III   Data Structures

Introduction

Sets are as fundamental to computer science as they are to mathematics. Whereas mathematical sets are unchanging, the sets manipulated by algorithms can grow, shrink, or otherwise change over time. We call such sets *dynamic*. The next five chapters present some basic techniques for representing finite dynamic sets and manipulating them on a computer.

Algorithms may require several different types of operations to be performed on sets. For example, many algorithms need only the ability to insert elements into, delete elements from, and test membership in a set. We call a dynamic set that supports these operations a *dictionary*. Other algorithms require more complicated operations. For example, min-priority queues, which Chapter 6 introduced in the context of the heap data structure, support the operations of inserting an element into and extracting the smallest element from a set. The best way to implement a dynamic set depends upon the operations that must be supported.

Elements of a dynamic set

In a typical implementation of a dynamic set, each element is represented by an object whose attributes can be examined and manipulated if we have a pointer to the object. (Section 10.3 discusses the implementation of objects and pointers in programming environments that do not contain them as basic data types.) Some kinds of dynamic sets assume that one of the object's attributes is an identifying *key*. If the keys are all different, we can think of the dynamic set as being a set of key values. The object may contain *satellite data*, which are carried around in other object attributes but are otherwise unused by the set implementation. It may

also have attributes that are manipulated by the set operations; these attributes may contain data or pointers to other objects in the set.

Some dynamic sets presuppose that the keys are drawn from a totally ordered set, such as the real numbers, or the set of all words under the usual alphabetic ordering. A total ordering allows us to define the minimum element of the set, for example, or to speak of the next element larger than a given element in a set.

Operations on dynamic sets

Operations on a dynamic set can be grouped into two categories: *queries*, which simply return information about the set, and *modifying operations*, which change the set. Here is a list of typical operations. Any specific application will usually require only a few of these to be implemented.

SEARCH(S, k)
  A query that, given a set S and a key value k, returns a pointer x to an element in S such that $x.key = k$, or NIL if no such element belongs to S.

INSERT(S, x)
  A modifying operation that augments the set S with the element pointed to by x. We usually assume that any attributes in element x needed by the set implementation have already been initialized.

DELETE(S, x)
  A modifying operation that, given a pointer x to an element in the set S, removes x from S. (Note that this operation takes a pointer to an element x, not a key value.)

MINIMUM(S)
  A query on a totally ordered set S that returns a pointer to the element of S with the smallest key.

MAXIMUM(S)
  A query on a totally ordered set S that returns a pointer to the element of S with the largest key.

SUCCESSOR(S, x)
  A query that, given an element x whose key is from a totally ordered set S, returns a pointer to the next larger element in S, or NIL if x is the maximum element.

PREDECESSOR(S, x)
  A query that, given an element x whose key is from a totally ordered set S, returns a pointer to the next smaller element in S, or NIL if x is the minimum element.

In some situations, we can extend the queries SUCCESSOR and PREDECESSOR so that they apply to sets with nondistinct keys. For a set on n keys, the normal presumption is that a call to MINIMUM followed by $n - 1$ calls to SUCCESSOR enumerates the elements in the set in sorted order.

We usually measure the time taken to execute a set operation in terms of the size of the set. For example, Chapter 13 describes a data structure that can support any of the operations listed above on a set of size n in time $O(\lg n)$.

Overview of Part III

Chapters 10–14 describe several data structures that we can use to implement dynamic sets; we shall use many of these later to construct efficient algorithms for a variety of problems. We already saw another important data structure—the heap—in Chapter 6.

Chapter 10 presents the essentials of working with simple data structures such as stacks, queues, linked lists, and rooted trees. It also shows how to implement objects and pointers in programming environments that do not support them as primitives. If you have taken an introductory programming course, then much of this material should be familiar to you.

Chapter 11 introduces hash tables, which support the dictionary operations IN-SERT, DELETE, and SEARCH. In the worst case, hashing requires $\Theta(n)$ time to perform a SEARCH operation, but the expected time for hash-table operations is $O(1)$. The analysis of hashing relies on probability, but most of the chapter requires no background in the subject.

Binary search trees, which are covered in Chapter 12, support all the dynamic-set operations listed above. In the worst case, each operation takes $\Theta(n)$ time on a tree with n elements, but on a randomly built binary search tree, the expected time for each operation is $O(\lg n)$. Binary search trees serve as the basis for many other data structures.

Chapter 13 introduces red-black trees, which are a variant of binary search trees. Unlike ordinary binary search trees, red-black trees are guaranteed to perform well: operations take $O(\lg n)$ time in the worst case. A red-black tree is a balanced search tree; Chapter 18 in Part V presents another kind of balanced search tree, called a B-tree. Although the mechanics of red-black trees are somewhat intricate, you can glean most of their properties from the chapter without studying the mechanics in detail. Nevertheless, you probably will find walking through the code to be quite instructive.

In Chapter 14, we show how to augment red-black trees to support operations other than the basic ones listed above. First, we augment them so that we can dynamically maintain order statistics for a set of keys. Then, we augment them in a different way to maintain intervals of real numbers.

10   Elementary Data Structures

In this chapter, we examine the representation of dynamic sets by simple data structures that use pointers. Although we can construct many complex data structures using pointers, we present only the rudimentary ones: stacks, queues, linked lists, and rooted trees. We also show ways to synthesize objects and pointers from arrays.

10.1   Stacks and queues

Stacks and queues are dynamic sets in which the element removed from the set by the DELETE operation is prespecified. In a *stack*, the element deleted from the set is the one most recently inserted: the stack implements a *last-in, first-out*, or *LIFO*, policy. Similarly, in a *queue*, the element deleted is always the one that has been in the set for the longest time: the queue implements a *first-in, first-out*, or *FIFO*, policy. There are several efficient ways to implement stacks and queues on a computer. In this section we show how to use a simple array to implement each.

Stacks

The INSERT operation on a stack is often called PUSH, and the DELETE operation, which does not take an element argument, is often called POP. These names are allusions to physical stacks, such as the spring-loaded stacks of plates used in cafeterias. The order in which plates are popped from the stack is the reverse of the order in which they were pushed onto the stack, since only the top plate is accessible.

As Figure 10.1 shows, we can implement a stack of at most n elements with an array $S[1 \mathinner{.\,.} n]$. The array has an attribute $S.top$ that indexes the most recently
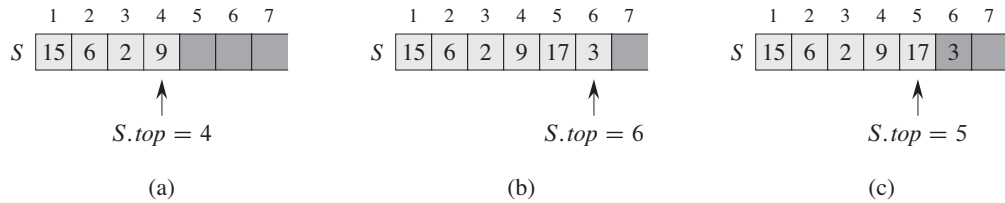
Figure 10.1 An array implementation of a stack S. Stack elements appear only in the lightly shaded positions. **(a)** Stack S has 4 elements. The top element is 9. **(b)** Stack S after the calls PUSH(S, 17) and PUSH(S, 3). **(c)** Stack S after the call POP(S) has returned the element 3, which is the one most recently pushed. Although element 3 still appears in the array, it is no longer in the stack; the top is element 17.

inserted element. The stack consists of elements $S[1 .. S.top]$, where $S[1]$ is the element at the bottom of the stack and $S[S.top]$ is the element at the top.

When $S.top = 0$, the stack contains no elements and is ***empty***. We can test to see whether the stack is empty by query operation STACK-EMPTY. If we attempt to pop an empty stack, we say the stack ***underflows***, which is normally an error. If $S.top$ exceeds n, the stack ***overflows***. (In our pseudocode implementation, we don't worry about stack overflow.)

We can implement each of the stack operations with just a few lines of code:

STACK-EMPTY(S)

```
1   if S.top == 0
2       return TRUE
3   else return FALSE
```

PUSH(S, x)

```
1   S.top = S.top + 1
2   S[S.top] = x
```

POP(S)

```
1   if STACK-EMPTY(S)
2       error "underflow"
3   else S.top = S.top − 1
4       return S[S.top + 1]
```

Figure 10.1 shows the effects of the modifying operations PUSH and POP. Each of the three stack operations takes $O(1)$ time.
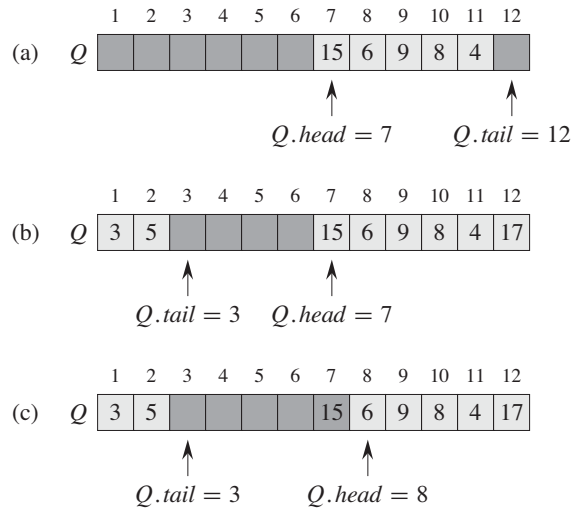
Figure 10.2   A queue implemented using an array $Q[1..12]$. Queue elements appear only in the lightly shaded positions. **(a)** The queue has 5 elements, in locations $Q[7..11]$. **(b)** The configuration of the queue after the calls ENQUEUE($Q, 17$), ENQUEUE($Q, 3$), and ENQUEUE($Q, 5$). **(c)** The configuration of the queue after the call DEQUEUE(Q) returns the key value 15 formerly at the head of the queue. The new head has key 6.

Queues

We call the INSERT operation on a queue ENQUEUE, and we call the DELETE operation DEQUEUE; like the stack operation POP, DEQUEUE takes no element argument. The FIFO property of a queue causes it to operate like a line of customers waiting to pay a cashier. The queue has a ***head*** and a ***tail***. When an element is enqueued, it takes its place at the tail of the queue, just as a newly arriving customer takes a place at the end of the line. The element dequeued is always the one at the head of the queue, like the customer at the head of the line who has waited the longest.

Figure 10.2 shows one way to implement a queue of at most $n - 1$ elements using an array $Q[1..n]$. The queue has an attribute $Q.head$ that indexes, or points to, its head. The attribute $Q.tail$ indexes the next location at which a newly arriving element will be inserted into the queue. The elements in the queue reside in locations $Q.head, Q.head + 1, \ldots, Q.tail - 1$, where we "wrap around" in the sense that location 1 immediately follows location n in a circular order. When $Q.head = Q.tail$, the queue is empty. Initially, we have $Q.head = Q.tail = 1$. If we attempt to dequeue an element from an empty queue, the queue underflows.

When $Q.head = Q.tail + 1$, the queue is full, and if we attempt to enqueue an element, then the queue overflows.

In our procedures ENQUEUE and DEQUEUE, we have omitted the error checking for underflow and overflow. (Exercise 10.1-4 asks you to supply code that checks for these two error conditions.) The pseudocode assumes that $n = Q.length$.

ENQUEUE(Q, x)

```
1   Q[Q.tail] = x
2   if Q.tail == Q.length
3       Q.tail = 1
4   else Q.tail = Q.tail + 1
```

DEQUEUE(Q)

```
1   x = Q[Q.head]
2   if Q.head == Q.length
3       Q.head = 1
4   else Q.head = Q.head + 1
5   return x
```

Figure 10.2 shows the effects of the ENQUEUE and DEQUEUE operations. Each operation takes $O(1)$ time.

Exercises

10.1-1
Using Figure 10.1 as a model, illustrate the result of each operation in the sequence PUSH($S, 4$), PUSH($S, 1$), PUSH($S, 3$), POP(S), PUSH($S, 8$), and POP(S) on an initially empty stack S stored in array $S[1 .. 6]$.

10.1-2
Explain how to implement two stacks in one array $A[1 .. n]$ in such a way that neither stack overflows unless the total number of elements in both stacks together is n. The PUSH and POP operations should run in $O(1)$ time.

10.1-3
Using Figure 10.2 as a model, illustrate the result of each operation in the sequence ENQUEUE($Q, 4$), ENQUEUE($Q, 1$), ENQUEUE($Q, 3$), DEQUEUE(Q), ENQUEUE($Q, 8$), and DEQUEUE(Q) on an initially empty queue Q stored in array $Q[1 .. 6]$.

10.1-4
Rewrite ENQUEUE and DEQUEUE to detect underflow and overflow of a queue.

10.1-5

Whereas a stack allows insertion and deletion of elements at only one end, and a queue allows insertion at one end and deletion at the other end, a ***deque*** (double-ended queue) allows insertion and deletion at both ends. Write four $O(1)$-time procedures to insert elements into and delete elements from both ends of a deque implemented by an array.

10.1-6

Show how to implement a queue using two stacks. Analyze the running time of the queue operations.

10.1-7

Show how to implement a stack using two queues. Analyze the running time of the stack operations.

10.2   Linked lists

A ***linked list*** is a data structure in which the objects are arranged in a linear order. Unlike an array, however, in which the linear order is determined by the array indices, the order in a linked list is determined by a pointer in each object. Linked lists provide a simple, flexible representation for dynamic sets, supporting (though not necessarily efficiently) all the operations listed on page 230.

As shown in Figure 10.3, each element of a ***doubly linked list*** L is an object with an attribute *key* and two other pointer attributes: *next* and *prev*. The object may also contain other satellite data. Given an element x in the list, $x.next$ points to its successor in the linked list, and $x.prev$ points to its predecessor. If $x.prev = $ NIL, the element x has no predecessor and is therefore the first element, or ***head***, of the list. If $x.next = $ NIL, the element x has no successor and is therefore the last element, or ***tail***, of the list. An attribute $L.head$ points to the first element of the list. If $L.head = $ NIL, the list is empty.

A list may have one of several forms. It may be either singly linked or doubly linked, it may be sorted or not, and it may be circular or not. If a list is ***singly linked***, we omit the *prev* pointer in each element. If a list is ***sorted***, the linear order of the list corresponds to the linear order of keys stored in elements of the list; the minimum element is then the head of the list, and the maximum element is the tail. If the list is ***unsorted***, the elements can appear in any order. In a ***circular list***, the *prev* pointer of the head of the list points to the tail, and the *next* pointer of the tail of the list points to the head. We can think of a circular list as a ring of
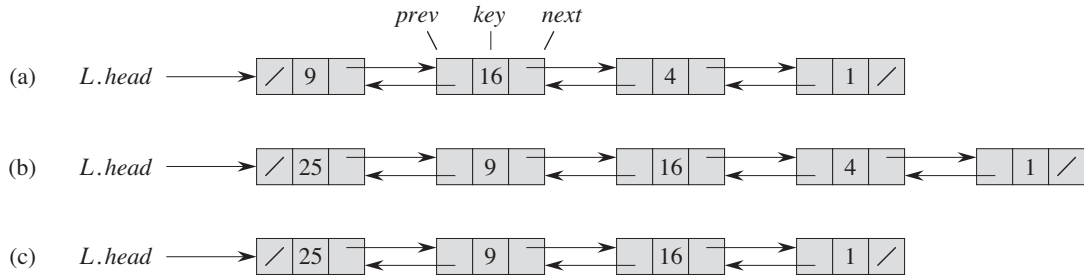
prev   key   next

(a)   L.head ⟶ [/ | 9 |] ⟷ [_ | 16 |] ⟷ [_ | 4 |] ⟷ [_ | 1 | /]

(b)   L.head ⟶ [/ | 25 |] ⟷ [_ | 9 |] ⟷ [_ | 16 |] ⟷ [_ | 4 |] ⟷ [_ | 1 | /]

(c)   L.head ⟶ [/ | 25 |] ⟷ [_ | 9 |] ⟷ [_ | 16 |] ⟷ [_ | 1 | /]

Figure 10.3   **(a)** A doubly linked list L representing the dynamic set $\{1, 4, 9, 16\}$. Each element in the list is an object with attributes for the key and pointers (shown by arrows) to the next and previous objects. The *next* attribute of the tail and the *prev* attribute of the head are NIL, indicated by a diagonal slash. The attribute $L.head$ points to the head. **(b)** Following the execution of LIST-INSERT(L, x), where $x.key = 25$, the linked list has a new object with key 25 as the new head. This new object points to the old head with key 9. **(c)** The result of the subsequent call LIST-DELETE(L, x), where x points to the object with key 4.

elements. In the remainder of this section, we assume that the lists with which we are working are unsorted and doubly linked.

Searching a linked list

The procedure LIST-SEARCH(L, k) finds the first element with key k in list L by a simple linear search, returning a pointer to this element. If no object with key k appears in the list, then the procedure returns NIL. For the linked list in Figure 10.3(a), the call LIST-SEARCH($L, 4$) returns a pointer to the third element, and the call LIST-SEARCH($L, 7$) returns NIL.

LIST-SEARCH(L, k)

1   $x = L.head$
2   **while** $x \neq$ NIL and $x.key \neq k$
3        $x = x.next$
4   **return** x

To search a list of n objects, the LIST-SEARCH procedure takes $\Theta(n)$ time in the worst case, since it may have to search the entire list.

Inserting into a linked list

Given an element x whose *key* attribute has already been set, the LIST-INSERT procedure "splices" x onto the front of the linked list, as shown in Figure 10.3(b).

LIST-INSERT(L, x)

1   $x.next = L.head$
2   **if** $L.head \neq$ NIL
3       $L.head.prev = x$
4   $L.head = x$
5   $x.prev =$ NIL

(Recall that our attribute notation can cascade, so that $L.head.prev$ denotes the *prev* attribute of the object that $L.head$ points to.) The running time for LIST-INSERT on a list of n elements is $O(1)$.

Deleting from a linked list

The procedure LIST-DELETE removes an element x from a linked list L. It must be given a pointer to x, and it then "splices" x out of the list by updating pointers. If we wish to delete an element with a given key, we must first call LIST-SEARCH to retrieve a pointer to the element.

LIST-DELETE(L, x)

1   **if** $x.prev \neq$ NIL
2       $x.prev.next = x.next$
3   **else** $L.head = x.next$
4   **if** $x.next \neq$ NIL
5       $x.next.prev = x.prev$

Figure 10.3(c) shows how an element is deleted from a linked list. LIST-DELETE runs in $O(1)$ time, but if we wish to delete an element with a given key, $\Theta(n)$ time is required in the worst case because we must first call LIST-SEARCH to find the element.

Sentinels

The code for LIST-DELETE would be simpler if we could ignore the boundary conditions at the head and tail of the list:

LIST-DELETE$'(L, x)$

1   $x.prev.next = x.next$
2   $x.next.prev = x.prev$

A *sentinel* is a dummy object that allows us to simplify boundary conditions. For example, suppose that we provide with list L an object $L.nil$ that represents NIL
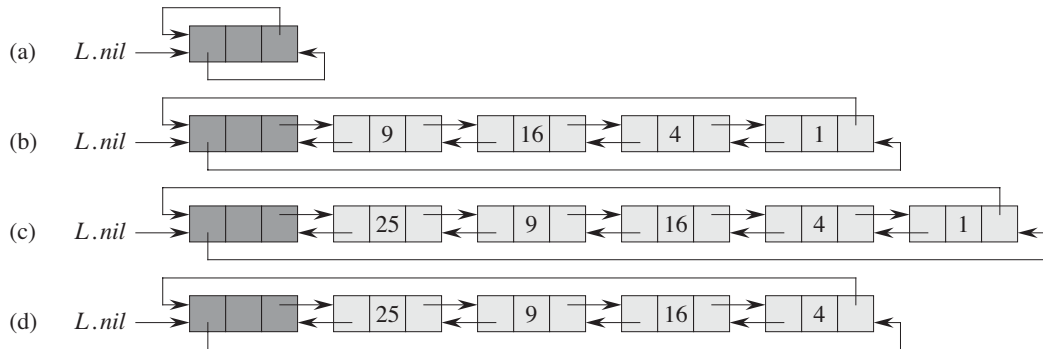
Figure 10.4   A circular, doubly linked list with a sentinel. The sentinel $L.nil$ appears between the head and tail. The attribute $L.head$ is no longer needed, since we can access the head of the list by $L.nil.next$. **(a)** An empty list. **(b)** The linked list from Figure 10.3(a), with key 9 at the head and key 1 at the tail. **(c)** The list after executing LIST-INSERT′(L, x), where $x.key = 25$. The new object becomes the head of the list. **(d)** The list after deleting the object with key 1. The new tail is the object with key 4.

but has all the attributes of the other objects in the list. Wherever we have a reference to NIL in list code, we replace it by a reference to the sentinel $L.nil$. As shown in Figure 10.4, this change turns a regular doubly linked list into a ***circular, doubly linked list with a sentinel***, in which the sentinel $L.nil$ lies between the head and tail. The attribute $L.nil.next$ points to the head of the list, and $L.nil.prev$ points to the tail. Similarly, both the *next* attribute of the tail and the *prev* attribute of the head point to $L.nil$. Since $L.nil.next$ points to the head, we can eliminate the attribute $L.head$ altogether, replacing references to it by references to $L.nil.next$. Figure 10.4(a) shows that an empty list consists of just the sentinel, and both $L.nil.next$ and $L.nil.prev$ point to $L.nil$.

The code for LIST-SEARCH remains the same as before, but with the references to NIL and $L.head$ changed as specified above:

LIST-SEARCH′(L, k)

```
1   x = L.nil.next
2   while x ≠ L.nil and x.key ≠ k
3       x = x.next
4   return x
```

We use the two-line procedure LIST-DELETE′ from before to delete an element from the list. The following procedure inserts an element into the list:

LIST-INSERT$'(L, x)$

1   $x.next = L.nil.next$
2   $L.nil.next.prev = x$
3   $L.nil.next = x$
4   $x.prev = L.nil$

Figure 10.4 shows the effects of LIST-INSERT$'$ and LIST-DELETE$'$ on a sample list.

Sentinels rarely reduce the asymptotic time bounds of data structure operations, but they can reduce constant factors. The gain from using sentinels within loops is usually a matter of clarity of code rather than speed; the linked list code, for example, becomes simpler when we use sentinels, but we save only $O(1)$ time in the LIST-INSERT$'$ and LIST-DELETE$'$ procedures. In other situations, however, the use of sentinels helps to tighten the code in a loop, thus reducing the coefficient of, say, n or n^2 in the running time.

We should use sentinels judiciously. When there are many small lists, the extra storage used by their sentinels can represent significant wasted memory. In this book, we use sentinels only when they truly simplify the code.

Exercises

10.2-1
Can you implement the dynamic-set operation INSERT on a singly linked list in $O(1)$ time? How about DELETE?

10.2-2
Implement a stack using a singly linked list L. The operations PUSH and POP should still take $O(1)$ time.

10.2-3
Implement a queue by a singly linked list L. The operations ENQUEUE and DE-QUEUE should still take $O(1)$ time.

10.2-4
As written, each loop iteration in the LIST-SEARCH$'$ procedure requires two tests: one for $x \neq L.nil$ and one for $x.key \neq k$. Show how to eliminate the test for $x \neq L.nil$ in each iteration.

10.2-5
Implement the dictionary operations INSERT, DELETE, and SEARCH using singly linked, circular lists. What are the running times of your procedures?

10.2-6
The dynamic-set operation UNION takes two disjoint sets S_1 and S_2 as input, and it returns a set $S = S_1 \cup S_2$ consisting of all the elements of S_1 and S_2. The sets S_1 and S_2 are usually destroyed by the operation. Show how to support UNION in $O(1)$ time using a suitable list data structure.

10.2-7
Give a $\Theta(n)$-time nonrecursive procedure that reverses a singly linked list of n elements. The procedure should use no more than constant storage beyond that needed for the list itself.

10.2-8 ★
Explain how to implement doubly linked lists using only one pointer value $x.np$ per item instead of the usual two (*next* and *prev*). Assume that all pointer values can be interpreted as k-bit integers, and define $x.np$ to be $x.np = x.next$ XOR $x.prev$, the k-bit "exclusive-or" of $x.next$ and $x.prev$. (The value NIL is represented by 0.) Be sure to describe what information you need to access the head of the list. Show how to implement the SEARCH, INSERT, and DELETE operations on such a list. Also show how to reverse such a list in $O(1)$ time.

10.3   Implementing pointers and objects

How do we implement pointers and objects in languages that do not provide them? In this section, we shall see two ways of implementing linked data structures without an explicit pointer data type. We shall synthesize objects and pointers from arrays and array indices.

A multiple-array representation of objects

We can represent a collection of objects that have the same attributes by using an array for each attribute. As an example, Figure 10.5 shows how we can implement the linked list of Figure 10.3(a) with three arrays. The array *key* holds the values of the keys currently in the dynamic set, and the pointers reside in the arrays *next* and *prev*. For a given array index x, the array entries *key*[x], *next*[x], and *prev*[x] represent an object in the linked list. Under this interpretation, a pointer x is simply a common index into the *key*, *next*, and *prev* arrays.

In Figure 10.3(a), the object with key 4 follows the object with key 16 in the linked list. In Figure 10.5, key 4 appears in *key*[2], and key 16 appears in *key*[5], and so *next*[5] = 2 and *prev*[2] = 5. Although the constant NIL appears in the *next*
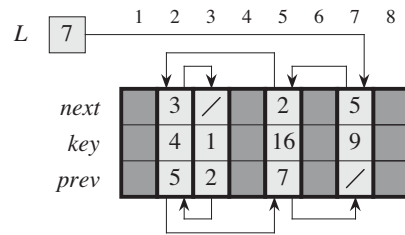
Figure 10.5   The linked list of Figure 10.3(a) represented by the arrays *key*, *next*, and *prev*. Each vertical slice of the arrays represents a single object. Stored pointers correspond to the array indices shown at the top; the arrows show how to interpret them. Lightly shaded object positions contain list elements. The variable L keeps the index of the head.

attribute of the tail and the *prev* attribute of the head, we usually use an integer (such as 0 or -1) that cannot possibly represent an actual index into the arrays. A variable L holds the index of the head of the list.

A single-array representation of objects

The words in a computer memory are typically addressed by integers from 0 to $M - 1$, where M is a suitably large integer. In many programming languages, an object occupies a contiguous set of locations in the computer memory. A pointer is simply the address of the first memory location of the object, and we can address other memory locations within the object by adding an offset to the pointer.

We can use the same strategy for implementing objects in programming environments that do not provide explicit pointer data types. For example, Figure 10.6 shows how to use a single array A to store the linked list from Figures 10.3(a) and 10.5. An object occupies a contiguous subarray $A[j \mathinner{\ldotp\ldotp} k]$. Each attribute of the object corresponds to an offset in the range from 0 to $k - j$, and a pointer to the object is the index j. In Figure 10.6, the offsets corresponding to *key*, *next*, and *prev* are 0, 1, and 2, respectively. To read the value of $i.prev$, given a pointer i, we add the value i of the pointer to the offset 2, thus reading $A[i + 2]$.

The single-array representation is flexible in that it permits objects of different lengths to be stored in the same array. The problem of managing such a heterogeneous collection of objects is more difficult than the problem of managing a homogeneous collection, where all objects have the same attributes. Since most of the data structures we shall consider are composed of homogeneous elements, it will be sufficient for our purposes to use the multiple-array representation of objects.
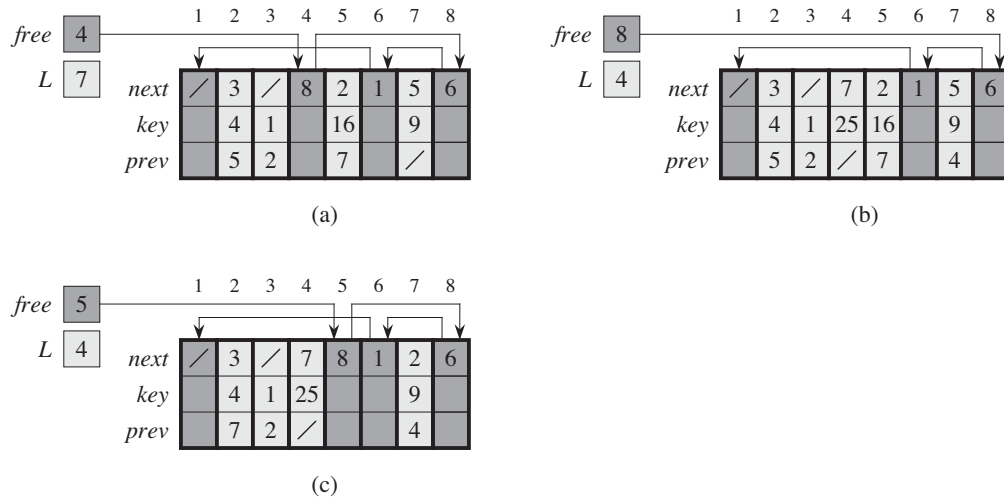
Figure 10.6   The linked list of Figures 10.3(a) and 10.5 represented in a single array A. Each list element is an object that occupies a contiguous subarray of length 3 within the array. The three attributes *key*, *next*, and *prev* correspond to the offsets 0, 1, and 2, respectively, within each object. A pointer to an object is the index of the first element of the object. Objects containing list elements are lightly shaded, and arrows show the list ordering.

Allocating and freeing objects

To insert a key into a dynamic set represented by a doubly linked list, we must allocate a pointer to a currently unused object in the linked-list representation. Thus, it is useful to manage the storage of objects not currently used in the linked-list representation so that one can be allocated. In some systems, a **garbage collector** is responsible for determining which objects are unused. Many applications, however, are simple enough that they can bear responsibility for returning an unused object to a storage manager. We shall now explore the problem of allocating and freeing (or deallocating) homogeneous objects using the example of a doubly linked list represented by multiple arrays.

Suppose that the arrays in the multiple-array representation have length m and that at some moment the dynamic set contains $n \leq m$ elements. Then n objects represent elements currently in the dynamic set, and the remaining $m-n$ objects are **free**; the free objects are available to represent elements inserted into the dynamic set in the future.

We keep the free objects in a singly linked list, which we call the **free list**. The free list uses only the *next* array, which stores the *next* pointers within the list. The head of the free list is held in the global variable *free*. When the dynamic set represented by linked list L is nonempty, the free list may be intertwined with list L, as shown in Figure 10.7. Note that each object in the representation is either in list L or in the free list, but not in both.

The free list acts like a stack: the next object allocated is the last one freed. We can use a list implementation of the stack operations PUSH and POP to implement the procedures for allocating and freeing objects, respectively. We assume that the global variable *free* used in the following procedures points to the first element of the free list.

(a) free = 4, L = 7

	1	2	3	4	5	6	7	8
next	/	3	/	8	2	1	5	6
key		4	1		16		9	
prev		5	2		7		/	

(b) free = 8, L = 4

	1	2	3	4	5	6	7	8
next	/	3	/	7	2	1	5	6
key		4	1	25	16		9	
prev		5	2	/	7		4	

(c) free = 5, L = 4

	1	2	3	4	5	6	7	8
next	/	3	/	7	8	1	2	6
key		4	1	25			9	
prev		7	2	/			4	

Figure 10.7 The effect of the ALLOCATE-OBJECT and FREE-OBJECT procedures. **(a)** The list of Figure 10.5 (lightly shaded) and a free list (heavily shaded). Arrows show the free-list structure. **(b)** The result of calling ALLOCATE-OBJECT() (which returns index 4), setting *key*[4] to 25, and calling LIST-INSERT(L, 4). The new free-list head is object 8, which had been *next*[4] on the free list. **(c)** After executing LIST-DELETE(L, 5), we call FREE-OBJECT(5). Object 5 becomes the new free-list head, with object 8 following it on the free list.

ALLOCATE-OBJECT()

```
1   if free == NIL
2        error "out of space"
3   else x = free
4        free = x.next
5        return x
```

FREE-OBJECT(x)

```
1   x.next = free
2   free = x
```

The free list initially contains all n unallocated objects. Once the free list has been exhausted, running the ALLOCATE-OBJECT procedure signals an error. We can even service several linked lists with just a single free list. Figure 10.8 shows two linked lists and a free list intertwined through *key*, *next*, and *prev* arrays.

The two procedures run in $O(1)$ time, which makes them quite practical. We can modify them to work for any homogeneous collection of objects by letting any one of the attributes in the object act like a *next* attribute in the free list.
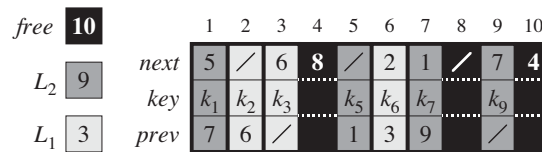
Figure 10.8   Two linked lists, L_1 (lightly shaded) and L_2 (heavily shaded), and a free list (darkened) intertwined.

Exercises

10.3-1
Draw a picture of the sequence $\langle 13, 4, 8, 19, 5, 11 \rangle$ stored as a doubly linked list using the multiple-array representation. Do the same for the single-array representation.

10.3-2
Write the procedures ALLOCATE-OBJECT and FREE-OBJECT for a homogeneous collection of objects implemented by the single-array representation.

10.3-3
Why don't we need to set or reset the *prev* attributes of objects in the implementation of the ALLOCATE-OBJECT and FREE-OBJECT procedures?

10.3-4
It is often desirable to keep all elements of a doubly linked list compact in storage, using, for example, the first m index locations in the multiple-array representation. (This is the case in a paged, virtual-memory computing environment.) Explain how to implement the procedures ALLOCATE-OBJECT and FREE-OBJECT so that the representation is compact. Assume that there are no pointers to elements of the linked list outside the list itself. (*Hint:* Use the array implementation of a stack.)

10.3-5
Let L be a doubly linked list of length n stored in arrays *key*, *prev*, and *next* of length m. Suppose that these arrays are managed by ALLOCATE-OBJECT and FREE-OBJECT procedures that keep a doubly linked free list F. Suppose further that of the m items, exactly n are on list L and $m - n$ are on the free list. Write a procedure COMPACTIFY-LIST(L, F) that, given the list L and the free list F, moves the items in L so that they occupy array positions $1, 2, \ldots, n$ and adjusts the free list F so that it remains correct, occupying array positions $n + 1, n + 2, \ldots, m$. The running time of your procedure should be $\Theta(n)$, and it should use only a constant amount of extra space. Argue that your procedure is correct.

10.4   Representing rooted trees

The methods for representing lists given in the previous section extend to any homogeneous data structure. In this section, we look specifically at the problem of representing rooted trees by linked data structures. We first look at binary trees, and then we present a method for rooted trees in which nodes can have an arbitrary number of children.

We represent each node of a tree by an object. As with linked lists, we assume that each node contains a *key* attribute. The remaining attributes of interest are pointers to other nodes, and they vary according to the type of tree.

Binary trees

Figure 10.9 shows how we use the attributes p, *left*, and *right* to store pointers to the parent, left child, and right child of each node in a binary tree T. If $x.p = \text{NIL}$, then x is the root. If node x has no left child, then $x.left = \text{NIL}$, and similarly for the right child. The root of the entire tree T is pointed to by the attribute $T.root$. If $T.root = \text{NIL}$, then the tree is empty.

Rooted trees with unbounded branching

We can extend the scheme for representing a binary tree to any class of trees in which the number of children of each node is at most some constant k: we replace the *left* and *right* attributes by $child_1, child_2, \ldots, child_k$. This scheme no longer works when the number of children of a node is unbounded, since we do not know how many attributes (arrays in the multiple-array representation) to allocate in advance. Moreover, even if the number of children k is bounded by a large constant but most nodes have a small number of children, we may waste a lot of memory.

Fortunately, there is a clever scheme to represent trees with arbitrary numbers of children. It has the advantage of using only $O(n)$ space for any n-node rooted tree. The **left-child, right-sibling representation** appears in Figure 10.10. As before, each node contains a parent pointer p, and $T.root$ points to the root of tree T. Instead of having a pointer to each of its children, however, each node x has only two pointers:

1. $x.left\text{-}child$ points to the leftmost child of node x, and

2. $x.right\text{-}sibling$ points to the sibling of x immediately to its right.

If node x has no children, then $x.left\text{-}child = \text{NIL}$, and if node x is the rightmost child of its parent, then $x.right\text{-}sibling = \text{NIL}$.
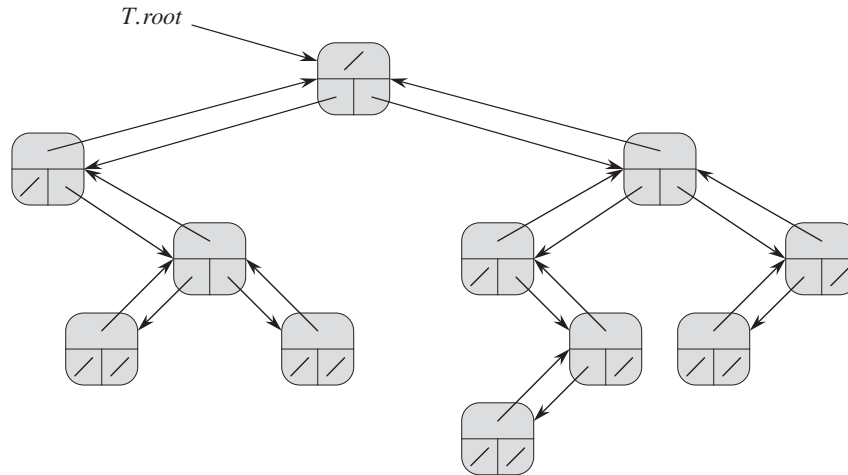
Figure 10.9   The representation of a binary tree T. Each node x has the attributes $x.p$ (top), $x.left$ (lower left), and $x.right$ (lower right). The *key* attributes are not shown.
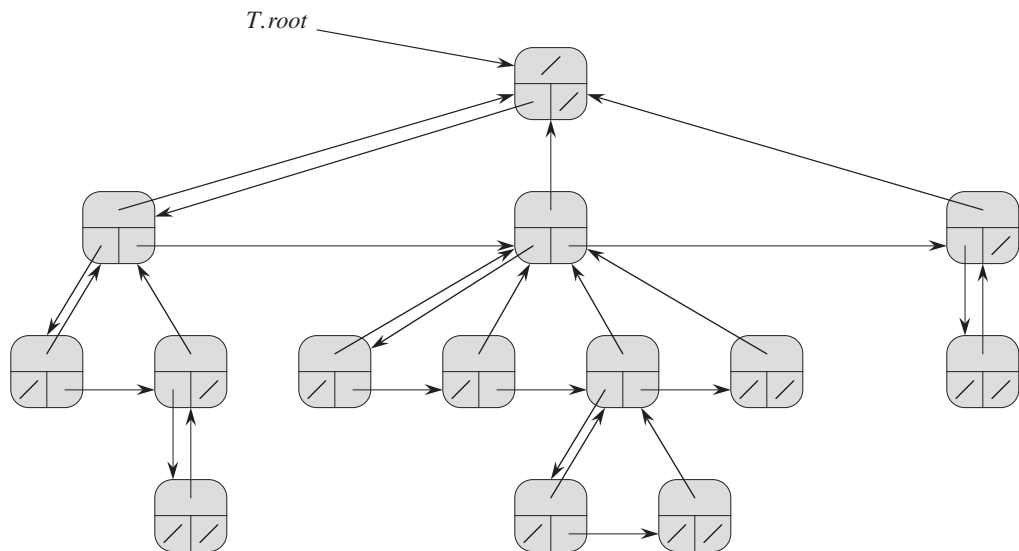

Figure 10.10   The left-child, right-sibling representation of a tree T. Each node x has attributes $x.p$ (top), $x.left\text{-}child$ (lower left), and $x.right\text{-}sibling$ (lower right). The *key* attributes are not shown.

Other tree representations

We sometimes represent rooted trees in other ways. In Chapter 6, for example, we represented a heap, which is based on a complete binary tree, by a single array plus the index of the last node in the heap. The trees that appear in Chapter 21 are traversed only toward the root, and so only the parent pointers are present; there are no pointers to children. Many other schemes are possible. Which scheme is best depends on the application.

Exercises

10.4-1
Draw the binary tree rooted at index 6 that is represented by the following attributes:

index	key	left	right
1	12	7	3
2	15	8	NIL
3	4	10	NIL
4	10	5	9
5	2	NIL	NIL
6	18	1	4
7	7	NIL	NIL
8	14	6	2
9	21	NIL	NIL
10	5	NIL	NIL

10.4-2
Write an $O(n)$-time recursive procedure that, given an n-node binary tree, prints out the key of each node in the tree.

10.4-3
Write an $O(n)$-time nonrecursive procedure that, given an n-node binary tree, prints out the key of each node in the tree. Use a stack as an auxiliary data structure.

10.4-4
Write an $O(n)$-time procedure that prints all the keys of an arbitrary rooted tree with n nodes, where the tree is stored using the left-child, right-sibling representation.

10.4-5 ★
Write an $O(n)$-time nonrecursive procedure that, given an n-node binary tree, prints out the key of each node. Use no more than constant extra space outside

of the tree itself and do not modify the tree, even temporarily, during the procedure.

10.4-6 ★

The left-child, right-sibling representation of an arbitrary rooted tree uses three pointers in each node: *left-child*, *right-sibling*, and *parent*. From any node, its parent can be reached and identified in constant time and all its children can be reached and identified in time linear in the number of children. Show how to use only two pointers and one boolean value in each node so that the parent of a node or all of its children can be reached and identified in time linear in the number of children.

Problems

10-1 *Comparisons among lists*

For each of the four types of lists in the following table, what is the asymptotic worst-case running time for each dynamic-set operation listed?

	unsorted, singly linked	sorted, singly linked	unsorted, doubly linked	sorted, doubly linked
SEARCH(L, k)				
INSERT(L, x)				
DELETE(L, x)				
SUCCESSOR(L, x)				
PREDECESSOR(L, x)				
MINIMUM(L)				
MAXIMUM(L)				

10-2   Mergeable heaps using linked lists

A ***mergeable heap*** supports the following operations: MAKE-HEAP (which creates an empty mergeable heap), INSERT, MINIMUM, EXTRACT-MIN, and UNION.[1] Show how to implement mergeable heaps using linked lists in each of the following cases. Try to make each operation as efficient as possible. Analyze the running time of each operation in terms of the size of the dynamic set(s) being operated on.

a. Lists are sorted.

b. Lists are unsorted.

c. Lists are unsorted, and dynamic sets to be merged are disjoint.

10-3   Searching a sorted compact list

Exercise 10.3-4 asked how we might maintain an n-element list compactly in the first n positions of an array. We shall assume that all keys are distinct and that the compact list is also sorted, that is, $key[i] < key[next[i]]$ for all $i = 1, 2, \ldots, n$ such that $next[i] \neq$ NIL. We will also assume that we have a variable L that contains the index of the first element on the list. Under these assumptions, you will show that we can use the following randomized algorithm to search the list in $O(\sqrt{n})$ expected time.

COMPACT-LIST-SEARCH(L, n, k)

```
 1  i = L
 2  while i ≠ NIL and key[i] < k
 3       j = RANDOM(1, n)
 4       if key[i] < key[j] and key[j] ≤ k
 5            i = j
 6            if key[i] == k
 7                 return i
 8       i = next[i]
 9  if i == NIL or key[i] > k
10       return NIL
11  else return i
```

If we ignore lines 3–7 of the procedure, we have an ordinary algorithm for searching a sorted linked list, in which index i points to each position of the list in

---

[1] Because we have defined a mergeable heap to support MINIMUM and EXTRACT-MIN, we can also refer to it as a ***mergeable min-heap***. Alternatively, if it supported MAXIMUM and EXTRACT-MAX, it would be a ***mergeable max-heap***.

turn. The search terminates once the index i "falls off" the end of the list or once $key[i] \geq k$. In the latter case, if $key[i] = k$, clearly we have found a key with the value k. If, however, $key[i] > k$, then we will never find a key with the value k, and so terminating the search was the right thing to do.

Lines 3–7 attempt to skip ahead to a randomly chosen position j. Such a skip benefits us if $key[j]$ is larger than $key[i]$ and no larger than k; in such a case, j marks a position in the list that i would have to reach during an ordinary list search. Because the list is compact, we know that any choice of j between 1 and n indexes some object in the list rather than a slot on the free list.

Instead of analyzing the performance of COMPACT-LIST-SEARCH directly, we shall analyze a related algorithm, COMPACT-LIST-SEARCH$'$, which executes two separate loops. This algorithm takes an additional parameter t which determines an upper bound on the number of iterations of the first loop.

COMPACT-LIST-SEARCH$'(L, n, k, t)$

```
 1   i = L
 2   for q = 1 to t
 3       j = RANDOM(1, n)
 4       if key[i] < key[j] and key[j] ≤ k
 5           i = j
 6           if key[i] == k
 7               return i
 8   while i ≠ NIL and key[i] < k
 9       i = next[i]
10   if i == NIL or key[i] > k
11       return NIL
12   else return i
```

To compare the execution of the algorithms COMPACT-LIST-SEARCH(L, n, k) and COMPACT-LIST-SEARCH$'(L, n, k, t)$, assume that the sequence of integers returned by the calls of RANDOM$(1, n)$ is the same for both algorithms.

a. Suppose that COMPACT-LIST-SEARCH(L, n, k) takes t iterations of the **while** loop of lines 2–8. Argue that COMPACT-LIST-SEARCH$'(L, n, k, t)$ returns the same answer and that the total number of iterations of both the **for** and **while** loops within COMPACT-LIST-SEARCH$'$ is at least t.

In the call COMPACT-LIST-SEARCH$'(L, n, k, t)$, let X_t be the random variable that describes the distance in the linked list (that is, through the chain of *next* pointers) from position i to the desired key k after t iterations of the **for** loop of lines 2–7 have occurred.

b. Argue that the expected running time of COMPACT-LIST-SEARCH$'(L, n, k, t)$ is $O(t + \mathrm{E}[X_t])$.

c. Show that $\mathrm{E}[X_t] \leq \sum_{r=1}^{n}(1 - r/n)^t$. (*Hint:* Use equation (C.25).)

d. Show that $\sum_{r=0}^{n-1} r^t \leq n^{t+1}/(t + 1)$.

e. Prove that $\mathrm{E}[X_t] \leq n/(t + 1)$.

f. Show that COMPACT-LIST-SEARCH$'(L, n, k, t)$ runs in $O(t + n/t)$ expected time.

g. Conclude that COMPACT-LIST-SEARCH runs in $O(\sqrt{n})$ expected time.

h. Why do we assume that all keys are distinct in COMPACT-LIST-SEARCH? Argue that random skips do not necessarily help asymptotically when the list contains repeated key values.

Chapter notes

Aho, Hopcroft, and Ullman [6] and Knuth [209] are excellent references for elementary data structures. Many other texts cover both basic data structures and their implementation in a particular programming language. Examples of these types of textbooks include Goodrich and Tamassia [147], Main [241], Shaffer [311], and Weiss [352, 353, 354]. Gonnet [145] provides experimental data on the performance of many data-structure operations.

The origin of stacks and queues as data structures in computer science is unclear, since corresponding notions already existed in mathematics and paper-based business practices before the introduction of digital computers. Knuth [209] cites A. M. Turing for the development of stacks for subroutine linkage in 1947.

Pointer-based data structures also seem to be a folk invention. According to Knuth, pointers were apparently used in early computers with drum memories. The A-1 language developed by G. M. Hopper in 1951 represented algebraic formulas as binary trees. Knuth credits the IPL-II language, developed in 1956 by A. Newell, J. C. Shaw, and H. A. Simon, for recognizing the importance and promoting the use of pointers. Their IPL-III language, developed in 1957, included explicit stack operations.

11   Hash Tables

Many applications require a dynamic set that supports only the dictionary operations INSERT, SEARCH, and DELETE. For example, a compiler that translates a programming language maintains a symbol table, in which the keys of elements are arbitrary character strings corresponding to identifiers in the language. A hash table is an effective data structure for implementing dictionaries. Although searching for an element in a hash table can take as long as searching for an element in a linked list—$\Theta(n)$ time in the worst case—in practice, hashing performs extremely well. Under reasonable assumptions, the average time to search for an element in a hash table is $O(1)$.

A hash table generalizes the simpler notion of an ordinary array. Directly addressing into an ordinary array makes effective use of our ability to examine an arbitrary position in an array in $O(1)$ time. Section 11.1 discusses direct addressing in more detail. We can take advantage of direct addressing when we can afford to allocate an array that has one position for every possible key.

When the number of keys actually stored is small relative to the total number of possible keys, hash tables become an effective alternative to directly addressing an array, since a hash table typically uses an array of size proportional to the number of keys actually stored. Instead of using the key as an array index directly, the array index is *computed* from the key. Section 11.2 presents the main ideas, focusing on "chaining" as a way to handle "collisions," in which more than one key maps to the same array index. Section 11.3 describes how we can compute array indices from keys using hash functions. We present and analyze several variations on the basic theme. Section 11.4 looks at "open addressing," which is another way to deal with collisions. The bottom line is that hashing is an extremely effective and practical technique: the basic dictionary operations require only $O(1)$ time on the average. Section 11.5 explains how "perfect hashing" can support searches in $O(1)$ *worst-case* time, when the set of keys being stored is static (that is, when the set of keys never changes once stored).

11.1   Direct-address tables

Direct addressing is a simple technique that works well when the universe U of keys is reasonably small. Suppose that an application needs a dynamic set in which each element has a key drawn from the universe $U = \{0, 1, \ldots, m-1\}$, where m is not too large. We shall assume that no two elements have the same key.

To represent the dynamic set, we use an array, or ***direct-address table***, denoted by $T[0 .. m-1]$, in which each position, or ***slot***, corresponds to a key in the universe U. Figure 11.1 illustrates the approach; slot k points to an element in the set with key k. If the set contains no element with key k, then $T[k] = \text{NIL}$.

The dictionary operations are trivial to implement:

DIRECT-ADDRESS-SEARCH(T, k)

1   **return** $T[k]$

DIRECT-ADDRESS-INSERT(T, x)

1   $T[x.key] = x$

DIRECT-ADDRESS-DELETE(T, x)

1   $T[x.key] = \text{NIL}$

Each of these operations takes only $O(1)$ time.
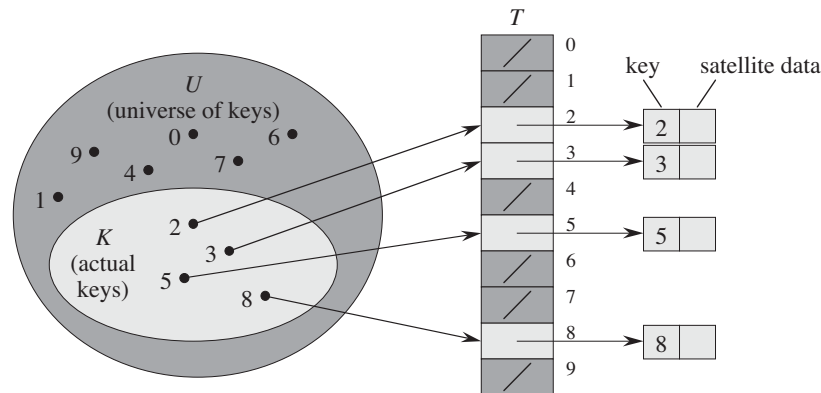
Figure 11.1   How to implement a dynamic set by a direct-address table T. Each key in the universe $U = \{0, 1, \ldots, 9\}$ corresponds to an index in the table. The set $K = \{2, 3, 5, 8\}$ of actual keys determines the slots in the table that contain pointers to elements. The other slots, heavily shaded, contain NIL.

For some applications, the direct-address table itself can hold the elements in the dynamic set. That is, rather than storing an element's key and satellite data in an object external to the direct-address table, with a pointer from a slot in the table to the object, we can store the object in the slot itself, thus saving space. We would use a special key within an object to indicate an empty slot. Moreover, it is often unnecessary to store the key of the object, since if we have the index of an object in the table, we have its key. If keys are not stored, however, we must have some way to tell whether the slot is empty.

Exercises

11.1-1
Suppose that a dynamic set S is represented by a direct-address table T of length m. Describe a procedure that finds the maximum element of S. What is the worst-case performance of your procedure?

11.1-2
A ***bit vector*** is simply an array of bits (0s and 1s). A bit vector of length m takes much less space than an array of m pointers. Describe how to use a bit vector to represent a dynamic set of distinct elements with no satellite data. Dictionary operations should run in $O(1)$ time.

11.1-3
Suggest how to implement a direct-address table in which the keys of stored elements do not need to be distinct and the elements can have satellite data. All three dictionary operations (INSERT, DELETE, and SEARCH) should run in $O(1)$ time. (Don't forget that DELETE takes as an argument a pointer to an object to be deleted, not a key.)

11.1-4 ★
We wish to implement a dictionary by using direct addressing on a *huge* array. At the start, the array entries may contain garbage, and initializing the entire array is impractical because of its size. Describe a scheme for implementing a direct-address dictionary on a huge array. Each stored object should use $O(1)$ space; the operations SEARCH, INSERT, and DELETE should take $O(1)$ time each; and initializing the data structure should take $O(1)$ time. (*Hint:* Use an additional array, treated somewhat like a stack whose size is the number of keys actually stored in the dictionary, to help determine whether a given entry in the huge array is valid or not.)

11.2   Hash tables

The downside of direct addressing is obvious: if the universe U is large, storing a table T of size $|U|$ may be impractical, or even impossible, given the memory available on a typical computer. Furthermore, the set K of keys *actually stored* may be so small relative to U that most of the space allocated for T would be wasted.

When the set K of keys stored in a dictionary is much smaller than the universe U of all possible keys, a hash table requires much less storage than a direct-address table. Specifically, we can reduce the storage requirement to $\Theta(|K|)$ while we maintain the benefit that searching for an element in the hash table still requires only $O(1)$ time. The catch is that this bound is for the *average-case time*, whereas for direct addressing it holds for the *worst-case time*.

With direct addressing, an element with key k is stored in slot k. With hashing, this element is stored in slot $h(k)$; that is, we use a ***hash function*** h to compute the slot from the key k. Here, h maps the universe U of keys into the slots of a ***hash table*** $T[0 \ldots m-1]$:

$$h : U \to \{0, 1, \ldots, m-1\} \ ,$$

where the size m of the hash table is typically much less than $|U|$. We say that an element with key k ***hashes*** to slot $h(k)$; we also say that $h(k)$ is the ***hash value*** of key k. Figure 11.2 illustrates the basic idea. The hash function reduces the range of array indices and hence the size of the array. Instead of a size of $|U|$, the array can have size m.
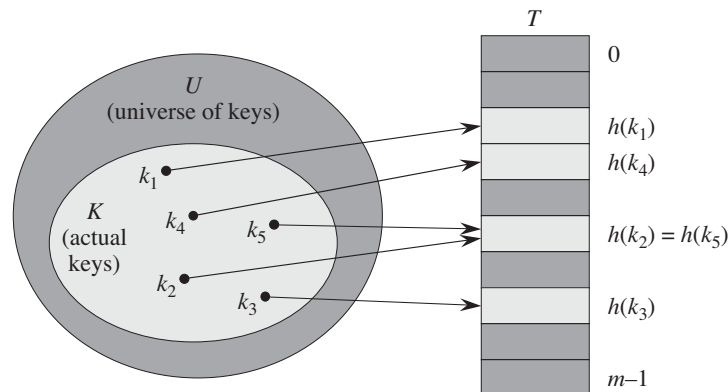
Figure 11.2   Using a hash function h to map keys to hash-table slots. Because keys k_2 and k_5 map to the same slot, they collide.
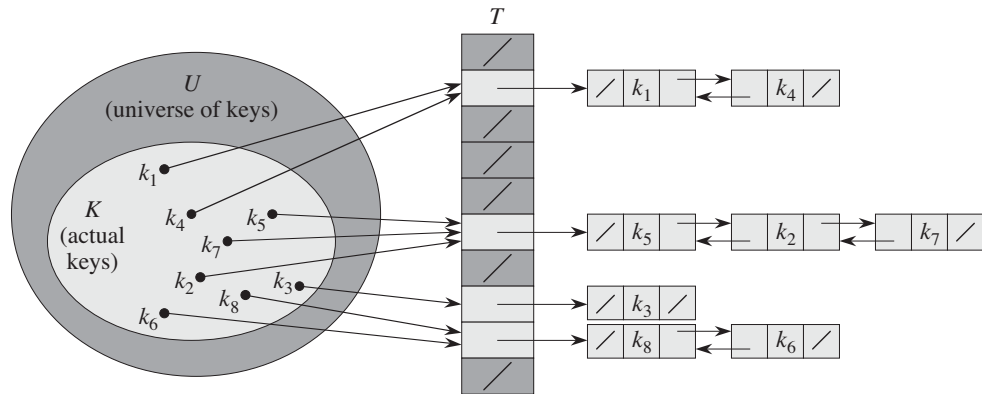
Figure 11.3 Collision resolution by chaining. Each hash-table slot $T[j]$ contains a linked list of all the keys whose hash value is j. For example, $h(k_1) = h(k_4)$ and $h(k_5) = h(k_7) = h(k_2)$. The linked list can be either singly or doubly linked; we show it as doubly linked because deletion is faster that way.

There is one hitch: two keys may hash to the same slot. We call this situation a ***collision***. Fortunately, we have effective techniques for resolving the conflict created by collisions.

Of course, the ideal solution would be to avoid collisions altogether. We might try to achieve this goal by choosing a suitable hash function h. One idea is to make h appear to be "random," thus avoiding collisions or at least minimizing their number. The very term "to hash," evoking images of random mixing and chopping, captures the spirit of this approach. (Of course, a hash function h must be deterministic in that a given input k should always produce the same output $h(k)$.) Because $|U| > m$, however, there must be at least two keys that have the same hash value; avoiding collisions altogether is therefore impossible. Thus, while a well-designed, "random"-looking hash function can minimize the number of collisions, we still need a method for resolving the collisions that do occur.

The remainder of this section presents the simplest collision resolution technique, called chaining. Section 11.4 introduces an alternative method for resolving collisions, called open addressing.

Collision resolution by chaining

In ***chaining***, we place all the elements that hash to the same slot into the same linked list, as Figure 11.3 shows. Slot j contains a pointer to the head of the list of all stored elements that hash to j; if there are no such elements, slot j contains NIL.

The dictionary operations on a hash table T are easy to implement when collisions are resolved by chaining:

CHAINED-HASH-INSERT(T, x)

1   insert x at the head of list $T[h(x.key)]$

CHAINED-HASH-SEARCH(T, k)

1   search for an element with key k in list $T[h(k)]$

CHAINED-HASH-DELETE(T, x)

1   delete x from the list $T[h(x.key)]$

The worst-case running time for insertion is $O(1)$. The insertion procedure is fast in part because it assumes that the element x being inserted is not already present in the table; if necessary, we can check this assumption (at additional cost) by searching for an element whose key is $x.key$ before we insert. For searching, the worst-case running time is proportional to the length of the list; we shall analyze this operation more closely below. We can delete an element in $O(1)$ time if the lists are doubly linked, as Figure 11.3 depicts. (Note that CHAINED-HASH-DELETE takes as input an element x and not its key k, so that we don't have to search for x first. If the hash table supports deletion, then its linked lists should be doubly linked so that we can delete an item quickly. If the lists were only singly linked, then to delete element x, we would first have to find x in the list $T[h(x.key)]$ so that we could update the *next* attribute of x's predecessor. With singly linked lists, both deletion and searching would have the same asymptotic running times.)

Analysis of hashing with chaining

How well does hashing with chaining perform? In particular, how long does it take to search for an element with a given key?

Given a hash table T with m slots that stores n elements, we define the **load factor** α for T as n/m, that is, the average number of elements stored in a chain. Our analysis will be in terms of α, which can be less than, equal to, or greater than 1.

The worst-case behavior of hashing with chaining is terrible: all n keys hash to the same slot, creating a list of length n. The worst-case time for searching is thus $\Theta(n)$ plus the time to compute the hash function—no better than if we used one linked list for all the elements. Clearly, we do not use hash tables for their worst-case performance. (Perfect hashing, described in Section 11.5, does provide good worst-case performance when the set of keys is static, however.)

The average-case performance of hashing depends on how well the hash function h distributes the set of keys to be stored among the m slots, on the average.

Section 11.3 discusses these issues, but for now we shall assume that any given element is equally likely to hash into any of the m slots, independently of where any other element has hashed to. We call this the assumption of **simple uniform hashing**.

For $j = 0, 1, \ldots, m - 1$, let us denote the length of the list $T[j]$ by n_j, so that

$$n = n_0 + n_1 + \cdots + n_{m-1} , \tag{11.1}$$

and the expected value of n_j is $\mathrm{E}[n_j] = \alpha = n/m$.

We assume that $O(1)$ time suffices to compute the hash value $h(k)$, so that the time required to search for an element with key k depends linearly on the length $n_{h(k)}$ of the list $T[h(k)]$. Setting aside the $O(1)$ time required to compute the hash function and to access slot $h(k)$, let us consider the expected number of elements examined by the search algorithm, that is, the number of elements in the list $T[h(k)]$ that the algorithm checks to see whether any have a key equal to k. We shall consider two cases. In the first, the search is unsuccessful: no element in the table has key k. In the second, the search successfully finds an element with key k.

Theorem 11.1
In a hash table in which collisions are resolved by chaining, an unsuccessful search takes average-case time $\Theta(1+\alpha)$, under the assumption of simple uniform hashing.

Proof Under the assumption of simple uniform hashing, any key k not already stored in the table is equally likely to hash to any of the m slots. The expected time to search unsuccessfully for a key k is the expected time to search to the end of list $T[h(k)]$, which has expected length $\mathrm{E}[n_{h(k)}] = \alpha$. Thus, the expected number of elements examined in an unsuccessful search is α, and the total time required (including the time for computing $h(k)$) is $\Theta(1 + \alpha)$. ∎

The situation for a successful search is slightly different, since each list is not equally likely to be searched. Instead, the probability that a list is searched is proportional to the number of elements it contains. Nonetheless, the expected search time still turns out to be $\Theta(1 + \alpha)$.

Theorem 11.2
In a hash table in which collisions are resolved by chaining, a successful search takes average-case time $\Theta(1+\alpha)$, under the assumption of simple uniform hashing.

Proof We assume that the element being searched for is equally likely to be any of the n elements stored in the table. The number of elements examined during a successful search for an element x is one more than the number of elements that

appear before x in x's list. Because new elements are placed at the front of the list, elements before x in the list were all inserted after x was inserted. To find the expected number of elements examined, we take the average, over the n elements x in the table, of 1 plus the expected number of elements added to x's list after x was added to the list. Let x_i denote the ith element inserted into the table, for $i = 1, 2, \ldots, n$, and let $k_i = x_i.key$. For keys k_i and k_j, we define the indicator random variable $X_{ij} = I\{h(k_i) = h(k_j)\}$. Under the assumption of simple uniform hashing, we have $\Pr\{h(k_i) = h(k_j)\} = 1/m$, and so by Lemma 5.1, $E[X_{ij}] = 1/m$. Thus, the expected number of elements examined in a successful search is

$$
\begin{aligned}
E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n} X_{ij}\right)\right] & \\
= \frac{1}{n}\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n} E[X_{ij}]\right) & \quad \text{(by linearity of expectation)} \\
= \frac{1}{n}\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n} \frac{1}{m}\right) & \\
= 1 + \frac{1}{nm}\sum_{i=1}^{n}(n - i) & \\
= 1 + \frac{1}{nm}\left(\sum_{i=1}^{n} n - \sum_{i=1}^{n} i\right) & \\
= 1 + \frac{1}{nm}\left(n^2 - \frac{n(n+1)}{2}\right) & \quad \text{(by equation (A.1))} \\
= 1 + \frac{n-1}{2m} & \\
= 1 + \frac{\alpha}{2} - \frac{\alpha}{2n} \, . &
\end{aligned}
$$

Thus, the total time required for a successful search (including the time for computing the hash function) is $\Theta(2 + \alpha/2 - \alpha/2n) = \Theta(1 + \alpha)$. ∎

What does this analysis mean? If the number of hash-table slots is at least proportional to the number of elements in the table, we have $n = O(m)$ and, consequently, $\alpha = n/m = O(m)/m = O(1)$. Thus, searching takes constant time on average. Since insertion takes $O(1)$ worst-case time and deletion takes $O(1)$ worst-case time when the lists are doubly linked, we can support all dictionary operations in $O(1)$ time on average.

Exercises

11.2-1

Suppose we use a hash function h to hash n distinct keys into an array T of length m. Assuming simple uniform hashing, what is the expected number of collisions? More precisely, what is the expected cardinality of $\{\{k, l\} : k \neq l$ and $h(k) = h(l)\}$?

11.2-2

Demonstrate what happens when we insert the keys $5, 28, 19, 15, 20, 33, 12, 17, 10$ into a hash table with collisions resolved by chaining. Let the table have 9 slots, and let the hash function be $h(k) = k \bmod 9$.

11.2-3

Professor Marley hypothesizes that he can obtain substantial performance gains by modifying the chaining scheme to keep each list in sorted order. How does the professor's modification affect the running time for successful searches, unsuccessful searches, insertions, and deletions?

11.2-4

Suggest how to allocate and deallocate storage for elements within the hash table itself by linking all unused slots into a free list. Assume that one slot can store a flag and either one element plus a pointer or two pointers. All dictionary and free-list operations should run in $O(1)$ expected time. Does the free list need to be doubly linked, or does a singly linked free list suffice?

11.2-5

Suppose that we are storing a set of n keys into a hash table of size m. Show that if the keys are drawn from a universe U with $|U| > nm$, then U has a subset of size n consisting of keys that all hash to the same slot, so that the worst-case searching time for hashing with chaining is $\Theta(n)$.

11.2-6

Suppose we have stored n keys in a hash table of size m, with collisions resolved by chaining, and that we know the length of each chain, including the length L of the longest chain. Describe a procedure that selects a key uniformly at random from among the keys in the hash table and returns it in expected time $O(L \cdot (1 + 1/\alpha))$.

11.3   Hash functions

In this section, we discuss some issues regarding the design of good hash functions and then present three schemes for their creation. Two of the schemes, hashing by division and hashing by multiplication, are heuristic in nature, whereas the third scheme, universal hashing, uses randomization to provide provably good performance.

What makes a good hash function?

A good hash function satisfies (approximately) the assumption of simple uniform hashing: each key is equally likely to hash to any of the m slots, independently of where any other key has hashed to. Unfortunately, we typically have no way to check this condition, since we rarely know the probability distribution from which the keys are drawn. Moreover, the keys might not be drawn independently.

Occasionally we do know the distribution. For example, if we know that the keys are random real numbers k independently and uniformly distributed in the range $0 \leq k < 1$, then the hash function

$$h(k) = \lfloor km \rfloor$$

satisfies the condition of simple uniform hashing.

In practice, we can often employ heuristic techniques to create a hash function that performs well. Qualitative information about the distribution of keys may be useful in this design process. For example, consider a compiler's symbol table, in which the keys are character strings representing identifiers in a program. Closely related symbols, such as `pt` and `pts`, often occur in the same program. A good hash function would minimize the chance that such variants hash to the same slot.

A good approach derives the hash value in a way that we expect to be independent of any patterns that might exist in the data. For example, the "division method" (discussed in Section 11.3.1) computes the hash value as the remainder when the key is divided by a specified prime number. This method frequently gives good results, assuming that we choose a prime number that is unrelated to any patterns in the distribution of keys.

Finally, we note that some applications of hash functions might require stronger properties than are provided by simple uniform hashing. For example, we might want keys that are "close" in some sense to yield hash values that are far apart. (This property is especially desirable when we are using linear probing, defined in Section 11.4.) Universal hashing, described in Section 11.3.3, often provides the desired properties.

Interpreting keys as natural numbers

Most hash functions assume that the universe of keys is the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers. Thus, if the keys are not natural numbers, we find a way to interpret them as natural numbers. For example, we can interpret a character string as an integer expressed in suitable radix notation. Thus, we might interpret the identifier `pt` as the pair of decimal integers $(112, 116)$, since $\mathtt{p} = 112$ and $\mathtt{t} = 116$ in the ASCII character set; then, expressed as a radix-128 integer, `pt` becomes $(112 \cdot 128) + 116 = 14452$. In the context of a given application, we can usually devise some such method for interpreting each key as a (possibly large) natural number. In what follows, we assume that the keys are natural numbers.

11.3.1 The division method

In the ***division method*** for creating hash functions, we map a key k into one of m slots by taking the remainder of k divided by m. That is, the hash function is

$$h(k) = k \bmod m \ .$$

For example, if the hash table has size $m = 12$ and the key is $k = 100$, then $h(k) = 4$. Since it requires only a single division operation, hashing by division is quite fast.

When using the division method, we usually avoid certain values of m. For example, m should not be a power of 2, since if $m = 2^p$, then $h(k)$ is just the p lowest-order bits of k. Unless we know that all low-order p-bit patterns are equally likely, we are better off designing the hash function to depend on all the bits of the key. As Exercise 11.3-3 asks you to show, choosing $m = 2^p - 1$ when k is a character string interpreted in radix 2^p may be a poor choice, because permuting the characters of k does not change its hash value.

A prime not too close to an exact power of 2 is often a good choice for m. For example, suppose we wish to allocate a hash table, with collisions resolved by chaining, to hold roughly $n = 2000$ character strings, where a character has 8 bits. We don't mind examining an average of 3 elements in an unsuccessful search, and so we allocate a hash table of size $m = 701$. We could choose $m = 701$ because it is a prime near $2000/3$ but not near any power of 2. Treating each key k as an integer, our hash function would be

$$h(k) = k \bmod 701 \ .$$

11.3.2 The multiplication method

The ***multiplication method*** for creating hash functions operates in two steps. First, we multiply the key k by a constant A in the range $0 < A < 1$ and extract the
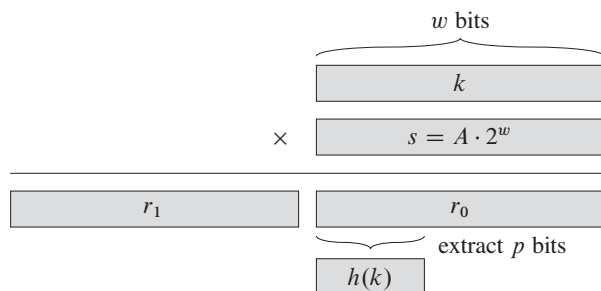
Figure 11.4 The multiplication method of hashing. The w-bit representation of the key k is multiplied by the w-bit value $s = A \cdot 2^w$. The p highest-order bits of the lower w-bit half of the product form the desired hash value $h(k)$.

fractional part of kA. Then, we multiply this value by m and take the floor of the result. In short, the hash function is

$$h(k) = \lfloor m\,(kA \bmod 1) \rfloor \;,$$

where "$kA \bmod 1$" means the fractional part of kA, that is, $kA - \lfloor kA \rfloor$.

An advantage of the multiplication method is that the value of m is not critical. We typically choose it to be a power of 2 ($m = 2^p$ for some integer p), since we can then easily implement the function on most computers as follows. Suppose that the word size of the machine is w bits and that k fits into a single word. We restrict A to be a fraction of the form $s/2^w$, where s is an integer in the range $0 < s < 2^w$. Referring to Figure 11.4, we first multiply k by the w-bit integer $s = A \cdot 2^w$. The result is a $2w$-bit value $r_1 2^w + r_0$, where r_1 is the high-order word of the product and r_0 is the low-order word of the product. The desired p-bit hash value consists of the p most significant bits of r_0.

Although this method works with any value of the constant A, it works better with some values than with others. The optimal choice depends on the characteristics of the data being hashed. Knuth [211] suggests that

$$A \approx (\sqrt{5} - 1)/2 = 0.6180339887\ldots \tag{11.2}$$

is likely to work reasonably well.

As an example, suppose we have $k = 123456$, $p = 14$, $m = 2^{14} = 16384$, and $w = 32$. Adapting Knuth's suggestion, we choose A to be the fraction of the form $s/2^{32}$ that is closest to $(\sqrt{5} - 1)/2$, so that $A = 2654435769/2^{32}$. Then $k \cdot s = 327706022297664 = (76300 \cdot 2^{32}) + 17612864$, and so $r_1 = 76300$ and $r_0 = 17612864$. The 14 most significant bits of r_0 yield the value $h(k) = 67$.

★ **11.3.3 Universal hashing**

If a malicious adversary chooses the keys to be hashed by some fixed hash function, then the adversary can choose n keys that all hash to the same slot, yielding an average retrieval time of $\Theta(n)$. Any fixed hash function is vulnerable to such terrible worst-case behavior; the only effective way to improve the situation is to choose the hash function *randomly* in a way that is *independent* of the keys that are actually going to be stored. This approach, called **universal hashing**, can yield provably good performance on average, no matter which keys the adversary chooses.

In universal hashing, at the beginning of execution we select the hash function at random from a carefully designed class of functions. As in the case of quicksort, randomization guarantees that no single input will always evoke worst-case behavior. Because we randomly select the hash function, the algorithm can behave differently on each execution, even for the same input, guaranteeing good average-case performance for any input. Returning to the example of a compiler's symbol table, we find that the programmer's choice of identifiers cannot now cause consistently poor hashing performance. Poor performance occurs only when the compiler chooses a random hash function that causes the set of identifiers to hash poorly, but the probability of this situation occurring is small and is the same for any set of identifiers of the same size.

Let $\mathcal{H}$ be a finite collection of hash functions that map a given universe U of keys into the range $\{0, 1, \ldots, m-1\}$. Such a collection is said to be **universal** if for each pair of distinct keys $k, l \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(k) = h(l)$ is at most $|\mathcal{H}|/m$. In other words, with a hash function randomly chosen from $\mathcal{H}$, the chance of a collision between distinct keys k and l is no more than the chance $1/m$ of a collision if $h(k)$ and $h(l)$ were randomly and independently chosen from the set $\{0, 1, \ldots, m-1\}$.

The following theorem shows that a universal class of hash functions gives good average-case behavior. Recall that n_i denotes the length of list $T[i]$.

Theorem 11.3
Suppose that a hash function h is chosen randomly from a universal collection of hash functions and has been used to hash n keys into a table T of size m, using chaining to resolve collisions. If key k is not in the table, then the expected length $\mathrm{E}\left[n_{h(k)}\right]$ of the list that key k hashes to is at most the load factor $\alpha = n/m$. If key k is in the table, then the expected length $\mathrm{E}\left[n_{h(k)}\right]$ of the list containing key k is at most $1 + \alpha$.

Proof We note that the expectations here are over the choice of the hash function and do not depend on any assumptions about the distribution of the keys. For each pair k and l of distinct keys, define the indicator random variable

$X_{kl} = \text{I}\{h(k) = h(l)\}$. Since by the definition of a universal collection of hash functions, a single pair of keys collides with probability at most $1/m$, we have $\Pr\{h(k) = h(l)\} \leq 1/m$. By Lemma 5.1, therefore, we have $\text{E}[X_{kl}] \leq 1/m$.

Next we define, for each key k, the random variable Y_k that equals the number of keys other than k that hash to the same slot as k, so that

$$Y_k = \sum_{\substack{l \in T \\ l \neq k}} X_{kl} \ .$$

Thus we have

$$
\begin{aligned}
\text{E}[Y_k] \ &= \ \text{E}\left[\sum_{\substack{l \in T \\ l \neq k}} X_{kl}\right] \\
&= \ \sum_{\substack{l \in T \\ l \neq k}} \text{E}[X_{kl}] \quad \text{(by linearity of expectation)} \\
&\leq \ \sum_{\substack{l \in T \\ l \neq k}} \frac{1}{m} \ .
\end{aligned}
$$

The remainder of the proof depends on whether key k is in table T.

- If $k \notin T$, then $n_{h(k)} = Y_k$ and $|\{l : l \in T \text{ and } l \neq k\}| = n$. Thus $\text{E}[n_{h(k)}] = \text{E}[Y_k] \leq n/m = \alpha$.

- If $k \in T$, then because key k appears in list $T[h(k)]$ and the count Y_k does not include key k, we have $n_{h(k)} = Y_k + 1$ and $|\{l : l \in T \text{ and } l \neq k\}| = n - 1$. Thus $\text{E}[n_{h(k)}] = \text{E}[Y_k] + 1 \leq (n-1)/m + 1 = 1 + \alpha - 1/m < 1 + \alpha$. ∎

The following corollary says universal hashing provides the desired payoff: it has now become impossible for an adversary to pick a sequence of operations that forces the worst-case running time. By cleverly randomizing the choice of hash function at run time, we guarantee that we can process every sequence of operations with a good average-case running time.

Corollary 11.4
Using universal hashing and collision resolution by chaining in an initially empty table with m slots, it takes expected time $\Theta(n)$ to handle any sequence of n INSERT, SEARCH, and DELETE operations containing $O(m)$ INSERT operations.

Proof   Since the number of insertions is $O(m)$, we have $n = O(m)$ and so $\alpha = O(1)$. The INSERT and DELETE operations take constant time and, by Theorem 11.3, the expected time for each SEARCH operation is $O(1)$. By linearity of

expectation, therefore, the expected time for the entire sequence of n operations is $O(n)$. Since each operation takes $\Omega(1)$ time, the $\Theta(n)$ bound follows.   ■

Designing a universal class of hash functions

It is quite easy to design a universal class of hash functions, as a little number theory will help us prove. You may wish to consult Chapter 31 first if you are unfamiliar with number theory.

We begin by choosing a prime number p large enough so that every possible key k is in the range 0 to $p - 1$, inclusive. Let $\mathbb{Z}_p$ denote the set $\{0, 1, \ldots, p - 1\}$, and let $\mathbb{Z}_p^*$ denote the set $\{1, 2, \ldots, p - 1\}$. Since p is prime, we can solve equations modulo p with the methods given in Chapter 31. Because we assume that the size of the universe of keys is greater than the number of slots in the hash table, we have $p > m$.

We now define the hash function h_{ab} for any $a \in \mathbb{Z}_p^*$ and any $b \in \mathbb{Z}_p$ using a linear transformation followed by reductions modulo p and then modulo m:

$$h_{ab}(k) = ((ak + b) \bmod p) \bmod m \ . \tag{11.3}$$

For example, with $p = 17$ and $m = 6$, we have $h_{3,4}(8) = 5$. The family of all such hash functions is

$$\mathcal{H}_{pm} = \left\{ h_{ab} : a \in \mathbb{Z}_p^* \text{ and } b \in \mathbb{Z}_p \right\} \ . \tag{11.4}$$

Each hash function h_{ab} maps $\mathbb{Z}_p$ to $\mathbb{Z}_m$. This class of hash functions has the nice property that the size m of the output range is arbitrary—not necessarily prime—a feature which we shall use in Section 11.5. Since we have $p - 1$ choices for a and p choices for b, the collection $\mathcal{H}_{pm}$ contains $p(p - 1)$ hash functions.

Theorem 11.5
The class $\mathcal{H}_{pm}$ of hash functions defined by equations (11.3) and (11.4) is universal.

Proof   Consider two distinct keys k and l from $\mathbb{Z}_p$, so that $k \neq l$. For a given hash function h_{ab} we let

$$r = (ak + b) \bmod p \ ,$$
$$s = (al + b) \bmod p \ .$$

We first note that $r \neq s$. Why? Observe that

$$r - s \equiv a(k - l) \pmod{p} \ .$$

It follows that $r \neq s$ because p is prime and both a and $(k - l)$ are nonzero modulo p, and so their product must also be nonzero modulo p by Theorem 31.6. Therefore, when computing any $h_{ab} \in \mathcal{H}_{pm}$, distinct inputs k and l map to distinct

values r and s modulo p; there are no collisions yet at the "mod p level." Moreover, each of the possible $p(p-1)$ choices for the pair (a, b) with $a \neq 0$ yields a *different* resulting pair (r, s) with $r \neq s$, since we can solve for a and b given r and s:

$$a = \left((r - s)((k - l)^{-1} \bmod p)\right) \bmod p ,$$
$$b = (r - ak) \bmod p ,$$

where $((k - l)^{-1} \bmod p)$ denotes the unique multiplicative inverse, modulo p, of $k - l$. Since there are only $p(p - 1)$ possible pairs (r, s) with $r \neq s$, there is a one-to-one correspondence between pairs (a, b) with $a \neq 0$ and pairs (r, s) with $r \neq s$. Thus, for any given pair of inputs k and l, if we pick (a, b) uniformly at random from $\mathbb{Z}_p^* \times \mathbb{Z}_p$, the resulting pair (r, s) is equally likely to be any pair of distinct values modulo p.

Therefore, the probability that distinct keys k and l collide is equal to the probability that $r \equiv s \pmod{m}$ when r and s are randomly chosen as distinct values modulo p. For a given value of r, of the $p - 1$ possible remaining values for s, the number of values s such that $s \neq r$ and $s \equiv r \pmod{m}$ is at most

$$\lceil p/m \rceil - 1 \leq ((p + m - 1)/m) - 1 \quad \text{(by inequality (3.6))}$$
$$= (p - 1)/m .$$

The probability that s collides with r when reduced modulo m is at most $((p - 1)/m)/(p - 1) = 1/m$.

Therefore, for any pair of distinct values $k, l \in \mathbb{Z}_p$,

$$\Pr\{h_{ab}(k) = h_{ab}(l)\} \leq 1/m ,$$

so that $\mathcal{H}_{pm}$ is indeed universal. ∎

Exercises

11.3-1
Suppose we wish to search a linked list of length n, where each element contains a key k along with a hash value $h(k)$. Each key is a long character string. How might we take advantage of the hash values when searching the list for an element with a given key?

11.3-2
Suppose that we hash a string of r characters into m slots by treating it as a radix-128 number and then using the division method. We can easily represent the number m as a 32-bit computer word, but the string of r characters, treated as a radix-128 number, takes many words. How can we apply the division method to compute the hash value of the character string without using more than a constant number of words of storage outside the string itself?

11.3-3

Consider a version of the division method in which $h(k) = k \bmod m$, where $m = 2^p - 1$ and k is a character string interpreted in radix 2^p. Show that if we can derive string x from string y by permuting its characters, then x and y hash to the same value. Give an example of an application in which this property would be undesirable in a hash function.

11.3-4

Consider a hash table of size $m = 1000$ and a corresponding hash function $h(k) = \lfloor m (kA \bmod 1) \rfloor$ for $A = (\sqrt{5} - 1)/2$. Compute the locations to which the keys 61, 62, 63, 64, and 65 are mapped.

11.3-5 ★

Define a family $\mathcal{H}$ of hash functions from a finite set U to a finite set B to be **ε-universal** if for all pairs of distinct elements k and l in U,

$$\Pr\{h(k) = h(l)\} \leq \epsilon \, ,$$

where the probability is over the choice of the hash function h drawn at random from the family $\mathcal{H}$. Show that an ϵ-universal family of hash functions must have

$$\epsilon \geq \frac{1}{|B|} - \frac{1}{|U|} \, .$$

11.3-6 ★

Let U be the set of n-tuples of values drawn from $\mathbb{Z}_p$, and let $B = \mathbb{Z}_p$, where p is prime. Define the hash function $h_b : U \to B$ for $b \in \mathbb{Z}_p$ on an input n-tuple $\langle a_0, a_1, \ldots, a_{n-1} \rangle$ from U as

$$h_b(\langle a_0, a_1, \ldots, a_{n-1} \rangle) = \left(\sum_{j=0}^{n-1} a_j b^j \right) \bmod p \, ,$$

and let $\mathcal{H} = \{h_b : b \in \mathbb{Z}_p\}$. Argue that $\mathcal{H}$ is $((n-1)/p)$-universal according to the definition of ϵ-universal in Exercise 11.3-5. (*Hint:* See Exercise 31.4-4.)

11.4   Open addressing

In **open addressing**, all elements occupy the hash table itself. That is, each table entry contains either an element of the dynamic set or NIL. When searching for an element, we systematically examine table slots until either we find the desired element or we have ascertained that the element is not in the table. No lists and

no elements are stored outside the table, unlike in chaining. Thus, in open addressing, the hash table can "fill up" so that no further insertions can be made; one consequence is that the load factor α can never exceed 1.

Of course, we could store the linked lists for chaining inside the hash table, in the otherwise unused hash-table slots (see Exercise 11.2-4), but the advantage of open addressing is that it avoids pointers altogether. Instead of following pointers, we *compute* the sequence of slots to be examined. The extra memory freed by not storing pointers provides the hash table with a larger number of slots for the same amount of memory, potentially yielding fewer collisions and faster retrieval.

To perform insertion using open addressing, we successively examine, or ***probe***, the hash table until we find an empty slot in which to put the key. Instead of being fixed in the order $0, 1, \ldots, m - 1$ (which requires $\Theta(n)$ search time), the sequence of positions probed *depends upon the key being inserted*. To determine which slots to probe, we extend the hash function to include the probe number (starting from 0) as a second input. Thus, the hash function becomes

$$h : U \times \{0, 1, \ldots, m - 1\} \to \{0, 1, \ldots, m - 1\} \ .$$

With open addressing, we require that for every key k, the ***probe sequence***

$$\langle h(k, 0), h(k, 1), \ldots, h(k, m - 1) \rangle$$

be a permutation of $\langle 0, 1, \ldots, m - 1 \rangle$, so that every hash-table position is eventually considered as a slot for a new key as the table fills up. In the following pseudocode, we assume that the elements in the hash table T are keys with no satellite information; the key k is identical to the element containing key k. Each slot contains either a key or NIL (if the slot is empty). The HASH-INSERT procedure takes as input a hash table T and a key k. It either returns the slot number where it stores key k or flags an error because the hash table is already full.

HASH-INSERT(T, k)

```
1   i = 0
2   repeat
3        j = h(k, i)
4        if T[j] == NIL
5             T[j] = k
6             return j
7        else i = i + 1
8   until i == m
9   error "hash table overflow"
```

The algorithm for searching for key k probes the same sequence of slots that the insertion algorithm examined when key k was inserted. Therefore, the search can

terminate (unsuccessfully) when it finds an empty slot, since k would have been inserted there and not later in its probe sequence. (This argument assumes that keys are not deleted from the hash table.) The procedure HASH-SEARCH takes as input a hash table T and a key k, returning j if it finds that slot j contains key k, or NIL if key k is not present in table T.

HASH-SEARCH(T, k)

1 $i = 0$
2 **repeat**
3      $j = h(k, i)$
4   **if** $T[j] == k$
5   **return** j
6      $i = i + 1$
7 **until** $T[j] ==$ NIL or $i == m$
8 **return** NIL

Deletion from an open-address hash table is difficult. When we delete a key from slot i, we cannot simply mark that slot as empty by storing NIL in it. If we did, we might be unable to retrieve any key k during whose insertion we had probed slot i and found it occupied. We can solve this problem by marking the slot, storing in it the special value DELETED instead of NIL. We would then modify the procedure HASH-INSERT to treat such a slot as if it were empty so that we can insert a new key there. We do not need to modify HASH-SEARCH, since it will pass over DELETED values while searching. When we use the special value DELETED, however, search times no longer depend on the load factor α, and for this reason chaining is more commonly selected as a collision resolution technique when keys must be deleted.

In our analysis, we assume ***uniform hashing***: the probe sequence of each key is equally likely to be any of the $m!$ permutations of $\langle 0, 1, \ldots, m - 1 \rangle$. Uniform hashing generalizes the notion of simple uniform hashing defined earlier to a hash function that produces not just a single number, but a whole probe sequence. True uniform hashing is difficult to implement, however, and in practice suitable approximations (such as double hashing, defined below) are used.

We will examine three commonly used techniques to compute the probe sequences required for open addressing: linear probing, quadratic probing, and double hashing. These techniques all guarantee that $\langle h(k, 0), h(k, 1), \ldots, h(k, m-1) \rangle$ is a permutation of $\langle 0, 1, \ldots, m - 1 \rangle$ for each key k. None of these techniques fulfills the assumption of uniform hashing, however, since none of them is capable of generating more than m^2 different probe sequences (instead of the $m!$ that uniform hashing requires). Double hashing has the greatest number of probe sequences and, as one might expect, seems to give the best results.

Linear probing

Given an ordinary hash function $h' : U \rightarrow \{0, 1, \ldots, m - 1\}$, which we refer to as an ***auxiliary hash function***, the method of ***linear probing*** uses the hash function

$$h(k, i) = (h'(k) + i) \bmod m$$

for $i = 0, 1, \ldots, m - 1$. Given key k, we first probe $T[h'(k)]$, i.e., the slot given by the auxiliary hash function. We next probe slot $T[h'(k) + 1]$, and so on up to slot $T[m - 1]$. Then we wrap around to slots $T[0], T[1], \ldots$ until we finally probe slot $T[h'(k) - 1]$. Because the initial probe determines the entire probe sequence, there are only m distinct probe sequences.

Linear probing is easy to implement, but it suffers from a problem known as ***primary clustering***. Long runs of occupied slots build up, increasing the average search time. Clusters arise because an empty slot preceded by i full slots gets filled next with probability $(i + 1)/m$. Long runs of occupied slots tend to get longer, and the average search time increases.

Quadratic probing

Quadratic probing uses a hash function of the form

$$h(k, i) = (h'(k) + c_1 i + c_2 i^2) \bmod m \, , \tag{11.5}$$

where h' is an auxiliary hash function, c_1 and c_2 are positive auxiliary constants, and $i = 0, 1, \ldots, m - 1$. The initial position probed is $T[h'(k)]$; later positions probed are offset by amounts that depend in a quadratic manner on the probe number i. This method works much better than linear probing, but to make full use of the hash table, the values of c_1, c_2, and m are constrained. Problem 11-3 shows one way to select these parameters. Also, if two keys have the same initial probe position, then their probe sequences are the same, since $h(k_1, 0) = h(k_2, 0)$ implies $h(k_1, i) = h(k_2, i)$. This property leads to a milder form of clustering, called ***secondary clustering***. As in linear probing, the initial probe determines the entire sequence, and so only m distinct probe sequences are used.

Double hashing

Double hashing offers one of the best methods available for open addressing because the permutations produced have many of the characteristics of randomly chosen permutations. ***Double hashing*** uses a hash function of the form

$$h(k, i) = (h_1(k) + i h_2(k)) \bmod m \, ,$$

where both h_1 and h_2 are auxiliary hash functions. The initial probe goes to position $T[h_1(k)]$; successive probe positions are offset from previous positions by the
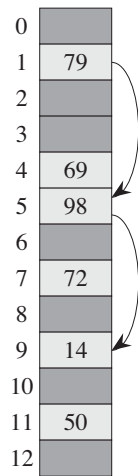
0	
1	79
2	
3	
4	69
5	98
6	
7	72
8	
9	14
10	
11	50
12	

Figure 11.5   Insertion by double hashing. Here we have a hash table of size 13 with $h_1(k) = k \bmod 13$ and $h_2(k) = 1 + (k \bmod 11)$. Since $14 \equiv 1 \pmod{13}$ and $14 \equiv 3 \pmod{11}$, we insert the key 14 into empty slot 9, after examining slots 1 and 5 and finding them to be occupied.

amount $h_2(k)$, modulo m. Thus, unlike the case of linear or quadratic probing, the probe sequence here depends in two ways upon the key k, since the initial probe position, the offset, or both, may vary. Figure 11.5 gives an example of insertion by double hashing.

The value $h_2(k)$ must be relatively prime to the hash-table size m for the entire hash table to be searched. (See Exercise 11.4-4.) A convenient way to ensure this condition is to let m be a power of 2 and to design h_2 so that it always produces an odd number. Another way is to let m be prime and to design h_2 so that it always returns a positive integer less than m. For example, we could choose m prime and let

$$
\begin{aligned}
h_1(k) &= k \bmod m \,, \\
h_2(k) &= 1 + (k \bmod m') \,,
\end{aligned}
$$

where m' is chosen to be slightly less than m (say, $m - 1$). For example, if $k = 123456$, $m = 701$, and $m' = 700$, we have $h_1(k) = 80$ and $h_2(k) = 257$, so that we first probe position 80, and then we examine every 257th slot (modulo m) until we find the key or have examined every slot.

When m is prime or a power of 2, double hashing improves over linear or quadratic probing in that $\Theta(m^2)$ probe sequences are used, rather than $\Theta(m)$, since each possible $(h_1(k), h_2(k))$ pair yields a distinct probe sequence. As a result, for

such values of m, the performance of double hashing appears to be very close to the performance of the "ideal" scheme of uniform hashing.

Although values of m other than primes or powers of 2 could in principle be used with double hashing, in practice it becomes more difficult to efficiently generate $h_2(k)$ in a way that ensures that it is relatively prime to m, in part because the relative density $\phi(m)/m$ of such numbers may be small (see equation (31.24)).

Analysis of open-address hashing

As in our analysis of chaining, we express our analysis of open addressing in terms of the load factor $\alpha = n/m$ of the hash table. Of course, with open addressing, at most one element occupies each slot, and thus $n \leq m$, which implies $\alpha \leq 1$.

We assume that we are using uniform hashing. In this idealized scheme, the probe sequence $\langle h(k, 0), h(k, 1), \ldots, h(k, m - 1)\rangle$ used to insert or search for each key k is equally likely to be any permutation of $\langle 0, 1, \ldots, m - 1\rangle$. Of course, a given key has a unique fixed probe sequence associated with it; what we mean here is that, considering the probability distribution on the space of keys and the operation of the hash function on the keys, each possible probe sequence is equally likely.

We now analyze the expected number of probes for hashing with open addressing under the assumption of uniform hashing, beginning with an analysis of the number of probes made in an unsuccessful search.

Theorem 11.6
Given an open-address hash table with load factor $\alpha = n/m < 1$, the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$, assuming uniform hashing.

Proof   In an unsuccessful search, every probe but the last accesses an occupied slot that does not contain the desired key, and the last slot probed is empty. Let us define the random variable X to be the number of probes made in an unsuccessful search, and let us also define the event A_i, for $i = 1, 2, \ldots$, to be the event that an ith probe occurs and it is to an occupied slot. Then the event $\{X \geq i\}$ is the intersection of events $A_1 \cap A_2 \cap \cdots \cap A_{i-1}$. We will bound $\Pr\{X \geq i\}$ by bounding $\Pr\{A_1 \cap A_2 \cap \cdots \cap A_{i-1}\}$. By Exercise C.2-5,

$$\Pr\{A_1 \cap A_2 \cap \cdots \cap A_{i-1}\} = \Pr\{A_1\} \cdot \Pr\{A_2 \mid A_1\} \cdot \Pr\{A_3 \mid A_1 \cap A_2\} \cdots$$
$$\Pr\{A_{i-1} \mid A_1 \cap A_2 \cap \cdots \cap A_{i-2}\} \ .$$

Since there are n elements and m slots, $\Pr\{A_1\} = n/m$. For $j > 1$, the probability that there is a jth probe and it is to an occupied slot, given that the first $j - 1$ probes were to occupied slots, is $(n - j + 1)/(m - j + 1)$. This probability follows

because we would be finding one of the remaining $(n - (j - 1))$ elements in one of the $(m - (j - 1))$ unexamined slots, and by the assumption of uniform hashing, the probability is the ratio of these quantities. Observing that $n < m$ implies that $(n - j)/(m - j) \le n/m$ for all j such that $0 \le j < m$, we have for all i such that $1 \le i \le m$,

$$
\begin{aligned}
\Pr\{X \ge i\} &= \frac{n}{m} \cdot \frac{n-1}{m-1} \cdot \frac{n-2}{m-2} \cdots \frac{n-i+2}{m-i+2} \\
&\le \left(\frac{n}{m}\right)^{i-1} \\
&= \alpha^{i-1} .
\end{aligned}
$$

Now, we use equation (C.25) to bound the expected number of probes:

$$
\begin{aligned}
\mathrm{E}[X] &= \sum_{i=1}^{\infty} \Pr\{X \ge i\} \\
&\le \sum_{i=1}^{\infty} \alpha^{i-1} \\
&= \sum_{i=0}^{\infty} \alpha^{i} \\
&= \frac{1}{1-\alpha} .
\end{aligned}
$$

$\blacksquare$

This bound of $1/(1-\alpha) = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots$ has an intuitive interpretation. We always make the first probe. With probability approximately α, the first probe finds an occupied slot, so that we need to probe a second time. With probability approximately α^2, the first two slots are occupied so that we make a third probe, and so on.

If α is a constant, Theorem 11.6 predicts that an unsuccessful search runs in $O(1)$ time. For example, if the hash table is half full, the average number of probes in an unsuccessful search is at most $1/(1 - .5) = 2$. If it is 90 percent full, the average number of probes is at most $1/(1 - .9) = 10$.

Theorem 11.6 gives us the performance of the HASH-INSERT procedure almost immediately.

Corollary 11.7
Inserting an element into an open-address hash table with load factor α requires at most $1/(1 - \alpha)$ probes on average, assuming uniform hashing.

Proof   An element is inserted only if there is room in the table, and thus $\alpha < 1$. Inserting a key requires an unsuccessful search followed by placing the key into the first empty slot found. Thus, the expected number of probes is at most $1/(1-\alpha)$.   ∎

We have to do a little more work to compute the expected number of probes for a successful search.

Theorem 11.8

Given an open-address hash table with load factor $\alpha < 1$, the expected number of probes in a successful search is at most

$$\frac{1}{\alpha} \ln \frac{1}{1-\alpha} \, ,$$

assuming uniform hashing and assuming that each key in the table is equally likely to be searched for.

Proof   A search for a key k reproduces the same probe sequence as when the element with key k was inserted. By Corollary 11.7, if k was the $(i+1)$st key inserted into the hash table, the expected number of probes made in a search for k is at most $1/(1 - i/m) = m/(m - i)$. Averaging over all n keys in the hash table gives us the expected number of probes in a successful search:

$$
\begin{aligned}
\frac{1}{n} \sum_{i=0}^{n-1} \frac{m}{m-i} \; &= \; \frac{m}{n} \sum_{i=0}^{n-1} \frac{1}{m-i} \\
&= \; \frac{1}{\alpha} \sum_{k=m-n+1}^{m} \frac{1}{k} \\
&\leq \; \frac{1}{\alpha} \int_{m-n}^{m} (1/x)\,dx \quad \text{(by inequality (A.12))} \\
&= \; \frac{1}{\alpha} \ln \frac{m}{m-n} \\
&= \; \frac{1}{\alpha} \ln \frac{1}{1-\alpha} \, .
\end{aligned}
$$
∎

If the hash table is half full, the expected number of probes in a successful search is less than 1.387. If the hash table is 90 percent full, the expected number of probes is less than 2.559.

Exercises

11.4-1
Consider inserting the keys $10, 22, 31, 4, 15, 28, 17, 88, 59$ into a hash table of
length $m = 11$ using open addressing with the auxiliary hash function $h'(k) = k$.
Illustrate the result of inserting these keys using linear probing, using quadratic
probing with $c_1 = 1$ and $c_2 = 3$, and using double hashing with $h_1(k) = k$ and
$h_2(k) = 1 + (k \bmod (m - 1))$.

11.4-2
Write pseudocode for HASH-DELETE as outlined in the text, and modify HASH-
INSERT to handle the special value DELETED.

11.4-3
Consider an open-address hash table with uniform hashing. Give upper bounds
on the expected number of probes in an unsuccessful search and on the expected
number of probes in a successful search when the load factor is $3/4$ and when it
is $7/8$.

11.4-4 ★
Suppose that we use double hashing to resolve collisions—that is, we use the hash
function $h(k, i) = (h_1(k) + i h_2(k)) \bmod m$. Show that if m and $h_2(k)$ have
greatest common divisor $d \geq 1$ for some key k, then an unsuccessful search for
key k examines $(1/d)$th of the hash table before returning to slot $h_1(k)$. Thus,
when $d = 1$, so that m and $h_2(k)$ are relatively prime, the search may examine the
entire hash table. (*Hint:* See Chapter 31.)

11.4-5 ★
Consider an open-address hash table with a load factor α. Find the nonzero value α
for which the expected number of probes in an unsuccessful search equals twice
the expected number of probes in a successful search. Use the upper bounds given
by Theorems 11.6 and 11.8 for these expected numbers of probes.

★ **11.5   Perfect hashing**

Although hashing is often a good choice for its excellent average-case perfor-
mance, hashing can also provide excellent *worst-case* performance when the set of
keys is ***static***: once the keys are stored in the table, the set of keys never changes.
Some applications naturally have static sets of keys: consider the set of reserved
words in a programming language, or the set of file names on a CD-ROM. We
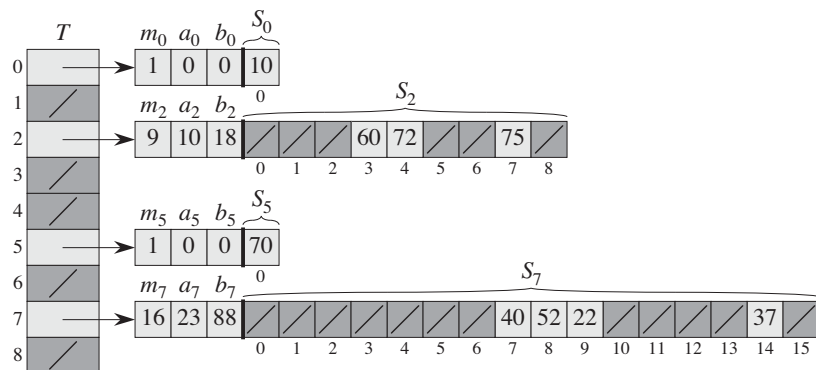
Figure 11.6 Using perfect hashing to store the set $K = \{10, 22, 37, 40, 52, 60, 70, 72, 75\}$. The outer hash function is $h(k) = ((ak + b) \bmod p) \bmod m$, where $a = 3$, $b = 42$, $p = 101$, and $m = 9$. For example, $h(75) = 2$, and so key 75 hashes to slot 2 of table T. A secondary hash table S_j stores all keys hashing to slot j. The size of hash table S_j is $m_j = n_j^2$, and the associated hash function is $h_j(k) = ((a_j k + b_j) \bmod p) \bmod m_j$. Since $h_2(75) = 7$, key 75 is stored in slot 7 of secondary hash table S_2. No collisions occur in any of the secondary hash tables, and so searching takes constant time in the worst case.

call a hashing technique ***perfect hashing*** if $O(1)$ memory accesses are required to perform a search in the worst case.

To create a perfect hashing scheme, we use two levels of hashing, with universal hashing at each level. Figure 11.6 illustrates the approach.

The first level is essentially the same as for hashing with chaining: we hash the n keys into m slots using a hash function h carefully selected from a family of universal hash functions.

Instead of making a linked list of the keys hashing to slot j, however, we use a small ***secondary hash table*** S_j with an associated hash function h_j. By choosing the hash functions h_j carefully, we can guarantee that there are no collisions at the secondary level.

In order to guarantee that there are no collisions at the secondary level, however, we will need to let the size m_j of hash table S_j be the square of the number n_j of keys hashing to slot j. Although you might think that the quadratic dependence of m_j on n_j may seem likely to cause the overall storage requirement to be excessive, we shall show that by choosing the first-level hash function well, we can limit the expected total amount of space used to $O(n)$.

We use hash functions chosen from the universal classes of hash functions of Section 11.3.3. The first-level hash function comes from the class $\mathcal{H}_{pm}$, where as in Section 11.3.3, p is a prime number greater than any key value. Those keys

hashing to slot j are re-hashed into a secondary hash table S_j of size m_j using a hash function h_j chosen from the class $\mathcal{H}_{p,m_j}$.[1]

We shall proceed in two steps. First, we shall determine how to ensure that the secondary tables have no collisions. Second, we shall show that the expected amount of memory used overall—for the primary hash table and all the secondary hash tables—is $O(n)$.

Theorem 11.9
Suppose that we store n keys in a hash table of size $m = n^2$ using a hash function h randomly chosen from a universal class of hash functions. Then, the probability is less than $1/2$ that there are any collisions.

Proof There are $\binom{n}{2}$ pairs of keys that may collide; each pair collides with probability $1/m$ if h is chosen at random from a universal family $\mathcal{H}$ of hash functions. Let X be a random variable that counts the number of collisions. When $m = n^2$, the expected number of collisions is

$$
\begin{aligned}
\mathrm{E}[X] &= \binom{n}{2} \cdot \frac{1}{n^2} \\
&= \frac{n^2 - n}{2} \cdot \frac{1}{n^2} \\
&< 1/2 .
\end{aligned}
$$

(This analysis is similar to the analysis of the birthday paradox in Section 5.4.1.) Applying Markov's inequality (C.30), $\Pr\{X \geq t\} \leq \mathrm{E}[X]/t$, with $t = 1$, completes the proof. ∎

In the situation described in Theorem 11.9, where $m = n^2$, it follows that a hash function h chosen at random from $\mathcal{H}$ is more likely than not to have *no* collisions. Given the set K of n keys to be hashed (remember that K is static), it is thus easy to find a collision-free hash function h with a few random trials.

When n is large, however, a hash table of size $m = n^2$ is excessive. Therefore, we adopt the two-level hashing approach, and we use the approach of Theorem 11.9 only to hash the entries within each slot. We use an outer, or first-level, hash function h to hash the keys into $m = n$ slots. Then, if n_j keys hash to slot j, we use a secondary hash table S_j of size $m_j = n_j^2$ to provide collision-free constant-time lookup.

---

[1] When $n_j = m_j = 1$, we don't really need a hash function for slot j; when we choose a hash function $h_{ab}(k) = ((ak + b) \bmod p) \bmod m_j$ for such a slot, we just use $a = b = 0$.

We now turn to the issue of ensuring that the overall memory used is $O(n)$. Since the size m_j of the jth secondary hash table grows quadratically with the number n_j of keys stored, we run the risk that the overall amount of storage could be excessive.

If the first-level table size is $m = n$, then the amount of memory used is $O(n)$ for the primary hash table, for the storage of the sizes m_j of the secondary hash tables, and for the storage of the parameters a_j and b_j defining the secondary hash functions h_j drawn from the class $\mathcal{H}_{p,m_j}$ of Section 11.3.3 (except when $n_j = 1$ and we use $a = b = 0$). The following theorem and a corollary provide a bound on the expected combined sizes of all the secondary hash tables. A second corollary bounds the probability that the combined size of all the secondary hash tables is superlinear (actually, that it equals or exceeds $4n$).

Theorem 11.10
Suppose that we store n keys in a hash table of size $m = n$ using a hash function h randomly chosen from a universal class of hash functions. Then, we have

$$\mathrm{E}\left[\sum_{j=0}^{m-1} n_j^2\right] < 2n \;,$$

where n_j is the number of keys hashing to slot j.

Proof   We start with the following identity, which holds for any nonnegative integer a:

$$a^2 = a + 2\binom{a}{2}. \tag{11.6}$$

We have

$$\mathrm{E}\left[\sum_{j=0}^{m-1} n_j^2\right]$$

$$= \mathrm{E}\left[\sum_{j=0}^{m-1}\left(n_j + 2\binom{n_j}{2}\right)\right] \qquad \text{(by equation (11.6))}$$

$$= \mathrm{E}\left[\sum_{j=0}^{m-1} n_j\right] + 2\,\mathrm{E}\left[\sum_{j=0}^{m-1}\binom{n_j}{2}\right] \qquad \text{(by linearity of expectation)}$$

$$= \mathrm{E}\left[n\right] + 2\,\mathrm{E}\left[\sum_{j=0}^{m-1}\binom{n_j}{2}\right] \qquad \text{(by equation (11.1))}$$

$$= \ n + 2\,\mathrm{E}\left[\sum_{j=0}^{m-1}\binom{n_j}{2}\right] \qquad \text{(since } n \text{ is not a random variable) .}$$

To evaluate the summation $\sum_{j=0}^{m-1}\binom{n_j}{2}$, we observe that it is just the total number of pairs of keys in the hash table that collide. By the properties of universal hashing, the expected value of this summation is at most

$$\binom{n}{2}\frac{1}{m} \ = \ \frac{n(n-1)}{2m}$$

$$= \ \frac{n-1}{2}\,,$$

since $m = n$. Thus,

$$\mathrm{E}\left[\sum_{j=0}^{m-1}n_j^2\right] \ \le \ n + 2\,\frac{n-1}{2}$$

$$= \ 2n - 1$$

$$< \ 2n\,. \qquad\blacksquare$$

Corollary 11.11
Suppose that we store n keys in a hash table of size $m = n$ using a hash function h randomly chosen from a universal class of hash functions, and we set the size of each secondary hash table to $m_j = n_j^2$ for $j = 0, 1, \ldots, m - 1$. Then, the expected amount of storage required for all secondary hash tables in a perfect hashing scheme is less than $2n$.

Proof   Since $m_j = n_j^2$ for $j = 0, 1, \ldots, m - 1$, Theorem 11.10 gives

$$\mathrm{E}\left[\sum_{j=0}^{m-1}m_j\right] \ = \ \mathrm{E}\left[\sum_{j=0}^{m-1}n_j^2\right]$$

$$< \ 2n\,, \qquad\qquad\qquad\qquad\qquad (11.7)$$

which completes the proof. $\blacksquare$

Corollary 11.12
Suppose that we store n keys in a hash table of size $m = n$ using a hash function h randomly chosen from a universal class of hash functions, and we set the size of each secondary hash table to $m_j = n_j^2$ for $j = 0, 1, \ldots, m - 1$. Then, the probability is less than $1/2$ that the total storage used for secondary hash tables equals or exceeds $4n$.

Proof   Again we apply Markov's inequality (C.30), $\Pr\{X \ge t\} \le \mathrm{E}[X]/t$, this time to inequality (11.7), with $X = \sum_{j=0}^{m-1} m_j$ and $t = 4n$:

$$\Pr\left\{\sum_{j=0}^{m-1} m_j \ge 4n\right\} \;\le\; \frac{\mathrm{E}\left[\sum_{j=0}^{m-1} m_j\right]}{4n}$$

$$< \; \frac{2n}{4n}$$

$$= \; 1/2 \;. \qquad\blacksquare$$

From Corollary 11.12, we see that if we test a few randomly chosen hash functions from the universal family, we will quickly find one that uses a reasonable amount of storage.

Exercises

11.5-1   ★
Suppose that we insert n keys into a hash table of size m using open addressing and uniform hashing. Let $p(n,m)$ be the probability that no collisions occur. Show that $p(n,m) \le e^{-n(n-1)/2m}$. (*Hint:* See equation (3.12).) Argue that when n exceeds $\sqrt{m}$, the probability of avoiding collisions goes rapidly to zero.

Problems

11-1   *Longest-probe bound for hashing*
Suppose that we use an open-addressed hash table of size m to store $n \le m/2$ items.

a. Assuming uniform hashing, show that for $i = 1, 2, \ldots, n$, the probability is at most 2^{-k} that the ith insertion requires strictly more than k probes.

b. Show that for $i = 1, 2, \ldots, n$, the probability is $O(1/n^2)$ that the ith insertion requires more than $2\lg n$ probes.

Let the random variable X_i denote the number of probes required by the ith insertion. You have shown in part (b) that $\Pr\{X_i > 2\lg n\} = O(1/n^2)$. Let the random variable $X = \max_{1 \le i \le n} X_i$ denote the maximum number of probes required by any of the n insertions.

c. Show that $\Pr\{X > 2\lg n\} = O(1/n)$.

d. Show that the expected length $\mathrm{E}[X]$ of the longest probe sequence is $O(\lg n)$.

11-2 Slot-size bound for chaining

Suppose that we have a hash table with n slots, with collisions resolved by chaining, and suppose that n keys are inserted into the table. Each key is equally likely to be hashed to each slot. Let M be the maximum number of keys in any slot after all the keys have been inserted. Your mission is to prove an $O(\lg n / \lg \lg n)$ upper bound on $E[M]$, the expected value of M.

a. Argue that the probability Q_k that exactly k keys hash to a particular slot is given by

$$Q_k = \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} \binom{n}{k}.$$

b. Let P_k be the probability that $M = k$, that is, the probability that the slot containing the most keys contains k keys. Show that $P_k \le nQ_k$.

c. Use Stirling's approximation, equation (3.18), to show that $Q_k < e^k / k^k$.

d. Show that there exists a constant $c > 1$ such that $Q_{k_0} < 1/n^3$ for $k_0 = c \lg n / \lg \lg n$. Conclude that $P_k < 1/n^2$ for $k \ge k_0 = c \lg n / \lg \lg n$.

e. Argue that

$$E[M] \le \Pr\left\{M > \frac{c \lg n}{\lg \lg n}\right\} \cdot n + \Pr\left\{M \le \frac{c \lg n}{\lg \lg n}\right\} \cdot \frac{c \lg n}{\lg \lg n}.$$

Conclude that $E[M] = O(\lg n / \lg \lg n)$.

11-3 Quadratic probing

Suppose that we are given a key k to search for in a hash table with positions $0, 1, \dots, m-1$, and suppose that we have a hash function h mapping the key space into the set $\{0, 1, \dots, m-1\}$. The search scheme is as follows:

1. Compute the value $j = h(k)$, and set $i = 0$.

2. Probe in position j for the desired key k. If you find it, or if this position is empty, terminate the search.

3. Set $i = i + 1$. If i now equals m, the table is full, so terminate the search. Otherwise, set $j = (i + j) \bmod m$, and return to step 2.

Assume that m is a power of 2.

a. Show that this scheme is an instance of the general "quadratic probing" scheme by exhibiting the appropriate constants c_1 and c_2 for equation (11.5).

b. Prove that this algorithm examines every table position in the worst case.

11-4   *Hashing and authentication*

Let $\mathcal{H}$ be a class of hash functions in which each hash function $h \in \mathcal{H}$ maps the universe U of keys to $\{0, 1, \ldots, m - 1\}$. We say that $\mathcal{H}$ is **k-universal** if, for every fixed sequence of k distinct keys $\langle x^{(1)}, x^{(2)}, \ldots, x^{(k)} \rangle$ and for any h chosen at random from $\mathcal{H}$, the sequence $\langle h(x^{(1)}), h(x^{(2)}), \ldots, h(x^{(k)}) \rangle$ is equally likely to be any of the m^k sequences of length k with elements drawn from $\{0, 1, \ldots, m - 1\}$.

a. Show that if the family $\mathcal{H}$ of hash functions is 2-universal, then it is universal.

b. Suppose that the universe U is the set of n-tuples of values drawn from $\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$, where p is prime. Consider an element $x = \langle x_0, x_1, \ldots, x_{n-1} \rangle \in U$. For any n-tuple $a = \langle a_0, a_1, \ldots, a_{n-1} \rangle \in U$, define the hash function h_a by

$$h_a(x) = \left(\sum_{j=0}^{n-1} a_j x_j \right) \bmod p \ .$$

Let $\mathcal{H} = \{h_a\}$. Show that $\mathcal{H}$ is universal, but not 2-universal. (*Hint:* Find a key for which all hash functions in $\mathcal{H}$ produce the same value.)

c. Suppose that we modify $\mathcal{H}$ slightly from part (b): for any $a \in U$ and for any $b \in \mathbb{Z}_p$, define

$$h'_{ab}(x) = \left(\sum_{j=0}^{n-1} a_j x_j + b \right) \bmod p$$

and $\mathcal{H}' = \{h'_{ab}\}$. Argue that $\mathcal{H}'$ is 2-universal. (*Hint:* Consider fixed n-tuples $x \in U$ and $y \in U$, with $x_i \neq y_i$ for some i. What happens to $h'_{ab}(x)$ and $h'_{ab}(y)$ as a_i and b range over $\mathbb{Z}_p$?)

d. Suppose that Alice and Bob secretly agree on a hash function h from a 2-universal family $\mathcal{H}$ of hash functions. Each $h \in \mathcal{H}$ maps from a universe of keys U to $\mathbb{Z}_p$, where p is prime. Later, Alice sends a message m to Bob over the Internet, where $m \in U$. She authenticates this message to Bob by also sending an authentication tag $t = h(m)$, and Bob checks that the pair (m, t) he receives indeed satisfies $t = h(m)$. Suppose that an adversary intercepts (m, t) en route and tries to fool Bob by replacing the pair (m, t) with a different pair (m', t'). Argue that the probability that the adversary succeeds in fooling Bob into accepting (m', t') is at most $1/p$, no matter how much computing power the adversary has, and even if the adversary knows the family $\mathcal{H}$ of hash functions used.

Chapter notes

Knuth [211] and Gonnet [145] are excellent references for the analysis of hashing algorithms. Knuth credits H. P. Luhn (1953) for inventing hash tables, along with the chaining method for resolving collisions. At about the same time, G. M. Amdahl originated the idea of open addressing.

Carter and Wegman introduced the notion of universal classes of hash functions in 1979 [58].

Fredman, Komlós, and Szemerédi [112] developed the perfect hashing scheme for static sets presented in Section 11.5. An extension of their method to dynamic sets, handling insertions and deletions in amortized expected time $O(1)$, has been given by Dietzfelbinger et al. [86].

12   Binary Search Trees

The search tree data structure supports many dynamic-set operations, including SEARCH, MINIMUM, MAXIMUM, PREDECESSOR, SUCCESSOR, INSERT, and DELETE. Thus, we can use a search tree both as a dictionary and as a priority queue.

Basic operations on a binary search tree take time proportional to the height of the tree. For a complete binary tree with n nodes, such operations run in $\Theta(\lg n)$ worst-case time. If the tree is a linear chain of n nodes, however, the same operations take $\Theta(n)$ worst-case time. We shall see in Section 12.4 that the expected height of a randomly built binary search tree is $O(\lg n)$, so that basic dynamic-set operations on such a tree take $\Theta(\lg n)$ time on average.

In practice, we can't always guarantee that binary search trees are built randomly, but we can design variations of binary search trees with good guaranteed worst-case performance on basic operations. Chapter 13 presents one such variation, red-black trees, which have height $O(\lg n)$. Chapter 18 introduces B-trees, which are particularly good for maintaining databases on secondary (disk) storage.

After presenting the basic properties of binary search trees, the following sections show how to walk a binary search tree to print its values in sorted order, how to search for a value in a binary search tree, how to find the minimum or maximum element, how to find the predecessor or successor of an element, and how to insert into or delete from a binary search tree. The basic mathematical properties of trees appear in Appendix B.

12.1   What is a binary search tree?

A binary search tree is organized, as the name suggests, in a binary tree, as shown in Figure 12.1. We can represent such a tree by a linked data structure in which each node is an object. In addition to a *key* and satellite data, each node contains attributes *left*, *right*, and *p* that point to the nodes corresponding to its left child,
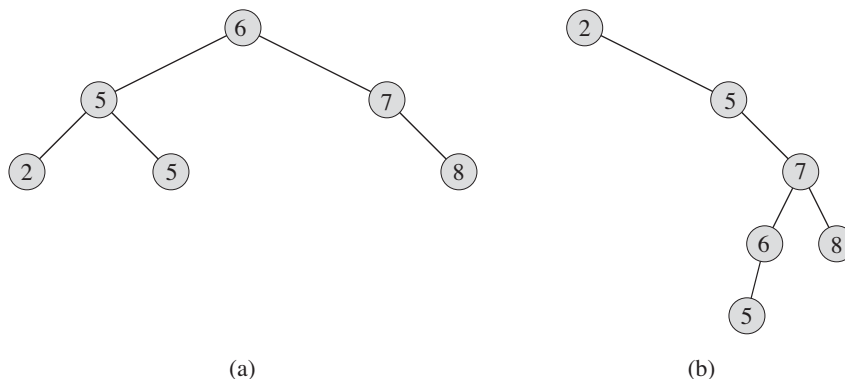
Figure 12.1   Binary search trees. For any node x, the keys in the left subtree of x are at most $x.key$, and the keys in the right subtree of x are at least $x.key$. Different binary search trees can represent the same set of values. The worst-case running time for most search-tree operations is proportional to the height of the tree. **(a)** A binary search tree on 6 nodes with height 2. **(b)** A less efficient binary search tree with height 4 that contains the same keys.

its right child, and its parent, respectively. If a child or the parent is missing, the appropriate attribute contains the value NIL. The root node is the only node in the tree whose parent is NIL.

The keys in a binary search tree are always stored in such a way as to satisfy the **binary-search-tree property**:

Let x be a node in a binary search tree. If y is a node in the left subtree of x, then $y.key \leq x.key$. If y is a node in the right subtree of x, then $y.key \geq x.key$.

Thus, in Figure 12.1(a), the key of the root is 6, the keys 2, 5, and 5 in its left subtree are no larger than 6, and the keys 7 and 8 in its right subtree are no smaller than 6. The same property holds for every node in the tree. For example, the key 5 in the root's left child is no smaller than the key 2 in that node's left subtree and no larger than the key 5 in the right subtree.

The binary-search-tree property allows us to print out all the keys in a binary search tree in sorted order by a simple recursive algorithm, called an **inorder tree walk**. This algorithm is so named because it prints the key of the root of a subtree between printing the values in its left subtree and printing those in its right subtree. (Similarly, a **preorder tree walk** prints the root before the values in either subtree, and a **postorder tree walk** prints the root after the values in its subtrees.) To use the following procedure to print all the elements in a binary search tree T, we call INORDER-TREE-WALK($T.root$).

INORDER-TREE-WALK(x)

```
1   if x ≠ NIL
2       INORDER-TREE-WALK(x.left)
3       print x.key
4       INORDER-TREE-WALK(x.right)
```

As an example, the inorder tree walk prints the keys in each of the two binary search trees from Figure 12.1 in the order $2, 5, 5, 6, 7, 8$. The correctness of the algorithm follows by induction directly from the binary-search-tree property.

It takes $\Theta(n)$ time to walk an n-node binary search tree, since after the initial call, the procedure calls itself recursively exactly twice for each node in the tree—once for its left child and once for its right child. The following theorem gives a formal proof that it takes linear time to perform an inorder tree walk.

Theorem 12.1
If x is the root of an n-node subtree, then the call INORDER-TREE-WALK(x) takes $\Theta(n)$ time.

Proof   Let $T(n)$ denote the time taken by INORDER-TREE-WALK when it is called on the root of an n-node subtree. Since INORDER-TREE-WALK visits all n nodes of the subtree, we have $T(n) = \Omega(n)$. It remains to show that $T(n) = O(n)$.

Since INORDER-TREE-WALK takes a small, constant amount of time on an empty subtree (for the test $x \neq$ NIL), we have $T(0) = c$ for some constant $c > 0$.

For $n > 0$, suppose that INORDER-TREE-WALK is called on a node x whose left subtree has k nodes and whose right subtree has $n - k - 1$ nodes. The time to perform INORDER-TREE-WALK(x) is bounded by $T(n) \leq T(k) + T(n-k-1) + d$ for some constant $d > 0$ that reflects an upper bound on the time to execute the body of INORDER-TREE-WALK(x), exclusive of the time spent in recursive calls.

We use the substitution method to show that $T(n) = O(n)$ by proving that $T(n) \leq (c + d)n + c$. For $n = 0$, we have $(c + d) \cdot 0 + c = c = T(0)$. For $n > 0$, we have

$$
\begin{aligned}
T(n) &\leq T(k) + T(n - k - 1) + d \\
&= ((c + d)k + c) + ((c + d)(n - k - 1) + c) + d \\
&= (c + d)n + c - (c + d) + c + d \\
&= (c + d)n + c ,
\end{aligned}
$$

which completes the proof.   ∎

Exercises

12.1-1
For the set of {1, 4, 5, 10, 16, 17, 21} of keys, draw binary search trees of heights 2, 3, 4, 5, and 6.

12.1-2
What is the difference between the binary-search-tree property and the min-heap property (see page 153)? Can the min-heap property be used to print out the keys of an n-node tree in sorted order in $O(n)$ time? Show how, or explain why not.

12.1-3
Give a nonrecursive algorithm that performs an inorder tree walk. (*Hint:* An easy solution uses a stack as an auxiliary data structure. A more complicated, but elegant, solution uses no stack but assumes that we can test two pointers for equality.)

12.1-4
Give recursive algorithms that perform preorder and postorder tree walks in $\Theta(n)$ time on a tree of n nodes.

12.1-5
Argue that since sorting n elements takes $\Omega(n \lg n)$ time in the worst case in the comparison model, any comparison-based algorithm for constructing a binary search tree from an arbitrary list of n elements takes $\Omega(n \lg n)$ time in the worst case.

12.2   Querying a binary search tree

We often need to search for a key stored in a binary search tree. Besides the SEARCH operation, binary search trees can support such queries as MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR. In this section, we shall examine these operations and show how to support each one in time $O(h)$ on any binary search tree of height h.

Searching

We use the following procedure to search for a node with a given key in a binary search tree. Given a pointer to the root of the tree and a key k, TREE-SEARCH returns a pointer to a node with key k if one exists; otherwise, it returns NIL.
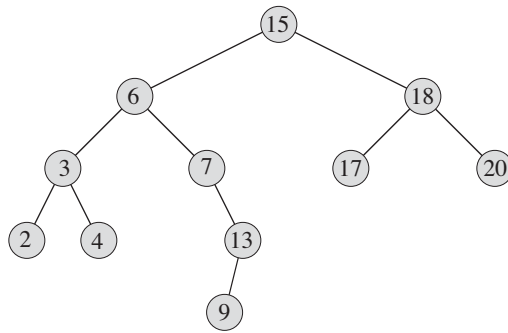
Figure 12.2   Queries on a binary search tree. To search for the key 13 in the tree, we follow the path $15 \rightarrow 6 \rightarrow 7 \rightarrow 13$ from the root. The minimum key in the tree is 2, which is found by following *left* pointers from the root. The maximum key 20 is found by following *right* pointers from the root. The successor of the node with key 15 is the node with key 17, since it is the minimum key in the right subtree of 15. The node with key 13 has no right subtree, and thus its successor is its lowest ancestor whose left child is also an ancestor. In this case, the node with key 15 is its successor.

TREE-SEARCH(x, k)

1   **if** x == NIL or k == $x.key$
2   **return** x
3   **if** $k < x.key$
4   **return** TREE-SEARCH($x.left, k$)
5   **else return** TREE-SEARCH($x.right, k$)

The procedure begins its search at the root and traces a simple path downward in the tree, as shown in Figure 12.2. For each node x it encounters, it compares the key k with $x.key$. If the two keys are equal, the search terminates. If k is smaller than $x.key$, the search continues in the left subtree of x, since the binary-search-tree property implies that k could not be stored in the right subtree. Symmetrically, if k is larger than $x.key$, the search continues in the right subtree. The nodes encountered during the recursion form a simple path downward from the root of the tree, and thus the running time of TREE-SEARCH is $O(h)$, where h is the height of the tree.

We can rewrite this procedure in an iterative fashion by "unrolling" the recursion into a **while** loop. On most computers, the iterative version is more efficient.

ITERATIVE-TREE-SEARCH(x, k)

```
1  while x ≠ NIL and k ≠ x.key
2      if k < x.key
3          x = x.left
4      else x = x.right
5  return x
```

Minimum and maximum

We can always find an element in a binary search tree whose key is a minimum by following *left* child pointers from the root until we encounter a NIL, as shown in Figure 12.2. The following procedure returns a pointer to the minimum element in the subtree rooted at a given node x, which we assume to be non-NIL:

TREE-MINIMUM(x)

```
1  while x.left ≠ NIL
2      x = x.left
3  return x
```

The binary-search-tree property guarantees that TREE-MINIMUM is correct. If a node x has no left subtree, then since every key in the right subtree of x is at least as large as $x.key$, the minimum key in the subtree rooted at x is $x.key$. If node x has a left subtree, then since no key in the right subtree is smaller than $x.key$ and every key in the left subtree is not larger than $x.key$, the minimum key in the subtree rooted at x resides in the subtree rooted at $x.left$.

The pseudocode for TREE-MAXIMUM is symmetric:

TREE-MAXIMUM(x)

```
1  while x.right ≠ NIL
2      x = x.right
3  return x
```

Both of these procedures run in $O(h)$ time on a tree of height h since, as in TREE-SEARCH, the sequence of nodes encountered forms a simple path downward from the root.

Successor and predecessor

Given a node in a binary search tree, sometimes we need to find its successor in the sorted order determined by an inorder tree walk. If all keys are distinct, the

successor of a node x is the node with the smallest key greater than $x.key$. The structure of a binary search tree allows us to determine the successor of a node without ever comparing keys. The following procedure returns the successor of a node x in a binary search tree if it exists, and NIL if x has the largest key in the tree:

TREE-SUCCESSOR(x)

```
1   if x.right ≠ NIL
2       return TREE-MINIMUM(x.right)
3   y = x.p
4   while y ≠ NIL and x == y.right
5       x = y
6       y = y.p
7   return y
```

We break the code for TREE-SUCCESSOR into two cases. If the right subtree of node x is nonempty, then the successor of x is just the leftmost node in x's right subtree, which we find in line 2 by calling TREE-MINIMUM$(x.right)$. For example, the successor of the node with key 15 in Figure 12.2 is the node with key 17.

On the other hand, as Exercise 12.2-6 asks you to show, if the right subtree of node x is empty and x has a successor y, then y is the lowest ancestor of x whose left child is also an ancestor of x. In Figure 12.2, the successor of the node with key 13 is the node with key 15. To find y, we simply go up the tree from x until we encounter a node that is the left child of its parent; lines 3–7 of TREE-SUCCESSOR handle this case.

The running time of TREE-SUCCESSOR on a tree of height h is $O(h)$, since we either follow a simple path up the tree or follow a simple path down the tree. The procedure TREE-PREDECESSOR, which is symmetric to TREE-SUCCESSOR, also runs in time $O(h)$.

Even if keys are not distinct, we define the successor and predecessor of any node x as the node returned by calls made to TREE-SUCCESSOR(x) and TREE-PREDECESSOR(x), respectively.

In summary, we have proved the following theorem.

Theorem 12.2
We can implement the dynamic-set operations SEARCH, MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR so that each one runs in $O(h)$ time on a binary search tree of height h.   ■

Exercises

12.2-1
Suppose that we have numbers between 1 and 1000 in a binary search tree, and we want to search for the number 363. Which of the following sequences could *not* be the sequence of nodes examined?

a. 2, 252, 401, 398, 330, 344, 397, 363.

b. 924, 220, 911, 244, 898, 258, 362, 363.

c. 925, 202, 911, 240, 912, 245, 363.

d. 2, 399, 387, 219, 266, 382, 381, 278, 363.

e. 935, 278, 347, 621, 299, 392, 358, 363.

12.2-2
Write recursive versions of TREE-MINIMUM and TREE-MAXIMUM.

12.2-3
Write the TREE-PREDECESSOR procedure.

12.2-4
Professor Bunyan thinks he has discovered a remarkable property of binary search trees. Suppose that the search for key k in a binary search tree ends up in a leaf. Consider three sets: A, the keys to the left of the search path; B, the keys on the search path; and C, the keys to the right of the search path. Professor Bunyan claims that any three keys $a \in A$, $b \in B$, and $c \in C$ must satisfy $a \leq b \leq c$. Give a smallest possible counterexample to the professor's claim.

12.2-5
Show that if a node in a binary search tree has two children, then its successor has no left child and its predecessor has no right child.

12.2-6
Consider a binary search tree T whose keys are distinct. Show that if the right subtree of a node x in T is empty and x has a successor y, then y is the lowest ancestor of x whose left child is also an ancestor of x. (Recall that every node is its own ancestor.)

12.2-7
An alternative method of performing an inorder tree walk of an n-node binary search tree finds the minimum element in the tree by calling TREE-MINIMUM and then making $n - 1$ calls to TREE-SUCCESSOR. Prove that this algorithm runs in $\Theta(n)$ time.

12.2-8

Prove that no matter what node we start at in a height-h binary search tree, k successive calls to TREE-SUCCESSOR take $O(k + h)$ time.

12.2-9

Let T be a binary search tree whose keys are distinct, let x be a leaf node, and let y be its parent. Show that $y.key$ is either the smallest key in T larger than $x.key$ or the largest key in T smaller than $x.key$.

12.3   Insertion and deletion

The operations of insertion and deletion cause the dynamic set represented by a binary search tree to change. The data structure must be modified to reflect this change, but in such a way that the binary-search-tree property continues to hold. As we shall see, modifying the tree to insert a new element is relatively straight-forward, but handling deletion is somewhat more intricate.

Insertion

To insert a new value v into a binary search tree T, we use the procedure TREE-INSERT. The procedure takes a node z for which $z.key = v$, $z.left =$ NIL, and $z.right =$ NIL. It modifies T and some of the attributes of z in such a way that it inserts z into an appropriate position in the tree.

TREE-INSERT(T, z)

```
 1   y = NIL
 2   x = T.root
 3   while x ≠ NIL
 4       y = x
 5       if z.key < x.key
 6           x = x.left
 7       else x = x.right
 8   z.p = y
 9   if y == NIL
10       T.root = z          // tree T was empty
11   elseif z.key < y.key
12       y.left = z
13   else y.right = z
```
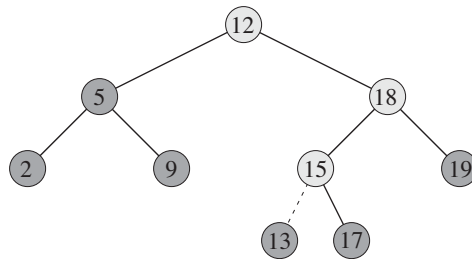
Figure 12.3   Inserting an item with key 13 into a binary search tree. Lightly shaded nodes indicate the simple path from the root down to the position where the item is inserted. The dashed line indicates the link in the tree that is added to insert the item.

Figure 12.3 shows how TREE-INSERT works. Just like the procedures TREE-SEARCH and ITERATIVE-TREE-SEARCH, TREE-INSERT begins at the root of the tree and the pointer x traces a simple path downward looking for a NIL to replace with the input item z. The procedure maintains the ***trailing pointer*** y as the parent of x. After initialization, the **while** loop in lines 3–7 causes these two pointers to move down the tree, going left or right depending on the comparison of $z.key$ with $x.key$, until x becomes NIL. This NIL occupies the position where we wish to place the input item z. We need the trailing pointer y, because by the time we find the NIL where z belongs, the search has proceeded one step beyond the node that needs to be changed. Lines 8–13 set the pointers that cause z to be inserted.

Like the other primitive operations on search trees, the procedure TREE-INSERT runs in $O(h)$ time on a tree of height h.

Deletion

The overall strategy for deleting a node z from a binary search tree T has three basic cases but, as we shall see, one of the cases is a bit tricky.

- If z has no children, then we simply remove it by modifying its parent to re-place z with NIL as its child.

- If z has just one child, then we elevate that child to take z's position in the tree by modifying z's parent to replace z by z's child.

- If z has two children, then we find z's successor y — which must be in z's right subtree — and have y take z's position in the tree. The rest of z's original right subtree becomes y's new right subtree, and z's left subtree becomes y's new left subtree. This case is the tricky one because, as we shall see, it matters whether y is z's right child.

The procedure for deleting a given node z from a binary search tree T takes as arguments pointers to T and z. It organizes its cases a bit differently from the three cases outlined previously by considering the four cases shown in Figure 12.4.

- If z has no left child (part (a) of the figure), then we replace z by its right child, which may or may not be NIL. When z's right child is NIL, this case deals with the situation in which z has no children. When z's right child is non-NIL, this case handles the situation in which z has just one child, which is its right child.

- If z has just one child, which is its left child (part (b) of the figure), then we replace z by its left child.

- Otherwise, z has both a left and a right child. We find z's successor y, which lies in z's right subtree and has no left child (see Exercise 12.2-5). We want to splice y out of its current location and have it replace z in the tree.

  - If y is z's right child (part (c)), then we replace z by y, leaving y's right child alone.

  - Otherwise, y lies within z's right subtree but is not z's right child (part (d)). In this case, we first replace y by its own right child, and then we replace z by y.

In order to move subtrees around within the binary search tree, we define a subroutine TRANSPLANT, which replaces one subtree as a child of its parent with another subtree. When TRANSPLANT replaces the subtree rooted at node u with the subtree rooted at node v, node u's parent becomes node v's parent, and u's parent ends up having v as its appropriate child.

TRANSPLANT(T, u, v)

```
1   if u.p == NIL
2       T.root = v
3   elseif u == u.p.left
4       u.p.left = v
5   else u.p.right = v
6   if v ≠ NIL
7       v.p = u.p
```

Lines 1–2 handle the case in which u is the root of T. Otherwise, u is either a left child or a right child of its parent. Lines 3–4 take care of updating $u.p.left$ if u is a left child, and line 5 updates $u.p.right$ if u is a right child. We allow v to be NIL, and lines 6–7 update $v.p$ if v is non-NIL. Note that TRANSPLANT does not attempt to update $v.left$ and $v.right$; doing so, or not doing so, is the responsibility of TRANSPLANT's caller.
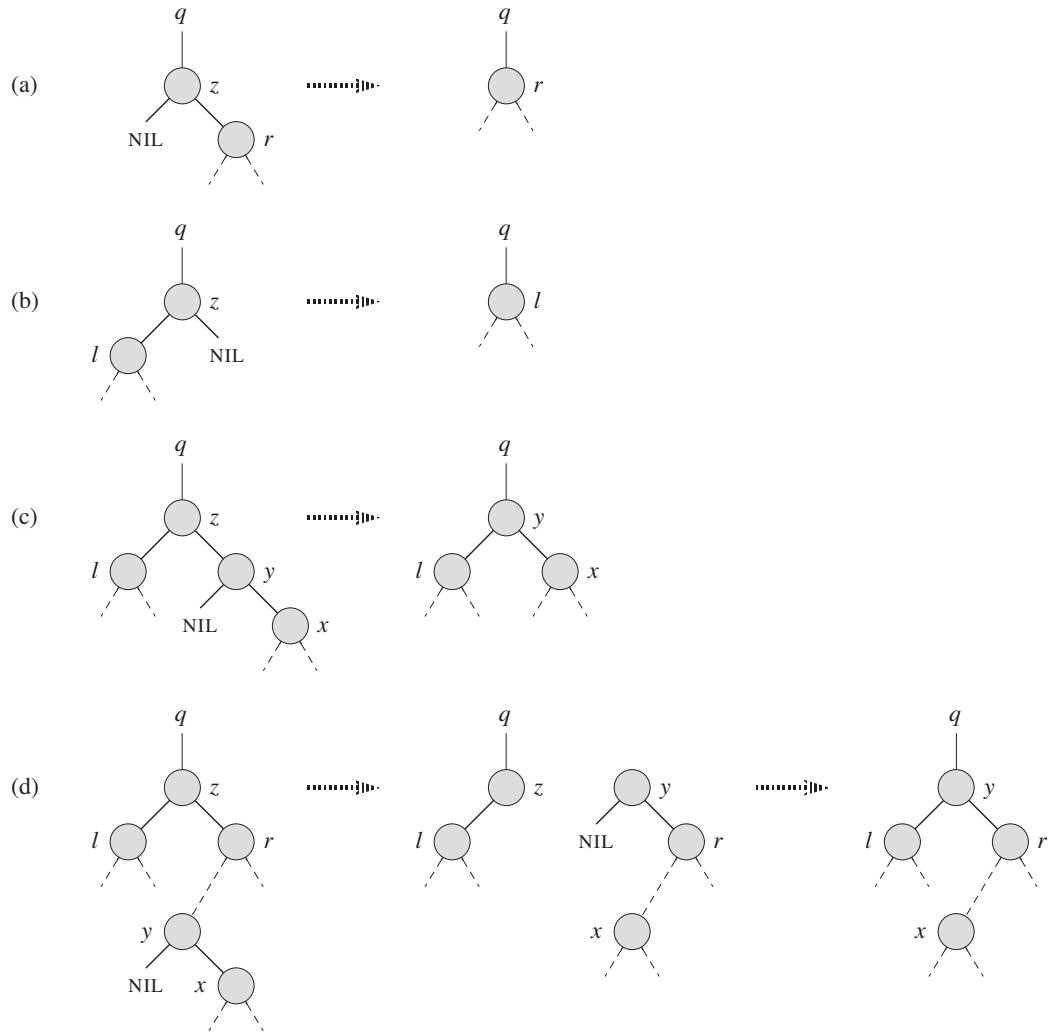
Figure 12.4   Deleting a node z from a binary search tree. Node z may be the root, a left child of node q, or a right child of q. **(a)** Node z has no left child. We replace z by its right child r, which may or may not be NIL. **(b)** Node z has a left child l but no right child. We replace z by l. **(c)** Node z has two children; its left child is node l, its right child is its successor y, and y's right child is node x. We replace z by y, updating y's left child to become l, but leaving x as y's right child. **(d)** Node z has two children (left child l and right child r), and its successor $y \neq r$ lies within the subtree rooted at r. We replace y by its own right child x, and we set y to be r's parent. Then, we set y to be q's child and the parent of l.

With the TRANSPLANT procedure in hand, here is the procedure that deletes node z from binary search tree T:

TREE-DELETE(T, z)

```
 1  if z.left == NIL
 2      TRANSPLANT(T, z, z.right)
 3  elseif z.right == NIL
 4      TRANSPLANT(T, z, z.left)
 5  else y = TREE-MINIMUM(z.right)
 6      if y.p ≠ z
 7          TRANSPLANT(T, y, y.right)
 8          y.right = z.right
 9          y.right.p = y
10      TRANSPLANT(T, z, y)
11      y.left = z.left
12      y.left.p = y
```

The TREE-DELETE procedure executes the four cases as follows. Lines 1–2 handle the case in which node z has no left child, and lines 3–4 handle the case in which z has a left child but no right child. Lines 5–12 deal with the remaining two cases, in which z has two children. Line 5 finds node y, which is the successor of z. Because z has a nonempty right subtree, its successor must be the node in that subtree with the smallest key; hence the call to TREE-MINIMUM$(z.right)$. As we noted before, y has no left child. We want to splice y out of its current location, and it should replace z in the tree. If y is z's right child, then lines 10–12 replace z as a child of its parent by y and replace y's left child by z's left child. If y is not z's left child, lines 7–9 replace y as a child of its parent by y's right child and turn z's right child into y's right child, and then lines 10–12 replace z as a child of its parent by y and replace y's left child by z's left child.

Each line of TREE-DELETE, including the calls to TRANSPLANT, takes constant time, except for the call to TREE-MINIMUM in line 5. Thus, TREE-DELETE runs in $O(h)$ time on a tree of height h.

In summary, we have proved the following theorem.

Theorem 12.3
We can implement the dynamic-set operations INSERT and DELETE so that each one runs in $O(h)$ time on a binary search tree of height h. ∎

Exercises

12.3-1
Give a recursive version of the Tree-Insert procedure.

12.3-2
Suppose that we construct a binary search tree by repeatedly inserting distinct values into the tree. Argue that the number of nodes examined in searching for a value in the tree is one plus the number of nodes examined when the value was first inserted into the tree.

12.3-3
We can sort a given set of n numbers by first building a binary search tree containing these numbers (using Tree-Insert repeatedly to insert the numbers one by one) and then printing the numbers by an inorder tree walk. What are the worst-case and best-case running times for this sorting algorithm?

12.3-4
Is the operation of deletion "commutative" in the sense that deleting x and then y from a binary search tree leaves the same tree as deleting y and then x? Argue why it is or give a counterexample.

12.3-5
Suppose that instead of each node x keeping the attribute $x.p$, pointing to x's parent, it keeps $x.succ$, pointing to x's successor. Give pseudocode for Search, Insert, and Delete on a binary search tree T using this representation. These procedures should operate in time $O(h)$, where h is the height of the tree T. (*Hint:* You may wish to implement a subroutine that returns the parent of a node.)

12.3-6
When node z in Tree-Delete has two children, we could choose node y as its predecessor rather than its successor. What other changes to Tree-Delete would be necessary if we did so? Some have argued that a fair strategy, giving equal priority to predecessor and successor, yields better empirical performance. How might Tree-Delete be changed to implement such a fair strategy?

⋆ **12.4   Randomly built binary search trees**

We have shown that each of the basic operations on a binary search tree runs in $O(h)$ time, where h is the height of the tree. The height of a binary search

tree varies, however, as items are inserted and deleted. If, for example, the n items are inserted in strictly increasing order, the tree will be a chain with height $n - 1$. On the other hand, Exercise B.5-4 shows that $h \geq \lfloor \lg n \rfloor$. As with quicksort, we can show that the behavior of the average case is much closer to the best case than to the worst case.

Unfortunately, little is known about the average height of a binary search tree when both insertion and deletion are used to create it. When the tree is created by insertion alone, the analysis becomes more tractable. Let us therefore define a ***randomly built binary search tree*** on n keys as one that arises from inserting the keys in random order into an initially empty tree, where each of the $n!$ permutations of the input keys is equally likely. (Exercise 12.4-3 asks you to show that this notion is different from assuming that every binary search tree on n keys is equally likely.) In this section, we shall prove the following theorem.

Theorem 12.4
The expected height of a randomly built binary search tree on n distinct keys is $O(\lg n)$.

Proof   We start by defining three random variables that help measure the height of a randomly built binary search tree. We denote the height of a randomly built binary search on n keys by X_n, and we define the ***exponential height*** $Y_n = 2^{X_n}$. When we build a binary search tree on n keys, we choose one key as that of the root, and we let R_n denote the random variable that holds this key's ***rank*** within the set of n keys; that is, R_n holds the position that this key would occupy if the set of keys were sorted. The value of R_n is equally likely to be any element of the set $\{1, 2, \ldots, n\}$. If $R_n = i$, then the left subtree of the root is a randomly built binary search tree on $i - 1$ keys, and the right subtree is a randomly built binary search tree on $n - i$ keys. Because the height of a binary tree is 1 more than the larger of the heights of the two subtrees of the root, the exponential height of a binary tree is twice the larger of the exponential heights of the two subtrees of the root. If we know that $R_n = i$, it follows that

$$Y_n = 2 \cdot \max(Y_{i-1}, Y_{n-i}) \ .$$

As base cases, we have that $Y_1 = 1$, because the exponential height of a tree with 1 node is $2^0 = 1$ and, for convenience, we define $Y_0 = 0$.

Next, define indicator random variables $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$, where

$$Z_{n,i} = \mathrm{I}\{R_n = i\} \ .$$

Because R_n is equally likely to be any element of $\{1, 2, \ldots, n\}$, it follows that $\Pr\{R_n = i\} = 1/n$ for $i = 1, 2, \ldots, n$, and hence, by Lemma 5.1, we have

$$\mathrm{E}[Z_{n,i}] = 1/n \ , \tag{12.1}$$

for $i = 1, 2, \ldots, n$. Because exactly one value of $Z_{n,i}$ is 1 and all others are 0, we also have

$$Y_n = \sum_{i=1}^{n} Z_{n,i} \left(2 \cdot \max(Y_{i-1}, Y_{n-i})\right) .$$

We shall show that $\mathrm{E}\left[Y_n\right]$ is polynomial in n, which will ultimately imply that $\mathrm{E}\left[X_n\right] = O(\lg n)$.

We claim that the indicator random variable $Z_{n,i} = \mathrm{I}\{R_n = i\}$ is independent of the values of Y_{i-1} and Y_{n-i}. Having chosen $R_n = i$, the left subtree (whose exponential height is Y_{i-1}) is randomly built on the $i - 1$ keys whose ranks are less than i. This subtree is just like any other randomly built binary search tree on $i - 1$ keys. Other than the number of keys it contains, this subtree's structure is not affected at all by the choice of $R_n = i$, and hence the random variables Y_{i-1} and $Z_{n,i}$ are independent. Likewise, the right subtree, whose exponential height is Y_{n-i}, is randomly built on the $n - i$ keys whose ranks are greater than i. Its structure is independent of the value of R_n, and so the random variables Y_{n-i} and $Z_{n,i}$ are independent. Hence, we have

$$
\begin{aligned}
\mathrm{E}\left[Y_n\right] &= \mathrm{E}\left[\sum_{i=1}^{n} Z_{n,i} \left(2 \cdot \max(Y_{i-1}, Y_{n-i})\right)\right] \\
&= \sum_{i=1}^{n} \mathrm{E}\left[Z_{n,i} \left(2 \cdot \max(Y_{i-1}, Y_{n-i})\right)\right] \quad \text{(by linearity of expectation)} \\
&= \sum_{i=1}^{n} \mathrm{E}\left[Z_{n,i}\right] \mathrm{E}\left[2 \cdot \max(Y_{i-1}, Y_{n-i})\right] \quad \text{(by independence)} \\
&= \sum_{i=1}^{n} \frac{1}{n} \cdot \mathrm{E}\left[2 \cdot \max(Y_{i-1}, Y_{n-i})\right] \quad \text{(by equation (12.1))} \\
&= \frac{2}{n} \sum_{i=1}^{n} \mathrm{E}\left[\max(Y_{i-1}, Y_{n-i})\right] \quad \text{(by equation (C.22))} \\
&\leq \frac{2}{n} \sum_{i=1}^{n} \left(\mathrm{E}\left[Y_{i-1}\right] + \mathrm{E}\left[Y_{n-i}\right]\right) \quad \text{(by Exercise C.3-4) .}
\end{aligned}
$$

Since each term $\mathrm{E}\left[Y_0\right], \mathrm{E}\left[Y_1\right], \ldots, \mathrm{E}\left[Y_{n-1}\right]$ appears twice in the last summation, once as $\mathrm{E}\left[Y_{i-1}\right]$ and once as $\mathrm{E}\left[Y_{n-i}\right]$, we have the recurrence

$$\mathrm{E}\left[Y_n\right] \leq \frac{4}{n} \sum_{i=0}^{n-1} \mathrm{E}\left[Y_i\right] . \tag{12.2}$$

Using the substitution method, we shall show that for all positive integers n, the recurrence (12.2) has the solution

$$E[Y_n] \le \frac{1}{4}\binom{n+3}{3}.$$

In doing so, we shall use the identity

$$\sum_{i=0}^{n-1}\binom{i+3}{3} = \binom{n+3}{4}. \tag{12.3}$$

(Exercise 12.4-1 asks you to prove this identity.)

For the base cases, we note that the bounds $0 = Y_0 = E[Y_0] \le (1/4)\binom{3}{3} = 1/4$ and $1 = Y_1 = E[Y_1] \le (1/4)\binom{1+3}{3} = 1$ hold. For the inductive case, we have that

$$
\begin{aligned}
E[Y_n] &\le \frac{4}{n}\sum_{i=0}^{n-1}E[Y_i] \\
&\le \frac{4}{n}\sum_{i=0}^{n-1}\frac{1}{4}\binom{i+3}{3} \qquad \text{(by the inductive hypothesis)} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\binom{i+3}{3} \\
&= \frac{1}{n}\binom{n+3}{4} \qquad \text{(by equation (12.3))} \\
&= \frac{1}{n}\cdot\frac{(n+3)!}{4!\,(n-1)!} \\
&= \frac{1}{4}\cdot\frac{(n+3)!}{3!\,n!} \\
&= \frac{1}{4}\binom{n+3}{3}.
\end{aligned}
$$

We have bounded $E[Y_n]$, but our ultimate goal is to bound $E[X_n]$. As Exercise 12.4-4 asks you to show, the function $f(x) = 2^x$ is convex (see page 1199). Therefore, we can employ Jensen's inequality (C.26), which says that

$$
\begin{aligned}
2^{E[X_n]} &\le E\left[2^{X_n}\right] \\
&= E[Y_n],
\end{aligned}
$$

as follows:

$$2^{E[X_n]} \le \frac{1}{4}\binom{n+3}{3}$$

$$= \frac{1}{4} \cdot \frac{(n+3)(n+2)(n+1)}{6}$$

$$= \frac{n^3 + 6n^2 + 11n + 6}{24} .$$

Taking logarithms of both sides gives $\mathrm{E}\,[X_n] = O(\lg n)$. ■

Exercises

12.4-1
Prove equation (12.3).

12.4-2
Describe a binary search tree on n nodes such that the average depth of a node in the tree is $\Theta(\lg n)$ but the height of the tree is $\omega(\lg n)$. Give an asymptotic upper bound on the height of an n-node binary search tree in which the average depth of a node is $\Theta(\lg n)$.

12.4-3
Show that the notion of a randomly chosen binary search tree on n keys, where each binary search tree of n keys is equally likely to be chosen, is different from the notion of a randomly built binary search tree given in this section. (*Hint:* List the possibilities when $n = 3$.)

12.4-4
Show that the function $f(x) = 2^x$ is convex.

12.4-5 ★
Consider RANDOMIZED-QUICKSORT operating on a sequence of n distinct input numbers. Prove that for any constant $k > 0$, all but $O(1/n^k)$ of the $n!$ input permutations yield an $O(n \lg n)$ running time.

Problems

12-1 Binary search trees with equal keys
Equal keys pose a problem for the implementation of binary search trees.

a. What is the asymptotic performance of TREE-INSERT when used to insert n items with identical keys into an initially empty binary search tree?

We propose to improve TREE-INSERT by testing before line 5 to determine whether $z.key = x.key$ and by testing before line 11 to determine whether $z.key = y.key$.

If equality holds, we implement one of the following strategies. For each strategy, find the asymptotic performance of inserting n items with identical keys into an initially empty binary search tree. (The strategies are described for line 5, in which we compare the keys of z and x. Substitute y for x to arrive at the strategies for line 11.)

b. Keep a boolean flag $x.b$ at node x, and set x to either $x.left$ or $x.right$ based on the value of $x.b$, which alternates between FALSE and TRUE each time we visit x while inserting a node with the same key as x.

c. Keep a list of nodes with equal keys at x, and insert z into the list.

d. Randomly set x to either $x.left$ or $x.right$. (Give the worst-case performance and informally derive the expected running time.)

12-2   Radix trees

Given two strings $a = a_0 a_1 \ldots a_p$ and $b = b_0 b_1 \ldots b_q$, where each a_i and each b_j is in some ordered set of characters, we say that string a is ***lexicographically less than*** string b if either

1. there exists an integer j, where $0 \le j \le \min(p, q)$, such that $a_i = b_i$ for all $i = 0, 1, \ldots, j - 1$ and $a_j < b_j$, or

2. $p < q$ and $a_i = b_i$ for all $i = 0, 1, \ldots, p$.

For example, if a and b are bit strings, then $10100 < 10110$ by rule 1 (letting $j = 3$) and $10100 < 101000$ by rule 2. This ordering is similar to that used in English-language dictionaries.

The ***radix tree*** data structure shown in Figure 12.5 stores the bit strings 1011, 10, 011, 100, and 0. When searching for a key $a = a_0 a_1 \ldots a_p$, we go left at a node of depth i if $a_i = 0$ and right if $a_i = 1$. Let S be a set of distinct bit strings whose lengths sum to n. Show how to use a radix tree to sort S lexicographically in $\Theta(n)$ time. For the example in Figure 12.5, the output of the sort should be the sequence 0, 011, 10, 100, 1011.

12-3   *Average node depth in a randomly built binary search tree*

In this problem, we prove that the average depth of a node in a randomly built binary search tree with n nodes is $O(\lg n)$. Although this result is weaker than that of Theorem 12.4, the technique we shall use reveals a surprising similarity between the building of a binary search tree and the execution of RANDOMIZED-QUICKSORT from Section 7.3.

We define the ***total path length*** $P(T)$ of a binary tree T as the sum, over all nodes x in T, of the depth of node x, which we denote by $d(x, T)$.
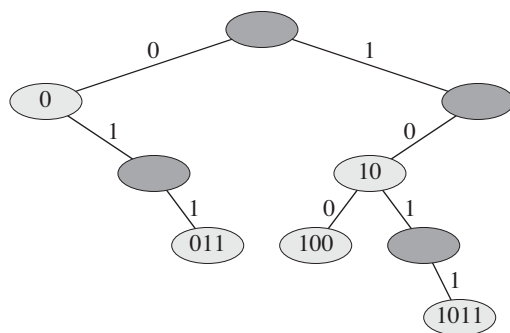
Figure 12.5 A radix tree storing the bit strings 1011, 10, 011, 100, and 0. We can determine each node's key by traversing the simple path from the root to that node. There is no need, therefore, to store the keys in the nodes; the keys appear here for illustrative purposes only. Nodes are heavily shaded if the keys corresponding to them are not in the tree; such nodes are present only to establish a path to other nodes.

a. Argue that the average depth of a node in T is

$$\frac{1}{n} \sum_{x \in T} d(x, T) = \frac{1}{n} P(T) .$$

Thus, we wish to show that the expected value of $P(T)$ is $O(n \lg n)$.

b. Let T_L and T_R denote the left and right subtrees of tree T, respectively. Argue that if T has n nodes, then

$$P(T) = P(T_L) + P(T_R) + n - 1 .$$

c. Let $P(n)$ denote the average total path length of a randomly built binary search tree with n nodes. Show that

$$P(n) = \frac{1}{n} \sum_{i=0}^{n-1} (P(i) + P(n - i - 1) + n - 1) .$$

d. Show how to rewrite $P(n)$ as

$$P(n) = \frac{2}{n} \sum_{k=1}^{n-1} P(k) + \Theta(n) .$$

e. Recalling the alternative analysis of the randomized version of quicksort given in Problem 7-3, conclude that $P(n) = O(n \lg n)$.

At each recursive invocation of quicksort, we choose a random pivot element to partition the set of elements being sorted. Each node of a binary search tree partitions the set of elements that fall into the subtree rooted at that node.

f. Describe an implementation of quicksort in which the comparisons to sort a set of elements are exactly the same as the comparisons to insert the elements into a binary search tree. (The order in which comparisons are made may differ, but the same comparisons must occur.)

12-4 *Number of different binary trees*

Let b_n denote the number of different binary trees with n nodes. In this problem, you will find a formula for b_n, as well as an asymptotic estimate.

a. Show that $b_0 = 1$ and that, for $n \geq 1$,

$$b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k} .$$

b. Referring to Problem 4-4 for the definition of a generating function, let $B(x)$ be the generating function

$$B(x) = \sum_{n=0}^{\infty} b_n x^n .$$

Show that $B(x) = x B(x)^2 + 1$, and hence one way to express $B(x)$ in closed form is

$$B(x) = \frac{1}{2x} \left(1 - \sqrt{1 - 4x} \right) .$$

The **Taylor expansion** of $f(x)$ around the point $x = a$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k ,$$

where $f^{(k)}(x)$ is the kth derivative of f evaluated at x.

c. Show that

$$b_n = \frac{1}{n+1} \binom{2n}{n}$$

(the nth ***Catalan number***) by using the Taylor expansion of $\sqrt{1 - 4x}$ around $x = 0$. (If you wish, instead of using the Taylor expansion, you may use the generalization of the binomial expansion (C.4) to nonintegral exponents n, where for any real number n and for any integer k, we interpret $\binom{n}{k}$ to be $n(n-1) \cdots (n-k+1)/k!$ if $k \geq 0$, and 0 otherwise.)

d. Show that

$$b_n = \frac{4^n}{\sqrt{\pi} n^{3/2}} (1 + O(1/n)) \ .$$

Chapter notes

Knuth [211] contains a good discussion of simple binary search trees as well as many variations. Binary search trees seem to have been independently discovered by a number of people in the late 1950s. Radix trees are often called "tries," which comes from the middle letters in the word *retrieval*. Knuth [211] also discusses them.

Many texts, including the first two editions of this book, have a somewhat simpler method of deleting a node from a binary search tree when both of its children are present. Instead of replacing node z by its successor y, we delete node y but copy its key and satellite data into node z. The downside of this approach is that the node actually deleted might not be the node passed to the delete procedure. If other components of a program maintain pointers to nodes in the tree, they could mistakenly end up with "stale" pointers to nodes that have been deleted. Although the deletion method presented in this edition of this book is a bit more complicated, it guarantees that a call to delete node z deletes node z and only node z.

Section 15.5 will show how to construct an optimal binary search tree when we know the search frequencies before constructing the tree. That is, given the frequencies of searching for each key and the frequencies of searching for values that fall between keys in the tree, we construct a binary search tree for which a set of searches that follows these frequencies examines the minimum number of nodes.

The proof in Section 12.4 that bounds the expected height of a randomly built binary search tree is due to Aslam [24]. Martínez and Roura [243] give randomized algorithms for insertion into and deletion from binary search trees in which the result of either operation is a random binary search tree. Their definition of a random binary search tree differs—only slightly—from that of a randomly built binary search tree in this chapter, however.

13   Red-Black Trees

Chapter 12 showed that a binary search tree of height h can support any of the basic dynamic-set operations—such as SEARCH, PREDECESSOR, SUCCESSOR, MINIMUM, MAXIMUM, INSERT, and DELETE—in $O(h)$ time. Thus, the set operations are fast if the height of the search tree is small. If its height is large, however, the set operations may run no faster than with a linked list. Red-black trees are one of many search-tree schemes that are "balanced" in order to guarantee that basic dynamic-set operations take $O(\lg n)$ time in the worst case.

13.1   Properties of red-black trees

A ***red-black tree*** is a binary search tree with one extra bit of storage per node: its ***color***, which can be either RED or BLACK. By constraining the node colors on any simple path from the root to a leaf, red-black trees ensure that no such path is more than twice as long as any other, so that the tree is approximately ***balanced***.

Each node of the tree now contains the attributes *color*, *key*, *left*, *right*, and *p*. If a child or the parent of a node does not exist, the corresponding pointer attribute of the node contains the value NIL. We shall regard these NILs as being pointers to leaves (external nodes) of the binary search tree and the normal, key-bearing nodes as being internal nodes of the tree.

A red-black tree is a binary tree that satisfies the following ***red-black properties***:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (NIL) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

Figure 13.1(a) shows an example of a red-black tree.

As a matter of convenience in dealing with boundary conditions in red-black tree code, we use a single sentinel to represent NIL (see page 238). For a red-black tree T, the sentinel $T.nil$ is an object with the same attributes as an ordinary node in the tree. Its *color* attribute is BLACK, and its other attributes—p, *left*, *right*, and *key*—can take on arbitrary values. As Figure 13.1(b) shows, all pointers to NIL are replaced by pointers to the sentinel $T.nil$.

We use the sentinel so that we can treat a NIL child of a node x as an ordinary node whose parent is x. Although we instead could add a distinct sentinel node for each NIL in the tree, so that the parent of each NIL is well defined, that approach would waste space. Instead, we use the one sentinel $T.nil$ to represent all the NILs—all leaves and the root's parent. The values of the attributes p, *left*, *right*, and *key* of the sentinel are immaterial, although we may set them during the course of a procedure for our convenience.

We generally confine our interest to the internal nodes of a red-black tree, since they hold the key values. In the remainder of this chapter, we omit the leaves when we draw red-black trees, as shown in Figure 13.1(c).

We call the number of black nodes on any simple path from, but not including, a node x down to a leaf the **black-height** of the node, denoted bh(x). By property 5, the notion of black-height is well defined, since all descending simple paths from the node have the same number of black nodes. We define the black-height of a red-black tree to be the black-height of its root.

The following lemma shows why red-black trees make good search trees.

Lemma 13.1
A red-black tree with n internal nodes has height at most $2 \lg(n + 1)$.

Proof   We start by showing that the subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal nodes. We prove this claim by induction on the height of x. If the height of x is 0, then x must be a leaf ($T.nil$), and the subtree rooted at x indeed contains at least $2^{bh(x)} - 1 = 2^0 - 1 = 0$ internal nodes. For the inductive step, consider a node x that has positive height and is an internal node with two children. Each child has a black-height of either bh(x) or bh(x) − 1, depending on whether its color is red or black, respectively. Since the height of a child of x is less than the height of x itself, we can apply the inductive hypothesis to conclude that each child has at least $2^{bh(x)-1} - 1$ internal nodes. Thus, the subtree rooted at x contains at least $(2^{bh(x)-1} - 1) + (2^{bh(x)-1} - 1) + 1 = 2^{bh(x)} - 1$ internal nodes, which proves the claim.

To complete the proof of the lemma, let h be the height of the tree. According to property 4, at least half the nodes on any simple path from the root to a leaf, not
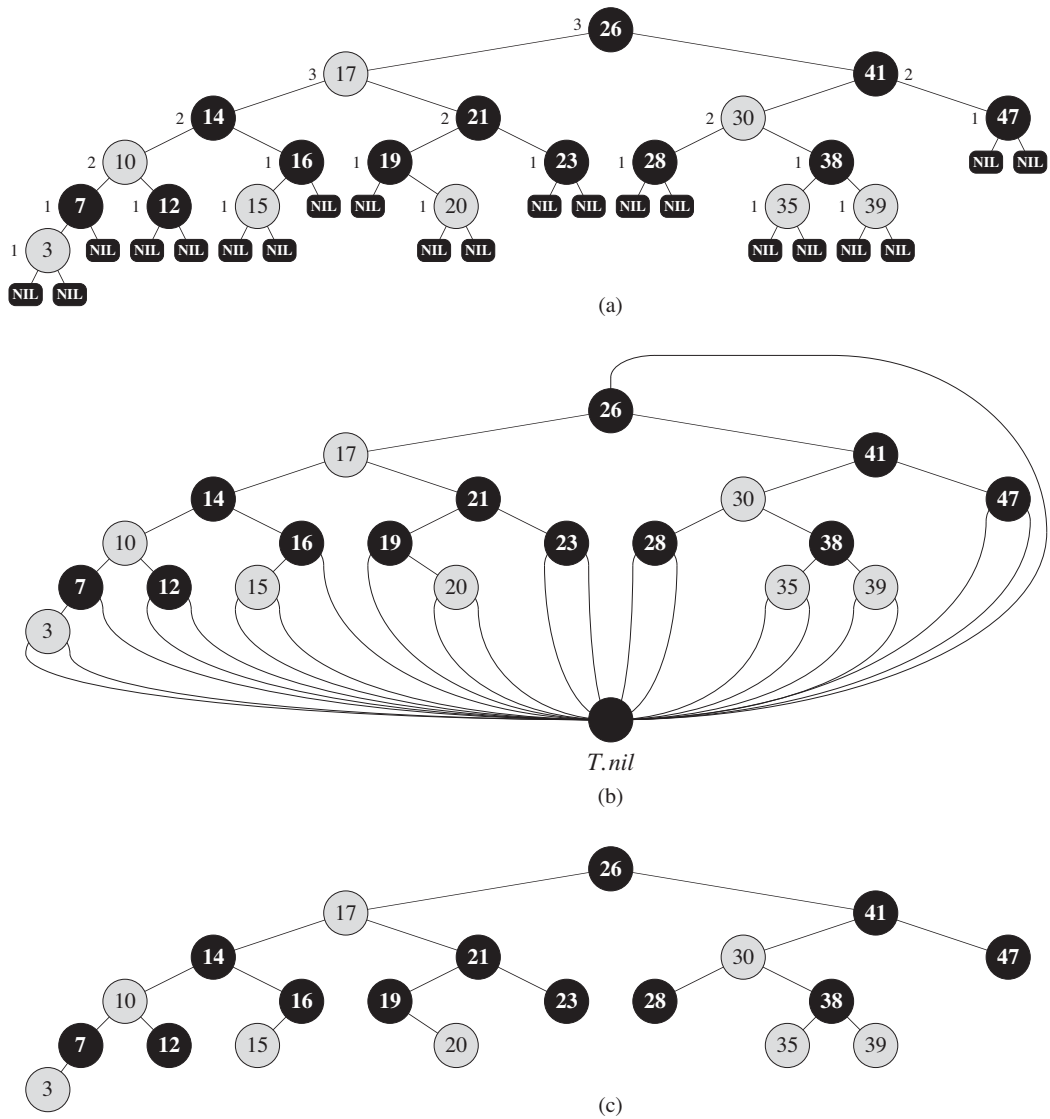
Figure 13.1 A red-black tree with black nodes darkened and red nodes shaded. Every node in a red-black tree is either red or black, the children of a red node are both black, and every simple path from a node to a descendant leaf contains the same number of black nodes. **(a)** Every leaf, shown as a NIL, is black. Each non-NIL node is marked with its black-height; NILs have black-height 0. **(b)** The same red-black tree but with each NIL replaced by the single sentinel $T.nil$, which is always black, and with black-heights omitted. The root's parent is also the sentinel. **(c)** The same red-black tree but with leaves and the root's parent omitted entirely. We shall use this drawing style in the remainder of this chapter.

including the root, must be black. Consequently, the black-height of the root must be at least $h/2$; thus,

$$n \geq 2^{h/2} - 1 .$$

Moving the 1 to the left-hand side and taking logarithms on both sides yields $\lg(n + 1) \geq h/2$, or $h \leq 2\lg(n + 1)$. ∎

   As an immediate consequence of this lemma, we can implement the dynamic-set operations SEARCH, MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR in $O(\lg n)$ time on red-black trees, since each can run in $O(h)$ time on a binary search tree of height h (as shown in Chapter 12) and any red-black tree on n nodes is a binary search tree with height $O(\lg n)$. (Of course, references to NIL in the algorithms of Chapter 12 would have to be replaced by $T.nil$.) Although the algorithms TREE-INSERT and TREE-DELETE from Chapter 12 run in $O(\lg n)$ time when given a red-black tree as input, they do not directly support the dynamic-set operations INSERT and DELETE, since they do not guarantee that the modified binary search tree will be a red-black tree. We shall see in Sections 13.3 and 13.4, however, how to support these two operations in $O(\lg n)$ time.

Exercises

13.1-1
In the style of Figure 13.1(a), draw the complete binary search tree of height 3 on the keys $\{1, 2, \ldots, 15\}$. Add the NIL leaves and color the nodes in three different ways such that the black-heights of the resulting red-black trees are 2, 3, and 4.

13.1-2
Draw the red-black tree that results after TREE-INSERT is called on the tree in Figure 13.1 with key 36. If the inserted node is colored red, is the resulting tree a red-black tree? What if it is colored black?

13.1-3
Let us define a ***relaxed red-black tree*** as a binary search tree that satisfies red-black properties 1, 3, 4, and 5. In other words, the root may be either red or black. Consider a relaxed red-black tree T whose root is red. If we color the root of T black but make no other changes to T, is the resulting tree a red-black tree?

13.1-4
Suppose that we "absorb" every red node in a red-black tree into its black parent, so that the children of the red node become children of the black parent. (Ignore what happens to the keys.) What are the possible degrees of a black node after all

its red children are absorbed? What can you say about the depths of the leaves of the resulting tree?

13.1-5

Show that the longest simple path from a node x in a red-black tree to a descendant leaf has length at most twice that of the shortest simple path from node x to a descendant leaf.

13.1-6

What is the largest possible number of internal nodes in a red-black tree with black-height k? What is the smallest possible number?

13.1-7

Describe a red-black tree on n keys that realizes the largest possible ratio of red internal nodes to black internal nodes. What is this ratio? What tree has the smallest possible ratio, and what is the ratio?

13.2   Rotations

The search-tree operations TREE-INSERT and TREE-DELETE, when run on a red-black tree with n keys, take $O(\lg n)$ time. Because they modify the tree, the result may violate the red-black properties enumerated in Section 13.1. To restore these properties, we must change the colors of some of the nodes in the tree and also change the pointer structure.

We change the pointer structure through ***rotation***, which is a local operation in a search tree that preserves the binary-search-tree property. Figure 13.2 shows the two kinds of rotations: left rotations and right rotations. When we do a left rotation on a node x, we assume that its right child y is not $T.nil$; x may be any node in the tree whose right child is not $T.nil$. The left rotation "pivots" around the link from x to y. It makes y the new root of the subtree, with x as y's left child and y's left child as x's right child.

The pseudocode for LEFT-ROTATE assumes that $x.right \neq T.nil$ and that the root's parent is $T.nil$.
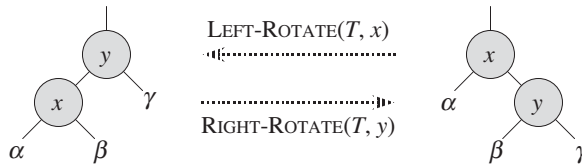
Figure 13.2   The rotation operations on a binary search tree. The operation LEFT-ROTATE(T, x) transforms the configuration of the two nodes on the right into the configuration on the left by changing a constant number of pointers. The inverse operation RIGHT-ROTATE(T, y) transforms the configuration on the left into the configuration on the right. The letters α, β, and γ represent arbitrary subtrees. A rotation operation preserves the binary-search-tree property: the keys in α precede $x.key$, which precedes the keys in β, which precede $y.key$, which precedes the keys in γ.

LEFT-ROTATE(T, x)

```
 1  y = x.right                // set y
 2  x.right = y.left           // turn y's left subtree into x's right subtree
 3  if y.left ≠ T.nil
 4      y.left.p = x
 5  y.p = x.p                  // link x's parent to y
 6  if x.p == T.nil
 7      T.root = y
 8  elseif x == x.p.left
 9      x.p.left = y
10  else x.p.right = y
11  y.left = x                 // put x on y's left
12  x.p = y
```

Figure 13.3 shows an example of how LEFT-ROTATE modifies a binary search tree. The code for RIGHT-ROTATE is symmetric. Both LEFT-ROTATE and RIGHT-ROTATE run in $O(1)$ time. Only pointers are changed by a rotation; all other attributes in a node remain the same.

Exercises

13.2-1
Write pseudocode for RIGHT-ROTATE.

13.2-2
Argue that in every n-node binary search tree, there are exactly $n - 1$ possible rotations.
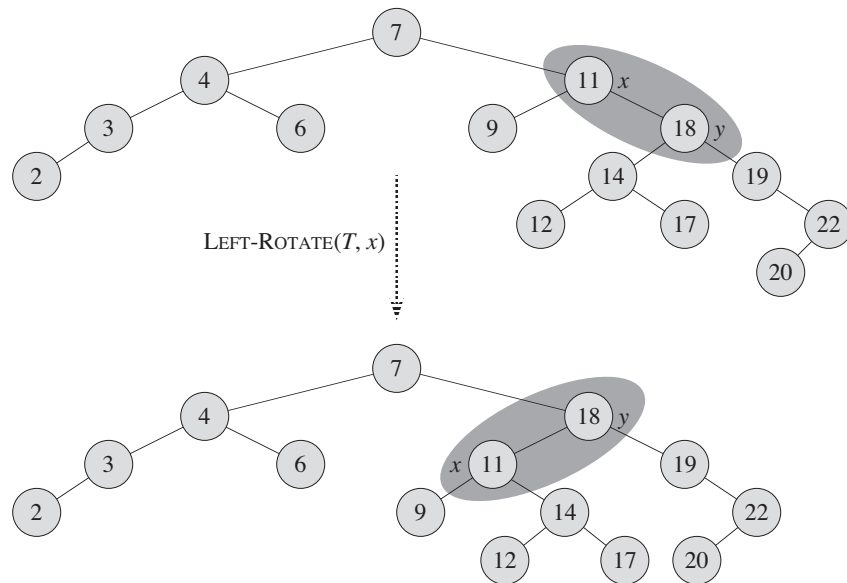
Figure 13.3   An example of how the procedure LEFT-ROTATE(T, x) modifies a binary search tree. Inorder tree walks of the input tree and the modified tree produce the same listing of key values.

13.2-3

Let a, b, and c be arbitrary nodes in subtrees α, β, and γ, respectively, in the left tree of Figure 13.2. How do the depths of a, b, and c change when a left rotation is performed on node x in the figure?

13.2-4

Show that any arbitrary n-node binary search tree can be transformed into any other arbitrary n-node binary search tree using $O(n)$ rotations. (*Hint:* First show that at most $n - 1$ right rotations suffice to transform the tree into a right-going chain.)

13.2-5   ★

We say that a binary search tree T_1 can be **right-converted** to binary search tree T_2 if it is possible to obtain T_2 from T_1 via a series of calls to RIGHT-ROTATE. Give an example of two trees T_1 and T_2 such that T_1 cannot be right-converted to T_2. Then, show that if a tree T_1 can be right-converted to T_2, it can be right-converted using $O(n^2)$ calls to RIGHT-ROTATE.

13.3   Insertion

We can insert a node into an n-node red-black tree in $O(\lg n)$ time. To do so, we
use a slightly modified version of the TREE-INSERT procedure (Section 12.3) to
insert node z into the tree T as if it were an ordinary binary search tree, and then we
color z red. (Exercise 13.3-1 asks you to explain why we choose to make node z
red rather than black.) To guarantee that the red-black properties are preserved, we
then call an auxiliary procedure RB-INSERT-FIXUP to recolor nodes and perform
rotations. The call RB-INSERT(T, z) inserts node z, whose *key* is assumed to have
already been filled in, into the red-black tree T.

RB-INSERT(T, z)

```
 1   y = T.nil
 2   x = T.root
 3   while x ≠ T.nil
 4       y = x
 5       if z.key < x.key
 6           x = x.left
 7       else x = x.right
 8   z.p = y
 9   if y == T.nil
10       T.root = z
11   elseif z.key < y.key
12       y.left = z
13   else y.right = z
14   z.left = T.nil
15   z.right = T.nil
16   z.color = RED
17   RB-INSERT-FIXUP(T, z)
```

The procedures TREE-INSERT and RB-INSERT differ in four ways. First, all
instances of NIL in TREE-INSERT are replaced by $T.nil$. Second, we set $z.left$
and $z.right$ to $T.nil$ in lines 14–15 of RB-INSERT, in order to maintain the
proper tree structure. Third, we color z red in line 16. Fourth, because col-
oring z red may cause a violation of one of the red-black properties, we call
RB-INSERT-FIXUP(T, z) in line 17 of RB-INSERT to restore the red-black prop-
erties.

RB-INSERT-FIXUP(T, z)

```
 1  while z.p.color == RED
 2      if z.p == z.p.p.left
 3          y = z.p.p.right
 4          if y.color == RED
 5              z.p.color = BLACK            // case 1
 6              y.color = BLACK              // case 1
 7              z.p.p.color = RED            // case 1
 8              z = z.p.p                    // case 1
 9          else if z == z.p.right
10              z = z.p                      // case 2
11              LEFT-ROTATE(T, z)            // case 2
12              z.p.color = BLACK            // case 3
13              z.p.p.color = RED            // case 3
14              RIGHT-ROTATE(T, z.p.p)       // case 3
15      else (same as then clause
                with "right" and "left" exchanged)
16  T.root.color = BLACK
```

To understand how RB-INSERT-FIXUP works, we shall break our examination of the code into three major steps. First, we shall determine what violations of the red-black properties are introduced in RB-INSERT when node z is inserted and colored red. Second, we shall examine the overall goal of the **while** loop in lines 1–15. Finally, we shall explore each of the three cases[1] within the **while** loop's body and see how they accomplish the goal. Figure 13.4 shows how RB-INSERT-FIXUP operates on a sample red-black tree.

Which of the red-black properties might be violated upon the call to RB-INSERT-FIXUP? Property 1 certainly continues to hold, as does property 3, since both children of the newly inserted red node are the sentinel $T.nil$. Property 5, which says that the number of black nodes is the same on every simple path from a given node, is satisfied as well, because node z replaces the (black) sentinel, and node z is red with sentinel children. Thus, the only properties that might be violated are property 2, which requires the root to be black, and property 4, which says that a red node cannot have a red child. Both possible violations are due to z being colored red. Property 2 is violated if z is the root, and property 4 is violated if z's parent is red. Figure 13.4(a) shows a violation of property 4 after the node z has been inserted.

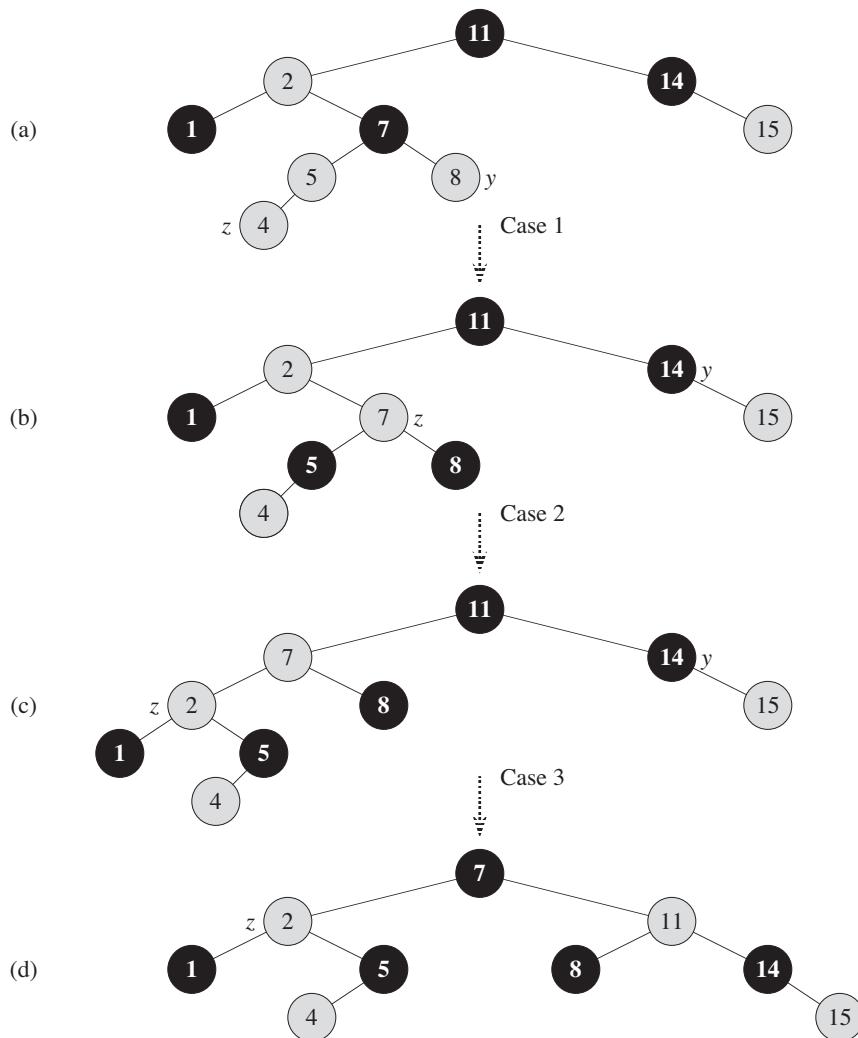[1]Case 2 falls through into case 3, and so these two cases are not mutually exclusive.

Figure 13.4   The operation of RB-INSERT-FIXUP. **(a)** A node z after insertion. Because both z and its parent $z.p$ are red, a violation of property 4 occurs. Since z's uncle y is red, case 1 in the code applies. We recolor nodes and move the pointer z up the tree, resulting in the tree shown in **(b)**. Once again, z and its parent are both red, but z's uncle y is black. Since z is the right child of $z.p$, case 2 applies. We perform a left rotation, and the tree that results is shown in **(c)**. Now, z is the left child of its parent, and case 3 applies. Recoloring and right rotation yield the tree in **(d)**, which is a legal red-black tree.

The **while** loop in lines 1–15 maintains the following three-part invariant at the start of each iteration of the loop:

a. Node z is red.

b. If $z.p$ is the root, then $z.p$ is black.

c. If the tree violates any of the red-black properties, then it violates at most one of them, and the violation is of either property 2 or property 4. If the tree violates property 2, it is because z is the root and is red. If the tree violates property 4, it is because both z and $z.p$ are red.

Part (c), which deals with violations of red-black properties, is more central to showing that RB-INSERT-FIXUP restores the red-black properties than parts (a) and (b), which we use along the way to understand situations in the code. Because we'll be focusing on node z and nodes near it in the tree, it helps to know from part (a) that z is red. We shall use part (b) to show that the node $z.p.p$ exists when we reference it in lines 2, 3, 7, 8, 13, and 14.

Recall that we need to show that a loop invariant is true prior to the first iteration of the loop, that each iteration maintains the loop invariant, and that the loop invariant gives us a useful property at loop termination.

We start with the initialization and termination arguments. Then, as we examine how the body of the loop works in more detail, we shall argue that the loop maintains the invariant upon each iteration. Along the way, we shall also demonstrate that each iteration of the loop has two possible outcomes: either the pointer z moves up the tree, or we perform some rotations and then the loop terminates.

Initialization: Prior to the first iteration of the loop, we started with a red-black tree with no violations, and we added a red node z. We show that each part of the invariant holds at the time RB-INSERT-FIXUP is called:

a. When RB-INSERT-FIXUP is called, z is the red node that was added.

b. If $z.p$ is the root, then $z.p$ started out black and did not change prior to the call of RB-INSERT-FIXUP.

c. We have already seen that properties 1, 3, and 5 hold when RB-INSERT-FIXUP is called.

If the tree violates property 2, then the red root must be the newly added node z, which is the only internal node in the tree. Because the parent and both children of z are the sentinel, which is black, the tree does not also violate property 4. Thus, this violation of property 2 is the only violation of red-black properties in the entire tree.

If the tree violates property 4, then, because the children of node z are black sentinels and the tree had no other violations prior to z being added, the

violation must be because both z and $z.p$ are red. Moreover, the tree violates no other red-black properties.

Termination: When the loop terminates, it does so because $z.p$ is black. (If z is the root, then $z.p$ is the sentinel $T.nil$, which is black.) Thus, the tree does not violate property 4 at loop termination. By the loop invariant, the only property that might fail to hold is property 2. Line 16 restores this property, too, so that when RB-INSERT-FIXUP terminates, all the red-black properties hold.

Maintenance: We actually need to consider six cases in the **while** loop, but three of them are symmetric to the other three, depending on whether line 2 determines z's parent $z.p$ to be a left child or a right child of z's grandparent $z.p.p$. We have given the code only for the situation in which $z.p$ is a left child. The node $z.p.p$ exists, since by part (b) of the loop invariant, if $z.p$ is the root, then $z.p$ is black. Since we enter a loop iteration only if $z.p$ is red, we know that $z.p$ cannot be the root. Hence, $z.p.p$ exists.

We distinguish case 1 from cases 2 and 3 by the color of z's parent's sibling, or "uncle." Line 3 makes y point to z's uncle $z.p.p.right$, and line 4 tests y's color. If y is red, then we execute case 1. Otherwise, control passes to cases 2 and 3. In all three cases, z's grandparent $z.p.p$ is black, since its parent $z.p$ is red, and property 4 is violated only between z and $z.p$.

Case 1: z's uncle y is red

Figure 13.5 shows the situation for case 1 (lines 5–8), which occurs when both $z.p$ and y are red. Because $z.p.p$ is black, we can color both $z.p$ and y black, thereby fixing the problem of z and $z.p$ both being red, and we can color $z.p.p$ red, thereby maintaining property 5. We then repeat the **while** loop with $z.p.p$ as the new node z. The pointer z moves up two levels in the tree.

Now, we show that case 1 maintains the loop invariant at the start of the next iteration. We use z to denote node z in the current iteration, and $z' = z.p.p$ to denote the node that will be called node z at the test in line 1 upon the next iteration.

a. Because this iteration colors $z.p.p$ red, node z' is red at the start of the next iteration.

b. The node $z'.p$ is $z.p.p.p$ in this iteration, and the color of this node does not change. If this node is the root, it was black prior to this iteration, and it remains black at the start of the next iteration.

c. We have already argued that case 1 maintains property 5, and it does not introduce a violation of properties 1 or 3.
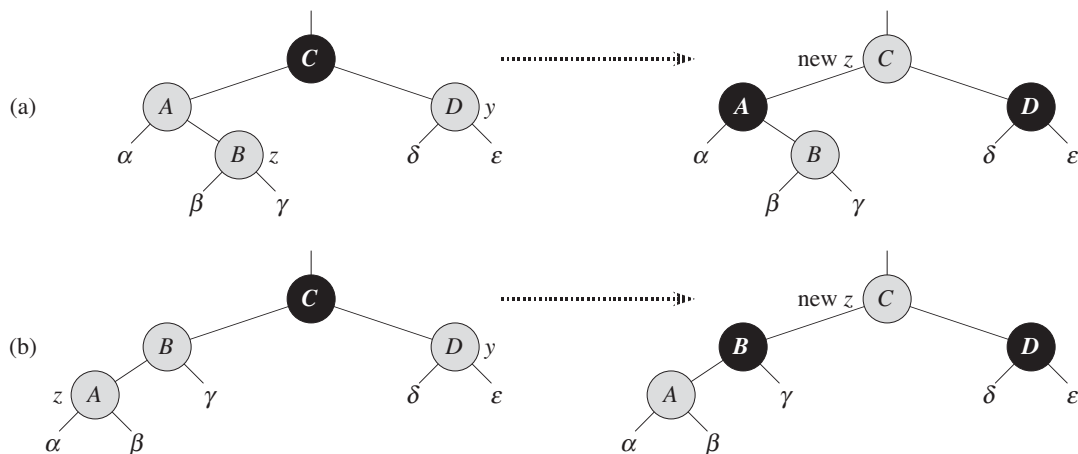
Figure 13.5   Case 1 of the procedure RB-INSERT-FIXUP. Property 4 is violated, since z and its parent $z.p$ are both red. We take the same action whether **(a)** z is a right child or **(b)** z is a left child. Each of the subtrees α, β, γ, δ, and ε has a black root, and each has the same black-height. The code for case 1 changes the colors of some nodes, preserving property 5: all downward simple paths from a node to a leaf have the same number of blacks. The **while** loop continues with node z's grandparent $z.p.p$ as the new z. Any violation of property 4 can now occur only between the new z, which is red, and its parent, if it is red as well.

If node z' is the root at the start of the next iteration, then case 1 corrected the lone violation of property 4 in this iteration. Since z' is red and it is the root, property 2 becomes the only one that is violated, and this violation is due to z'.

If node z' is not the root at the start of the next iteration, then case 1 has not created a violation of property 2. Case 1 corrected the lone violation of property 4 that existed at the start of this iteration. It then made z' red and left $z'.p$ alone. If $z'.p$ was black, there is no violation of property 4. If $z'.p$ was red, coloring z' red created one violation of property 4 between z' and $z'.p$.

Case 2: z's uncle y is black and z is a right child
Case 3: z's uncle y is black and z is a left child

In cases 2 and 3, the color of z's uncle y is black. We distinguish the two cases according to whether z is a right or left child of $z.p$. Lines 10–11 constitute case 2, which is shown in Figure 13.6 together with case 3. In case 2, node z is a right child of its parent. We immediately use a left rotation to transform the situation into case 3 (lines 12–14), in which node z is a left child. Because
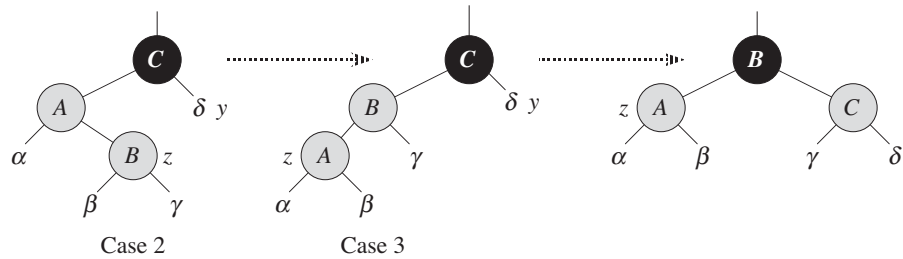
Figure 13.6   Cases 2 and 3 of the procedure RB-INSERT-FIXUP. As in case 1, property 4 is violated in either case 2 or case 3 because z and its parent $z.p$ are both red. Each of the subtrees α, β, γ, and δ has a black root (α, β, and γ from property 4, and δ because otherwise we would be in case 1), and each has the same black-height. We transform case 2 into case 3 by a left rotation, which preserves property 5: all downward simple paths from a node to a leaf have the same number of blacks. Case 3 causes some color changes and a right rotation, which also preserve property 5. The **while** loop then terminates, because property 4 is satisfied: there are no longer two red nodes in a row.

both z and $z.p$ are red, the rotation affects neither the black-height of nodes nor property 5. Whether we enter case 3 directly or through case 2, z's uncle y is black, since otherwise we would have executed case 1. Additionally, the node $z.p.p$ exists, since we have argued that this node existed at the time that lines 2 and 3 were executed, and after moving z up one level in line 10 and then down one level in line 11, the identity of $z.p.p$ remains unchanged. In case 3, we execute some color changes and a right rotation, which preserve property 5, and then, since we no longer have two red nodes in a row, we are done. The **while** loop does not iterate another time, since $z.p$ is now black.

We now show that cases 2 and 3 maintain the loop invariant. (As we have just argued, $z.p$ will be black upon the next test in line 1, and the loop body will not execute again.)

a. Case 2 makes z point to $z.p$, which is red. No further change to z or its color occurs in cases 2 and 3.

b. Case 3 makes $z.p$ black, so that if $z.p$ is the root at the start of the next iteration, it is black.

c. As in case 1, properties 1, 3, and 5 are maintained in cases 2 and 3.

   Since node z is not the root in cases 2 and 3, we know that there is no violation of property 2. Cases 2 and 3 do not introduce a violation of property 2, since the only node that is made red becomes a child of a black node by the rotation in case 3.

   Cases 2 and 3 correct the lone violation of property 4, and they do not introduce another violation.

Having shown that each iteration of the loop maintains the invariant, we have shown that RB-INSERT-FIXUP correctly restores the red-black properties.

Analysis

What is the running time of RB-INSERT? Since the height of a red-black tree on n nodes is $O(\lg n)$, lines 1–16 of RB-INSERT take $O(\lg n)$ time. In RB-INSERT-FIXUP, the **while** loop repeats only if case 1 occurs, and then the pointer z moves two levels up the tree. The total number of times the **while** loop can be executed is therefore $O(\lg n)$. Thus, RB-INSERT takes a total of $O(\lg n)$ time. Moreover, it never performs more than two rotations, since the **while** loop terminates if case 2 or case 3 is executed.

Exercises

13.3-1
In line 16 of RB-INSERT, we set the color of the newly inserted node z to red. Observe that if we had chosen to set z's color to black, then property 4 of a red-black tree would not be violated. Why didn't we choose to set z's color to black?

13.3-2
Show the red-black trees that result after successively inserting the keys $41, 38, 31, 12, 19, 8$ into an initially empty red-black tree.

13.3-3
Suppose that the black-height of each of the subtrees $\alpha, \beta, \gamma, \delta, \varepsilon$ in Figures 13.5 and 13.6 is k. Label each node in each figure with its black-height to verify that the indicated transformation preserves property 5.

13.3-4
Professor Teach is concerned that RB-INSERT-FIXUP might set $T.nil.color$ to RED, in which case the test in line 1 would not cause the loop to terminate when z is the root. Show that the professor's concern is unfounded by arguing that RB-INSERT-FIXUP never sets $T.nil.color$ to RED.

13.3-5
Consider a red-black tree formed by inserting n nodes with RB-INSERT. Argue that if $n > 1$, the tree has at least one red node.

13.3-6
Suggest how to implement RB-INSERT efficiently if the representation for red-black trees includes no storage for parent pointers.

13.4 Deletion

Like the other basic operations on an n-node red-black tree, deletion of a node takes time $O(\lg n)$. Deleting a node from a red-black tree is a bit more complicated than inserting a node.

The procedure for deleting a node from a red-black tree is based on the TREE-DELETE procedure (Section 12.3). First, we need to customize the TRANSPLANT subroutine that TREE-DELETE calls so that it applies to a red-black tree:

RB-TRANSPLANT(T, u, v)

1   **if** $u.p == T.nil$
2       $T.root = v$
3   **elseif** $u == u.p.left$
4       $u.p.left = v$
5   **else** $u.p.right = v$
6   $v.p = u.p$

The procedure RB-TRANSPLANT differs from TRANSPLANT in two ways. First, line 1 references the sentinel $T.nil$ instead of NIL. Second, the assignment to $v.p$ in line 6 occurs unconditionally: we can assign to $v.p$ even if v points to the sentinel. In fact, we shall exploit the ability to assign to $v.p$ when $v = T.nil$.

The procedure RB-DELETE is like the TREE-DELETE procedure, but with additional lines of pseudocode. Some of the additional lines keep track of a node y that might cause violations of the red-black properties. When we want to delete node z and z has fewer than two children, then z is removed from the tree, and we want y to be z. When z has two children, then y should be z's successor, and y moves into z's position in the tree. We also remember y's color before it is removed from or moved within the tree, and we keep track of the node x that moves into y's original position in the tree, because node x might also cause violations of the red-black properties. After deleting node z, RB-DELETE calls an auxiliary procedure RB-DELETE-FIXUP, which changes colors and performs rotations to restore the red-black properties.

RB-DELETE(T, z)

```
 1  y = z
 2  y-original-color = y.color
 3  if z.left == T.nil
 4      x = z.right
 5      RB-TRANSPLANT(T, z, z.right)
 6  elseif z.right == T.nil
 7      x = z.left
 8      RB-TRANSPLANT(T, z, z.left)
 9  else y = TREE-MINIMUM(z.right)
10      y-original-color = y.color
11      x = y.right
12      if y.p == z
13          x.p = y
14      else RB-TRANSPLANT(T, y, y.right)
15          y.right = z.right
16          y.right.p = y
17      RB-TRANSPLANT(T, z, y)
18      y.left = z.left
19      y.left.p = y
20      y.color = z.color
21  if y-original-color == BLACK
22      RB-DELETE-FIXUP(T, x)
```

    Although RB-DELETE contains almost twice as many lines of pseudocode as TREE-DELETE, the two procedures have the same basic structure. You can find each line of TREE-DELETE within RB-DELETE (with the changes of replacing NIL by $T.nil$ and replacing calls to TRANSPLANT by calls to RB-TRANSPLANT), executed under the same conditions.

    Here are the other differences between the two procedures:

- We maintain node y as the node either removed from the tree or moved within the tree. Line 1 sets y to point to node z when z has fewer than two children and is therefore removed. When z has two children, line 9 sets y to point to z's successor, just as in TREE-DELETE, and y will move into z's position in the tree.

- Because node y's color might change, the variable *y-original-color* stores y's color before any changes occur. Lines 2 and 10 set this variable immediately after assignments to y. When z has two children, then $y \neq z$ and node y moves into node z's original position in the red-black tree; line 20 gives y the same color as z. We need to save y's original color in order to test it at the

end of RB-DELETE; if it was black, then removing or moving y could cause violations of the red-black properties.

- As discussed, we keep track of the node x that moves into node y's original position. The assignments in lines 4, 7, and 11 set x to point to either y's only child or, if y has no children, the sentinel $T.nil$. (Recall from Section 12.3 that y has no left child.)

- Since node x moves into node y's original position, the attribute $x.p$ is always set to point to the original position in the tree of y's parent, even if x is, in fact, the sentinel $T.nil$. Unless z is y's original parent (which occurs only when z has two children and its successor y is z's right child), the assignment to $x.p$ takes place in line 6 of RB-TRANSPLANT. (Observe that when RB-TRANSPLANT is called in lines 5, 8, or 14, the second parameter passed is the same as x.)

  When y's original parent is z, however, we do not want $x.p$ to point to y's original parent, since we are removing that node from the tree. Because node y will move up to take z's position in the tree, setting $x.p$ to y in line 13 causes $x.p$ to point to the original position of y's parent, even if $x = T.nil$.

- Finally, if node y was black, we might have introduced one or more violations of the red-black properties, and so we call RB-DELETE-FIXUP in line 22 to restore the red-black properties. If y was red, the red-black properties still hold when y is removed or moved, for the following reasons:

  1. No black-heights in the tree have changed.
  2. No red nodes have been made adjacent. Because y takes z's place in the tree, along with z's color, we cannot have two adjacent red nodes at y's new position in the tree. In addition, if y was not z's right child, then y's original right child x replaces y in the tree. If y is red, then x must be black, and so replacing y by x cannot cause two red nodes to become adjacent.
  3. Since y could not have been the root if it was red, the root remains black.

If node y was black, three problems may arise, which the call of RB-DELETE-FIXUP will remedy. First, if y had been the root and a red child of y becomes the new root, we have violated property 2. Second, if both x and $x.p$ are red, then we have violated property 4. Third, moving y within the tree causes any simple path that previously contained y to have one fewer black node. Thus, property 5 is now violated by any ancestor of y in the tree. We can correct the violation of property 5 by saying that node x, now occupying y's original position, has an "extra" black. That is, if we add 1 to the count of black nodes on any simple path that contains x, then under this interpretation, property 5 holds. When we remove or move the black node y, we "push" its blackness onto node x. The problem is that now node x is neither red nor black, thereby violating property 1. Instead,

node x is either "doubly black" or "red-and-black," and it contributes either 2 or 1, respectively, to the count of black nodes on simple paths containing x. The *color* attribute of x will still be either RED (if x is red-and-black) or BLACK (if x is doubly black). In other words, the extra black on a node is reflected in x's pointing to the node rather than in the *color* attribute.

We can now see the procedure RB-DELETE-FIXUP and examine how it restores the red-black properties to the search tree.

RB-DELETE-FIXUP(T, x)

```
 1   while x ≠ T.root and x.color == BLACK
 2       if x == x.p.left
 3           w = x.p.right
 4           if w.color == RED
 5               w.color = BLACK                               // case 1
 6               x.p.color = RED                               // case 1
 7               LEFT-ROTATE(T, x.p)                           // case 1
 8               w = x.p.right                                 // case 1
 9           if w.left.color == BLACK and w.right.color == BLACK
10               w.color = RED                                 // case 2
11               x = x.p                                       // case 2
12           else if w.right.color == BLACK
13                   w.left.color = BLACK                      // case 3
14                   w.color = RED                             // case 3
15                   RIGHT-ROTATE(T, w)                        // case 3
16                   w = x.p.right                             // case 3
17               w.color = x.p.color                           // case 4
18               x.p.color = BLACK                             // case 4
19               w.right.color = BLACK                         // case 4
20               LEFT-ROTATE(T, x.p)                           // case 4
21               x = T.root                                    // case 4
22       else (same as then clause with "right" and "left" exchanged)
23   x.color = BLACK
```

The procedure RB-DELETE-FIXUP restores properties 1, 2, and 4. Exercises 13.4-1 and 13.4-2 ask you to show that the procedure restores properties 2 and 4, and so in the remainder of this section, we shall focus on property 1. The goal of the **while** loop in lines 1–22 is to move the extra black up the tree until

1. x points to a red-and-black node, in which case we color x (singly) black in line 23;

2. x points to the root, in which case we simply "remove" the extra black; or

3. having performed suitable rotations and recolorings, we exit the loop.

Within the **while** loop, x always points to a nonroot doubly black node. We determine in line 2 whether x is a left child or a right child of its parent $x.p$. (We have given the code for the situation in which x is a left child; the situation in which x is a right child—line 22—is symmetric.) We maintain a pointer w to the sibling of x. Since node x is doubly black, node w cannot be $T.nil$, because otherwise, the number of blacks on the simple path from $x.p$ to the (singly black) leaf w would be smaller than the number on the simple path from $x.p$ to x.

The four cases[2] in the code appear in Figure 13.7. Before examining each case in detail, let's look more generally at how we can verify that the transformation in each of the cases preserves property 5. The key idea is that in each case, the transformation applied preserves the number of black nodes (including x's extra black) from (and including) the root of the subtree shown to each of the subtrees $\alpha, \beta, \ldots, \zeta$. Thus, if property 5 holds prior to the transformation, it continues to hold afterward. For example, in Figure 13.7(a), which illustrates case 1, the number of black nodes from the root to either subtree α or β is 3, both before and after the transformation. (Again, remember that node x adds an extra black.) Similarly, the number of black nodes from the root to any of γ, δ, ε, and ζ is 2, both before and after the transformation. In Figure 13.7(b), the counting must involve the value c of the *color* attribute of the root of the subtree shown, which can be either RED or BLACK. If we define count(RED) $= 0$ and count(BLACK) $= 1$, then the number of black nodes from the root to α is $2 + \text{count}(c)$, both before and after the transformation. In this case, after the transformation, the new node x has *color* attribute c, but this node is really either red-and-black (if $c = $ RED) or doubly black (if $c = $ BLACK). You can verify the other cases similarly (see Exercise 13.4-5).

Case 1: *x's sibling w is red*

Case 1 (lines 5–8 of RB-DELETE-FIXUP and Figure 13.7(a)) occurs when node w, the sibling of node x, is red. Since w must have black children, we can switch the colors of w and $x.p$ and then perform a left-rotation on $x.p$ without violating any of the red-black properties. The new sibling of x, which is one of w's children prior to the rotation, is now black, and thus we have converted case 1 into case 2, 3, or 4.

Cases 2, 3, and 4 occur when node w is black; they are distinguished by the colors of w's children.

---

[2]As in RB-INSERT-FIXUP, the cases in RB-DELETE-FIXUP are not mutually exclusive.

Case 2: x's sibling w is black, and both of w's children are black

In case 2 (lines 10–11 of RB-Delete-Fixup and Figure 13.7(b)), both of w's children are black. Since w is also black, we take one black off both x and w, leaving x with only one black and leaving w red. To compensate for removing one black from x and w, we would like to add an extra black to $x.p$, which was originally either red or black. We do so by repeating the **while** loop with $x.p$ as the new node x. Observe that if we enter case 2 through case 1, the new node x is red-and-black, since the original $x.p$ was red. Hence, the value c of the *color* attribute of the new node x is RED, and the loop terminates when it tests the loop condition. We then color the new node x (singly) black in line 23.

Case 3: x's sibling w is black, w's left child is red, and w's right child is black

Case 3 (lines 13–16 and Figure 13.7(c)) occurs when w is black, its left child is red, and its right child is black. We can switch the colors of w and its left child $w.left$ and then perform a right rotation on w without violating any of the red-black properties. The new sibling w of x is now a black node with a red right child, and thus we have transformed case 3 into case 4.

Case 4: x's sibling w is black, and w's right child is red

Case 4 (lines 17–21 and Figure 13.7(d)) occurs when node x's sibling w is black and w's right child is red. By making some color changes and performing a left rotation on $x.p$, we can remove the extra black on x, making it singly black, without violating any of the red-black properties. Setting x to be the root causes the **while** loop to terminate when it tests the loop condition.

Analysis

What is the running time of RB-Delete? Since the height of a red-black tree of n nodes is $O(\lg n)$, the total cost of the procedure without the call to RB-Delete-Fixup takes $O(\lg n)$ time. Within RB-Delete-Fixup, each of cases 1, 3, and 4 lead to termination after performing a constant number of color changes and at most three rotations. Case 2 is the only case in which the **while** loop can be repeated, and then the pointer x moves up the tree at most $O(\lg n)$ times, performing no rotations. Thus, the procedure RB-Delete-Fixup takes $O(\lg n)$ time and performs at most three rotations, and the overall time for RB-Delete is therefore also $O(\lg n)$.
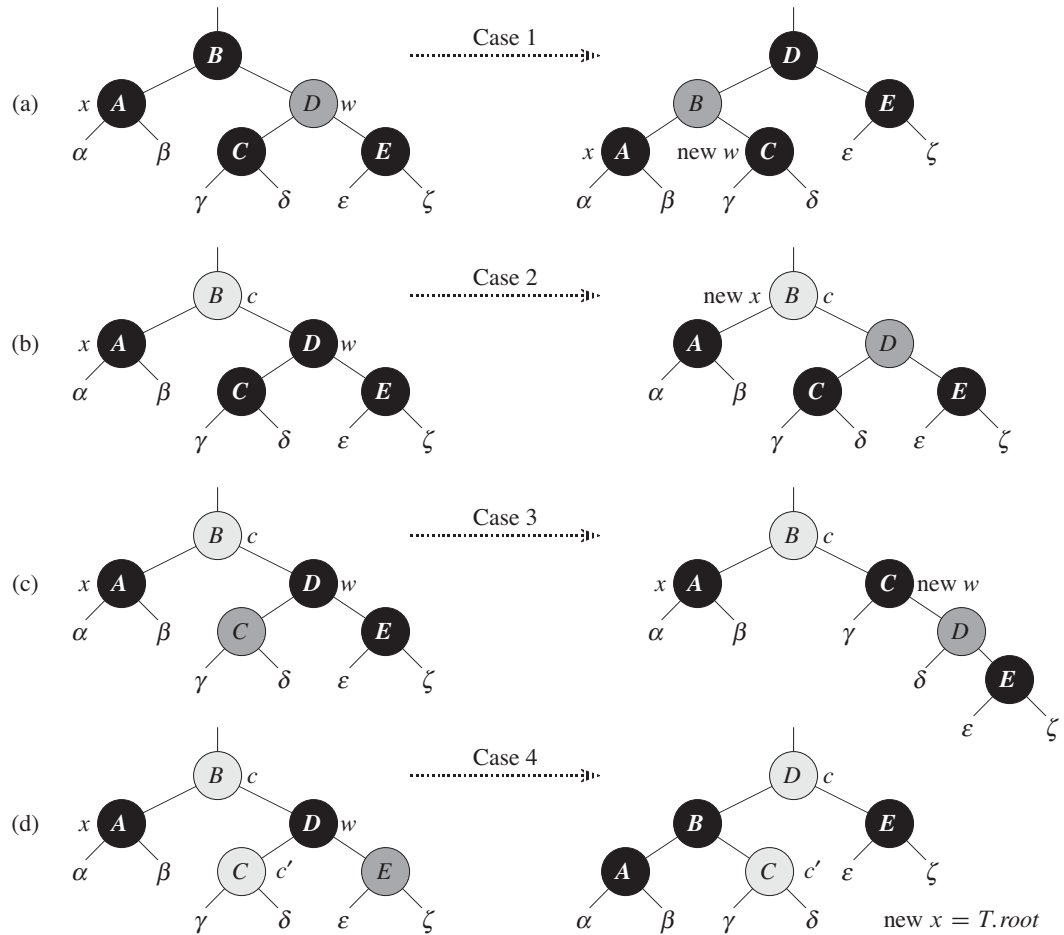
Figure 13.7 The cases in the **while** loop of the procedure RB-DELETE-FIXUP. Darkened nodes have *color* attributes BLACK, heavily shaded nodes have *color* attributes RED, and lightly shaded nodes have *color* attributes represented by c and c', which may be either RED or BLACK. The letters $\alpha, \beta, \ldots, \zeta$ represent arbitrary subtrees. Each case transforms the configuration on the left into the configuration on the right by changing some colors and/or performing a rotation. Any node pointed to by x has an extra black and is either doubly black or red-and-black. Only case 2 causes the loop to repeat. **(a)** Case 1 is transformed to case 2, 3, or 4 by exchanging the colors of nodes B and D and performing a left rotation. **(b)** In case 2, the extra black represented by the pointer x moves up the tree by coloring node D red and setting x to point to node B. If we enter case 2 through case 1, the **while** loop terminates because the new node x is red-and-black, and therefore the value c of its *color* attribute is RED. **(c)** Case 3 is transformed to case 4 by exchanging the colors of nodes C and D and performing a right rotation. **(d)** Case 4 removes the extra black represented by x by changing some colors and performing a left rotation (without violating the red-black properties), and then the loop terminates.

Exercises

13.4-1
Argue that after executing RB-DELETE-FIXUP, the root of the tree must be black.

13.4-2
Argue that if in RB-DELETE both x and $x.p$ are red, then property 4 is restored by the call to RB-DELETE-FIXUP(T, x).

13.4-3
In Exercise 13.3-2, you found the red-black tree that results from successively inserting the keys $41, 38, 31, 12, 19, 8$ into an initially empty tree. Now show the red-black trees that result from the successive deletion of the keys in the order $8, 12, 19, 31, 38, 41$.

13.4-4
In which lines of the code for RB-DELETE-FIXUP might we examine or modify the sentinel $T.nil$?

13.4-5
In each of the cases of Figure 13.7, give the count of black nodes from the root of the subtree shown to each of the subtrees $\alpha, \beta, \ldots, \zeta$, and verify that each count remains the same after the transformation. When a node has a *color* attribute c or c', use the notation count(c) or count(c') symbolically in your count.

13.4-6
Professors Skelton and Baron are concerned that at the start of case 1 of RB-DELETE-FIXUP, the node $x.p$ might not be black. If the professors are correct, then lines 5–6 are wrong. Show that $x.p$ must be black at the start of case 1, so that the professors have nothing to worry about.

13.4-7
Suppose that a node x is inserted into a red-black tree with RB-INSERT and then is immediately deleted with RB-DELETE. Is the resulting red-black tree the same as the initial red-black tree? Justify your answer.

Problems

13-1 *Persistent dynamic sets*

During the course of an algorithm, we sometimes find that we need to maintain past versions of a dynamic set as it is updated. We call such a set ***persistent***. One way to implement a persistent set is to copy the entire set whenever it is modified, but this approach can slow down a program and also consume much space. Sometimes, we can do much better.

Consider a persistent set S with the operations INSERT, DELETE, and SEARCH, which we implement using binary search trees as shown in Figure 13.8(a). We maintain a separate root for every version of the set. In order to insert the key 5 into the set, we create a new node with key 5. This node becomes the left child of a new node with key 7, since we cannot modify the existing node with key 7. Similarly, the new node with key 7 becomes the left child of a new node with key 8 whose right child is the existing node with key 10. The new node with key 8 becomes, in turn, the right child of a new root r' with key 4 whose left child is the existing node with key 3. We thus copy only part of the tree and share some of the nodes with the original tree, as shown in Figure 13.8(b).

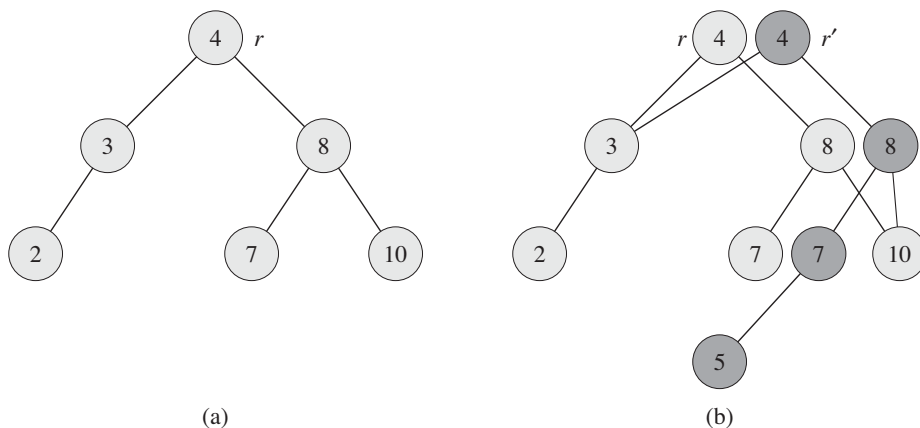Assume that each tree node has the attributes *key*, *left*, and *right* but no parent. (See also Exercise 13.3-6.)

(a)   (b)

Figure 13.8 **(a)** A binary search tree with keys $2, 3, 4, 7, 8, 10$. **(b)** The persistent binary search tree that results from the insertion of key 5. The most recent version of the set consists of the nodes reachable from the root r', and the previous version consists of the nodes reachable from r. Heavily shaded nodes are added when key 5 is inserted.

a. For a general persistent binary search tree, identify the nodes that we need to change to insert a key k or delete a node y.

b. Write a procedure PERSISTENT-TREE-INSERT that, given a persistent tree T and a key k to insert, returns a new persistent tree T' that is the result of inserting k into T.

c. If the height of the persistent binary search tree T is h, what are the time and space requirements of your implementation of PERSISTENT-TREE-INSERT? (The space requirement is proportional to the number of new nodes allocated.)

d. Suppose that we had included the parent attribute in each node. In this case, PERSISTENT-TREE-INSERT would need to perform additional copying. Prove that PERSISTENT-TREE-INSERT would then require $\Omega(n)$ time and space, where n is the number of nodes in the tree.

e. Show how to use red-black trees to guarantee that the worst-case running time and space are $O(\lg n)$ per insertion or deletion.

13-2   *Join operation on red-black trees*

The *join* operation takes two dynamic sets S_1 and S_2 and an element x such that for any $x_1 \in S_1$ and $x_2 \in S_2$, we have $x_1.key \leq x.key \leq x_2.key$. It returns a set $S = S_1 \cup \{x\} \cup S_2$. In this problem, we investigate how to implement the join operation on red-black trees.

a. Given a red-black tree T, let us store its black-height as the new attribute $T.bh$. Argue that RB-INSERT and RB-DELETE can maintain the bh attribute without requiring extra storage in the nodes of the tree and without increasing the asymptotic running times. Show that while descending through T, we can determine the black-height of each node we visit in $O(1)$ time per node visited.

We wish to implement the operation RB-JOIN(T_1, x, T_2), which destroys T_1 and T_2 and returns a red-black tree $T = T_1 \cup \{x\} \cup T_2$. Let n be the total number of nodes in T_1 and T_2.

b. Assume that $T_1.bh \geq T_2.bh$. Describe an $O(\lg n)$-time algorithm that finds a black node y in T_1 with the largest key from among those nodes whose black-height is $T_2.bh$.

c. Let T_y be the subtree rooted at y. Describe how $T_y \cup \{x\} \cup T_2$ can replace T_y in $O(1)$ time without destroying the binary-search-tree property.

d. What color should we make x so that red-black properties 1, 3, and 5 are maintained? Describe how to enforce properties 2 and 4 in $O(\lg n)$ time.

e. Argue that no generality is lost by making the assumption in part (b). Describe the symmetric situation that arises when $T_1.bh \leq T_2.bh$.

f. Argue that the running time of RB-JOIN is $O(\lg n)$.

13-3 AVL trees

An **AVL tree** is a binary search tree that is **height balanced**: for each node x, the heights of the left and right subtrees of x differ by at most 1. To implement an AVL tree, we maintain an extra attribute in each node: $x.h$ is the height of node x. As for any other binary search tree T, we assume that $T.root$ points to the root node.

a. Prove that an AVL tree with n nodes has height $O(\lg n)$. (*Hint:* Prove that an AVL tree of height h has at least F_h nodes, where F_h is the hth Fibonacci number.)

b. To insert into an AVL tree, we first place a node into the appropriate place in binary search tree order. Afterward, the tree might no longer be height balanced. Specifically, the heights of the left and right children of some node might differ by 2. Describe a procedure BALANCE(x), which takes a subtree rooted at x whose left and right children are height balanced and have heights that differ by at most 2, i.e., $|x.right.h - x.left.h| \leq 2$, and alters the subtree rooted at x to be height balanced. (*Hint:* Use rotations.)

c. Using part (b), describe a recursive procedure AVL-INSERT(x, z) that takes a node x within an AVL tree and a newly created node z (whose key has already been filled in), and adds z to the subtree rooted at x, maintaining the property that x is the root of an AVL tree. As in TREE-INSERT from Section 12.3, assume that $z.key$ has already been filled in and that $z.left = $ NIL and $z.right = $ NIL; also assume that $z.h = 0$. Thus, to insert the node z into the AVL tree T, we call AVL-INSERT$(T.root, z)$.

d. Show that AVL-INSERT, run on an n-node AVL tree, takes $O(\lg n)$ time and performs $O(1)$ rotations.

13-4 Treaps

If we insert a set of n items into a binary search tree, the resulting tree may be horribly unbalanced, leading to long search times. As we saw in Section 12.4, however, randomly built binary search trees tend to be balanced. Therefore, one strategy that, on average, builds a balanced tree for a fixed set of items would be to randomly permute the items and then insert them in that order into the tree.

What if we do not have all the items at once? If we receive the items one at a time, can we still randomly build a binary search tree out of them?
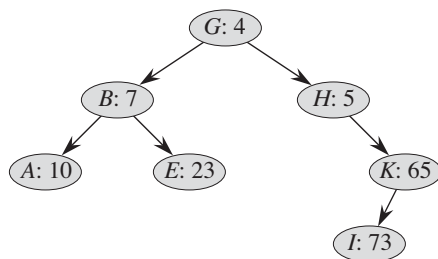
Figure 13.9 A treap. Each node x is labeled with $x.key : x.priority$. For example, the root has key G and priority 4.

We will examine a data structure that answers this question in the affirmative. A **treap** is a binary search tree with a modified way of ordering the nodes. Figure 13.9 shows an example. As usual, each node x in the tree has a key value $x.key$. In addition, we assign $x.priority$, which is a random number chosen independently for each node. We assume that all priorities are distinct and also that all keys are distinct. The nodes of the treap are ordered so that the keys obey the binary-search-tree property and the priorities obey the min-heap order property:

- If v is a left child of u, then $v.key < u.key$.

- If v is a right child of u, then $v.key > u.key$.

- If v is a child of u, then $v.priority > u.priority$.

(This combination of properties is why the tree is called a "treap": it has features of both a binary search tree and a heap.)

It helps to think of treaps in the following way. Suppose that we insert nodes $x_1, x_2, \ldots, x_n$, with associated keys, into a treap. Then the resulting treap is the tree that would have been formed if the nodes had been inserted into a normal binary search tree in the order given by their (randomly chosen) priorities, i.e., $x_i.priority < x_j.priority$ means that we had inserted x_i before x_j.

a. Show that given a set of nodes $x_1, x_2, \ldots, x_n$, with associated keys and priorities, all distinct, the treap associated with these nodes is unique.

b. Show that the expected height of a treap is $\Theta(\lg n)$, and hence the expected time to search for a value in the treap is $\Theta(\lg n)$.

Let us see how to insert a new node into an existing treap. The first thing we do is assign to the new node a random priority. Then we call the insertion algorithm, which we call TREAP-INSERT, whose operation is illustrated in Figure 13.10.
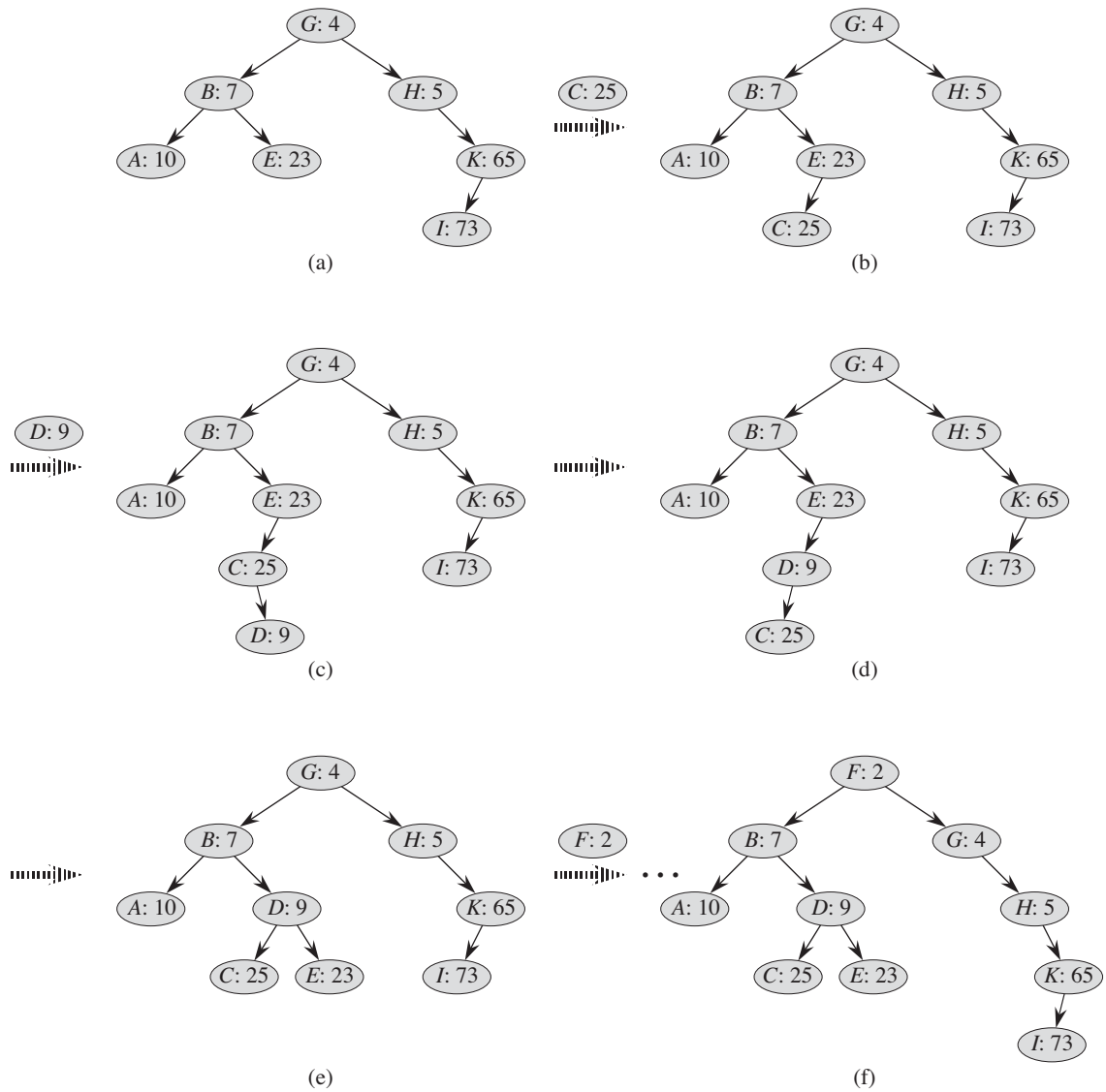
Figure 13.10 The operation of TREAP-INSERT. **(a)** The original treap, prior to insertion. **(b)** The treap after inserting a node with key C and priority 25. **(c)–(d)** Intermediate stages when inserting a node with key D and priority 9. **(e)** The treap after the insertion of parts (c) and (d) is done. **(f)** The treap after inserting a node with key F and priority 2.
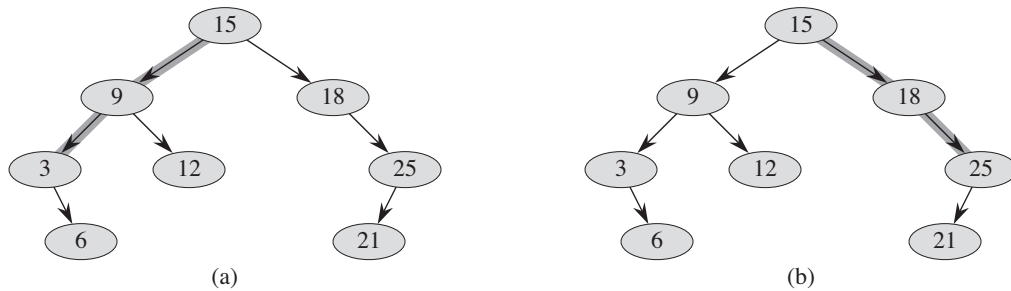
Figure 13.11   Spines of a binary search tree. The left spine is shaded in **(a)**, and the right spine is shaded in **(b)**.

c. Explain how TREAP-INSERT works. Explain the idea in English and give pseudocode. (*Hint:* Execute the usual binary-search-tree insertion procedure and then perform rotations to restore the min-heap order property.)

d. Show that the expected running time of TREAP-INSERT is $\Theta(\lg n)$.

TREAP-INSERT performs a search and then a sequence of rotations. Although these two operations have the same expected running time, they have different costs in practice. A search reads information from the treap without modifying it. In contrast, a rotation changes parent and child pointers within the treap. On most computers, read operations are much faster than write operations. Thus we would like TREAP-INSERT to perform few rotations. We will show that the expected number of rotations performed is bounded by a constant.

In order to do so, we will need some definitions, which Figure 13.11 depicts. The *left spine* of a binary search tree T is the simple path from the root to the node with the smallest key. In other words, the left spine is the simple path from the root that consists of only left edges. Symmetrically, the *right spine* of T is the simple path from the root consisting of only right edges. The *length* of a spine is the number of nodes it contains.

e. Consider the treap T immediately after TREAP-INSERT has inserted node x. Let C be the length of the right spine of the left subtree of x. Let D be the length of the left spine of the right subtree of x. Prove that the total number of rotations that were performed during the insertion of x is equal to $C + D$.

We will now calculate the expected values of C and D. Without loss of generality, we assume that the keys are $1, 2, \ldots, n$, since we are comparing them only to one another.

For nodes x and y in treap T, where $y \neq x$, let $k = x.key$ and $i = y.key$. We define indicator random variables

$$X_{ik} = \text{I}\{y \text{ is in the right spine of the left subtree of } x\} .$$

f. Show that $X_{ik} = 1$ if and only if $y.priority > x.priority$, $y.key < x.key$, and, for every z such that $y.key < z.key < x.key$, we have $y.priority < z.priority$.

g. Show that

$$\Pr\{X_{ik} = 1\} = \frac{(k-i-1)!}{(k-i+1)!}$$

$$= \frac{1}{(k-i+1)(k-i)} .$$

h. Show that

$$\text{E}[C] = \sum_{j=1}^{k-1} \frac{1}{j(j+1)}$$

$$= 1 - \frac{1}{k} .$$

i. Use a symmetry argument to show that

$$\text{E}[D] = 1 - \frac{1}{n-k+1} .$$

j. Conclude that the expected number of rotations performed when inserting a node into a treap is less than 2.

Chapter notes

The idea of balancing a search tree is due to Adel'son-Vel'skiĭ and Landis [2], who introduced a class of balanced search trees called "AVL trees" in 1962, described in Problem 13-3. Another class of search trees, called "2-3 trees," was introduced by J. E. Hopcroft (unpublished) in 1970. A 2-3 tree maintains balance by manipulating the degrees of nodes in the tree. Chapter 18 covers a generalization of 2-3 trees introduced by Bayer and McCreight [35], called "B-trees."

Red-black trees were invented by Bayer [34] under the name "symmetric binary B-trees." Guibas and Sedgewick [155] studied their properties at length and introduced the red/black color convention. Andersson [15] gives a simpler-to-code

variant of red-black trees. Weiss [351] calls this variant AA-trees. An AA-tree is similar to a red-black tree except that left children may never be red.

Treaps, the subject of Problem 13-4, were proposed by Seidel and Aragon [309]. They are the default implementation of a dictionary in LEDA [253], which is a well-implemented collection of data structures and algorithms.

There are many other variations on balanced binary trees, including weight-balanced trees [264], k-neighbor trees [245], and scapegoat trees [127]. Perhaps the most intriguing are the "splay trees" introduced by Sleator and Tarjan [320], which are "self-adjusting." (See Tarjan [330] for a good description of splay trees.) Splay trees maintain balance without any explicit balance condition such as color. Instead, "splay operations" (which involve rotations) are performed within the tree every time an access is made. The amortized cost (see Chapter 17) of each operation on an n-node tree is $O(\lg n)$.

Skip lists [286] provide an alternative to balanced binary trees. A skip list is a linked list that is augmented with a number of additional pointers. Each dictionary operation runs in expected time $O(\lg n)$ on a skip list of n items.

14   Augmenting Data Structures

Some engineering situations require no more than a "textbook" data structure—such as a doubly linked list, a hash table, or a binary search tree—but many others require a dash of creativity. Only in rare situations will you need to create an entirely new type of data structure, though. More often, it will suffice to augment a textbook data structure by storing additional information in it. You can then program new operations for the data structure to support the desired application. Augmenting a data structure is not always straightforward, however, since the added information must be updated and maintained by the ordinary operations on the data structure.

This chapter discusses two data structures that we construct by augmenting red-black trees. Section 14.1 describes a data structure that supports general order-statistic operations on a dynamic set. We can then quickly find the ith smallest number in a set or the rank of a given element in the total ordering of the set. Section 14.2 abstracts the process of augmenting a data structure and provides a theorem that can simplify the process of augmenting red-black trees. Section 14.3 uses this theorem to help design a data structure for maintaining a dynamic set of intervals, such as time intervals. Given a query interval, we can then quickly find an interval in the set that overlaps it.

14.1   Dynamic order statistics

Chapter 9 introduced the notion of an order statistic. Specifically, the ith order statistic of a set of n elements, where $i \in \{1, 2, \ldots, n\}$, is simply the element in the set with the ith smallest key. We saw how to determine any order statistic in $O(n)$ time from an unordered set. In this section, we shall see how to modify red-black trees so that we can determine any order statistic for a dynamic set in $O(\lg n)$ time. We shall also see how to compute the ***rank*** of an element—its position in the linear order of the set—in $O(\lg n)$ time.
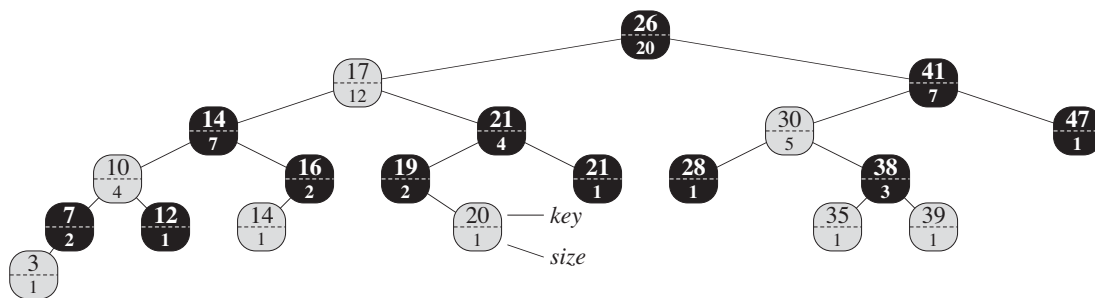
Figure 14.1   An order-statistic tree, which is an augmented red-black tree. Shaded nodes are red, and darkened nodes are black. In addition to its usual attributes, each node x has an attribute $x.size$, which is the number of nodes, other than the sentinel, in the subtree rooted at x.

Figure 14.1 shows a data structure that can support fast order-statistic operations. An ***order-statistic tree*** T is simply a red-black tree with additional information stored in each node. Besides the usual red-black tree attributes $x.key$, $x.color$, $x.p$, $x.left$, and $x.right$ in a node x, we have another attribute, $x.size$. This attribute contains the number of (internal) nodes in the subtree rooted at x (including x itself), that is, the size of the subtree. If we define the sentinel's size to be 0—that is, we set $T.nil.size$ to be 0—then we have the identity

$$x.size = x.left.size + x.right.size + 1 .$$

We do not require keys to be distinct in an order-statistic tree. (For example, the tree in Figure 14.1 has two keys with value 14 and two keys with value 21.) In the presence of equal keys, the above notion of rank is not well defined. We remove this ambiguity for an order-statistic tree by defining the rank of an element as the position at which it would be printed in an inorder walk of the tree. In Figure 14.1, for example, the key 14 stored in a black node has rank 5, and the key 14 stored in a red node has rank 6.

Retrieving an element with a given rank

Before we show how to maintain this size information during insertion and deletion, let us examine the implementation of two order-statistic queries that use this additional information. We begin with an operation that retrieves an element with a given rank. The procedure OS-SELECT(x, i) returns a pointer to the node containing the ith smallest key in the subtree rooted at x. To find the node with the ith smallest key in an order-statistic tree T, we call OS-SELECT$(T.root, i)$.

OS-SELECT(x, i)

1   $r = x.left.size + 1$
2   **if** $i == r$
3   **return** x
4   **elseif** $i < r$
5   **return** OS-SELECT$(x.left, i)$
6   **else return** OS-SELECT$(x.right, i - r)$

In line 1 of OS-SELECT, we compute r, the rank of node x within the subtree rooted at x. The value of $x.left.size$ is the number of nodes that come before x in an inorder tree walk of the subtree rooted at x. Thus, $x.left.size + 1$ is the rank of x within the subtree rooted at x. If $i = r$, then node x is the ith smallest element, and so we return x in line 3. If $i < r$, then the ith smallest element resides in x's left subtree, and so we recurse on $x.left$ in line 5. If $i > r$, then the ith smallest element resides in x's right subtree. Since the subtree rooted at x contains r elements that come before x's right subtree in an inorder tree walk, the ith smallest element in the subtree rooted at x is the $(i - r)$th smallest element in the subtree rooted at $x.right$. Line 6 determines this element recursively.

To see how OS-SELECT operates, consider a search for the 17th smallest element in the order-statistic tree of Figure 14.1. We begin with x as the root, whose key is 26, and with $i = 17$. Since the size of 26's left subtree is 12, its rank is 13. Thus, we know that the node with rank 17 is the $17 - 13 = 4$th smallest element in 26's right subtree. After the recursive call, x is the node with key 41, and $i = 4$. Since the size of 41's left subtree is 5, its rank within its subtree is 6. Thus, we know that the node with rank 4 is the 4th smallest element in 41's left subtree. After the recursive call, x is the node with key 30, and its rank within its subtree is 2. Thus, we recurse once again to find the $4 - 2 = 2$nd smallest element in the subtree rooted at the node with key 38. We now find that its left subtree has size 1, which means it is the second smallest element. Thus, the procedure returns a pointer to the node with key 38.

Because each recursive call goes down one level in the order-statistic tree, the total time for OS-SELECT is at worst proportional to the height of the tree. Since the tree is a red-black tree, its height is $O(\lg n)$, where n is the number of nodes. Thus, the running time of OS-SELECT is $O(\lg n)$ for a dynamic set of n elements.

Determining the rank of an element

Given a pointer to a node x in an order-statistic tree T, the procedure OS-RANK returns the position of x in the linear order determined by an inorder tree walk of T.

OS-RANK(T, x)

```
1   r = x.left.size + 1
2   y = x
3   while y ≠ T.root
4       if y == y.p.right
5           r = r + y.p.left.size + 1
6       y = y.p
7   return r
```

The procedure works as follows. We can think of node x's rank as the number of nodes preceding x in an inorder tree walk, plus 1 for x itself. OS-RANK maintains the following loop invariant:

> At the start of each iteration of the **while** loop of lines 3–6, r is the rank of $x.key$ in the subtree rooted at node y.

We use this loop invariant to show that OS-RANK works correctly as follows:

Initialization: Prior to the first iteration, line 1 sets r to be the rank of $x.key$ within the subtree rooted at x. Setting $y = x$ in line 2 makes the invariant true the first time the test in line 3 executes.

Maintenance: At the end of each iteration of the **while** loop, we set $y = y.p$. Thus we must show that if r is the rank of $x.key$ in the subtree rooted at y at the start of the loop body, then r is the rank of $x.key$ in the subtree rooted at $y.p$ at the end of the loop body. In each iteration of the **while** loop, we consider the subtree rooted at $y.p$. We have already counted the number of nodes in the subtree rooted at node y that precede x in an inorder walk, and so we must add the nodes in the subtree rooted at y's sibling that precede x in an inorder walk, plus 1 for $y.p$ if it, too, precedes x. If y is a left child, then neither $y.p$ nor any node in $y.p$'s right subtree precedes x, and so we leave r alone. Otherwise, y is a right child and all the nodes in $y.p$'s left subtree precede x, as does $y.p$ itself. Thus, in line 5, we add $y.p.left.size + 1$ to the current value of r.

Termination: The loop terminates when $y = T.root$, so that the subtree rooted at y is the entire tree. Thus, the value of r is the rank of $x.key$ in the entire tree.

As an example, when we run OS-RANK on the order-statistic tree of Figure 14.1 to find the rank of the node with key 38, we get the following sequence of values of $y.key$ and r at the top of the **while** loop:

iteration	$y.key$	r
1	38	2
2	30	4
3	41	4
4	26	17

The procedure returns the rank 17.

Since each iteration of the **while** loop takes $O(1)$ time, and y goes up one level in the tree with each iteration, the running time of OS-RANK is at worst proportional to the height of the tree: $O(\lg n)$ on an n-node order-statistic tree.

Maintaining subtree sizes

Given the *size* attribute in each node, OS-SELECT and OS-RANK can quickly compute order-statistic information. But unless we can efficiently maintain these attributes within the basic modifying operations on red-black trees, our work will have been for naught. We shall now show how to maintain subtree sizes for both insertion and deletion without affecting the asymptotic running time of either operation.

We noted in Section 13.3 that insertion into a red-black tree consists of two phases. The first phase goes down the tree from the root, inserting the new node as a child of an existing node. The second phase goes up the tree, changing colors and performing rotations to maintain the red-black properties.

To maintain the subtree sizes in the first phase, we simply increment $x.size$ for each node x on the simple path traversed from the root down toward the leaves. The new node added gets a *size* of 1. Since there are $O(\lg n)$ nodes on the traversed path, the additional cost of maintaining the *size* attributes is $O(\lg n)$.

In the second phase, the only structural changes to the underlying red-black tree are caused by rotations, of which there are at most two. Moreover, a rotation is a local operation: only two nodes have their *size* attributes invalidated. The link around which the rotation is performed is incident on these two nodes. Referring to the code for LEFT-ROTATE(T, x) in Section 13.2, we add the following lines:

```
13   y.size = x.size
14   x.size = x.left.size + x.right.size + 1
```

Figure 14.2 illustrates how the attributes are updated. The change to RIGHT-ROTATE is symmetric.

Since at most two rotations are performed during insertion into a red-black tree, we spend only $O(1)$ additional time updating *size* attributes in the second phase. Thus, the total time for insertion into an n-node order-statistic tree is $O(\lg n)$, which is asymptotically the same as for an ordinary red-black tree.

Deletion from a red-black tree also consists of two phases: the first operates on the underlying search tree, and the second causes at most three rotations and otherwise performs no structural changes. (See Section 13.4.) The first phase either removes one node y from the tree or moves upward it within the tree. To update the subtree sizes, we simply traverse a simple path from node y (starting from its original position within the tree) up to the root, decrementing the *size*
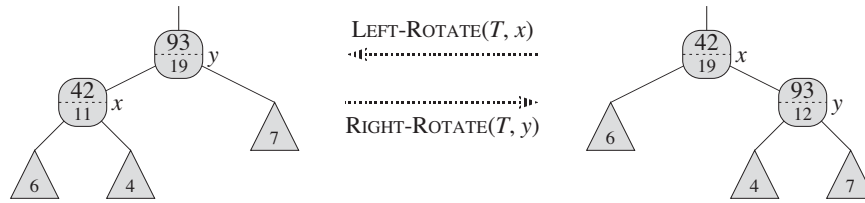
Figure 14.2 Updating subtree sizes during rotations. The link around which we rotate is incident on the two nodes whose *size* attributes need to be updated. The updates are local, requiring only the *size* information stored in x, y, and the roots of the subtrees shown as triangles.

attribute of each node on the path. Since this path has length $O(\lg n)$ in an n-node red-black tree, the additional time spent maintaining *size* attributes in the first phase is $O(\lg n)$. We handle the $O(1)$ rotations in the second phase of deletion in the same manner as for insertion. Thus, both insertion and deletion, including maintaining the *size* attributes, take $O(\lg n)$ time for an n-node order-statistic tree.

Exercises

14.1-1
Show how OS-SELECT$(T.root, 10)$ operates on the red-black tree T of Figure 14.1.

14.1-2
Show how OS-RANK(T, x) operates on the red-black tree T of Figure 14.1 and the node x with $x.key = 35$.

14.1-3
Write a nonrecursive version of OS-SELECT.

14.1-4
Write a recursive procedure OS-KEY-RANK(T, k) that takes as input an order-statistic tree T and a key k and returns the rank of k in the dynamic set represented by T. Assume that the keys of T are distinct.

14.1-5
Given an element x in an n-node order-statistic tree and a natural number i, how can we determine the ith successor of x in the linear order of the tree in $O(\lg n)$ time?

14.1-6

Observe that whenever we reference the *size* attribute of a node in either OS-SELECT or OS-RANK, we use it only to compute a rank. Accordingly, suppose we store in each node its rank in the subtree of which it is the root. Show how to maintain this information during insertion and deletion. (Remember that these two operations can cause rotations.)

14.1-7

Show how to use an order-statistic tree to count the number of inversions (see Problem 2-4) in an array of size n in time $O(n \lg n)$.

14.1-8   ⋆

Consider n chords on a circle, each defined by its endpoints. Describe an $O(n \lg n)$-time algorithm to determine the number of pairs of chords that intersect inside the circle. (For example, if the n chords are all diameters that meet at the center, then the correct answer is $\binom{n}{2}$.) Assume that no two chords share an endpoint.

14.2   How to augment a data structure

The process of augmenting a basic data structure to support additional functionality occurs quite frequently in algorithm design. We shall use it again in the next section to design a data structure that supports operations on intervals. In this section, we examine the steps involved in such augmentation. We shall also prove a theorem that allows us to augment red-black trees easily in many cases.

   We can break the process of augmenting a data structure into four steps:

1. Choose an underlying data structure.
2. Determine additional information to maintain in the underlying data structure.
3. Verify that we can maintain the additional information for the basic modifying operations on the underlying data structure.
4. Develop new operations.

As with any prescriptive design method, you should not blindly follow the steps in the order given. Most design work contains an element of trial and error, and progress on all steps usually proceeds in parallel. There is no point, for example, in determining additional information and developing new operations (steps 2 and 4) if we will not be able to maintain the additional information efficiently. Nevertheless, this four-step method provides a good focus for your efforts in augmenting a data structure, and it is also a good way to organize the documentation of an augmented data structure.

We followed these steps in Section 14.1 to design our order-statistic trees. For step 1, we chose red-black trees as the underlying data structure. A clue to the suitability of red-black trees comes from their efficient support of other dynamic-set operations on a total order, such as MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR.

For step 2, we added the *size* attribute, in which each node x stores the size of the subtree rooted at x. Generally, the additional information makes operations more efficient. For example, we could have implemented OS-SELECT and OS-RANK using just the keys stored in the tree, but they would not have run in $O(\lg n)$ time. Sometimes, the additional information is pointer information rather than data, as in Exercise 14.2-1.

For step 3, we ensured that insertion and deletion could maintain the *size* attributes while still running in $O(\lg n)$ time. Ideally, we should need to update only a few elements of the data structure in order to maintain the additional information. For example, if we simply stored in each node its rank in the tree, the OS-SELECT and OS-RANK procedures would run quickly, but inserting a new minimum element would cause a change to this information in every node of the tree. When we store subtree sizes instead, inserting a new element causes information to change in only $O(\lg n)$ nodes.

For step 4, we developed the operations OS-SELECT and OS-RANK. After all, the need for new operations is why we bother to augment a data structure in the first place. Occasionally, rather than developing new operations, we use the additional information to expedite existing ones, as in Exercise 14.2-1.

Augmenting red-black trees

When red-black trees underlie an augmented data structure, we can prove that insertion and deletion can always efficiently maintain certain kinds of additional information, thereby making step 3 very easy. The proof of the following theorem is similar to the argument from Section 14.1 that we can maintain the *size* attribute for order-statistic trees.

Theorem 14.1 (Augmenting a red-black tree)
Let f be an attribute that augments a red-black tree T of n nodes, and suppose that the value of f for each node x depends on only the information in nodes x, $x.left$, and $x.right$, possibly including $x.left.f$ and $x.right.f$. Then, we can maintain the values of f in all nodes of T during insertion and deletion without asymptotically affecting the $O(\lg n)$ performance of these operations.

Proof   The main idea of the proof is that a change to an f attribute in a node x propagates only to ancestors of x in the tree. That is, changing $x.f$ may re-

quire $x.p.f$ to be updated, but nothing else; updating $x.p.f$ may require $x.p.p.f$ to be updated, but nothing else; and so on up the tree. Once we have updated $T.root.f$, no other node will depend on the new value, and so the process terminates. Since the height of a red-black tree is $O(\lg n)$, changing an f attribute in a node costs $O(\lg n)$ time in updating all nodes that depend on the change.

Insertion of a node x into T consists of two phases. (See Section 13.3.) The first phase inserts x as a child of an existing node $x.p$. We can compute the value of $x.f$ in $O(1)$ time since, by supposition, it depends only on information in the other attributes of x itself and the information in x's children, but x's children are both the sentinel $T.nil$. Once we have computed $x.f$, the change propagates up the tree. Thus, the total time for the first phase of insertion is $O(\lg n)$. During the second phase, the only structural changes to the tree come from rotations. Since only two nodes change in a rotation, the total time for updating the f attributes is $O(\lg n)$ per rotation. Since the number of rotations during insertion is at most two, the total time for insertion is $O(\lg n)$.

Like insertion, deletion has two phases. (See Section 13.4.) In the first phase, changes to the tree occur when the deleted node is removed from the tree. If the deleted node had two children at the time, then its successor moves into the position of the deleted node. Propagating the updates to f caused by these changes costs at most $O(\lg n)$, since the changes modify the tree locally. Fixing up the red-black tree during the second phase requires at most three rotations, and each rotation requires at most $O(\lg n)$ time to propagate the updates to f. Thus, like insertion, the total time for deletion is $O(\lg n)$.   ∎

In many cases, such as maintaining the *size* attributes in order-statistic trees, the cost of updating after a rotation is $O(1)$, rather than the $O(\lg n)$ derived in the proof of Theorem 14.1. Exercise 14.2-3 gives an example.

Exercises

14.2-1
Show, by adding pointers to the nodes, how to support each of the dynamic-set queries MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR in $O(1)$ worst-case time on an augmented order-statistic tree. The asymptotic performance of other operations on order-statistic trees should not be affected.

14.2-2
Can we maintain the black-heights of nodes in a red-black tree as attributes in the nodes of the tree without affecting the asymptotic performance of any of the red-black tree operations? Show how, or argue why not. How about maintaining the depths of nodes?

14.2-3 ★

Let $\otimes$ be an associative binary operator, and let a be an attribute maintained in each node of a red-black tree. Suppose that we want to include in each node x an additional attribute f such that $x.f = x_1.a \otimes x_2.a \otimes \cdots \otimes x_m.a$, where $x_1, x_2, \ldots, x_m$ is the inorder listing of nodes in the subtree rooted at x. Show how to update the f attributes in $O(1)$ time after a rotation. Modify your argument slightly to apply it to the *size* attributes in order-statistic trees.

14.2-4 ★

We wish to augment red-black trees with an operation RB-ENUMERATE(x, a, b) that outputs all the keys k such that $a \leq k \leq b$ in a red-black tree rooted at x. Describe how to implement RB-ENUMERATE in $\Theta(m + \lg n)$ time, where m is the number of keys that are output and n is the number of internal nodes in the tree. (*Hint:* You do not need to add new attributes to the red-black tree.)

14.3   Interval trees

In this section, we shall augment red-black trees to support operations on dynamic sets of intervals. A ***closed interval*** is an ordered pair of real numbers $[t_1, t_2]$, with $t_1 \leq t_2$. The interval $[t_1, t_2]$ represents the set $\{t \in \mathbb{R} : t_1 \leq t \leq t_2\}$. ***Open*** and ***half-open*** intervals omit both or one of the endpoints from the set, respectively. In this section, we shall assume that intervals are closed; extending the results to open and half-open intervals is conceptually straightforward.

Intervals are convenient for representing events that each occupy a continuous period of time. We might, for example, wish to query a database of time intervals to find out what events occurred during a given interval. The data structure in this section provides an efficient means for maintaining such an interval database.

We can represent an interval $[t_1, t_2]$ as an object i, with attributes $i.low = t_1$ (the ***low endpoint***) and $i.high = t_2$ (the ***high endpoint***). We say that intervals i and i' ***overlap*** if $i \cap i' \neq \emptyset$, that is, if $i.low \leq i'.high$ and $i'.low \leq i.high$. As Figure 14.3 shows, any two intervals i and i' satisfy the ***interval trichotomy***; that is, exactly one of the following three properties holds:

a. i and i' overlap,

b. i is to the left of i' (i.e., $i.high < i'.low$),

c. i is to the right of i' (i.e., $i'.high < i.low$).

An ***interval tree*** is a red-black tree that maintains a dynamic set of elements, with each element x containing an interval $x.int$. Interval trees support the following operations:
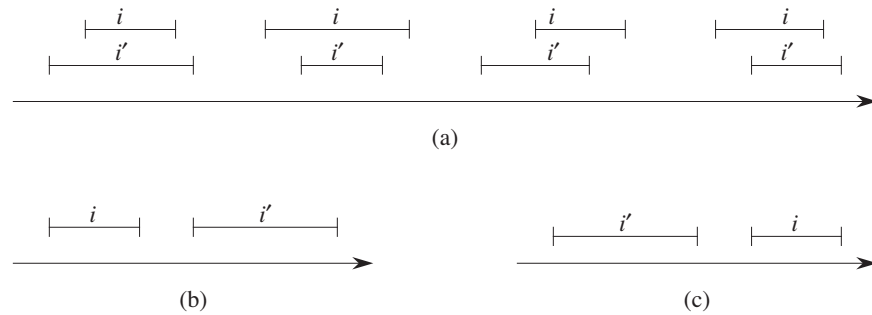
(a)

(b)   (c)

Figure 14.3   The interval trichotomy for two closed intervals i and i'. **(a)** If i and i' overlap, there are four situations; in each, $i.low \leq i'.high$ and $i'.low \leq i.high$. **(b)** The intervals do not overlap, and $i.high < i'.low$. **(c)** The intervals do not overlap, and $i'.high < i.low$.

INTERVAL-INSERT(T, x) adds the element x, whose *int* attribute is assumed to contain an interval, to the interval tree T.

INTERVAL-DELETE(T, x) removes the element x from the interval tree T.

INTERVAL-SEARCH(T, i) returns a pointer to an element x in the interval tree T such that $x.int$ overlaps interval i, or a pointer to the sentinel $T.nil$ if no such element is in the set.

Figure 14.4 shows how an interval tree represents a set of intervals. We shall track the four-step method from Section 14.2 as we review the design of an interval tree and the operations that run on it.

Step 1: Underlying data structure

We choose a red-black tree in which each node x contains an interval $x.int$ and the key of x is the low endpoint, $x.int.low$, of the interval. Thus, an inorder tree walk of the data structure lists the intervals in sorted order by low endpoint.

Step 2: Additional information

In addition to the intervals themselves, each node x contains a value $x.max$, which is the maximum value of any interval endpoint stored in the subtree rooted at x.

Step 3: Maintaining the information

We must verify that insertion and deletion take $O(\lg n)$ time on an interval tree of n nodes. We can determine $x.max$ given interval $x.int$ and the *max* values of node x's children:
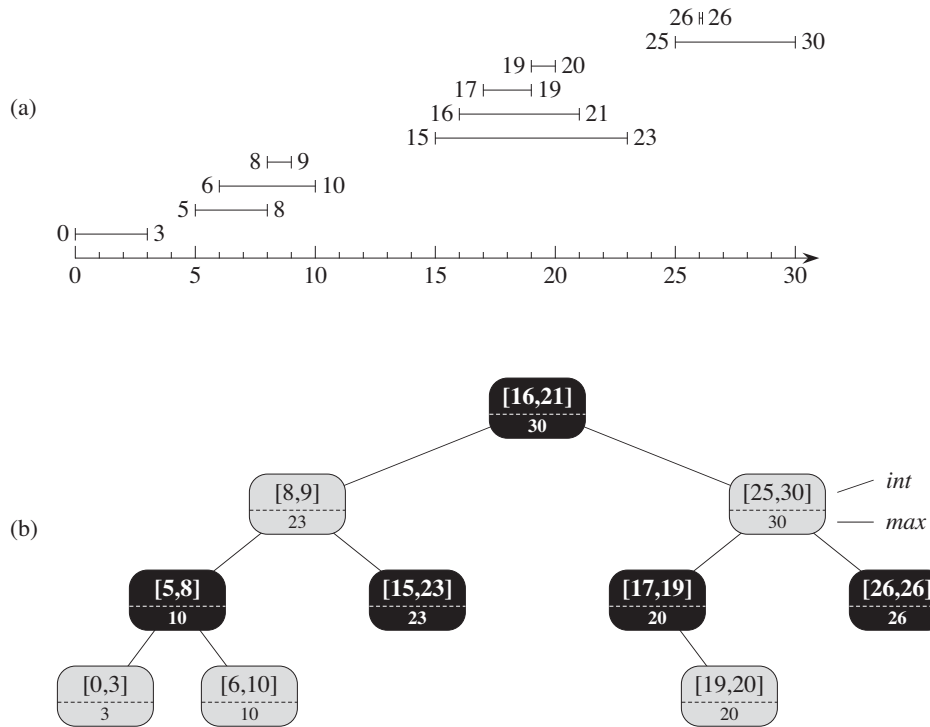
Figure 14.4   An interval tree. **(a)** A set of 10 intervals, shown sorted bottom to top by left endpoint. **(b)** The interval tree that represents them. Each node x contains an interval, shown above the dashed line, and the maximum value of any interval endpoint in the subtree rooted at x, shown below the dashed line. An inorder tree walk of the tree lists the nodes in sorted order by left endpoint.

$$x.max = \max(x.int.high, x.left.max, x.right.max) .$$

Thus, by Theorem 14.1, insertion and deletion run in $O(\lg n)$ time. In fact, we can update the *max* attributes after a rotation in $O(1)$ time, as Exercises 14.2-3 and 14.3-1 show.

Step 4: Developing new operations

The only new operation we need is INTERVAL-SEARCH(T, i), which finds a node in tree T whose interval overlaps interval i. If there is no interval that overlaps i in the tree, the procedure returns a pointer to the sentinel $T.nil$.

INTERVAL-SEARCH(T, i)

```
1   x = T.root
2   while x ≠ T.nil and i does not overlap x.int
3       if x.left ≠ T.nil and x.left.max ≥ i.low
4           x = x.left
5       else x = x.right
6   return x
```

The search for an interval that overlaps i starts with x at the root of the tree and proceeds downward. It terminates when either it finds an overlapping interval or x points to the sentinel $T.nil$. Since each iteration of the basic loop takes $O(1)$ time, and since the height of an n-node red-black tree is $O(\lg n)$, the INTERVAL-SEARCH procedure takes $O(\lg n)$ time.

Before we see why INTERVAL-SEARCH is correct, let's examine how it works on the interval tree in Figure 14.4. Suppose we wish to find an interval that overlaps the interval $i = [22, 25]$. We begin with x as the root, which contains $[16, 21]$ and does not overlap i. Since $x.left.max = 23$ is greater than $i.low = 22$, the loop continues with x as the left child of the root—the node containing $[8, 9]$, which also does not overlap i. This time, $x.left.max = 10$ is less than $i.low = 22$, and so the loop continues with the right child of x as the new x. Because the interval $[15, 23]$ stored in this node overlaps i, the procedure returns this node.

As an example of an unsuccessful search, suppose we wish to find an interval that overlaps $i = [11, 14]$ in the interval tree of Figure 14.4. We once again begin with x as the root. Since the root's interval $[16, 21]$ does not overlap i, and since $x.left.max = 23$ is greater than $i.low = 11$, we go left to the node containing $[8, 9]$. Interval $[8, 9]$ does not overlap i, and $x.left.max = 10$ is less than $i.low = 11$, and so we go right. (Note that no interval in the left subtree overlaps i.) Interval $[15, 23]$ does not overlap i, and its left child is $T.nil$, so again we go right, the loop terminates, and we return the sentinel $T.nil$.

To see why INTERVAL-SEARCH is correct, we must understand why it suffices to examine a single path from the root. The basic idea is that at any node x, if $x.int$ does not overlap i, the search always proceeds in a safe direction: the search will definitely find an overlapping interval if the tree contains one. The following theorem states this property more precisely.

Theorem 14.2

Any execution of INTERVAL-SEARCH(T, i) either returns a node whose interval overlaps i, or it returns $T.nil$ and the tree T contains no node whose interval overlaps i.
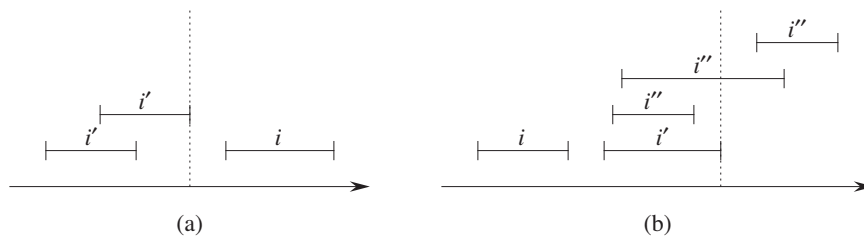
Figure 14.5   Intervals in the proof of Theorem 14.2. The value of $x.left.max$ is shown in each case as a dashed line. **(a)** The search goes right. No interval i' in x's left subtree can overlap i. **(b)** The search goes left. The left subtree of x contains an interval that overlaps i (situation not shown), or x's left subtree contains an interval i' such that $i'.high = x.left.max$. Since i does not overlap i', neither does it overlap any interval i'' in x's right subtree, since $i'.low \leq i''.low$.

Proof   The **while** loop of lines 2–5 terminates either when $x = T.nil$ or i overlaps $x.int$. In the latter case, it is certainly correct to return x. Therefore, we focus on the former case, in which the **while** loop terminates because $x = T.nil$.

We use the following invariant for the **while** loop of lines 2–5:

> If tree T contains an interval that overlaps i, then the subtree rooted at x contains such an interval.

We use this loop invariant as follows:

Initialization: Prior to the first iteration, line 1 sets x to be the root of T, so that the invariant holds.

Maintenance: Each iteration of the **while** loop executes either line 4 or line 5. We shall show that both cases maintain the loop invariant.

If line 5 is executed, then because of the branch condition in line 3, we have $x.left = T.nil$, or $x.left.max < i.low$. If $x.left = T.nil$, the subtree rooted at $x.left$ clearly contains no interval that overlaps i, and so setting x to $x.right$ maintains the invariant. Suppose, therefore, that $x.left \neq T.nil$ and $x.left.max < i.low$. As Figure 14.5(a) shows, for each interval i' in x's left subtree, we have

$$i'.high \leq x.left.max$$
$$< i.low \ .$$

By the interval trichotomy, therefore, i' and i do not overlap. Thus, the left subtree of x contains no intervals that overlap i, so that setting x to $x.right$ maintains the invariant.

If, on the other hand, line 4 is executed, then we will show that the contrapositive of the loop invariant holds. That is, if the subtree rooted at $x.left$ contains no interval overlapping i, then no interval anywhere in the tree overlaps i. Since line 4 is executed, then because of the branch condition in line 3, we have $x.left.max \geq i.low$. Moreover, by definition of the max attribute, x's left subtree must contain some interval i' such that

$$
\begin{aligned}
i'.high &= x.left.max \\
&\geq i.low .
\end{aligned}
$$

(Figure 14.5(b) illustrates the situation.) Since i and i' do not overlap, and since it is not true that $i'.high < i.low$, it follows by the interval trichotomy that $i.high < i'.low$. Interval trees are keyed on the low endpoints of intervals, and thus the search-tree property implies that for any interval i'' in x's right subtree,

$$
\begin{aligned}
i.high &< i'.low \\
&\leq i''.low .
\end{aligned}
$$

By the interval trichotomy, i and i'' do not overlap. We conclude that whether or not any interval in x's left subtree overlaps i, setting x to $x.left$ maintains the invariant.

Termination: If the loop terminates when $x = T.nil$, then the subtree rooted at x contains no interval overlapping i. The contrapositive of the loop invariant implies that T contains no interval that overlaps i. Hence it is correct to return $x = T.nil$. ∎

Thus, the INTERVAL-SEARCH procedure works correctly.

Exercises

14.3-1
Write pseudocode for LEFT-ROTATE that operates on nodes in an interval tree and updates the *max* attributes in $O(1)$ time.

14.3-2
Rewrite the code for INTERVAL-SEARCH so that it works properly when all intervals are open.

14.3-3
Describe an efficient algorithm that, given an interval i, returns an interval overlapping i that has the minimum low endpoint, or $T.nil$ if no such interval exists.

14.3-4

Given an interval tree T and an interval i, describe how to list all intervals in T that overlap i in $O(\min(n, k \lg n))$ time, where k is the number of intervals in the output list. (*Hint:* One simple method makes several queries, modifying the tree between queries. A slightly more complicated method does not modify the tree.)

14.3-5

Suggest modifications to the interval-tree procedures to support the new operation INTERVAL-SEARCH-EXACTLY(T, i), where T is an interval tree and i is an interval. The operation should return a pointer to a node x in T such that $x.int.low = i.low$ and $x.int.high = i.high$, or $T.nil$ if T contains no such node. All operations, including INTERVAL-SEARCH-EXACTLY, should run in $O(\lg n)$ time on an n-node interval tree.

14.3-6

Show how to maintain a dynamic set Q of numbers that supports the operation MIN-GAP, which gives the magnitude of the difference of the two closest numbers in Q. For example, if $Q = \{1, 5, 9, 15, 18, 22\}$, then MIN-GAP$(Q)$ returns $18 - 15 = 3$, since 15 and 18 are the two closest numbers in Q. Make the operations INSERT, DELETE, SEARCH, and MIN-GAP as efficient as possible, and analyze their running times.

14.3-7 ★

VLSI databases commonly represent an integrated circuit as a list of rectangles. Assume that each rectangle is rectilinearly oriented (sides parallel to the x- and y-axes), so that we represent a rectangle by its minimum and maximum x- and y-coordinates. Give an $O(n \lg n)$-time algorithm to decide whether or not a set of n rectangles so represented contains two rectangles that overlap. Your algorithm need not report all intersecting pairs, but it must report that an overlap exists if one rectangle entirely covers another, even if the boundary lines do not intersect. (*Hint:* Move a "sweep" line across the set of rectangles.)

Problems

14-1   Point of maximum overlap

Suppose that we wish to keep track of a *point of maximum overlap* in a set of intervals—a point with the largest number of intervals in the set that overlap it.

a. Show that there will always be a point of maximum overlap that is an endpoint of one of the segments.

b. Design a data structure that efficiently supports the operations INTERVAL-INSERT, INTERVAL-DELETE, and FIND-POM, which returns a point of maximum overlap. (*Hint:* Keep a red-black tree of all the endpoints. Associate a value of $+1$ with each left endpoint, and associate a value of -1 with each right endpoint. Augment each node of the tree with some extra information to maintain the point of maximum overlap.)

14-2 *Josephus permutation*

We define the ***Josephus problem*** as follows. Suppose that n people form a circle and that we are given a positive integer $m \leq n$. Beginning with a designated first person, we proceed around the circle, removing every mth person. After each person is removed, counting continues around the circle that remains. This process continues until we have removed all n people. The order in which the people are removed from the circle defines the ***(n, m)-Josephus permutation*** of the integers $1, 2, \ldots, n$. For example, the $(7, 3)$-Josephus permutation is $\langle 3, 6, 2, 7, 5, 1, 4 \rangle$.

a. Suppose that m is a constant. Describe an $O(n)$-time algorithm that, given an integer n, outputs the (n, m)-Josephus permutation.

b. Suppose that m is not a constant. Describe an $O(n \lg n)$-time algorithm that, given integers n and m, outputs the (n, m)-Josephus permutation.

Chapter notes

In their book, Preparata and Shamos [282] describe several of the interval trees that appear in the literature, citing work by H. Edelsbrunner (1980) and E. M. McCreight (1981). The book details an interval tree that, given a static database of n intervals, allows us to enumerate all k intervals that overlap a given query interval in $O(k + \lg n)$ time.