\frac{3}{4}M$	$\frac{23}{12}-\frac{M}{4}$	0	$-\frac{23}{12}+\frac{M}{4}$		
							$\uparrow$				
0	S_3	-8	0	0	-4	-3	1	4	-1	11	
3	y_2	$-2/3$	1	0	$-1/3$	0	0	$1/3$	0	$5/3$	
5	y_3	$-\frac{14}{3}$	0	1	$-4/3$	-1	0	$4/3$	0	$14/3$	
	Z_j	$-\frac{76}{3}$	3	5	$-\frac{23}{3}$	-5	0	$\frac{23}{3}$	0		
	C_j-Z_j	$\frac{70}{3}$	0	0	$\frac{23}{3}$	5	0	$-\frac{23}{3}+M$	M		

$\therefore$ optimal solution is

$y_1 = 0, \ y_2 = \frac{5}{3}, \ y_3 = \frac{14}{3}$

& $W_{min} = \frac{85}{3}$

The optimal solution of primal problem

is $x_1 = \frac{23}{5}, \ x_2 = 5, \ x_3 = 0$

& $Z_{max} = \frac{85}{3}$

Dual Simplex Method

In this method the solution starts with a basic infeasible optimal solution and work towards feasibility.

Step1. Convert the problem into maximization type.
Step2. Write all the constraints into $\le$ type.
Step3. Convert all the constraints into equal to type by adding slacks.
Step4. Find the initial solution and express it in the form of a table known as **dual simplex table**.

Step4. Compute $c_j - z_j$ row.

Case1. If any $c_j - z_j > 0$ then **method fails**.

Case2. If all $c_j - z_j \le 0$ and all solution values are ≥ 0 then the solution obtained is a **optimal feasible solution.**

Case3. if all $c_j - z_j \le 0$ and any solution value is negative then the solution obtained is **optimal but infeasible** and therefore we can go to **step5** to improve it towards feasibility.

Step5. The row containing least negative solution value is the **key row** and contains the out going variable.

Case1. If all elements of key row are ≥ 0 then the problem **does not have a feasible solution**.

Case2. If at least one element is negative, then find the ratios between the corresponding values of $c_j - z_j$ and these values (ignoring positive and zero elements of key row).
The column containing the smallest of these ratios contains the incoming variable.

Perform the usual simplex operation to improve the solution till an optimal feasible solution obtained.

Dual simplex Method

E-x. Use dual simplex method to solve

1 $\quad$ Max $z = -3x_1 - 2x_2$

Sub to $\quad x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

Soln $\quad$ Converting to maximization cannonical form.

Max $z = -3x_1 - 2x_2$

Sub to $\quad -x_1 - x_2 \leq -1$

$x_1 + x_2 \leq 7$

$-x_1 - 2x_2 \leq -10$

$x_3 \leq 3$

$x_1, x_2 \geq 0$

Introducing slacks and obtaining initial solution.

Max $z = -3x_1 - 2x_2$

sub to $\quad -x_1 - x_2 + s_1 = -1$

$x_1 + x_2 + s_2 = 7$

$-x_1 - 2x_2 + s_3 = -10$

$x_2 + s_4 = 3$

$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

put $x_1 = 0,\ x_2 = 0,$

$s_1 = -1,\ s_2 = 7,\ s_3 = -10,\ s_4 = 3$

	C_j	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	Sol^n
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \rightarrow$
0	s_4	0	1	0	0	0	1	3
	Z_j	0	0	0	0	0	0	
	$C_j - Z_j$	-3	-2	0	0	0	0	
	$\frac{C_j - Z_j}{a_{3j}}$	3	1	$-$	$-$	$-$	$-$	
			$\uparrow$					
0	s_1	$-1/2$	0	1	0	$-1/2$	0	4
0	s_2	$1/2$	0	0	1	$1/2$	0	2
-2	x_2	$1/2$	1	0	0	$-1/2$	0	5
0	s_4	$(-1/2)$	0	0	0	$1/2$	1	$-2 \rightarrow$
	Z_j	-1	-2	0	0	1	0	
	$C_j - Z_j$	-2	0	0	0	-1	0	
	$\frac{C_j - Z_j}{a_{4j}}$	4	$-$	$-$	$-$	$-$	$-$	
		$\uparrow$						

		x_1	x_2	s_1	s_2	s_3	s_4	Sol^n
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	0	1	3
-3	x_1	1	0	0	0	-1	-2	4
	Z_j	-3	-2	0	0	3	4	
	$C_j - Z_j$	0	0	0	0	-3	-4	

$\therefore$ optimal feasible solution is

$x_1 = 4,\ x_2 = 3$

& $Z_{max} = -18$

Note that s_2 is in the basis with $s_2 = 0 \Rightarrow$ It is optimal feasible degenerate solution.

(2) Min $Z = x_1 + x_2$

$$2x_1 + x_2 \geq 2$$
$$-x_1 - x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Sol^n Converting into maximization cannonical form.

Max $W = -Z = -x_1 - x_2$

sub to $-2x_1 - x_2 \leq -2$

$$x_1 + x_2 \leq -1$$
$$x_1, x_2 \geq 0$$

Introducing slacks to obtain initial solution.

$$\text{Max } W = -x_1 - x_2$$

$$\text{sub to } \quad -2x_1 - x_2 + s_1 = -2$$

$$x_1 + x_2 + s_2 = -1$$

$$x_1, x_2, s_1, s_2 \geq 0$$

	c_j	-1	-1	0	0	
		x_1	x_2	s_1	s_2	Sol^n
0	s_1	(-2)	-1	1	0	$-2 \rightarrow$
0	s_2	1	1	0	1	-1
	z_j	0	0	0	0	
	$c_j - z_j$	-1	-1	0	0	
	$\dfrac{c_j - z_j}{a_{ij}}$	$1/2$	1	$-$	$-$	
		$\uparrow$				
-1	x_1	1	$1/2$	$-1/2$	0	1
0	s_2	0	$1/2$	$1/2$	1	$-2 \rightarrow$
	z_j	-1	$-1/2$	$1/2$	0	
	$c_j - z_j$	0	$-1/2$	$-1/2$	0	

Since all $c_j - z_j \leq 0$, solution is optimal, but $s_2 = -2 \Rightarrow$ It is infeasible.

Here s_2 is outgoing variable but s_2-row contains no negative coefficient

$\Rightarrow$ no other variable enters the basis

$\Rightarrow$ Problem does not have feasible optimal solution.