Inverse Z-Transform

Find the inverse z-transform in the given region of convergence.

(1) $\dfrac{z}{z-a}$ (i) $|z| > |a|$ (ii) $|z| < |a|$

Soln Let $F(z) = \dfrac{z}{z-a}$

(i) $|z| > |a|$

$$F(z) = \frac{z}{z-a} = z \, \frac{1}{z} \, \frac{1}{\left(1-\frac{a}{z}\right)}$$

$$= \frac{1}{1-\frac{a}{z}} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$= \sum_{k=0}^{\infty} a^k z^{-k}$$

for $\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$

Coefficient $z^{-k} = a^k$, $k \geq 0$

$\Rightarrow z^{-1}\{F(z)\} = a^k$, $k \geq 0$

(ii) $|z| < |a|$

$$F(z) = \frac{z}{z-a} = \frac{z}{-a} \, \frac{1}{1-\frac{z}{a}}$$

$$= -\frac{z}{a} \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^k$$

$$\Rightarrow F(Z) = -\frac{Z}{a} \sum_{K=0}^{\infty} \frac{Z^K}{a^K}$$

$$= \sum_{K=0}^{\infty} -\frac{1}{a^{K+1}} Z^{K+1}$$

for $\left|\frac{Z}{a}\right| < 1 \Rightarrow |Z| < |a|$

coefficient of $Z^{K+1} = -\frac{1}{a^{K+1}}, \quad K \geq 0$

coefficient of $Z^{K} = -\frac{1}{a^{K}}, \quad K-1 \geq 0$

coefficient of $Z^{-K} = -\frac{1}{a^{-K}}, \quad -K-1 \geq 0$

$$= -a^K, \quad K \leq -1$$

$$\Rightarrow Z^{-1}\{F(Z)\} = -a^K, \quad K \leq -1$$

(2) $\frac{Z}{(Z-a)^2}$ (i) $|Z| > |a|$ (ii) $|Z| < |a|$

Sol^n Let $F(Z) = \frac{Z}{(Z-a)^2}$

(i) $|Z| > |a|$

$$F(Z) = \frac{Z}{(Z-a)^2}$$

$$= \frac{Z}{Z^2} \frac{1}{\left(1-\frac{a}{Z}\right)^2}$$

$$= \frac{1}{Z} \frac{1}{\left(1-\frac{a}{Z}\right)^2}$$

$$F(Z) = \frac{1}{Z} \sum_{K=0}^{\infty} (K+1) \left(\frac{a}{Z}\right)^K$$

$$= \frac{1}{Z} \sum_{K=0}^{\infty} (K+1)\, a^K Z^{-K}$$

$$= \sum_{K=0}^{\infty} (K+1)\, a^K Z^{-K-1}$$

for $\left|\frac{a}{Z}\right| < 1 \Rightarrow |Z| > |a|$

coeff of $Z^{-K-1} = (K+1)\, a^K, \quad K \geq 0$

coeff of $Z^{-(K+1)} = (K+1)\, a^K, \quad K \geq 0$

coeff of $Z^{-K} = K\, a^{K-1}, \quad K-1 \geq 0$

$\Leftrightarrow K \geq 1$

$\therefore Z^{-1}\{F(Z)\} = K\, a^{K-1}, \quad K \geq 1$

(ii) $|Z| < |a|$

$$F(Z) = \frac{Z}{(Z-a)^2}$$

$$= \frac{Z}{(-a)^2} \frac{1}{\left(1 - \frac{Z}{a}\right)^2}$$

$$= \frac{Z}{a^2} \sum_{K=0}^{\infty} (K+1) \left(\frac{Z}{a}\right)^K$$

$$= \frac{Z}{a^2} \sum_{K=0}^{\infty} (K+1) \frac{Z^K}{a^K}$$

$$\Rightarrow \quad F(Z) = \sum_{K=0}^{\infty} (K+1) \frac{1}{a^{K+2}} Z^{K+1}, \quad \text{for } \left|\frac{Z}{a}\right| < 1$$

$$\Rightarrow |Z| < |a|$$

$$\text{coeff of } Z^{K+1} = \frac{K+1}{a^{K+2}}, \quad K \geq 0$$

$$\text{coeff of } Z^{K} = \frac{K}{a^{K+1}}, \quad K-1 \geq 0$$

$$\text{coeff of } Z^{-K} = \frac{-K}{a^{-K+1}}, \quad -K-1 \geq 0$$

$$= -K a^{K-1}, \quad K \leq -1$$

$$\Rightarrow Z^{-1}\{F(Z)\} = -K a^{K-1}, \quad K \leq -1$$

3. $\frac{1}{(Z+2)^3}$ (i) $|Z| > 2$ (ii) $|Z| < 2$

Sol^n Let $F(Z) = \frac{1}{(Z+2)^3}$

(i) For $|Z| > 2$

$$F(Z) = \frac{1}{(Z+2)^3} = \frac{1}{Z^3} \frac{1}{\left(1+\frac{2}{Z}\right)^3}$$

$$= \frac{1}{Z^3} \sum_{K=0}^{\infty} (-1)^K \frac{(K+1)(K+2)}{2!} \left(\frac{2}{Z}\right)^K$$

$$= \frac{1}{Z^3} \sum_{K=0}^{\infty} (-1)^K \frac{(K+1)(K+2)}{2} \frac{2^K}{Z^K}$$

$$F(z) = \sum_{k=0}^{\infty} (-1)^k (k+1)(k+2) \frac{2^{k-1}}{z^{k+3}}$$

$$= \sum_{k=0}^{\infty} (-1)^k (k+1)(k+2)\, 2^{k-1} z^{-(k+3)}$$

for $\left|\frac{2}{z}\right| < 1 \Rightarrow |z| > 2$

coeff of $z^{-(k+3)} = (-1)^k (k+1)(k+2)\, 2^{k-1}, \quad k \geq 0$

coeff of $z^{-k} = (-1)^{k-3} (k-2)(k-1)\, 2^{k-4},$

$$k-3 \geq 0$$

$$= -(-1)^k (k-2)(k-1)\, 2^{k-4},$$

$$k \geq 3$$

$\therefore Z^{-1}\{F(z)\} = -(-1)^k (k-2)(k-1)\, 2^{k-4}, \quad k \geq 3$

(ii) For $|z| < 2$

$$F(z) = \frac{1}{(z+2)^3} = \frac{1}{2^3} \frac{1}{\left(1+\frac{z}{2}\right)^3}$$

$$= \frac{1}{2^3} \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(k+2)}{2!} \left(\frac{z}{2}\right)^k$$

$$= \frac{1}{2^3} \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(k+2)}{2} \frac{z^k}{2^k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(k+2)}{2^{k+4}} z^k$$

for $\left|\frac{Z}{2}\right| < 1 \Rightarrow |Z| < 2$

$\therefore$ coeff of $Z^K = (-1)^K \frac{(K+1)(K+2)}{2^{K+4}}$, $K \geq 0$

$\therefore$ coeff of $Z^{-K} = (-1)^{-K} \frac{(-K+1)(-K+2)}{2^{-K+4}}$, $-K \geq 0$

$= (-1)^K (K-1)(K-2)\, 2^{K-4}$, $K \leq 0$

$\therefore Z^{-1}\{F(Z)\} = (-1)^K (K-1)(K-2)\, 2^{K-4}$, $K \leq 0$

(4) $\frac{1}{Z^2+Z-6}$ (i) $|Z| < 2$ (ii) $2 < |Z| < 3$ (iii) $|Z| > 3$

Sol^n Let $F(Z) = \frac{1}{Z^2+Z-6}$

$= \frac{1}{(Z+3)(Z-2)}$

$= \frac{A}{(Z+3)} + \frac{B}{(Z-2)}$

$\Rightarrow A(Z-2) + B(Z+3) = 1$

$\Rightarrow A = -\frac{1}{5}$, $B = \frac{1}{5}$

$F(Z) = \frac{1}{5}\frac{1}{(Z-2)} - \frac{1}{5}\frac{1}{(Z+3)}$ ——— (1)

(i) For $|z|<2$

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k$$

$$= -\frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k} = \sum_{k=0}^{\infty} -\frac{1}{2^{k+1}} z^k$$

for $\left|\frac{z}{2}\right|<1 \Rightarrow |z|<2$

coeff of $z^k = -\frac{1}{2^{k+1}}$, $k \geq 0$

coeff of $z^{-k} = -\frac{1}{2^{-k+1}} = -2^{k-1}$, $-k \geq 0$
$\Rightarrow k \leq 0$

$$\frac{1}{(z+3)} = \frac{1}{3}\frac{1}{\left(1+\frac{z}{3}\right)} = \frac{1}{3}\sum_{k=0}^{\infty}(-1)^k\frac{z^k}{3^k}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{1}{3^{k+1}} z^k$$

for $\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$

coeff of $z^k = (-1)^k\frac{1}{3^{k+1}}$, $k \geq 0$

coeff of $z^{-k} = (-1)^{-k}\frac{1}{3^{-k+1}} = (-1)^k 3^{k-1}$,
$-k \geq 0 \Rightarrow k \leq 0$

$\therefore$ for $|z|<2$

$$z^{-1}\{F(z)\} = -\frac{1}{5}2^{k-1} - \frac{1}{5}(-1)^k 3^{k-1}, \quad k \leq 0$$

$$= -\frac{1}{5}\left(2^{K-1} + (-1)^K 3^{K-1}\right), \quad K \le 0$$

(ii) For $2 < |z| < 3$

$$\frac{1}{(z-2)} = \frac{1}{z}\frac{1}{\left(1-\frac{2}{z}\right)} = \frac{1}{z}\sum_{K=0}^{\infty}\left(\frac{2}{z}\right)^K$$

$$= \frac{1}{z}\sum_{K=0}^{\infty}\frac{2^K}{z^K} = \sum_{K=0}^{\infty} 2^K z^{-(K+1)}$$

for $\left|\frac{2}{z}\right| < 1 \Rightarrow 2 < |z| \Rightarrow |z| > 2$

coeff of $z^{-(K+1)} = 2^K, \quad K \ge 0$

coeff of $z^{-K} = 2^{K-1}, \quad K-1 \ge 0$

$\Rightarrow K \ge 1$

$$\therefore Z^{-1}\{F(z)\} = \begin{cases} \frac{1}{5} 2^{K-1} & K \ge 1 \\ -\frac{1}{5}(-1)^K 3^{K-1} & K \le 0 \end{cases}$$

$$= \frac{1}{5}\begin{cases} 2^{K-1} & K \ge 1 \\ -(-1)^K 3^{K-1} & K \le 0 \end{cases}$$

(ii) For $|z| > 3$

$$\frac{1}{z+3} = \frac{1}{z}\frac{1}{\left(1+\frac{3}{z}\right)} = \frac{1}{z}\sum_{K=0}^{\infty}(-1)^K\left(\frac{3}{z}\right)^K$$

$$= \frac{1}{z}\sum_{K=0}^{\infty}(-1)^K\frac{3^K}{z^K} = \sum_{K=0}^{\infty}(-1)^K\frac{3^K}{z^{K+1}}$$

$$\frac{1}{z+3} = \sum_{k=0}^{\infty} (-1)^k 3^k z^{-(k+1)}, \text{ for } \left|\frac{3}{z}\right|<1 \Rightarrow |z|>3$$

$$\text{coeff } z^{-(k+1)} = (-1)^k 3^k, \quad k \geq 0$$

$$\text{coeff of } z^{-k} = (-1)^{k-1} 3^{k-1}, \quad k-1 \geq 0$$

$$= -(-1)^k 3^{k-1}, \quad k \geq 1$$

$\therefore$ for $|z|>3$

$$z^{-1}\{F(z)\} = \frac{1}{5} 2^{k-1} + \frac{1}{5}(-1)^k 3^{k-1}$$

$$= \frac{1}{5}\left(2^{k-1} + (-1)^k 3^{k-1}\right), \quad k \geq 1$$