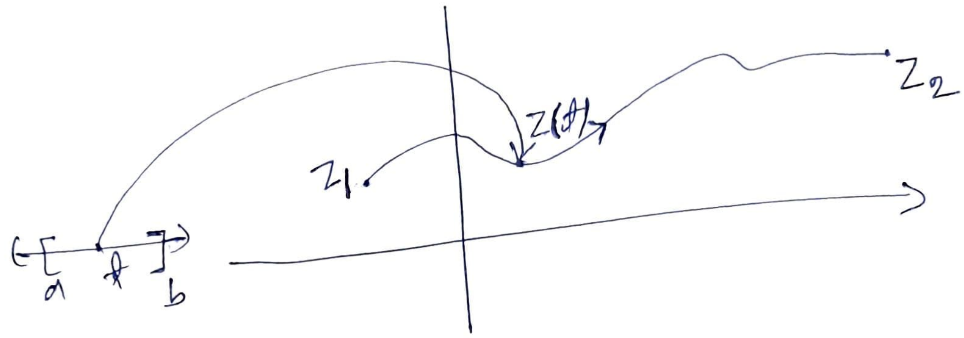
Integration of Complex variable Functions

Line (Path) or Contour in a complex plane:-

A curve (path) C is defined as

$$C: z = x(t) + i\,y(t), \quad a \le t \le b$$

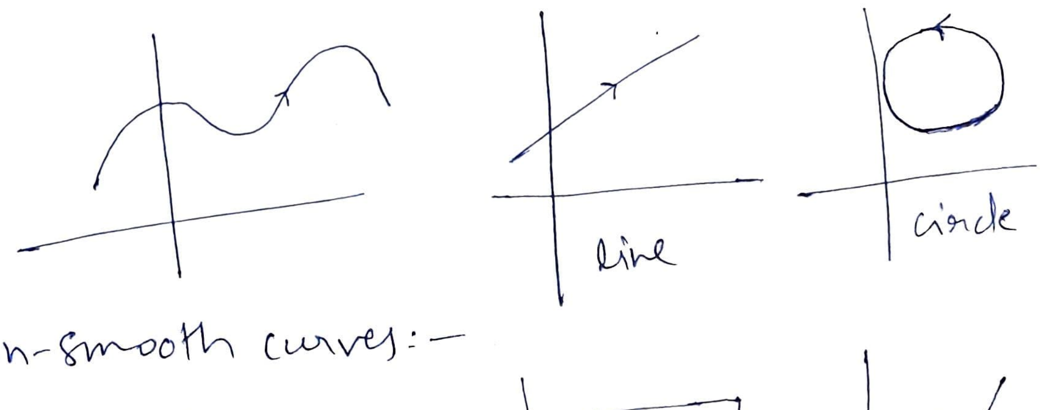
extending from $z(a) = z_1$ to $z(b) = z_2$



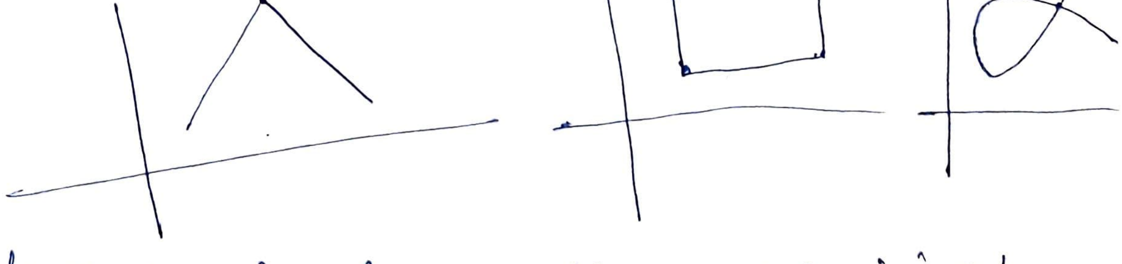
$C: z(t)$ is said to be smooth if

$\frac{dz}{dt}$ exist at all points.

e.g.



Non-smooth curves:-

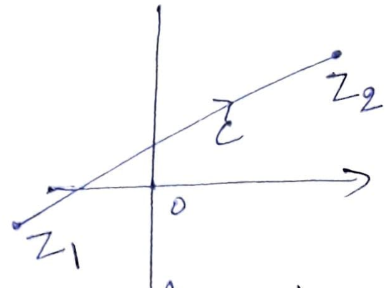


Def:- A set of smooth curves joined end to end is referred as a contour.

Note :-

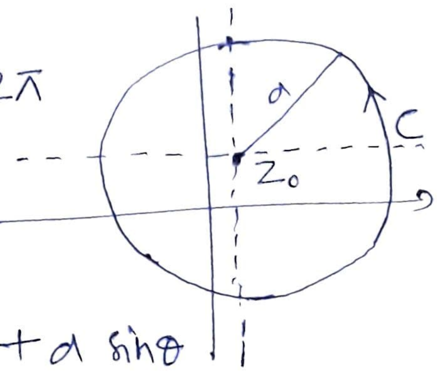
① Parametric form of a line from z_1 to z_2 is

$$C: z = (1-t)z_1 + t z_2, \quad 0 \le t \le 1$$



② Parametric form of a circle with centre $z_0 = x_0 + i y_0$ and radius 'a' is

$$C: z = z_0 + a e^{i\theta}, \quad 0 \le \theta \le 2\pi$$



$\Leftrightarrow x + iy = x_0 + i y_0 + a(\cos\theta + i\sin\theta)$

$\Leftrightarrow x = x_0 + a\cos\theta, \quad y = y_0 + a\sin\theta$

Particular case: (i) If centre $z_0 = 0$

$$C: z = a e^{i\theta}, \quad 0 \le \theta \le 2\pi$$

$\Leftrightarrow x = a\cos\theta, \quad y = a\sin\theta$

(ii) For upper semicircle:

$$C: z = z_0 + a e^{i\theta}, \quad 0 \le \theta \le \pi$$

(iii) For left half of the circle

$$z = z_0 + a e^{i\theta}, \quad \pi/2 \le \theta \le \frac{3\pi}{2}$$

Line or path or contour Integration of complex variable functions :-

Let $C : z(t) = x(t) + i\, y(t)$, $a \le t \le b$ be a smooth curve in complex plane and $f(z) = u(x,y) + i\, v(x,y)$ be a function.

We know that :

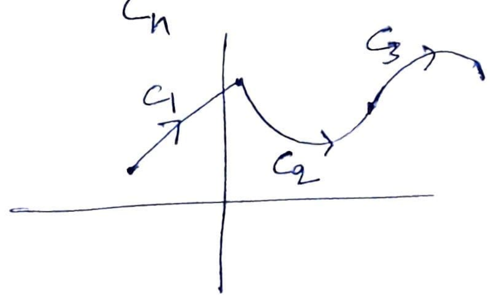
$$z = x + iy \; ; \quad dz = dx + i\, dy$$

$$\therefore \int_C f(z)\, dz = \int_C (u + iv)(dx + i\, dy)$$

$$= \int_a^b f(z(t))\, z'(t)\, dt$$

Note: If C is not smooth and $C = C_1 + C_2 + \cdots + C_n$, where $C_1, C_2, \ldots\ldots, C_n$ are smooth then

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \cdots + \int_{C_n} f(z)\, dz$$



Ex.

① Evaluate $\int_C \bar{z}\, dz$ where C is the curve (i) $z = t^2 + it$ from 0 to $4+2i$

(ii) straight line from 0 to $4+2i$

(iii) upper half of the circle $|z|=1$

Soln: Let $I = \int_C \bar{z}\, dz$

(i) $C:\ z = t^2 + it$ from $z=0$ to $z=4+2i$

$\therefore\ dz = (2t+i)\, dt$

$\bar{z} = t^2 - it$

$z=0 \Rightarrow t=0$

$z=4+2i \Rightarrow t=2$

$$\therefore\ I = \int_0^2 (t^2 - it)(2t+i)\, dt$$

$$= \int_0^2 2t^3 + it^2 - 2it^2 + t\, dt$$

$$= \int_0^2 2t^3 - it^2 + t\, dt$$

$$= \left[2\frac{t^4}{4} - i\frac{t^3}{3} + \frac{t^2}{2} \right]_0^2$$

$$= 8 - i\frac{8}{3} + 2$$

$$= 10 - \frac{8}{3} i$$

Or

C: $z = t^2 + it$ from $z=0$ to $z = 4+2i$

$\Rightarrow \quad x = t^2, \quad y = t$

$dx = 2t\,dt, \quad dy = dt$

$z = 0 \Rightarrow t = 0$ & $z = 4+2i \Rightarrow t = 2$

$\therefore \quad I = \int_C \bar{z}\, dz$

$= \int_C \overline{x+iy}\,(dx + i\,dy)$

$= \int_C (x - iy)(dx + i\,dy)$

$= \int_0^2 (t^2 - it)(2t\,dt + i\,dt)$

$= \int_0^2 (t^2 - it)(2t + i)\,dt$

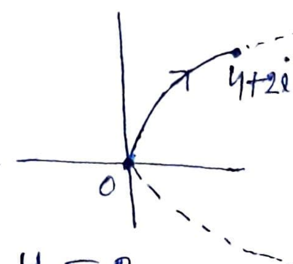
$\vdots$

$= 10 - \frac{8}{3} i$

Or

C: $z = t^2 + it$ from $z=0$ to $z = 4+2i$

$\Rightarrow \quad x = t^2, \quad y = t$



$\Rightarrow \quad x = y^2, \quad dx = 2y\,dy$

$z = 0 \Rightarrow y = 0$ & $z = 4+2i \Rightarrow y = 2$

$\therefore \quad I = \int_C \bar{z} \, dz$

$= \int_C (x - iy)(dx + i\,dy)$

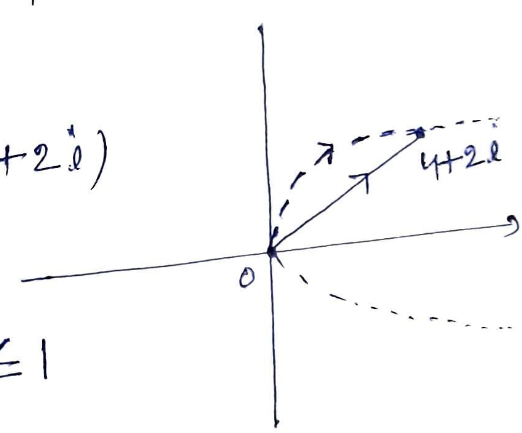
$= \int_0^2 (y^2 - iy)(2y\,dy + i\,dy)$

$= \int_0^2 (y^2 - iy)(2y + i)\,dy$

$\vdots$

$= 10 - \frac{8}{3} i$

(ii) C: straight line from $z = 0$ to $z = 4 + 2i$



$\therefore$ C: $z = (1-t)\,0 + t\,(4+2i)$

$= (4+2i)\,t, \quad 0 \leq t \leq 1$

$\therefore \bar{z} = (4-2i)\,t$

$dz = (4+2i)\,dt$

$\therefore \quad I = \int_C \bar{z}\,dz$

$= \int_0^1 (4-2i)\,t\,(4+2i)\,dt$

$$\Rightarrow I = 20 \int_0^1 t \, dt$$

$$= 20 \left[\frac{t^2}{2} \right]_0^1 = 20 \times \frac{1}{2} = 10$$

or

$$C: \quad z = (1-t)\cdot 0 + t(4+2i), \quad 0 \le t \le 1$$

$$= (4+2i)\,t$$

$$= 4t + 2t\,i$$

$$\Rightarrow x = 4t, \quad y = 2t$$

$$\Rightarrow dx = 4\,dt, \quad dy = 2\,dt$$

$$\therefore \quad I = \int_C \bar{z} \, dz$$

$$= \int_C (x - iy)(dx + i\,dy)$$

$$= \int_0^1 (4t - i2t)(4\,dt + i\,2\,dt)$$

$$= \int_0^1 (4-2i)\,t\,(4+2i)\,dt$$

$$\vdots$$

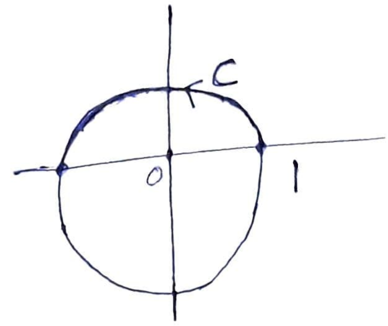
$$= 10$$

(iii) C: upper half of the circle

$|z|=1$

$\therefore C: \ z = 0 + 1\, e^{i\theta}$

$= e^{i\theta}, \quad 0 \le \theta \le \pi$



$\therefore \bar{z} = e^{-i\theta}$

$dz = i\, e^{i\theta}\, d\theta$

$\therefore \quad I = \int_C \bar{z}\, dz$

$= \int_0^{\pi} e^{-i\theta} \cdot i\, e^{i\theta}\, d\theta$

$= i \int_0^{\pi} d\theta = i\, [\theta]_0^{\pi} = \pi i$

(2) Evaluate $\int_0^{2+i} z^2\, dz$ along the

(i) line $x = 2y$ (ii) along the parabola

$2y^2 = x$.

Sol^n $\quad I = \int_0^{2+i} z^2\, dz$

$= \int_0^{2+i} (x+iy)^2\,(dx + i\,dy)$

(i) $\quad C: \ x = 2y$

$dx = 2\,dy$

$z = 0 \Rightarrow y = 0$

$z = 2 + i \Rightarrow y = 1$

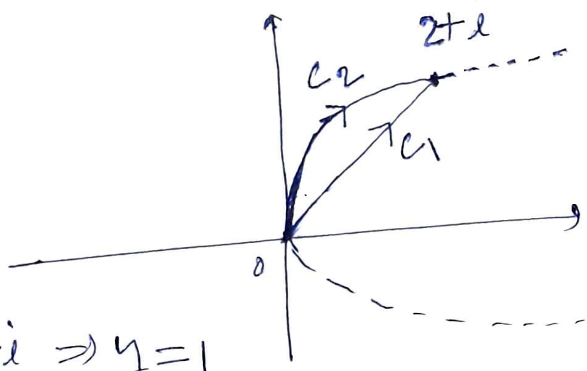
$$\therefore I = \int_0^1 (2y + iy)^2 (2\,dy + i\,dy)$$

$$= \int_0^1 (2+i)^2 y^2 (2+i)\, dy$$

$$= (2+i)^3 \int_0^1 y^2\, dy$$

$$= (2+i)^3 \left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{2 + 11i}{3}$$



(ii) $C: \; x = 2y^2$

$dx = 4y\, dy$

$z = 0 \Rightarrow y = 0$ & $z = 2+i \Rightarrow y = 1$

$$\therefore I = \int_0^1 (2y^2 + iy)^2 (4y\, dy + i\, dy)$$

$$= \int_0^1 (2y^2 + iy)^2 (4y + i)\, dy$$

$$= \int_0^1 (4y^4 + 4iy^3 - y^2)(4y + i)\, dy$$

$$I = \int_0^1 16y^5 + 4iy^4 + 16iy^4 - 4y^3 - 4y^3 - iy^2 \, dy$$

$$= \int_0^1 16y^5 + 20iy^4 - 8y^3 - iy^2 \, dy$$

$$= \left[16\frac{y^6}{6} + 4iy^5 - 2y^4 - i\frac{y^3}{3} \right]_0^1$$

$$= \frac{8}{3} + 4i - 2 - \frac{i}{3}$$

$$= \frac{2}{3} + \frac{11}{3}i = \frac{2+11i}{3}$$

Note that: For both curves line and parabola having same initial and final points; the integral values are equal. why?

Result :- If $f(z)$ is Analytic in a domain containing a curve C extending from z_1 to z_2, then the integral of $f(z)$ along C is independent of the curve and depends only on initial and final points z_1 and z_2.

i.e. $$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = \left[\int f(z)\,dz \right]_{z=z_1}^{z_2}$$

Note :-

① $\bar{z}$, $|z|$ are not Analytic at all points on complex plane.

② constant, z, polynomial in z, e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are Analytic at all points in complex plane.

③ If $f(z)$ and $g(z)$ are two Analytic functions then

(i) $f(z)+g(z)$ is Analytic

(ii) $f(z)\cdot g(z)$ is Analytic

(iii) $f(g(z))$ is Analytic

(iv) $\dfrac{f(z)}{g(z)}$ is not Analytic only at $g(z)=0$

② Evaluate $\int_0^{2+i} z^2\, dz$ along the curves (i) line $x=2y$ (ii) parabola $2y^2=x$.

Sol^n Let $I=\int_0^{2+i} z^2\, dz$

$f(z)=z^2$ is Analytic at all points.

$\therefore$ integral of $f(z)$ is independent of the curves.

$\therefore$ for both the curves (i) and (ii),

$$I = \int_0^{2+i} z^2 \, dz$$

$$= \left[\frac{z^3}{3} \right]_0^{2+i}$$

$$= \frac{1}{3}(2+i)^3 = \frac{2+11i}{3}$$

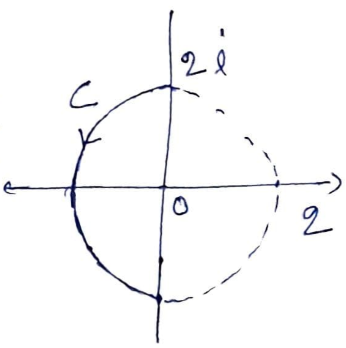
③ Evaluate $\int_C |z| \, dz$ where C is the left half of the circle $|z|=2$.

<u>Sol^n</u> $I = \int_C |z| \, dz$

C: left half circle $|z|=2$

$\therefore$ $z = 0 + 2e^{i\theta}$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$= 2e^{i\theta}$



$dz = 2i \, e^{i\theta} \, d\theta$

$|z| = 2$

$$\therefore I = \int_{\pi/2}^{3\pi/2} 2 \cdot 2i \, e^{i\theta} \, d\theta$$

$e^{i3\pi/2}$
$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$
$= -i$

$$= 4i \left[\frac{e^{i\theta}}{i} \right]_{\pi/2}^{3\pi/2} = 4\left[e^{i\frac{3\pi}{2}} - e^{i\pi/2} \right]$$

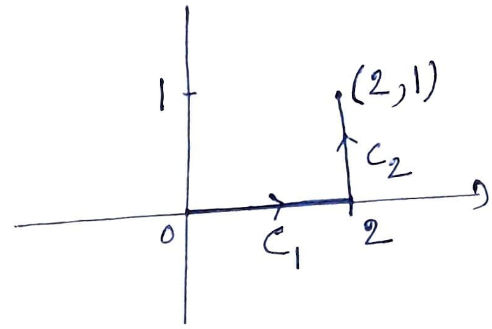
$$= 4[-i - i] = -8i$$

(4) Evaluate $\int_0^{2+i} \bar{z}^2\, dz$ along the line from A to B and then from B to C where $A=(0,0)$, $B=(2,0)$, $C=(2,1)$.

<u>Sol^n</u> $$I = \int_0^{2+i} (\bar{z})^2\, dz$$

C: line from $(0,0)$ to $(2,0)$ and then line from $(2,0)$ to $(2,1)$

$C = C_1 + C_2$



C_1: line $y=0$ from $(0,0)$ to $(2,0)$

$\therefore\ dy = 0\, dx$

& $x=0$ to $x=2$

$$I_1 = \int_{C_1} (\bar{z})^2\, dz$$

$$= \int_{C_1} (x - iy)^2 (dx + i\, dy)$$

$$= \int_0^2 (x)^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

C_2: line from $(2,0)$ to $(2,1)$

i.e. $x = 2$

$\therefore dx = 0\, dy$

& from $y = 0$ to $y = 1$

$$\therefore I_2 = \int_{C_2} (\bar{z})^2 \, dz$$

$$= \int_0^1 (x - iy)^2 (dx + i\, dy)$$

$$= \int_0^1 (2 - iy)^2 \, i\, dy$$

$$= i \int_0^1 4 - 4iy - y^2 \, dy$$

$$= i \left[4y - 2iy^2 - \frac{y^3}{3} \right]_0^1$$

$$= i \left[4 - 2i - \frac{1}{3} \right]$$

$$= i \left[\frac{11}{3} - 2i \right]$$

$$= \frac{11}{3} i + 2$$

$$\therefore I = I_1 + I_2$$

$$= \frac{8}{3} + \frac{11}{3} i + 2$$

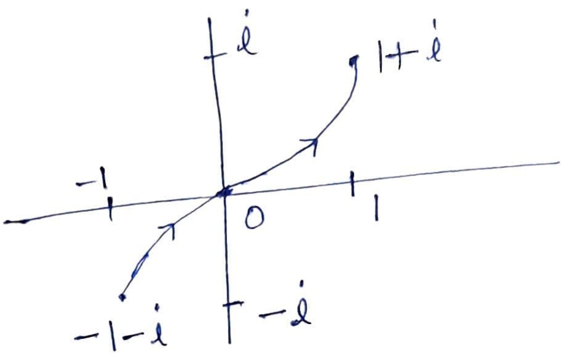
$$= \frac{14}{3} + \frac{11}{3} i$$

(5) Evaluate $\int_C f(z)\, dz$ where

$$f(z) = \begin{cases} 4y & y>0 \\ 1 & y<0 \end{cases}$$

and C is the arc from $-1-i$ to $1+i$ of the cubical curve $y=x^3$.

Solⁿ C: $y=x^3$ from $-1-i$ to $1+i$



$$I = \int_C f(z)\, dz$$

$$= \int_{-1-i}^{0} 1\, dz + \int_{0}^{1+i} 4y\, dz$$

along $y = x^3$

$$\therefore dy = 3x^2\, dx$$

$$I = \int_{-1-i}^{0} 1\ (dx + i\,dy) + \int_{0}^{1+i} 4y\ (dx + i\,dy)$$

$$= \int_{-1}^{0} 1\ (dx + i\,3x^2 dx) + \int_{0}^{1} 4x^3\ (dx + i\,3x^2 dx)$$

$$= \int_{-1}^{0} 1+3ix^2\,dx + \int_0^1 4x^3(1+3ix^2)\,dx$$

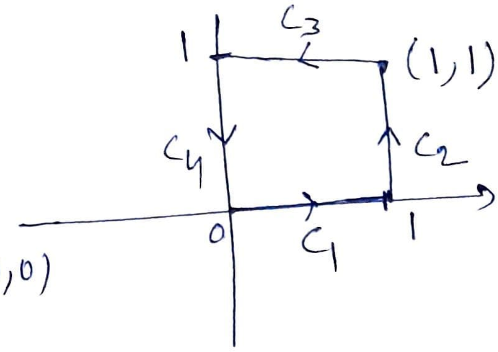
$$= \int_{-1}^{0} 1+3ix^2\,dx + \int_0^1 4x^3+12ix^5\,dx$$

$$= \left[x+ix^3\right]_{-1}^{0} + \left[x^4+2ix^6\right]_0^1$$

$$= 1+i+1+2i = 2+3i$$

(6) Evaluate $\int_C |z|^2\,dz$ where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

Soln C: square with vertices $(0,0)$, $(1,0)$, $(1,1)$ & $(0,1)$



$C = C_1+C_2+C_3+C_4$

C_1: line $y=0$ from $(0,0)$ to $(1,0)$.

$\therefore\ dy = 0\,dx$

& from $x=0$ to $x=1$

| $z = x+iy$
| $|z| = \sqrt{x^2+y^2}$
| $\therefore |z|^2 = x^2+y^2$

$$I_1 = \int_{C_1} |z|^2\,dz$$

$$= \int_{C_1} (x^2+y^2)(dx+i\,dy)$$

$$I_1 = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

C_2: line $x=1$ from $(1,0)$ to $(1,1)$

$\therefore dx = 0 \, dy$

& for $y=0$ to $y=1$

$$\therefore I_2 = \int_{C_2} (x^2+y^2)(dx + i\,dy)$$

$$= \int_0^1 (1+y^2) \, i \, dy$$

$$= \left[y + \frac{y^3}{3}\right]_0^1 i = \left[1+\frac{1}{3}\right] i$$

$$= \frac{4}{3} i$$

C_3: line $y=1$ from $(1,1)$ to $(0,1)$

$\therefore dy = 0 \, dx$

& for $x=1$ to $x=0$

$$\therefore I_3 = \int_{C_3} (x^2+y^2)(dx + i\,dy)$$

$$= \int_1^0 (x^2+1) \, dx$$

$$= \left[\frac{x^3}{3} + x\right]_1^0 = -\left(\frac{1}{3}+1\right) = -\frac{4}{3}$$

C_4: line $x = 0$ from $(0,1)$ to $(0,0)$

$\therefore dx = 0\, dy$

& for $y=1$ to $y=0$

$$\therefore I_4 = \int_{C_4} (x^2+y^2)(dx + i\,dy)$$

$$= \int_1^0 y^2\, i\, dy$$

$$= \left[\frac{y^3}{3}\right]_1^0 i = -\frac{1}{3} i$$

$$\therefore I = I_1 + I_2 + I_3 + I_4$$

$$= \frac{1}{3} + \frac{4}{3} i - \frac{4}{3} - \frac{1}{3} i$$

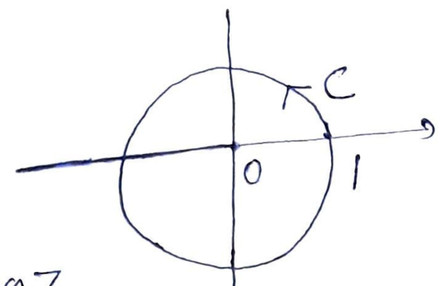
$$= -1 + i$$

(7) Evaluate $\int_C \log z \, dz$ where C is the unit circle in the z-plane.

Sol^n $I = \int_C \log z \, dz$

$f(z) = \log z$ is not Analytic at $z = 0$ and $z =$ negative real numbers.

C: Unit circle i.e. $|z| = 1$



$\therefore$ There is some not Analytic point of $\log z$ lies on the circle.

parametric form of circle $|z|=1$ is $z = 0 + 1\,e^{i\theta}$

$= e^{i\theta}, \quad 0 \le \theta \le 2\pi$

$dz = i\,e^{i\theta}\,d\theta$

$\log(z) = i\theta$

$I = \int_0^{2\pi} i\theta \cdot i\,e^{i\theta}\,d\theta$

$= -\int_0^{2\pi} \theta\,e^{i\theta}\,d\theta$

$= -\left[\theta \dfrac{e^{i\theta}}{i} + e^{i\theta}\right]_0^{2\pi}$

$= -\left[\dfrac{2\pi}{i} e^{i2\pi} + e^{i2\pi} - 0 - 1\right]$

$= -\left[\dfrac{2\pi}{i} + 1 - 1\right] = -\dfrac{2\pi}{i} = 2\pi i$

θ	$e^{i\theta}$
1	$\dfrac{e^{i\theta}}{i}$
0	$\dfrac{e^{i\theta}}{-1}$

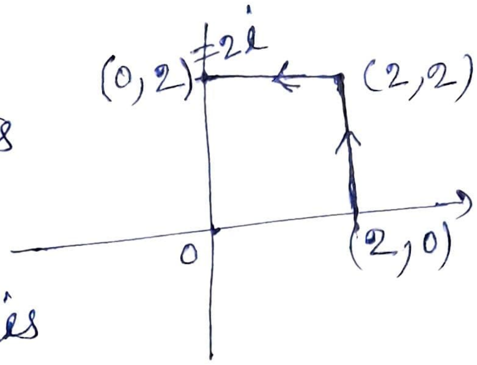
$e^{i2\pi}$
$= \cos 2\pi + i\sin 2\pi$
$= 1$

⑧ Evaluate $\int_C z^2+3z\, dz$ along the straight line from (2,0) to (2,2) and then from (2,2) to (0,2).

Soln $I = \int_C z^2+3z\, dz$

C: line from (2,0) to (2,2) and the from (2,2) to (0,2).

$f(z) = z^2+3z$ is Analytic at all points of complex plane.



$\therefore$ Integral of $f(z)$ is independent of the curves and depends on initial and final points.

$$I = \int_2^{2i} z^2+3z\, dz$$

$$= \left[\frac{z^3}{3} + \frac{3}{2}z^2\right]_2^{2i}$$

$$= -\frac{8}{3}i - 6 - \frac{8}{3} - 6$$

$$= -\frac{44}{3} - \frac{8}{3}i$$