30 March 2022
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Linear Programming Problems (L.P.P.)

- Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Let Niki works x_1 hours at Job I &
x_2 hours at Job II

Maximum Income $= 40x_1 + 30x_2$

$x_1 + x_2 \le 12$ [Working hours]

$2x_1 + x_2 \le 16$ [preparation time]

$x_1 \geq 0, \; x_2 \geq 0$

(1) Solve the foll. L.P.P. by simplex method

Maximise $Z = 40x_1 + 30x_2$

subject to

$x_1 + x_2 \le 12$

$2x_1 + x_2 \le 16$

$x_1, x_2 \geq 0$

Sol:

The standard form of L.P.P. is

Maximise $Z = 40x_1 + 30x_2 + 0S_1 + 0S_2$

subject to

$x_1 + x_2 + S_1 = 12$

$2x_1 + x_2 + S_2 = 16$

$x_1, x_2, S_1, S_2 \geq 0$

		$C_j \to$	40	30	0	0		
	C_B	x_B	x_1	x_2	S_1	S_2	b	$\theta = \frac{b}{\text{key col.}}$
$R_1(0)$	0	S_1	1	1	1	0	12	$\frac{12}{1} = 12$
$R_2(0)$	0	S_2	2	1	0	1	16	$\frac{16}{2} = 8 \leftarrow$
		z_j	0	0	0	0	0	
		$c_j - z_j$	40 ↑	30	0	0		
$R_1(N)$	0	S_1	0	1/2	1	−1/2	4	$\frac{4}{1/2} = 8 \leftarrow$
$R_2(N)$	40	x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	8	$\frac{8}{1/2} = 16$
		z_j	40	20	0	20	320	
		$c_j - z_j$	0	10 ↑	0	−20		
	30	x_2	0	1	2	−1	8	
	40	x_1	1	0	−1/2	$\frac{3}{4}$	4	
		z_j	40	30	40	0	400	
		$c_j - z_j$	0	0	−40	0		

$R_1(N) = R_1(0) - 1\,R_2(N)$

$R_2(N) = \frac{R_2(0)}{\text{key element}}$

$R_1(N) = \frac{R_1(0)}{\text{key elem}}$

$R_2(N) = R_2(0) - \frac{1}{2} R_1(N)$

z_j	40	30	40	0	400
$c_j - z_j$	0	0	-40	0	

$\because c_j - z_j \leq 0 \ \forall j$ $\therefore Z_{max} = 400$ at $x_1 = 4$, $x_2 = 8$

Break meet at 4:15 pm.

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Dual Simplex Method

(2) Use Dual simplex method to solve foll. L.P.P.

Minimise $z = 2x_1 + 2x_2 + 4x_3$

subject to
$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Sol:

Maximise $z' = -2x_1 - 2x_2 - 4x_3$

subject to
$$-2x_1 - 3x_2 - 5x_3 \leq -2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

The standard form of L.P.P. is

Maximise $z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to
$$-2x_1 - 3x_2 - 5x_3 + s_1 = -2$$
$$3x_1 + x_2 + 7x_3 + s_2 = 3$$
$$x_1 + 4x_2 + 6x_3 + s_3 = 5$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

	$C_j \rightarrow$	-2	-2	-4	0	0	0	
C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	-2	$\boxed{-3}$	-5	1	0	0	-2 ←
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5
	z_j	0	0	0	0	0	0	0
	$c_j - z_j$	-2	-2	-4	0	0	0	

	Z_j							
	$C_j - Z_j$	-2	-2	-4	0	0	0	
	$\theta = \frac{C_j - Z_j}{\text{key Row}}$	$\frac{-2}{-2}$	$\frac{-2}{-3}$ ↑	$\frac{-4}{-5}$	$\frac{0}{1}$	$\frac{0}{0}$	$\frac{0}{0}$	
-2	x_2	$\frac{2}{3}$	1	$\frac{5}{3}$	$\frac{-1}{3}$	0	0	$\frac{2}{3}$
0	S_2	$7/3$	0	$\frac{16}{3}$	$1/3$	1	0	$7/3$
0	S_3	$-5/3$	0	$-2/3$	$4/3$	0	1	$7/3$
	Z_j	$-4/3$	-2	$-10/3$	$2/3$	0	0	$-4/3$
	$C_j - Z_j$	$-2/3$	0	$-2/3$	$-2/3$	0	0	

$R_2(N) = R_2(0) - 1 R_1(N)$

$R_3(N) = R_3(0) - 4 \times R_1(N)$

θ: $C_j - Z_j \le 0 \ \forall j$ & $b_i \geq 0$, $Z_{max} = -\frac{4}{3}$ at $x_1 = 0, x_2 = \frac{2}{3}$, $x_3 = 0$

Also $\quad Z_{min} = \frac{4}{3}$ at $x_1 = 0, x_2 = \frac{2}{3}, x_3 = 0$

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Simplex	Big M	Dual simplex
(1) Max	(1) max	(1) max.
(2) All constr '$\leq$'	(2) Atleast one constr. should be '$\geq$' or '='	(2) All constr. '$\leq$' type
(3) All $b_i \geq 0$	(3) $b_i \geq 0$	(3) At least one b_i should be negative

Duality

Problem A

$\text{Max. } z = 6x_1 + 10x_2$

subject to

$2x_1 + 4x_2 \le 18$

$2x_1 + x_2 \le 8$

$x_1 + 3x_2 \le 20$

$x_1, x_2 \geq 0$

n = No. of variables = 2

m = No. of constraints = 3

coefficient matrix $\begin{bmatrix} 2 & 4 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$

Problem B

$\text{Min } W = 18y_1 + 8y_2 + 20y_3$

Subject to

$3y_1 + 2y_2 + y_3 \geq 6$

$4y_1 + y_2 + 3y_3 \geq 10$

$y_1, y_2, y_3 \geq 0$

n = No. of vari = 3

m = No. of constr. = 2

coeff matrix $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}$

Dual of given primal is obtained as follows

1. Max. $\Longleftrightarrow$ min.
2. R.H.solution $\Longleftrightarrow$ cost coefficient
3. '$\geq$' $\Longleftrightarrow$ '$\leq$'
4. coeff matrix $\Longleftrightarrow$ $(\text{coeff matrix})^t$

Note:

1. If L.P.P is Maximisation type then all constraints must be '$\leq$' type
2. If L.P.P. is minimisation type then all constraints must be '$\geq$' type.

1 Write the dual of foll. LPP.

$\max z = 5x_1 + 2x_2$

subject to $3x_1 + 4x_2 \le 5$

$x_1 + x_2 \geq -7$

$x_1, x_2 \geq 0$

Sol:

$\text{Max } z = 8x_1 + 2x_2$

subject to $3x_1 + 4x_2 \le 5$

$-x_1 - x_2 \le 7$

$x_1, x_2 \geq 0$

Interchange

$[8 \quad 2]$

$\begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}^t \quad \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

The dual is given by

Min. $W = 5y_1 + 7y_2$

subject to $3y_1 - y_2 \geq 8$

$4y_1 - y_2 \geq 2$

$y_1, y_2 \geq 0$

② construct the Dual of foll. LPP

$x \geq 5$

$x \leq 5$

$\Rightarrow \boxed{x = 5}$

Min. $z = 3x_1 + 2x_2$

subject to $2x_1 + 7x_2 = 5$

$-x_1 + x_2 \geq 8$

$x_1, x_2 \geq 0$

$2x_1 + 7x_2 = 5$

$2x_1 + 7x_2 \geq 5$ $\quad$ $2x_1 + 7x_2 \leq 5$

$-2x_1 - 7x_2 \geq -5$

Sol:

Min. $Z = 3x_1 + 2x_2$

subject to $2x_1 + 7x_2 \geq 5$

$-2x_1 - 7x_2 \geq -5$

$-x_1 + x_2 \geq 8$

$x_1, x_2 \geq 0$

$$[3 \;\; 2] \qquad \begin{bmatrix} 2 & 7 \\ -2 & -7 \\ -1 & 1 \end{bmatrix}^t \qquad \begin{bmatrix} 5 \\ -5 \\ 8 \end{bmatrix}$$

The Dual of the LPP is

Max. $W = 5y_1 - 5y_2 + 8y_3$

subject to $2y_1 - 2y_2 - y_3 \leq 3$

$7y_1 - 7y_2 + y_3 \leq 2$

$y_1, y_2, y_3 \geq 0$

Let $y_4 = y_1 - y_2$

max. $W = 5y_4 + 8y_3$

subject to $2y_4 - y_3 \leq 3$

$7y_4 + y_3 \leq 2$

$y_3 \geq 0$, y_4 unrestricted

Q Obtain the dual of foll. LPP.

Maximise $z = 2x_1 - x_2 + 3x_3$

subject to $x_1 - 2x_2 + x_3 \geq 4$

$2x_1 + x_3 \leq 10$

$x_1 + x_2 + 3x_3 = 20$

$x_1, x_3 \geq 0$, x_2 unrestricted.

Sol: Let $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$

$$\text{Max. } Z = 2x_1 - x_2' + x_2'' + 3x_3$$

subject to
$$x_1 - 2x_2' + 2x_2'' + x_3 \geq 4$$
$$2x_1 + x_3 \leq 10$$
$$x_1 + x_2' - x_2'' + 3x_3 = 20$$
$$x_1, x_2', x_2'', x_3 \geq 0$$

$x_1 + x_2' - x_2'' + 3x_3 \geq 20$ $\quad$ $x_1 + x_2' - x_2'' + 3x_3 \leq 20$

$-x_1 - x_2' + x_2'' - 3x_3 \leq -20$

$$\text{Max. } Z = 2x_1 - x_2' + x_2'' + 3x_3$$

subject to
$$-x_1 + 2x_2' - 2x_2'' - x_3 \leq -4$$
$$2x_1 + 0x_2' + 0x_2'' + x_3 \leq 10$$
$$-x_1 - x_2' + x_2'' - 3x_3 \leq -20$$
$$x_1 + x_2' - x_2'' + 3x_3 \leq 20$$
$$x_1, x_2', x_2'', x_3 \geq 0$$

The Dual of the L.P.P. is

$$\text{Min. } W = -4y_1 + 10y_2 - 20y_3 + 20y_4$$

$$-y_1 + 2y_2 - y_3 + y_4 \geq 2$$
$$2y_1 + 0y_2 - y_3 + y_4 \geq -1$$
$$-2y_1 + 0y_2 + y_3 - y_4 \geq 1$$
$$-y_1 + y_2 - 3y_3 + 3y_4 \geq 3$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Let $y_5 = y_3 - y_4$

$$\text{Min. } W = -4y_1 + 10y_2 - 20y_5$$

subject to
$$-y_1 + 2y_3 - y_5 \geq 2$$
$$\left.\begin{aligned} 2y_1 + 0y_2 - y_5 &\geq -1 \\ -2y_1 + 0y_2 + y_5 &\geq 1 \end{aligned}\right\}$$
$$-y_1 + y_2 - 3y_5 \geq 3$$

$y_1, y_2 \geq 0$, y_5 unrestricted

$$\left.\begin{aligned} -2y_1 + 0y_2 + y_5 &\leq 1 \\ -2y_1 + 0y_2 + y_5 &\geq 1 \end{aligned}\right\}$$

$$-2y_1 + 0y_2 + y_5 = 1$$

$$\text{Min } W = -4y_1 + 10y_2 - 20y_5$$

subject to
$$-y_1 + 2y_3 - y_5 \geq 2$$
$$-2y_1 + 0y_2 + y_5 = 1$$
$$-y_1 + y_2 - 3y_5 \geq 3$$

$y_1, y_2 \geq 0$, y_5 unrestricted.

Q obtain the Dual of foll. LPP

$$\text{Max. } Z = x_1 - 2x_2 + 3x_3$$

subject to $-2x_1 + x_2 + 3x_3 = 2$

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Sol:

$$\text{Min. } W = 2y_1 + y_2$$

subject to $-2y_1 + 2y_2 \geq 1$

$$y_1 + 3y_2 \geq -2$$

$$3y_1 + 4y_2 \geq 3$$

y_1, y_2 unrestricted.

* — Duality / Principle of Duality — *

① Using Duality solve the foll. LPP.

OR

Using principle of Duality solve the foll. L.P.P.

$$\text{Min. } Z = 2x_1 + 4x_2 + 3x_3$$

subject to
$$-x_1 + x_2 + x_3 \geq 2$$
$$2x_1 + x_2 \geq 1$$
$$x_1, x_2, x_3 \geq 0$$

Sol: The Dual of the L.P.P. is

$$\text{Max. } W = 2y_1 + y_2$$

subject to
$$-y_1 + 2y_2 \leq 2$$
$$y_1 + y_2 \leq 4$$
$$y_1 + 0y_2 \leq 3$$
$$y_1, y_2 \geq 0$$

The standard form of LPP is

$$\text{Max. } W = 2y_1 + y_2 + 0S_1 + 0S_2 + 0S_3$$

subject to
$$-y_1 + 2y_2 + S_1 = 2$$
$$y_1 + y_2 + S_2 = 4$$
$$y_1 + 0y_2 + S_3 = 3$$
$$y_1, y_2, S_1, S_2, S_3 \geq 0$$

	$C_j \rightarrow$	2	1	0	0	0		
C_B	y_b	y_1	y_2	S_1	S_2	S_3	b	$\theta = \frac{b}{\text{key col.}}$
0	S_1	$\boxed{-1}$	2	1	0	0	2	$\frac{2}{-1} = -2$
0	S_2	$\boxed{1}$	1	0	1	0	4	$4/1 = 4$
0	S_3	$\boxed{1}$	0	0	0	1	3	$3/1 = 3$ ←
	Z_j	0	0	0	0	0	0	
	$C_j - Z_j$	2 ↑	1	0	0	0		
0	S_1	0	$\boxed{2}$	1	0	1	5	$\frac{5}{2}$
0	S_2	0	$\boxed{1}$	0	1	-1	1	$\frac{1}{1}$ ←
2	y_1	1	$\boxed{0}$	0	0	1	3	$\frac{3}{0}$
	Z_j	2	0	0	0	2	6	
	$C_j - Z_j$	0	1 ↑	0	0	-2		

	z_j	2	0	0	0	1	3
	$c_j - z_j$	0	1	0	0	-2	
0	s_1	0	0	1	-2	3	3
1	y_2	0	1	0	1	-1	1
2	y_1	1	0	0	0	1	3
	z_j	2	1	0	1	+1	7
	$c_j - z_j$	0	0	0	-1	-1	

$R_1(N) = R_1(O) - 2R_2(N)$

$\because c_j - z_j \le 0 \ \forall j \quad \therefore W_{max} = 7$ at $y_1 = 3, y_2 = 1$

$Z_{min} = 7$ at $x_1 = 0, x_2 = 1, x_3 = 1$

Basic solutions

① Determine all basic solutions to the foll. problem

Maximise $z = x_1 - 2x_2 + 4x_3$

subject to $x_1 + 2x_2 + 3x_3 = 7$

$3x_1 + 4x_2 + 6x_3 = 15$

Sol:

Sr. No. of Basic solution	Non Basic Variable = 0	Basic Variable	Equation & value of Basic Variable	Is the sol. feasible Is all $x_j \geq 0$	Is the sol. non degenerate? Is all $x_j > 0$ (only Basic variable)	Value of z	Is the sol. optimal
1	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 7$ $3x_1 + 4x_2 = 15$ $x_1 = 1, x_2 = 3$	Yes	Yes	-5	No
2	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 7$ $3x_1 + 6x_3 = 15$ $x_1 = 1, x_3 = 2$	Yes	Yes	9	Yes
3	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 7$ $4x_2 + 6x_3 = 15$ unbounded sol.	—	—	—	—

② Consider

Max. $z = 2x_1 - 2x_2 + 4x_3 - 5x_4$

subject to $x_1 + 4x_2 - 2x_3 + 8x_4 = 2$

$-x_1 + 2x_2 + 3x_3 + 4x_4 = 1$

Determine ⓐ all Basic sol.

ⓑ all feasible basic sol.

ⓒ optimal feasible Basic sol.

Sol:

Sr. No. of Basic Sol.	Non Basic Variable = 0	Basic variable	Eqn. & value of Basic variable	Is the sol. feasible (Is all $x_j \geq 0$)	Is the sol. non degenerate Is Basic var. > 0	Value of z	Is the sol. optimal?
1	$x_3 = 0$ $x_4 = 0$	x_1, x_2	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	Yes	No	-1	No
2	$x_2 = 0$ $x_4 = 0$	x_1, x_3	$x_1 = 8, x_3 = 3$	Yes	Yes	28	Yes
3	$x_1 = 0$ $x_4 = 0$	x_2, x_3	$x_2 = 1/2, x_3 = 0$	Yes	No	-1	No
4	$x_2 = 0$ $x_3 = 0$	x_1, x_4	$x_1 = 0, x_4 = 1/4$	Yes	No	$-\frac{5}{4}$	No

4	$x_2=0$, $x_3=0$	x_1, x_4	4			$-\frac{3}{4}$	NO
5	$x_1=0$, $x_3=0$	x_2, x_4	Unbounded	—	—	—	—
6	$x_1=0$, $x_2=0$	x_3, x_4	$x_3=0$, $x_4=1/4$	Yes	NO	$-\frac{5}{4}$	No