Date
Page

1) Sol

$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$

$u(x,y) + iv(x,y) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$

$\therefore u = \frac{1}{2}\log(x^2+y^2) \quad V = \tan^{-1}\frac{kx}{y}$

f(z) is analytic

$\therefore u_x = V_y$ & $u_y = -V_x$

i.e CR eq^n are satisfied

$u_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$

$u_y = \frac{y}{(x^2+y^2)}$

$V_x = \frac{1}{1+\frac{k^2x^2}{y^2}} \cdot \frac{k}{y} = \frac{ky}{y^2+k^2x^2}$

$V_y = \frac{1}{1+\frac{k^2x^2}{y^2}} \cdot \left(-\frac{kx}{y^2}\right) = -\frac{kx}{y^2+k^2x^2}$

Since $u_x = V_y$ & $u_y = -V_x$

$\frac{x}{x^2+y^2} = \frac{-kx}{k^2x^2+y^2}$ and $\frac{y}{x^2+y^2} = -\frac{ky}{k^2x^2+y^2}$

Both the eq^n are satisfied when $\underline{k=-1}$

Q2) $f(z) = \frac{1}{z}$; At origin $(z=0)$, $f(z)$ is not defined i.e not analytic

$f(z) = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$$\left.\begin{aligned} u &= \frac{1}{r}\cos\theta & v &= -\frac{1}{r}\sin\theta \\ u_r &= -\frac{\cos\theta}{r^2} & v_r &= \frac{1}{r^2}\sin\theta \\ u_\theta &= -\frac{\sin\theta}{r} & v_\theta &= -\frac{1}{r}\cos\theta \end{aligned}\right\} \text{for } z \neq 0$$

$\frac{1}{r}v_\theta = -\frac{1}{r^2}\cos\theta = u_r$

or $u_r = \frac{1}{r}v_\theta$

And $-rv_r = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} = u_\theta$

$\therefore u_\theta = -rv_r$

$\therefore$ CR eq^n are satisfied ---- (for $z \neq 0$)

$\therefore f(z) = \frac{1}{z}$ is analytic except at $z=0$ (origin)

Q3) $u = y^3 - 3x^2y$

$u_x = -6xy$ $\qquad$ $u_y = 3y^2 - 3x^2$

$u_{xx} = -6y$ $\qquad$ $u_{yy} = 6y$

$u_{xx} + u_{yy} = -6y + 6y$

$\therefore u = yx^3 - 3x^2y$ is a harmonic fn

we have v which is harmonic conjugate at u such that $f(z) = u + iv$ is analytic.

$\therefore u_x = v_y$ $\qquad$ $u_y = -v_x$

$\therefore \dfrac{\partial v}{\partial y} = -6xy$ & $\therefore \dfrac{\partial v}{\partial x} = 3x^2 - 3y^2$

$\therefore f'(z) = \dfrac{\partial v}{\partial y} + i\dfrac{\partial v}{\partial x}$

$f'(z) = -6xy + i(3x^2 - 3y^2)$

Using Mihe Thomson method

$f'(z) = i(3z^2)$

Integrating w.r.t z

$\therefore f(z) = i3\left(\dfrac{z^3}{3}\right) + c = iz^3 + c$

$\therefore f(z) = i(x+iy)^3 + c$

$= i[x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3] + c$

$= i[(x^2 - 3xy^2) + i(3x^2y - y^3)] + c$

$\therefore f(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2)$ ... (for $c=0$)

Harmonic conjugate $(v) = x^3 - 3xy^2$

Analytic function $f(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2)$

classmate

Date

Page

Q.4) Yet the analytic function be $f(z) = u + iv$

$\therefore v = \dfrac{x}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^e}$$

$$\frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^e)^2}$$

$$f'(z) = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{-2xy}{(x^2+y^2)^2} + i\frac{(y^2-x^2)}{(x^2+y^2)^2}$$

Using Milne - Thomsons method

$$f'(z) = \frac{-iz^2}{z^4} = \frac{-i}{z^2}$$

integration w.r.t z

$$f(z) = -i\left(\frac{-1}{z}\right) + c = \frac{i}{z} + c = \frac{i}{(x+iy)} + c$$

$$f(z) = \frac{ix+y}{x^2+y^2} + c$$

or $f(z) = \left(\dfrac{y}{x^2+y^2} + c\right) + i\left(\dfrac{x}{x^2+y^2}\right)$ => Analytic fn

$\therefore$ Real part $= \dfrac{y}{x^2+y^2} + c$

Q5)
a) $x^2-y^2+x=c$

$\therefore u=x^2-y^2+x$

$u_x=2x+1$

$u_y=-2y$

$f'(z)=u_x+iv_x=u_x-iu_y$

$f'(z)=2x+1+2iy$

$=2z+1$

$\therefore f(z)=\int 2z+1$

$=z^2+z+c$

$=(x+iy)^2+(x+iy)+c$

$=(x^2-y^2+2xiy)+(x+iy)+c$

$=x^2-y^2+x+2ixy+iy+c$

Imaginary part $=2xy+y$

Req orthogonal trajectory $\rightarrow$ $2xy+y=c$

classmate

Date

Page

b] $r^2\cos 2\theta = \alpha$

$u = r^2\cos 2\theta$

$u_r = 2r\cos 2\theta$

$u_\theta = -2r^2\sin 2\theta$

$f'(z) = e^{-i\theta}(u_r + iv_r)$

$= e^{-i\theta}\left(u_r - \frac{i}{r}u_\theta\right)$

$= e^{-i\theta}(2r\cos 2\theta - 2ir\sin 2\theta)$

$\therefore f'(z) = 2re^{-i\theta}(\cos 2\theta - i\sin 2\theta)$

Using Milne Thomson method,

$f'(z) = 2z$

$\therefore f(z) = z^2 + c$

$\therefore f(z) = (x+iy)^2 + c$

$f(z) = x^2 - y^2 + 2ixy + c$

or

$f(z) = r^2\cos^2\theta - r^2\sin^2\theta + 2ir\cos\theta\, r\sin\theta + c$

$f'(z) = r^2\cos 2\theta + ir^2\sin 2\theta + c$

$\therefore$ Req orthogonal trajectory

$r^2\sin 2\theta = c$

classmate

Date

Page

Q6) $r = b\,\mathrm{cosec}\,\theta \rightarrow r\sin\theta = b$

$r = a\sec\theta \rightarrow r\cos\theta = a$

Let $u = r\cos\theta$ & $v = r\sin\theta$

we form fn $f(z) = u + iv$

$\therefore f(z) = r\cos\theta + ir\sin\theta$

$u_r = \cos\theta$ $\quad v_r = \sin\theta$

$u_\theta = -r\sin\theta$ $\quad v_\theta = r\cos\theta$

$u_r = \frac{1}{r}v_\theta$ & $u_\theta = -r v_r$

CR eqn are sat satisfied

$\therefore f(z)$ is analytic

$\therefore r\sin\theta = b$ & $r\cos\theta = a$ forms orthogonal trajectories or $r - b\,\mathrm{cosec}\,\theta$ & $r = a\sec\theta$ forms orthogonal trajectories