Predicate Logic

Extend propositional logic by the following new features.

- Variables: $x, y, z, \ldots$
- Predicates (i.e., propositional functions):
  $P(x), Q(x), R(y), M(x, y), \ldots$.
- Quantifiers: $\forall, \exists$.

Propositional functions are a generalization of propositions.

- Can contain variables and predicates, e.g., $P(x)$.
- Variables stand for (and can be replaced by) elements from their domain.

Quantifiers

- We need quantifiers to formally express the meaning of the words “all” and “some”.
- The two most important quantifiers are:
  - Universal quantifier, “For all”. Symbol: $\forall$
  - Existential quantifier, “There exists”. Symbol: $\exists$
- $\forall x\ P(x)$ asserts that $P(x)$ is true for **every** x in the domain.
- $\exists x\ P(x)$ asserts that $P(x)$ is true for **some** x in the domain.
- The quantifiers are said to bind the variable x in these expressions.
- Variables in the scope of some quantifier are called bound variables. All other variables in the expression are called free variables.
- A propositional function that does not contain any free variables is a proposition and has a truth value.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, $P(x)$" or "For every x, $P(x)$".
- The truth value depends not only on P, but also on the domain U.
- **Example:** Let $P(x)$ denote $x > 0$.
  - If U is the integers then $\forall x\, P(x)$ is false.
  - If U is the positive integers then $\forall x\, P(x)$ is true.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, $P(x)$" or "There is an x such that, $P(x)$", or "For at least one x, $P(x)$".
- The truth value depends not only on P, but also on the domain U.
- **Example:** Let $P(x)$ denote $x < 0$.
  - If U is the integers then $\exists x\, P(x)$ is true.
  - If U is the positive integers then $\exists x\, P(x)$ is false.

Precedence of Quantifiers

- Quantifiers $\forall$ and $\exists$ have higher precedence then all logical operators.
- $\forall x\ P(x) \wedge Q(x)$ means $(\forall x\ P(x)) \wedge Q(x)$. In particular, this expression contains a free variable.
- $\forall x\ (P(x) \wedge Q(x))$ means something different.

De Morgan's Law for Quantifiers

The rules for negating quantifiers are:

- $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

- Whoever can read is literate

$$\forall x\ [R(x) \supset L(x)]$$

- Dolphins are not literate

$$\forall x\ [D(x) \supset \neg L(x)]$$

- Some Dolphins are intelligent

$$\exists x\ [D(x) \wedge I(x)]$$

1. *Marcus was a man*

Man(Marcus)

2. *Marcus was a Pompeian*

Pompeian(Marcus)

3. *All Pompeians were Romans*

$\forall x\ [\text{Pompeian}(x) \supset \text{Roman}(x)]$

4. *Caesar was a ruler*

Ruler(Caesar)

5. *All Romans were either loyal to Caesar or hated him*

$\forall x\ [\text{Roman}(y) \supset (\text{LoyalTo}(x,\text{Caesar}) \vee \text{Hate}(x,\text{Caesar}))]$

6. *Everyone is loyal to someone*

$\forall x\ \exists y\ \text{LoyalTo}(x,y)$

Let $Q(x, y)$ denote the statement "x is the capital of y."
What are these truth values?

a) Q(Denver, Colorado)
b) Q(Detroit, Michigan)
c) Q(Massachusetts, Boston)
d) Q(New York, New York)

Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$

b) $\forall x(C(x) \wedge F(x))$

c) $\exists x(C(x) \rightarrow F(x))$

d) $\exists x(C(x) \wedge F(x))$

Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

a) $\forall x(R(x) \rightarrow H(x))$

b) $\forall x(R(x) \land H(x))$

c) $\exists x(R(x) \rightarrow H(x))$

d) $\exists x(R(x) \land H(x))$

Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.

b) All Swedish movies are serious.

c) No one can keep a secret.

d) There is someone in this class who does not have a good attitude.

Let $P(x, y)$ be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

Use quantifiers and predicates with more than one variable to express these statements.

a) Every computer science student needs a course in discrete mathematics.

b) There is a student in this class who owns a personal computer.

c) Every student in this class has taken at least one computer science course.

d) There is a student in this class who has taken at least one course in computer science.

e) Every student in this class has been in every building on campus.

f) There is a student in this class who has been in every room of at least one building on campus.

g) Every student in this class has been in at least one room of every building on campus.

“Every student has an Internet account.” “Homer does not have an Internet account.” “Maggie has an Internet account.”

$P(x) =$ "x is a student"
$Q(x) =$ "x has an Internet account"

We can then rewrite the given statements using the above interpretations.

	Step	**Reason**
1.	$\forall x(P(x) \to Q(x))$	*Premise*
2.	$\neg Q(\text{Homer})$	*Premise*
3.	$Q(\text{Maggie})$	*Premise*
4.	$P(\text{Homer}) \to Q(\text{Homer})$	Universal instantiation from (1)
5.	$\neg P(\text{Homer})$	Modus tollens from (2) and (4)

Step (5) means that "Homer is not a student".