Propositional Logic & Predicate Logic

Ms. Sunita Suralkar
CMPN, VESIT

What is Logic?

Logic is a truth-preserving system of inference

Truth-preserving: If the initial statements are true, the inferred statements will be true

System: a set of mechanistic transformations, based on syntax alone

Inference: the process of deriving (inferring) new statements from old statements

Propositional Logic

- A *proposition* is a statement that is either true or false
- Examples:
  - This class is CS122 (true)
  - Today is Sunday (false)
  - It is currently raining in Singapore (???)
- Every proposition is true or false, but its *truth value* (true or false) may be unknown

Propositional Logic

- A propositional statement is one of:
  - A simple proposition
    - denoted by a capital letter, e.g. ‘A’.
  - A negation of a propositional statement
    - e.g. $\neg A$: “not A”
  - Two propositional statements joined by a *connective*
    - e.g. $A \wedge B$: “A and B”
    - e.g. $A \vee B$: “A or B”
  - If a connective joins complex statements, parenthesis are added
    - e.g. $A \wedge (B \vee C)$

Truth Tables

- The truth value of a compound propositional statement is determined by its truth table
- Truth tables define the truth value of a connective for every possible truth value of its terms

Logical Negation

- Negation of proposition A is ¬A
  - A: It is snowing.
  - ¬A: It is not snowing
  - A: Newton knew Einstein.
  - ¬A: Newton did not know Einstein.
  - A: I am not registered for CS195.
  - ¬A: I am registered for CS195.

Truth Table

A	$\neg A$
0	1
1	0

Logical AND (conjunction)

- Conjunction of A and B is $A \wedge B$
  - A: CS160 teaches logic.
  - B: CS160 teaches Java.
  - $A \wedge B$: CS160 teaches logic and Java.
- Combining conjunction and negation
  - A: I like fish.
  - B: I like sushi.
  - I like fish but not sushi: $A \wedge \neg B$

Truth Table

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

Logical OR (disjunction)

- Disjunction of A and B is A v B
  - A: Today is Friday.
  - B: It is snowing.
  - A v B: Today is Friday or it is snowing.
- This statement is true if any of the following
  - Today is Friday
  - It is snowing
  - Both
- Otherwise it is false

Truth Table

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive OR

- The "or" connective ∨ is inclusive: it is true if either *or both* arguments are true
- There is also an exclusive or (either or): ⊕

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

The *exclusive or* of A and B is the proposition that is true when exactly one of A and B is true and false otherwise.

Conditional & Biconditional Implication

- The conditional implication connective is $\rightarrow$
- The biconditional implication connective is $\leftrightarrow$
- These, too, are defined by truth tables

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

A	B	$A \leftrightarrow B$
0	0	1
0	1	0
1	0	0
1	1	1

Conditional Implication

- A: A programming homework is due.
- B: It is Tuesday.
- $A \rightarrow B$:
  - If a programming homework is due, then it must be Tuesday.
- Is this the same?
  - If it is Tuesday, then a programming homework is due.

Bi-conditional

- A: You can take the flight.
- B: You have a valid ticket.
- $A \leftrightarrow B$
  - You can take the flight if and only if you have a valid ticket (and vice versa).

Compound Truth Tables

- Truth tables can also be used to determine the truth values of compound statements, such as $(A \vee B) \wedge (\neg A)$

A	B	$\neg A$	$A \vee B$	$(A \vee B) \wedge (\neg A)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

Tautology & Contradiction

- A *tautology* is a compound proposition that is always true.
- A *contradiction* is a compound proposition that is always false.
- A *contingency* is neither a tautology nor a contradiction.
- A compound proposition is *satisfiable* if there is at least one assignment of truth values to the variables that makes the statement true.

Examples

A	¬A	A∨¬A	A∧¬A
0	1	1	0
1	0	1	0

Result is always true, no matter what A is

Therefore, it is a ***tautology***

Result is always false, no matter what A is

Therefore, it is a ***contradiction***

Logical Equivalence

- Two compound propositions, p and q, are logically equivalent if $p \leftrightarrow q$ is a tautology.
- Notation: $p \equiv q$
- De Morgan's Laws:
  - $\neg (p \wedge q) \equiv \neg p \vee \neg q$
  - $\neg (p \vee q) \equiv \neg p \wedge \neg q$
- How so? Let's build a truth table!

Prove $\neg(p \land q) \equiv \neg p \lor \neg q$

p	q	$\neg p$	$\neg q$	$(p \land q)$	$\neg(p \land q)$	$\neg p \lor \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

Other Equivalences

- Show $p \to q \equiv \neg p \vee q$
- Show Distributive Law:
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Show $p \rightarrow q \equiv \neg p \vee q$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Show $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

More Equivalences

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption

Equivalences with Conditionals and Biconditionals, Precedence

- Conditionals
  - $p \to q \equiv \neg p \vee q$
  - $p \to q \equiv \neg q \to \neg p$
  - $\neg(p \to q) \equiv p \wedge \neg q$
- Biconditionals
  - $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
  - $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
  - $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
- Precedence:
  - $\neg$ highest
  - $\wedge$ higher than $\vee$
  - $\wedge$ and $\vee$ higher than $\to$ and $\leftrightarrow$
  - equal precedence: left to right

Prove Biconditional Equivalence

p	q	$\neg q$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$p \leftrightarrow \neg q$
0	0	1	1	0	0
0	1	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	0

Contrapositive

- The *contrapositive* of an implication $p \rightarrow q$ is:

$$\neg q \rightarrow \neg p$$

The contrapositive is equivalent to the original implication.

Prove it!

so now we have:

$$p \rightarrow q \equiv \neg p \vee q \equiv \neg q \rightarrow \neg p$$

a) Draw the truth table for the following propositional formula:

$$(p \vee \neg q) \Rightarrow (q \wedge r)$$

b) Formulate propositional formulas which are logically equivalent to the formulas $(p \wedge q)$ and $(p \Rightarrow q)$ using only the propositional connectives $\neg$ (negation) and $\vee$ (disjunction).

c) Formalise the following English sentences as propositional logic formulas:

   i) "When the front and back doors are closed then the light is off".

   ii) "Either the lift doors are open or the lift is moving and the lift doors are closed".

Predicate Logic

- Some statements cannot be expressed in propositional logic, such as:
  - All men are mortal.
  - Some trees have needles.
  - X > 3.
- Predicate logic can express these statements and make inferences on them.

Statements in Predicate Logic

P(x,y)

- Two parts:
  - A predicate P describes a relation or property.
  - Variables (x,y) can take arbitrary values from some domain.
- Still have two truth values for statements (T and F)
- When we assign values to x and y, then P has a truth value.

Example

- Let $Q(x,y)$ denote "$x=y+3$".
  - What are truth values of:
    - $Q(1,2)$ false
    - $Q(3,0)$ true

- Let $R(x,y)$ denote x beats y in Rock/Paper/Scissors with 2 players with following rules:
  - Rock smashes scissors, Scissors cuts paper, Paper covers rock.
  - What are the truth values of:
    - R(rock, paper) false
    - R(scissors, paper) true

Quantifiers

- Quantification expresses the extent to which a predicate is true over a set of elements.
- Two forms:
  - Universal, for all: $\forall$
  - Existential, there is, or, for some: $\exists$

Universal Quantifier

- $P(x)$ is true for all values in the domain
  $\forall x \in D, P(x)$
- For every x in D, $P(x)$ is true.
- An element x for which $P(x)$ is false is called a *counterexample*.
- Given $P(x)$ as "$x+1>x$" and the domain of R, what is the truth value of:

$$\forall x\, P(x)$$

true

Existential Quantifier

- $P(x)$ is true for <u>at least one value</u> in the domain.

  $$\exists x \in D, P(x)$$

- For some x in D, $P(x)$ is true.
- Let the domain of x be "animals", $M(x)$ be "x is a mammal" and $E(x)$ be "x lays eggs", what is the truth value of:

  $$\exists x\, (M(x) \wedge E(x))$$

English to Logic

- Some person in this class has visited the Grand Canyon.
- Domain of x is the set of all persons
- C(x): x is a person in this class
- V(x): x has visited the Grand Canyon
- $\exists x(C(x) \wedge V(x))$

English to Logic

- For every one there is someone to love.
- Domain of x and y is the set of all persons
- $L(x, y)$: x loves y
- $\forall x \exists y\, L(x,y)$
- Is it necessary to explicitly include that x and y must be different people (i.e. $x \neq y$)?
  - Just because x and y are different variable names doesn't mean that they can't take the same values

Evaluating Expressions: Precedence and Variable Bindings

- Precedence:
  - Quantifiers and negation are evaluated before operators
  - $\wedge$ higher than $\vee$
  - $\wedge$ and $\vee$ higher than $\rightarrow$ and $\leftrightarrow$
  - equal precedence: left to right
- Bound:
  - Variables can be given specific values or
  - Can be constrained by quantifiers

Predicate Logic Equivalences

Statements are *logically equivalent* iff they have the same truth value under all possible bindings.

For example:

$$\forall x\big(P(x) \wedge Q(x)\big) \equiv \forall x P(x) \wedge \forall x Q(x)$$

In English: "Given the domain of students in CS160, all students have passed M124 course (P) and are registered at CSU (Q); hence, all students have passed M124 and all students are registered at CSU.

Other Equivalences

- Someone likes skiing (P) or likes swimming (Q); hence, there exists someone who likes skiing or there exists someone who likes skiing.

$$\exists x\left(P(x) \vee Q(x)\right) \equiv \exists x P(x) \vee \exists x Q(x)$$

- Not everyone likes to go to the dentist; hence there is someone who does not like to go to the dentist.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

- There does not exist someone who likes to go to the dentist; hence everyone does not like to go to the dentist.

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$