Q 1) $f(t) = \begin{cases} t^2 & 0<t<2 \\ t-1 & 2<t<3 \\ 7 & t>3. \end{cases}$

$$L(f(t)) = \int_0^\infty e^{-st} f(t).dt = \int_0^2 e^{-st} t^2 dt + \int_2^3 e^{-st}(t-1)\,dt.$$

$$+ \int_3^\infty e^{-st}(7)\,dt.$$

$$I_1 = \int_0^2 e^{-st} t^2 dt. = \left[-t^2\left(\frac{e^{-st}}{-s}\right) - \frac{e^{-st}(2t)}{s^2} - \frac{2e^{-st}}{s^3}\right]_0^2$$

t^2	e^{-st}
$2t$	$e^{-st}/-s$
t	e^{-st}/s^2
0	$e^{-st}/-s^3$

$$= \frac{e^{-2s}}{s}\left(-4 - \frac{4}{s} - \frac{2}{s^2}\right)$$

$$I_2 = \int_2^3 e^{-st}(t-1)\,dt$$

$t-1$	e^{-st}
t	$e^{-st}/-s$
0	e^{-st}/s^2

$$= \left[\frac{(t-1)e^{-st}}{-s} - \frac{1(e^{-st})}{s^2}\right]_2^3$$

$$= \frac{2e^{-st}}{s} + \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s^2}.$$

$$= \frac{-2e^{-2s}}{s} + \frac{e^{-3s}}{s^2} + \frac{e^{-2s} - e^{-3s}}{s^2}$$

$$= \frac{-2e^{-2s} + e^{-3s}}{s} + \frac{e^{-2s} - e^{-3s}}{s^2}.$$

$$I_3 = 7\int_3^{\infty} e^{-st}\,dt.$$

$$= 7 \times \left[\frac{e^{-st}}{-s}\right]_3^{\infty} = 7\left[\frac{-1}{s} + \frac{e^{-3s}}{s}\right]$$

$$= 7\left[\frac{e^{-3s} - 1}{s}\right]$$

$$I = I_1 + I_2 + I_3$$

$$= \frac{e^{-2s}}{s}\left(-4 - \frac{4}{s} - \frac{2}{s^2}\right) + \frac{e^{-2s} - 2e^{-3s}}{s} + \frac{e^{-2s} - e^{-3s}}{s^2}$$

$$+ 7\left[\frac{e^{-3s} - 1}{s}\right]$$

$$\boxed{\therefore I = \frac{e^{-2s}}{s}\left(-4 - \frac{4}{s} - \frac{2}{s^2}\right) + \frac{e^{-2s} - 2e^{-3s}}{s} + \frac{e^{-2s} - e^{-3s}}{s^2} + 7\left[\frac{e^{-3s} - 1}{s}\right]}$$

f(2) $\int_0^\infty e^{-t}\left(\frac{\sin 3t + \sin 2t}{t}\right)dt.$ $L(\sin 3t) = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.ds.$

$L(\sin 2t) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}.ds$

$\therefore L\left(\frac{\sin 3t + \sin 2t}{t}\right) = \int_s^\infty \frac{3}{s^2+9} + \int_s^\infty \frac{2}{s^2+4}.ds.$

$= \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty + \left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) + \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right).$

$= \pi - \left[\tan^{-1}\left(\frac{s}{3}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]$

Now, let $\tan^{-1}\left(\frac{s}{3}\right) = \alpha$ & $\tan^{-1}\left(\frac{s}{2}\right) = \beta.$

$\therefore \tan\alpha = \frac{s}{3}$ & $\tan\beta = \frac{s}{2}.$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{s/3 + s/2}{1 - s^2/6} = \frac{5s/\cancel{6}}{6-s^2/\cancel{6}} = \frac{5s}{6-s^2}.$

$\alpha+\beta = \tan^{-1}\left(5s/6-s^2\right)$

$\therefore L\left(\frac{\sin 3t + \sin 2t}{t}\right) = \pi - \tan^{-1}\left(\frac{5s}{6-s^2}\right)$

$\int_0^\infty e^{-t}\left(\frac{\sin 3t + \sin 2t}{t}\right) = \pi - \tan^{-1}\left(\frac{5(1)}{6-1^2}\right) = \pi - \frac{\pi}{4}. = \frac{3\pi}{4}.$

$\therefore \int_0^\infty e^{-t}\left(\frac{\sin 3t + \sin 2t}{t}\right) = \frac{3\pi}{4}.$

Q3) $L\left(\int_0^t t^{-1} e^{-u} \sin u\, du\right) = L\left(t^{-1}\int_0^t e^{-u} \sin u\, du\right)$

$= L\left(e^{-t} \sin t\right) = \dfrac{1}{(s+1)^2+1} = \dfrac{1}{s^2+2s+2}$.

$L\left(\int_0^t e^{-u} \sin u\, du\right) = \dfrac{1}{s} \times \dfrac{1}{s^2+2s+2}$.

$L\left(t^{-1}\int_0^t e^{-u} \sin u\, du\right) = \int_s^\infty \dfrac{1}{s} \times \dfrac{1}{s^2+2s+2}\, ds$

Divide by t.

$\dfrac{1}{s} \times \dfrac{1}{s^2+2s+2} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+2s+2}$.

$1 = A(s^2+2s+2) + Bs^2 + Cs$.

coefficient s^2 $\quad A+B=0 \quad \therefore B = -1/2$

coeffient s $\quad 2A+C=0 \quad \therefore C=-1$

coeffient const $\quad 1 = 2A \quad \therefore A = 1/2$

$\dfrac{1}{s} \times \dfrac{1}{s^2+2s+2} = \dfrac{1}{2}s + \dfrac{-1/2\, s}{s^2+2s+2} = \dfrac{1}{2} \times \dfrac{1}{s} + \dfrac{-1}{2} \times \dfrac{s+2}{s^2+2s+2}$

$= \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s+2}{s^2+2s+2}\right)$

$= A\dfrac{(s^2+2s+2)-(s+2)}{s(s^2+2s+2)} = \dfrac{1}{2}\int_s^\infty\left(\dfrac{1}{2} - \dfrac{s+2}{s^2+2s+2}\right) ds.$

$= \dfrac{1}{2}\int_s^\infty \dfrac{1}{s} - \dfrac{1}{2}\left\{\dfrac{2s+2}{s^2+2s+2} + \dfrac{2}{s^2+2s+2}\right\} ds. = \left[\dfrac{1}{2}\left(\log s - \dfrac{1}{2}\left\{\log s^2+2s+2 + 2\tan^{-1}(s+1)\right\}\right)\right]_s^\infty$

$= \dfrac{1}{4}\left(2\log s - \log(s^2+2s+2)\right)_s^\infty + 2\cot^{-1}(s+1)$

$= \dfrac{1}{4}\left[0 - \log\dfrac{s^2}{s^2+2s+2}\right] + 2\cot^{-1}(s+1)$

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Q4) $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$

$$\cos t = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$$

$$\cos t^{1/2} = 1 - \frac{(t^{1/2})^2}{2!} + \frac{(t^{1/2})^4}{4!} \ldots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} - \frac{1}{2}\frac{t}{t^{1/2}} + \frac{1}{24} t^{3/2}$$

$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = L[t]^{-1/2} - \frac{1}{2} L\left(t^{1/2}\right) + \frac{1}{24} L(t)^{3/2}$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{(-1/2+1)}} + \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{(1/2+1)}} + \frac{1}{24}\times\left[\Gamma\left(\frac{3}{2}+1\right) / s^{(3/2+1)}\right]$$

$$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{\frac{1}{2}\Gamma 3/2}{s^{1/2+1}} + \frac{1}{24}\frac{\Gamma 5/2}{s^{(1/2+2)}}$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} + \frac{1}{4}\frac{\sqrt{\pi}}{s^{(1/2+1)}} + \frac{1}{24}\frac{3/2 \times 1/2\sqrt{\pi}}{s^{(1/2+2)}}$$

$$= \sqrt{\pi}\left[\frac{1}{s^{1/2}} - \frac{1/4}{s^{1/2}\times s} + \frac{1/32}{s^{1/2}\times s^2}\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left(1 - \frac{1}{4s} + \frac{1}{32s^2}\right)$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} \times e^{-1/4s}$$

$$\boxed{\therefore\ L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\sqrt{\pi}}{s^{1/2}} \times e^{-1/4s}}$$

$$\therefore \int_0^{1} t^{-1} e^{-u} \sin u \, du = \frac{-1}{4} \log \frac{s^2+2s+2}{s^2} + 2\cot^{-1}(s+1).$$

Q5) $L\left(t\sqrt{1+\sin t}\right)$

$$\sqrt{1+\sin t} = \sqrt{\sin^2\frac{t}{2} + \cos^2\frac{t}{2} + 2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \sqrt{\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right)^2}$$

$$\therefore L\left(\sqrt{1+\sin t}\right) = L\left(t\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right)\right)$$

$$= L\left(\sin\frac{t}{2}\right) + L\left(\cos\frac{t}{2}\right)$$

$$= \frac{1}{2}\frac{1}{s^2+1/4} + \frac{s}{s^2+1/4}.$$

$$L\left(\sqrt{1+\sin t}\right) = \frac{4s+2}{4s^2+1}$$

$$\therefore L\left(t\sqrt{1+\sin t}\right) = \frac{-d}{ds}\left[\frac{4s+2}{4S^2+1}\right] = -2\frac{d}{ds}\left[\frac{2s+1}{4s^2+1}\right]$$

$$= -2\left[\frac{(4s^2+1)2 - (2s+1)\cdot 8s}{(4s^2+1)^2}\right]$$

$$= -2\left[\frac{8s^2+2-16s^2-8s}{(4s^2+1)^2}\right] = -2\left[\frac{-8s^2-8s+2}{(4s^2+1)^2}\right]$$

$$= \frac{\cancel{16s^2+16s-4}}{} \quad \frac{16s^2+16s-4}{(4s^2+1)^2}$$

$$\boxed{\therefore L\left(t\sqrt{1+\sin t}\right) = \frac{16s^2+16s-4}{(4s^2+1)^2}}$$