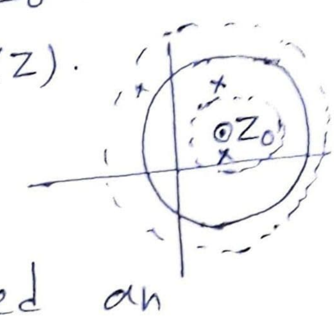
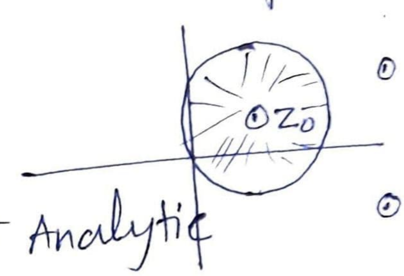
Singular Point :-

If $f(z)$ is not Analytic at z_0 and Analytic at at-least one point in every open disk around z_0, then z_0 is called a singular point of $f(z)$.



Isolated singular point :-

A singular point z_0 is called an isolated singular point if there exist an open disk around z_0 where $f(z)$ is Analytic at all points except at z_0.

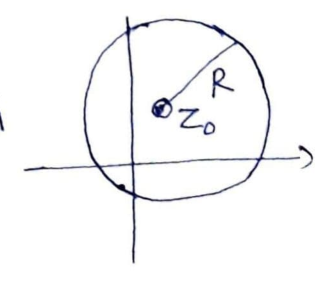


Note:- If $f(z)$ is not Analytic at finite numbers of points, then all are isolated singular points of $f(z)$.

Types of Isolated singular points :-

Let z_0 be an isolated singular point of $f(z)$, then $f(z)$ is is Analytic in some deleted neighborhood $0<|z-z_0|<R$ and therefore has Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$



i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

for $0 < |z-z_0| < R$

① If all $b_n = 0$

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

Then z_0 is called a Removable singular point of $f(z)$.

② If finite numbers of b_n's are nonzero and $b_m \neq 0$, $b_{m+1} = 0$, $b_{m+2} = 0$, ...

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$

$$\cdots + \frac{b_m}{(z-z_0)^m}$$

Then z_0 is called a Pole of order m of $f(z)$.

Note:- A pole of order one is referred as simple pole.

③ If infinite numbers of $b_n \neq 0$

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$

Then z_0 is called an Essential singular point of $f(z)$.

Note:- If z_0 is a Removable or an Essential singular point of $f(z)$ then it can be identified only by series expansion of $f(z)$ about the point $z=z_0$ in the domain $0<|z-z_0|<R$.

If z_0 is a pole then it can be identified using some more results.

Results :-

(1) z_0 is a pole of order m of $f(z)$ iff $f(z)=\dfrac{g(z)}{(z-z_0)^m}$ where $g(z)$ is Analytic at z_0 and $g(z_0)\neq 0$.

(2) If $f(z)=\dfrac{g(z)}{h(z)}$ where $g(z)$ is Analytic at z_0, $g(z_0)\neq 0$,
$h(z_0)=0$, $h'(z_0)=0$, $h''(z_0)=0, \ldots, h^{(m-1)}(z_0)=0$
and $h^{(m)}(z_0)\neq 0$,
Then z_0 is a pole of order m of $f(z)$

Note :- If $f(z)=\dfrac{g(z)}{h(z)}$ where $g(z)$ is Analytic at z_0, $g(z_0)\neq 0$, $h(z)$ is a polynomial and

z_0 is a root of the equation $h(z) = 0$ and repeated m times then z_0 is a pole of order m of $f(z)$.

Residue's of $f(z)$ at an Isolated singular points :—

Let z_0 be an isolated singular point of $f(z)$. Then $f(z)$ has Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$\text{for } 0 < |z-z_0| < R$$

Residue of $f(z)$ at $z = z_0$

$$= \operatorname*{Res}_{z=z_0} f(z) = \text{coefficient of } \frac{1}{(z-z_0)} \text{ term}$$

$$= b_1$$

Note:— Residue at Removable or Essential singular point can be obtained only from the series expansion.

Residue at a pole can be obtained using some more results.

Residues at Poles :-

① Let z_0 is a pole of order m of $f(z)$

and $$f(z) = \frac{g(z)}{(z-z_0)^m}$$

where $g(z)$ is Analytic at z_0, $g(z_0) \neq 0$

Then
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(z) \Big|_{z=z_0}$$

② Let z_0 is a pole of order one of $f(z)$

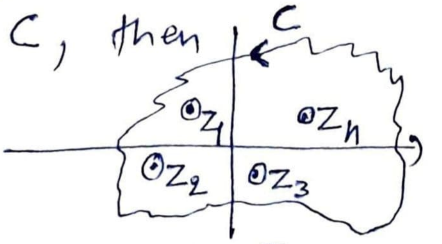
and $$f(z) = \frac{g(z)}{h(z)}$$

where $g(z)$ is Analytic at z_0, $g(z_0) \neq 0$

Then
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g(z)}{h'(z)} \Big|_{z=z_0}$$

Cauchy Residue Theorem :-

If $f(z)$ is Analytic on and inside a closed contour C except at finite numbers of points $z_1, z_2, \dots, z_n$ lying completely inside C, then



$$\int_C f(z)\,dz$$
$$= 2\pi i \left[\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \dots + \operatorname*{Res}_{z=z_n} f(z) \right]$$

Integration of complex variable using Residue Theory :-

Ex. Evaluate $\int_C \frac{2z-1}{z^4-2z^3-3z^2} dz$

where (i) $|z-\frac{1}{2}|=1$ (ii) $|z+1|=2$

(iii) $|z-1|=3$ (iv) $|z-2i|=1$

Solⁿ Let $I = \int_C \frac{2z-1}{z^4-2z^3-3z^2} dz$

$f(z) = \frac{2z-1}{z^4-2z^3-3z^2}$ is not Analytic

at $z^4-2z^3-3z^2=0$

$\Rightarrow z^2(z^2-2z-3)=0$

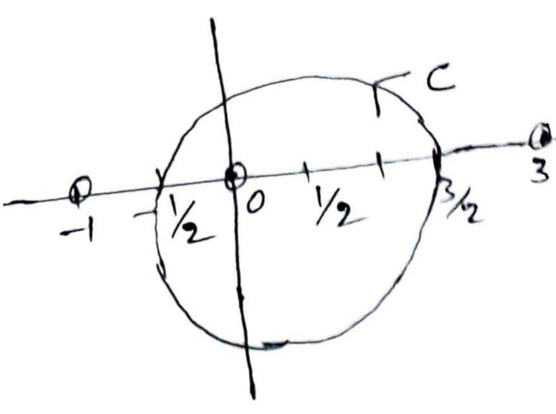
$\Rightarrow z^2=0, \quad z^2-2z-3=0$

$\Rightarrow z=0, 0, \quad z=3, -1$

(i) $C: |z-\frac{1}{2}|=1$

$z=0$ lies inside C

$z=0$ is a pole of order two.



$$\therefore \operatorname*{Res}_{z=0} f(z) = \operatorname*{Res}_{z=0} \frac{2z-1}{z^4-2z^3-3z^2}$$

$$= \operatorname*{Res}_{z=0} \frac{2z-1}{z^2(z^2-2z-3)}$$

$$= \frac{1}{1!} \frac{d}{dz}\left[\frac{2z-1}{z^2-2z-3}\right]\Bigg|_{z=0}$$

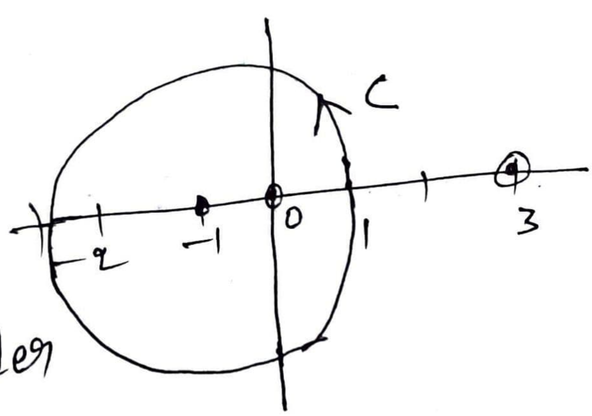
$$= \frac{2(z^2-2z-3)-(2z-1)(2z-2)}{(z^2-2z-3)^2}\Bigg|_{z=0}$$

$$= \frac{2(-3)-(-1)(-2)}{(-3)^2} = -\frac{8}{9}$$

$$\therefore \int_c f(z)\,dz = 2\pi i\left[-\frac{8}{9}\right] = -\frac{16\pi i}{9}$$

(ii) $C: |z+1| = 2$

$z = 0, -1$ lies inside C.



$z = 0$ is a pole of order two

$z = -1$ is a pole of order one.

$$\underset{z=0}{\text{Res}}\, f(z) = -\frac{8}{9}$$

$$\underset{z=-1}{\text{Res}}\, f(z) = \underset{z=-1}{\text{Res}} \frac{2z-1}{z^4-2z^3-3z^2}$$

$$= \cancel{\underset{z\to}{\text{Res}}} \left.\frac{2z-1}{4z^3-6z^2-6z}\right|_{z=-1}$$

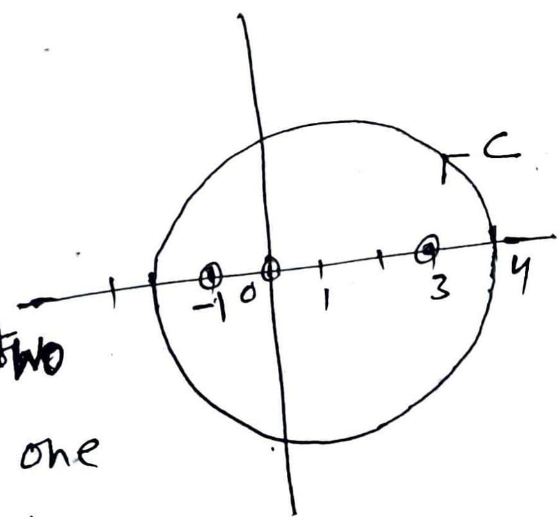
$$= \frac{-2-1}{-4-6+6} = \frac{3}{4}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left[-\frac{8}{9}+\frac{3}{4}\right]$$

$$= -\frac{5}{18}\pi i$$

(iii) $C: |z-1| = 3$

$z = 0, -1, 3$ lies inside C.



$z=0$ is a pole of order two

$z=-1$ is a pole of order one

$z=3$ is a pole of order one

$$\operatorname*{Res}_{z=0} f(z) = -\frac{8}{9}$$

$$\operatorname*{Res}_{z=-1} f(z) = \frac{3}{4}$$

$$\operatorname*{Res}_{z=3} f(z) = \operatorname*{Res}_{z=3} \frac{2z-1}{z^4-2z^3-3z^2}$$

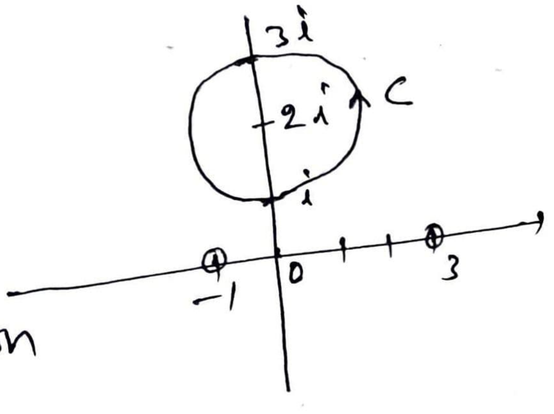
$$= \operatorname*{Res}_{z=3} \frac{2z-1}{4z^3-6z^2-6z}\Bigg|_{z=3}$$

$$= \frac{6-1}{4\times 27-6\times 9-18} = \frac{5}{36}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left[-\frac{8}{9}+\frac{3}{4}+\frac{5}{36}\right]$$

$$= 0$$

(iv) $C: |z-2i| = 1$



All $z = 0, -1, 3$ lies outside C.

$\therefore$ $f(z)$ is Analytic on and inside C;

$\Rightarrow$ By CIT

$$\int_C f(z)\,dz = 0$$

(2) Evaluate $\int_C \frac{2z^3+z^2+4}{z^4+4z^2}\,dz$

$C: |z-2-2i|=3$

Sol^n Let $I = \int_C \frac{2z^3+z^2+4}{z^4+4z^2}\,dz$

$f(z) = \frac{2z^3+z^2+4}{z^4+4z^2}$ is

not Analytic at

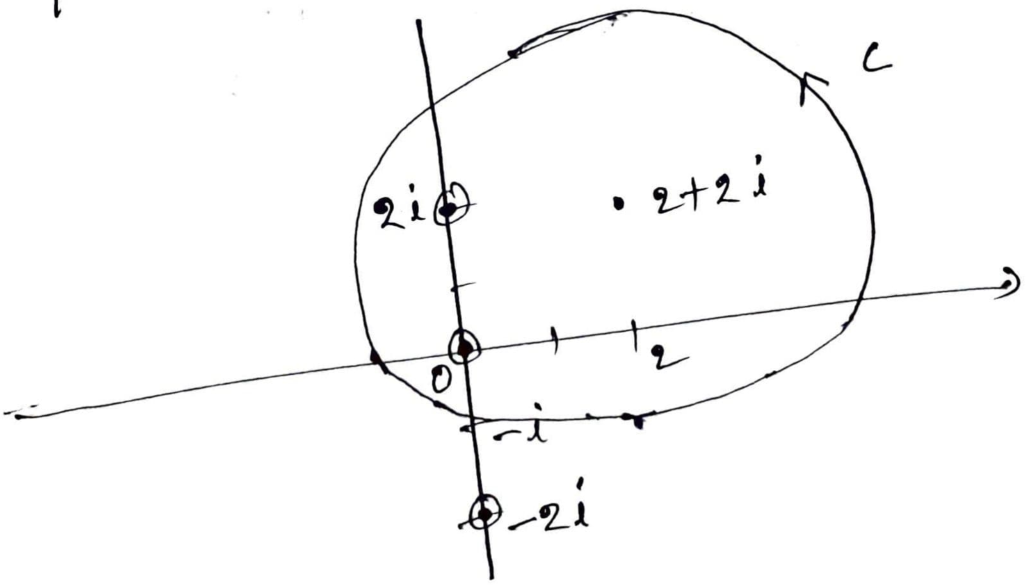
$z^4+4z^2=0$

$\Rightarrow z^2(z^2+4)=0$

$\Rightarrow z^2=0, \quad z^2+4=0$

$\Rightarrow z=0, 0, 2i, -2i$

$C: |z-(2+2i)|=3$



$z = 0, 2i$ lies inside c.

$z=0$ is a pole of order two

$z = 2i$ is a pole of order one.

$$\underset{z=0}{\text{Res}}\, f(z) = \underset{z=0}{\text{Res}} \frac{2z^3+z^2+4}{z^4+4z^2}$$

$$= \underset{z=0}{\text{Res}} \frac{2z^3+z^2+4}{z^2(z^2+4)}$$

$$= \frac{1}{1!}\frac{d}{dz}\left[\frac{2z^3+z^2+4}{z^2+4}\right]\Bigg|_{z=0}$$

$$= \frac{(6z^2+2z)(z^2+4) - (2z^3+z^2+4)\,2z}{(z^2+4)^2}\Bigg|_{z=0}$$

$$= 0$$

$$\underset{z=2i}{\text{Res}}\, f(z) = \underset{z=2i}{\text{Res}} \frac{2z^3+z^2+4}{z^4+4z^2}$$

$$= \frac{2z^3+z^2+4}{4z^3+8z}\Bigg|_{z=2i} = \frac{2(2i)^3+(2i)^2+4}{4(2i)^3+8\times 2i}$$

$$= 1$$

$$\therefore \int_c f(z)\, dz = 2\pi i\,[0+1] = 2\pi i$$

(4) Evaluate $\int_C z^5 e^{-1/z^2} dz$; $C: |z| = 1$

Sol^n $I = \int_C z^5 e^{-1/z^2} dz$

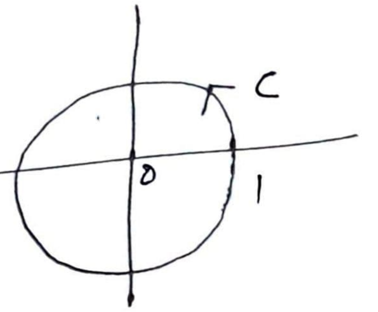
$f(z) = z^5 e^{-1/z^2}$

$f(z)$ is not Analytic at

$z^2 = 0 \Rightarrow z = 0$

$C: |z| = 1$

$z = 0$ lies inside c.



$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$

$\therefore e^{-1/z^2} = 1 - \frac{1}{1!}\frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} - \frac{1}{3!}\frac{1}{z^6} + \cdots$

$\therefore f(z) = z^5 e^{-1/z^2}$

$= z^5 - z^3 + \frac{1}{2!} z - \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \cdots$

$\Rightarrow z = 0$ is an essential singular point.

$\operatorname{Res}_{z=0} f(z) = -\frac{1}{3!}$

$\therefore \int_C f(z)\, dz = 2\pi i \left[-\frac{1}{3!}\right] = -\frac{\pi i}{3}$

(7) Evaluate $\int_C \tan 2\pi z \, dz$; $|z|=1$

Sol^n Let $I = \int_C \tan 2\pi z \, dz$

$$f(z) = \tan 2\pi z = \frac{\sin 2\pi z}{\cos 2\pi z}$$

$f(z)$ is not Analytic at

$\cos 2\pi z = 0$

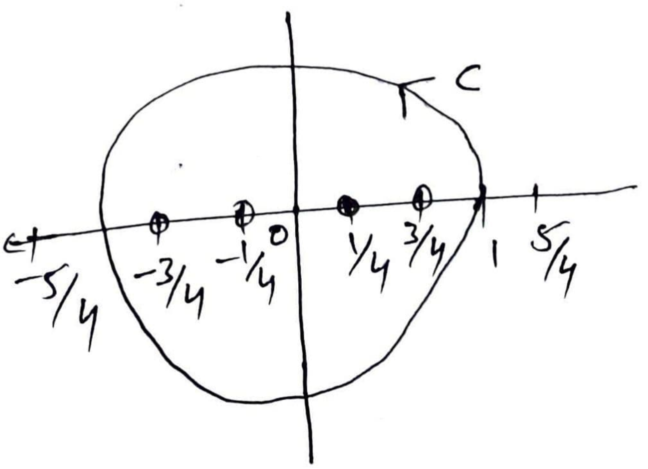
$$\Rightarrow 2\pi z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$

$$\Rightarrow z = \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{5}{4}, \dots$$

C ; $|z|=1$

$z = 1/4, -1/4, 3/4, -3/4$

lies inside C.



Let $h(z) = \cos 2\pi z$

$$h'(z) = -2\pi \sin 2\pi z$$

$$\therefore h'(z)\Big|_{z=1/4} = -2\pi \sin \pi/2 = -2\pi \neq 0$$

$\Rightarrow z = 1/4$ is a pole of order one.

$|||^{ly}$ $h'(z)\Big|_{z=-1/4,\,3/4,\,-3/4} \neq 0$

$\Rightarrow$ All $z = 1/4, -1/4, 3/4, -3/4$ are poles of order one.

$\therefore \operatorname*{Res}_{z=1/4} f(z) = \dfrac{\sin \cancel{2\pi z}}{-2\pi \sin \cancel{2\pi z}}\Bigg|_{z=1/4}$

$= -\dfrac{1}{2\pi}$

$|||^{ly}$ $\operatorname*{Res}_{z=-1/4} f(z) = -\dfrac{1}{2\pi}$

$\operatorname*{Res}_{z=3/4} f(z) = -\dfrac{1}{2\pi}$

$\operatorname*{Res}_{z=-3/4} f(z) = -\dfrac{1}{2\pi}$

$\therefore \int_c \tan 2\pi z \, dz$

$= 2\pi i \left[-\dfrac{1}{2\pi} - \dfrac{1}{2\pi} - \dfrac{1}{2\pi} - \dfrac{1}{2\pi} \right]$

$= -4i$

Ex. Evaluate $\int_C \frac{\cot z}{z}\, dz$; $C: |z|=1$

Sol^n Let $I = \int_C \frac{\cot z}{z}\, dz$

$$f(z) = \frac{\cot z}{z} = \frac{\cos z}{z \sin z}$$

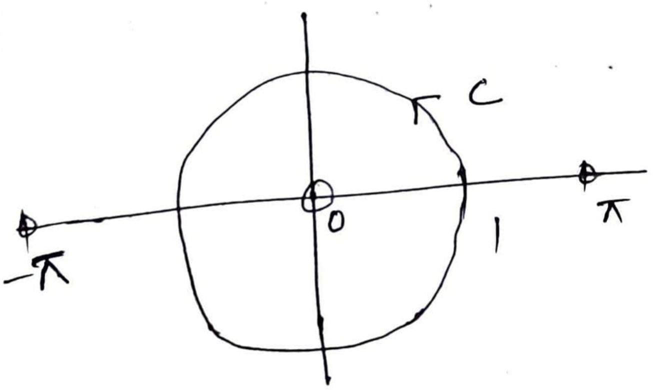
$f(z)$ is not Analytic at

$$z \sin z = 0$$

$\Rightarrow z = 0, \quad \sin z = 0$

$\Rightarrow z = 0, \pm\pi, \pm 2\pi, \ldots$

$C: |z| = 1$



$z = 0$ lies inside C.

$$f(z) = \frac{\cos z}{z \sin z}$$

Let $h(z) = z \sin z$

$h'(z) = \sin z + z \cos z$

$h'(0) = 0$

$$h''(z) = \cos z + \cos z - z \sin z$$

$$h''(0) = 2 \neq 0$$

$\Rightarrow z = 0$ is a pole of order two

$$f(z) = \frac{\cos z}{z \sin z}$$

$$= \frac{\cos z}{z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}$$

$$= \frac{\cos z}{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]}$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = \frac{1}{1!} \frac{d}{dz}\left[\frac{\cos z}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}\right]\Bigg|_{z=0}$$

$$= \frac{-\sin z\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right] - \cos z\left[-\frac{2z}{3!} + \frac{4z^3}{5!} - \cdots\right]}{\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]^2}\Bigg|_{z=0}$$

$$= 0$$

$$\therefore \int_C \frac{\cot z}{z}\, dz = 2\pi i[0] = 0$$