(3) The marks obtained by a number of students for a certain subject are assumed to be normally distributed with mean 65 and s.d. 5. If 3 students are selected at random from this set, what is the probability that exactly 2 of them will have marks more than 60.

Soln X: marks of a student.

$X \sim N(\mu, \sigma^2)$

$\mu = 65, \quad \sigma = 5$

$P\{$A student marks is more than 60$\}$

$= P\{X > 60\} = 1 - P\{X \leq 60\}$

$= 1 - P\{Z \leq \frac{60-65}{5}\} = 1 - P\{Z \leq -1.00\}$

$= 1 - 0.1587 = 0.8413$

Y: No. of students out of 3 selected one having marks more than 60.

$\Rightarrow Y \sim B(n, p); \; n = 3, \quad p = 0.8413$

$\therefore q = 1 - p = 0.1587$

$\therefore P\{$2 of them having marks more 60$\}$

$= P\{Y = 2\} = {}^3C_2 \, (0.8413)^2 \, (0.1587) = 0.337$

(4) A machine automatically packs a chemical fertilizer in polythene packets. It is observed that 10% of the packets weigh less than 2.5 Kg and 12% of the packets weigh more than 2.8 Kg. Assuming that weight of the packets is normally distributed, find the mean wt. and variace of the packets.

Soln X: wt. of a packet.

$X \sim N(\mu, \sigma^2)$

$\mu = ?\ ,\quad \sigma^2 = ?$

$P\{X < 2.5\} = \frac{10}{100} = 0.1$ —— (i)

& $P\{X > 2.8\} = \frac{12}{100} = 0.12$ —— (ii)

$\Rightarrow P\{Z < \frac{2.5 - \mu}{\sigma}\} = 0.1$

$\Rightarrow \frac{2.5 - \mu}{\sigma} = -1.28$

$\Rightarrow \mu - 1.28\sigma = 2.5$ —— (1)

from (ii) $1 - P\{X \leq 2.8\} = 0.12$

$\Rightarrow P\{X \leq 2.8\} = 0.88$

$\Rightarrow P\{Z \leq \frac{2.8 - \mu}{\sigma}\} = 0.88$

$$\Rightarrow \frac{2.8 - \mu}{\sigma} = 1.18$$

$$\Rightarrow \mu + 1.18\sigma = 2.8 \quad \text{--------} \quad (2)$$

$$\Rightarrow \mu = 2.66, \quad \sigma = 0.122$$

$\therefore$ mean wt. of packet $= \mu = 2.66$

$\quad var(X) = \sigma^2 = 0.0159$

(5) In an examination, suppose there are 100 questions of 1 marks each. Each questions has three choices of answers of which one correct. one passes the examination if he/she scores at least 40 marks.

(i) Find the probability that a candidate who chooses the answer to each questions randomly will pass the examination.

(ii) Estimate the least number of questions required in that paper so that the pass marks remain 40% but the probability of passing by random choice of answers will not exceed 1%.

Soln (i) X: marks obtained by random choice of answers out of 100.

$X \sim B(n, p)$; $n = 100$, $p = 1/3$

P{ A candidate passes the examination}

$= P\{X \geq 40\}$

$= 1 - P\{X < 40\}$

$n = 100$ is large enough, therefore approx. by Normal distribution;

$X \sim N(\mu, \sigma^2)$,

$\mu = np = 100 \times \frac{1}{3} = \frac{100}{3}$

$\sigma^2 = npq = 100 \times \frac{1}{3} \times \frac{2}{3} = \frac{200}{9}$

$\Rightarrow \sigma = \frac{\sqrt{200}}{3}$

$P\{X \geq 40\}$

$= 1 - P\{X < 40\}$ {X is a discrete random variable}

$= 1 - P\{X \leq 40 - 0.5\}$

$= 1 - P\left\{Z \leq \dfrac{39.5 - \frac{100}{3}}{\frac{\sqrt{200}}{3}}\right\}$

$= 1 - P\{Z \leq 1.31\}$

$= 1 - 0.9049 = 0.0951$

(ii) Let n be the number of questions required.

X: marks obtained by random choice out of 'n' questions.

$\therefore \quad X \sim B(n, p)$

$n = ? \quad , \quad p = 1/3$

Approximating by Normal distribution,

$X \sim N(\mu, \sigma^2)$

$\mu = np = n/3$

$\sigma^2 = npq = n \frac{1}{3} \times \frac{2}{3} = \frac{2n}{9}$

$\therefore \quad \sigma = \frac{\sqrt{2n}}{3}$

$\therefore$ P{A candidate passes the examination} = 1%

$\Rightarrow \quad P\{X \geq 40\% \text{ of } n\} = \frac{1}{100} = 0.01$ $\quad \boxed{\frac{40}{100} n = 0.4n}$

$\Rightarrow \quad P\{X \geq 0.4n\} = 0.01$

$\Rightarrow \quad 1 - P\{X < 0.4n\} = 0.01$

$\Rightarrow \quad P\{X < 0.4n\} = 0.99$

$\Rightarrow \quad P\{X \leq 0.4n - 0.5\} = 0.99$

$\Rightarrow \quad P\left\{Z \leq \frac{0.4n - 0.5 - \frac{n}{3}}{\frac{\sqrt{2n}}{3}}\right\} = 0.99$

$\Rightarrow \quad \frac{0.4n - 0.5 - \frac{n}{3}}{\frac{\sqrt{2n}}{3}} = 2.33$

$\Rightarrow \quad \frac{0.2n - 1.5}{\sqrt{2n}} = 2.33$

$\Rightarrow \quad 0.2n - 1.5 = 2.33\sqrt{2n}$

$\Rightarrow \quad (0.2n-1.5)^2 = (2.33)^2 \times 2n$

$\Rightarrow \quad 0.04n^2 - 0.6n + 2.25 = 10.8578n$

$\Rightarrow \quad 0.04n^2 - 11.4578n + 2.25 = 0$

$\Rightarrow \quad n = 286.3 \simeq 286$

(6) In an examination, the marks obtained by students in Mathematics, physics and chemistry are normally distributed with means 40, 46, 44 and with standard deviation 15, 12, 16 respectively. Find the probability of a student securing total marks (i) 180 or above (ii) 90 or below.

Solⁿ X_1 : Marks obtained in Mathematics

X_2 : ——————— " ——————— Physics

X_3 : ——————— " ——————— Chemistry

$X_1 \sim N(\mu_1, \sigma_1^2)$, $\mu_1 = 40$, $\sigma_1 = 15$

$X_2 \sim N(\mu_2, \sigma_2^2)$, $\mu_2 = 46$, $\sigma_2 = 12$

$X_3 \sim N(\mu_3, \sigma_3^2)$, $\mu_3 = 44$, $\sigma_3 = 16$

T : Total marks obtained in Mathematics, physics and chemistry.

$T = X_1 + X_2 + X_3$

$\therefore T \sim N(\mu, \sigma^2)$

$\mu = E(T) = E(X_1 + X_2 + X_3)$

$= E(X_1) + E(X_2) + E(X_3)$

$= 40 + 46 + 44 = 130$

$\sigma^2 = Var(T) = Var(X_1 + X_2 + X_3)$ [X_1, X_2, X_3 are independent]

$= Var(X_1) + Var(X_2) + Var(X_3)$

$= \sigma_1^2 + \sigma_2^2 + \sigma_3^2$

$= 15^2 + 12^2 + 16^2 = 625$

$\therefore \sigma = 25$

(i) P{Total marks 180 or above}

$= P\{T \geq 180\} = 1 - P\{T < 180\}$

$= 1 - P\{Z \leq \frac{180-130}{25}\} = 1 - P\{Z \leq 2.00\}$

$= 1 - 0.9772 = 0.0228$

(ii) P{Total marks 90 or below}

$= P\{T \leq 90\}$

$= P\{Z \leq \frac{90-130}{25}\}$

$= P\{Z \leq -1.60\} = 0.0548$

Central Limit Theorem (CLT) :-

If $X_1, X_2, \dots, X_n$ are n independent identically distributed (iid) random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$, then the sum $T = X_1 + X_2 + \dots + X_n$ is approximately Normal with mean $E(T) = n\mu$ and $Var(T) = n\sigma^2$ for sufficiently large 'n'

i.e.

$$T = X_1 + X_2 + \dots + X_n \sim N(\mu_T, \sigma_T^2) \text{ as } n \to \infty$$

where $\mu_T = n\mu$, $\sigma_T^2 = n\sigma^2$

Distribution of mean n random variables:-

If $X_1, X_2, \dots, X_n$ are n iid with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$, then by CLT,

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n) \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

as $n \to \infty$

Note: CLT hold good for $n \geq 20$.

① A charter plane company was asked to carry 100 units of a particular product. one unit product has mean wt. 48 kg and variance 9 kg². suppose that plane available has carrying capacity of 5000 kg. Is it safe to carry products.

Sol^n X: Total wt. of 100 units of product

$n = 100$,

μ = mean wt. of 1 unit = 48 kg

σ^2 = variance of 1 unit = 9 kg²

∴ by CLT

$X \sim N(\mu_X, \sigma_X^2)$

$\mu_X = n\mu = 100 \times 48 = 4800$

$\sigma_X^2 = n\sigma^2 = 100 \times 9 = 900$

$\therefore \sigma_X = 30$

∴ P{It is safe to carry products}

$= P\{X \leq 5000\}$

$= P\{Z \leq \frac{5000 - 4800}{30}\}$

$= P\{Z \leq 6.67\} = 1$

⇒ It is safe to carry products.

(2) A parent offer his son to pay Rs 1000 if the mean score of 50 throws of a die exceeds 3. What is the probability that the parent will pay the money?

Sol

Soln: X : Mean score of 50 throws of a die.

$n = 50$ is large

$\therefore X \sim N(\mu_X, \sigma_X^2)$

Let X_i : i^{th} throw of of a die.

X_i :	1	2	3	4	5	6
$p(x)$:	1/6	1/6	1/6	1/6	1/6	1/6

$$\mu = E(X_i) = \frac{1}{6}(1+2+\cdots+6) = 3.5$$

$$E(X_i^2) = \frac{1}{6}(1+2^2+\cdots+6^2) = \frac{91}{6}$$

$$\therefore \sigma^2 = Var(X_i) = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$
$$= 2.917$$

$$\therefore \mu_X = \mu = 3.5,$$

$$\sigma_X^2 = \frac{\sigma^2}{n} = \frac{\frac{35}{12}}{50} = \frac{7}{120}$$

$$\therefore \sigma_X = 0.2415$$

$\therefore$ P{ parent will pay the money}

$= P\{X > 3\}$

$= 1 - P\{X \le 3\}$

$= 1 - P\{Z \le \frac{3-3.5}{0.2415}\}$

$= 1 - P\{Z \le -2.07\}$

$= 1 - 0.0192$

$= 0.9808$