Rules of Inference

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TABLE 1 **Quantifiers.**

Statement	*When True?*	*When False?*
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x\, P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x\, P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Rules of Inference

- ***An argument***: a sequence of statements that end with a conclusion
- Some forms of argument ("valid") never lead from correct statements to an incorrect conclusion. Some other forms of argument ("fallacies") can lead from true statements to an incorrect conclusion.
- ***A logical argument*** consists of a list of (possibly compound) propositions called premises/hypotheses and a single proposition called the conclusion.
- ***Logical rules of inference***: methods that depend on logic alone for deriving a new statement from a set of other statements. (Templates for constructing valid arguments.)

- Example: A logical argument

  If I dance all night, then I get tired.

  I danced all night.

  Therefore I got tired.

- Logical representation of underlying variables:

  p: I dance all night. q: I get tired.

- Logical analysis of argument:

$$
\begin{array}{ll}
 & p \rightarrow q \qquad \text{premise 1} \\
 & \underline{p \qquad\qquad \text{premise 2}} \\
\therefore & q \qquad\qquad \text{conclusion}
\end{array}
$$

- A form of logical argument is ***valid*** if whenever every premise is true, the conclusion is also true. A form of argument that is not valid is called a ***fallacy***.

Inference Rules: General Form

- An *Inference Rule* is
  - A pattern establishing that if we know that a set of *premise* statements of certain forms are all true, then we can validly deduce that a certain related *conclusion* statement is true.

$$\begin{array}{l} \quad\textit{premise 1} \\ \quad\textit{premise 2} \\ \quad\ldots \\ \hline \therefore\ \textit{conclusion} \end{array}$$

"$\therefore$" means "therefore"

Inference Rules & Implications

- Each valid logical inference rule corresponds to an implication that is a tautology.

$$\boxed{\begin{array}{ll} & \textit{premise 1} \\ & \textit{premise 2} \\ & \ldots \\ \hline \therefore & \textit{conclusion} \end{array}}$$

Inference rule

- Corresponding tautology:

$$((\textit{premise 1}) \wedge (\textit{premise 2}) \wedge \cdots) \rightarrow \textit{conclusion}$$

Modus Ponens

- $$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \hline \therefore q \end{array}$$

  Rule of ***Modus ponens*** (a.k.a. *law of detachment*)

  "the mode of affirming"

- $(p \land (p \rightarrow q)) \rightarrow q$ is a tautology

p	q	$p \rightarrow q$	$p \land (p \rightarrow q)$	$(p \land (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- Notice that the first row is the only one where premises are all true

Modus Ponens: Example

$$\text{If}\ \left\{\begin{array}{ll} p \to q & : \text{“If it snows today then we will go skiing”} \\ p & : \text{“It is snowing today”} \end{array}\right\}\ \text{assumed TRUE}$$

$$\text{Then}\ \therefore q \quad : \text{“We will go skiing”} \quad \text{is TRUE}$$

$$\text{If}\ \left\{\begin{array}{ll} p \to q & : \text{“If } n \text{ is divisible by 3 then } n^2 \text{ is divisible by 3”} \\ p & : \text{“} n \text{ is divisible by 3”} \end{array}\right\}\ \text{assumed TRUE}$$

$$\text{Then}\ \therefore q \quad : \text{“} n^2 \text{ is divisible by 3”} \quad \text{is TRUE}$$

Modus Tollens

- $$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$ Rule of ***Modus tollens***

"the mode of denying"

- $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology
- Example

If $\left\{ \begin{array}{ll} p \rightarrow q & : \text{“If this jewel is really a diamond then it will scratch glass”} \\ \neg q & : \text{“The jewel doesn't scratch glass”} \end{array} \right\}$ assumed TRUE

Then $\therefore \neg p$: "The jewel is not a diamond" is TRUE

More Inference Rules

- $$\frac{p}{\therefore p \vee q}$$ Rule of ***Addition***

  Tautology: $p \rightarrow (p \vee q)$

- $$\frac{p \wedge q}{\therefore p}$$ Rule of ***Simplification***

  Tautology: $(p \wedge q) \rightarrow p$

- $$\frac{\begin{matrix} p \\ q \end{matrix}}{\therefore p \wedge q}$$ Rule of ***Conjunction***

  Tautology: $[(p) \wedge (q)] \rightarrow p \wedge q$

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing now. Therefore, it is either below freezing or raining now.
  - It is below freezing and raining now. Therefore, it is below freezing now.
- *p*: It is below freezing now.
  q: It is raining now.
  - $p \to (p \vee q)$ (rule of addition)
  - $(p \wedge q) \to p$ (rule of simplification)

Hypothetical Syllogism

- $$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$ Rule of ***Hypothetical syllogism***

  Tautology:

  $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

- Example: State the rule of inference used in the argument:

  "If $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will not have a barbecue today}}_{q}$. If $\underbrace{\text{we do not have a barbecue today}}_{q}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$. Therefore, if $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$."

Disjunctive Syllogism

- $$\boxed{\begin{array}{l} \quad p \vee q \\ \quad \neg p \\ \hline \therefore\ q \end{array}}$$

  Rule of ***Disjunctive syllogism***

  Tautology: $[(p \vee q) \wedge (\neg p)] \rightarrow q$

- Example
  - Ed's wallet is in his back pocket or it is on his desk. $(p \vee q)$
    p q
  - Ed's wallet is not in his back pocket. $(\neg p)$
  - Therefore, Ed's wallet is on his desk. (q)

Resolution

- $$\boxed{\begin{array}{r} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}}$$

  Rule of ***Resolution***

  Tautology:

  $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$

- When $q = r$:

  $[(p \vee q) \wedge (\neg p \vee q)] \rightarrow q$

- When $r = \mathbf{F}$:

  $[(p \vee q) \wedge (\neg p)] \rightarrow q$ (Disjunctive syllogism)

Resolution: Example

$$\boxed{\begin{array}{r} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}}$$

- Example: Use resolution to show that the hypotheses "$\underbrace{\textit{Jasmine is skiing}}_{r}$ *or* $\underbrace{\textit{it is not snowing}}_{\neg p}$" and "$\underbrace{\textit{It is snowing}}_{p}$ *or* $\underbrace{\textit{Bart is playing hockey}}_{q}$" imply that "$\underbrace{\textit{Jasmine is skiing}}_{r}$ *or* $\underbrace{\textit{Bart is playing hockey}}_{q}$"

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$$

Formal Proofs

- A formal proof of a conclusion *C*, given premises $p_1, p_2, \ldots, p_n$ consists of a sequence of *steps*, each of which applies some inference rule to premises or previously-proven statements to yield a new true statement (the *conclusion*).
- A proof demonstrates that *if* the premises are true, *then* the conclusion is true.

Formal Proofs Example:

- Suppose we have the following premises:
  "It is not sunny and it is cold."
  "We will swim only if it is sunny."
  "If we do not swim, then we will canoe."
  "If we canoe, then we will be home by sunset."

- Given these premises, prove the conclusion **"We will be home by sunset"** using inference rules.

Example:

- Step 1: Identify the propositions (Let us adopt the following abbreviations)
  - *sunny* = **"It is sunny"**; *cold* = **"It is cold"**; *swim* = **"We will swim"**; *canoe* = **"We will canoe"**; *sunset* = **"We will be home by sunset"**.
- Step 2: Identify the argument. (Build the argument form)

$$\begin{array}{l} \neg sunny \land cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore\ sunset \end{array}$$

It is not sunny and it is cold.

We will swim only if it is sunny.

If we do not swim, then we will canoe.

If we canoe, then we will be home by sunset.

We will be home by sunset.

Example cont….

- Step 3: Verify the reasoning using the rules of inference

$$\begin{array}{l} \neg sunny \wedge cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore\ sunset \end{array}$$

Step	Proved by
1. $\neg sunny \wedge cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $swim \rightarrow sunny$	Premise #2.
4. $\neg swim$	Modus tollens on 2 and 3.
5. $\neg swim \rightarrow canoe$	Premise #3.
6. $canoe$	Modus ponens on 4 and 5.
7. $canoe \rightarrow sunset$	Premise #4.
8. $sunset$	Modus ponens on 6 and 7.

Common Fallacies

- A ***fallacy*** is an inference rule or other proof method that is not logically valid.
  - A fallacy may yield a false conclusion!
- *Fallacy of affirming the conclusion*:
  - "$p \rightarrow q$ is true, and q is true, so p must be true." (No, because **F** $\rightarrow$ **T** is true.)
- Example
  - If David Cameron (DC) is president of the US, then he is at least 40 years old. ($p \rightarrow q$)
  - DC is at least 40 years old. (q)
  - Therefore, DC is president of the US. (p)

Common Fallacies

- *Fallacy of denying the hypothesis*:
  - "$p \rightarrow q$ is true, and p is false, so q must be false." (No, again because **F** $\rightarrow$ **T** is true.)
- Example
  - If a person does arithmetic well then his/her checkbook will balance. ($p \rightarrow q$)
  - I cannot do arithmetic well. ($\neg p$)
  - Therefore my checkbook does not balance. ($\neg q$)

Inference Rules for Quantifiers

- $$\frac{\forall x\, P(x)}{\therefore P(c)}$$ **Universal instantiation** (substitute *any* specific member *c* in the domain)

- $$\frac{P(c)}{\therefore \forall x\, P(x)}$$ (for an arbitrary element *c* of the domain) **Universal generalization**

- $$\frac{\exists x\, P(x)}{\therefore P(c)}$$ **Existential instantiation** (substitute an element *c* for which *P*(*c*) is true)

- $$\frac{P(c)}{\therefore \exists x\, P(x)}$$ (for some element *c* in the domain) **Existential generalization**

Example

- Every man has two legs. John Smith is a man. Therefore, John Smith has two legs.
- <u>Proof</u>
  - Define the predicates:
    - $M(x)$: x is a man
    - $L(x)$: x has two legs
    - J: John Smith, a member of the universe
  - The argument becomes

    1. $\forall x\,[M(x) \rightarrow L(x)]$
    2. $M(J)$

    $\therefore L(J)$

What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."

b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."

c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."

$P(x) =$"I work on day x"
$Q(x) =$"It is sunny on day x"
$R(x) =$"It is partly sunny on day x"

We can then rewrite the given statements using the above interpretations.

	Step	Reason
1.	$\forall x(P(x) \to Q(x) \vee R(x))$	*Premise*
2.	$P(\text{Monday}) \vee P(\text{Friday})$	*Premise*
3.	$\neg Q(\text{Tuesday})$	*Premise*
4.	$\neg R(\text{Friday})$	*Premise*
5.	$\neg Q(\text{Tuesday}) \wedge \neg R(\text{Friday})$	Conjunction from (3) and (4)

Step (5) means that "It was not sunny on Tuesday and It was not partly sunny on Friday.".

$P(x) =$ "x are insects"
$Q(x) =$ "x have six legs"
$R(x, y) =$ "x eats y"

We can then rewrite the given statements using the above interpretations.

	Step	Reason
1.	$\forall x (P(x) \to Q(x))$	*Premise*
2.	$P(\text{Dragonflies})$	*Premise*
3.	$\neg Q(\text{Spiders})$	*Premise*
4.	$R(\text{Spiders}, \text{Dragonflies})$	*Premise*
5.	$P(\text{Dragonflies}) \to Q(\text{Dragonflies})$	Universal instantiation from (1)
6.	$P(\text{Spiders}) \to Q(\text{Spiders})$	Universal instantiation from (1)
7.	$Q(\text{Dragonflies})$	Modus ponens from (2) and (5)
8.	$\neg P(\text{Spiders})$	Modus tollens from (3) and (6)

Step (7) means that "Dragonflies have six legs".
Step (8) means that "Spiders are not insects".

For each of these arguments, explain which rules of inference are used for each step.

a) "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."

b) "Each of five roommates, Melissa, Aaron, Ralph, Veneesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

c) "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

d) "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

"Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."

$P(x) =$ "x is a student"
$Q(x) =$ "x has an Internet account"

We can then rewrite the given statements using the above interpretations.

	Step	Reason
1.	$\forall x(P(x) \to Q(x))$	*Premise*
2.	$\neg Q(\text{Homer})$	*Premise*
3.	$Q(\text{Maggie})$	*Premise*
4.	$P(\text{Homer}) \to Q(\text{Homer})$	Universal instantiation from (1)
5.	$\neg P(\text{Homer})$	Modus tollens from (2) and (4)

Step (5) means that "Homer is not a student".