Simple Proof of The Routh Stability Criterion Based on Order Reduction of Polynomials and Principle of Argument

Naoki Matsumoto
Department of Electronics and Communication, Meiji University
Tama, Kawasaki, 214-8571 Japan
E-mail: matsumot@isc.meiji.ac.jp

ABSTRACT

Simple proof for the Routh stability criterion is derived by applying the principle of the argument to the explicit polynomial order reduction formula which generates each row of the Routh table. By this proof, we can easily reach to the famous result that the number of times of sign changes in the first column elements in the Routh table coincides with the number of unstable zeros of a given polynomial. Our proof is easily extendible to the stability tests for complex coefficients systems and time-delay systems.

1. INTRODUCTION

Usually, the third grade or the fourth grade students belong to the department of engineering learn the Routh stability criterion and the Jury stability criterion in the course of electrical circuits theory, classical feedback control theory, and digital signal processing. So far, a lot of efforts have been devoted to develop simple proofs for these stability criteria [1]-[7]. However, elementary proofs which even undergraduate students can understand have been still few. Recently, Ho, Datta, and Bhattacharyya [3] have given a simple proof to this famous theorem by using extended Hermite-Bieler theorem. In this paper, we give another simple proof to this famous theorem. In our proof, we define an explicit order reduction formula which is a polynomial version of the well known Routh algorithm. The proof is completed by applying the principle of the argument, and the detailed discussion of Hermite-Bieler theorem like [1][3][4] is not required. The results in this paper are extendible to the stability tests for the complex coefficient polynomials in a quite natural manner.

2. ORDER REDUCTION AND ORDER AUGMENTATION FORMULA FOR POLYNOMIALS

Let $f_n(s)$ be a real coefficient polynomial of complex variable s of order n.

$$f_n(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{1}$$

Without loss of generality, we assume that $a_0 > 0$.

Definition 1: $f_n(s)$ in (1) is an *nth* order Hurwitz polynomial if all the roots of $f_n(s)$ exist in the left half-plane (imaginary axis is excluded) in the s-plane.

Now we define the order reduction formula, which in later we use to generate each row of the Routh table. Let $f_k(s)$ be a kth order real coefficient polynomial of complex variable s.

$$f_k(s) = \alpha_0 s^k + \alpha_1 s^{k-1} + \cdots + \alpha_{k-1}s + \alpha_k \tag{2}$$

The order reduction formula for $f_k(s)$ is defined as

$$f_{k-1}(s) = f_k(s) - \mu_k s\{f_k(s) - (-1)^k f_k(-s)\} \tag{3a}$$

$$\mu_k = \frac{\alpha_0}{2\alpha_1} \tag{3b}$$

If $f_k(s)$ in (2) is a Hurwitz polynomial, every coefficient of $f_k(s)$ is positive. Throughout this paper, we assume that μ_k in Eq. (3b) is finite and $\mu_k \neq 0$. Substituting (2) into (3), we obtain the reduced order polynomial $f_{k-1}(s)$ as

$$f_{k-1}(s) = \beta_0 s^{k-1} + \beta_1 s^{k-2} + \cdots + \beta_{k-2}s + \beta_{k-1} \tag{4a}$$

where

$$\beta_{2i} = \alpha_{2i+1} \quad i = 0,\ 1,\ 2,\cdots \tag{4b}$$

$$\beta_{2i+1} = -\frac{1}{\alpha_1}\begin{bmatrix}\alpha_0 & \alpha_{2i+2} \\ \alpha_1 & \alpha_{2i+3}\end{bmatrix},\ i = 0,\ 1,\ 2,\cdots \tag{4c}$$

We should note here that Eqs. (4b) and (4c) are the well known Routh algorithm itself. The order reduction formula in (3) is a polynomial representation of the Routh algorithm. From (4b) and (4c) it follows that $f_k(s)$ and $f_{k-1}(s)$ have the same even polynomial part when k is odd and have the same odd polynomial part when k is even. Furthermore, the last coefficients of $f_k(s)$ and $f_{k-1}(s)$ become always same, namely $\alpha_k = \beta_{k-1}$. By repeating the order reduction in (3), we obtain a sequence of the reduced order polynomials $\{f_n(s), f_{n-1}(s), \cdots, f_2(s), f_1(s)\}$ and a sequence of the constants $\{\mu_n, \mu_{n-1}, \cdots, \mu_2, \mu_1\}$. Each row of the Routh table consists of the coefficients of the even polynomial part or the odd polynomial part of $f_k(s)$. The last polynomial $f_1(s)$ becomes in the form of $f_1(s) = (2\mu_1 s + 1)a_n$ (see Eq. (2), (3) and (4)). The matrix representation of the order reduction formula (3) becomes as follows.

$$\begin{bmatrix} f_{k-1}(s) \\ f_{k-1}(-s) \end{bmatrix} = \begin{bmatrix} 1-\mu_k s & (-1)^k \mu_k s \\ -(-1)^k \mu_k s & 1+\mu_k s \end{bmatrix} \begin{bmatrix} f_k(s) \\ f_k(-s) \end{bmatrix} \tag{5}$$

The reverse representation of (5) is given by

$$\begin{bmatrix} f_k(s) \\ f_k(-s) \end{bmatrix} = \begin{bmatrix} 1+\mu_k s & -(-1)^k \mu_k s \\ (-1)^k \mu_k s & 1-\mu_k s \end{bmatrix} \begin{bmatrix} f_{k-1}(s) \\ f_{k-1}(-s) \end{bmatrix} \tag{6}$$

From (6), we have the order augmentation formula which reconstructs $f_k(s)$ form $f_{k-1}(-s)$, namely

$$\begin{aligned} f_k(s) &= (1+\mu_k s) f_{k-1}(s) \\ &\quad \times \left\{ 1 - (-1)^k \frac{\mu_k s}{1+\mu_k s} \frac{f_{k-1}(-s)}{f_{k-1}(s)} \right\} \\ &= (1+\mu_k s) f_{k-1}(s) g_{k-1}(s) \end{aligned} \tag{7}$$

where

$$\mu_k = \frac{\alpha_0}{2\alpha_1} = \frac{\alpha_0}{2\beta_0} \tag{8a}$$

$$g_{k-1}(s) = 1 - (-1)^k \frac{\mu_k s}{1+\mu_k s} \frac{f_{k-1}(-s)}{f_{k-1}(s)} \tag{8b}$$

3. THE PRINCIPLE OF ARGUMENT

First we note that the Nyquist locus of $g_{k-1}(s)$ lies only on the right half-plane of the s-plane. We obtain the following property with respect to $g_{k-1}(s)$ in (8b).

Property 1: If $\mu_k \neq 0$, $g_{k-1}(s)$ in (8) satisfies

(*i*) $\mathrm{Re}\{g_{k-1}(j\omega)\} > 0 \quad (-\infty < \omega < +\infty)$ (9a)

(*ii*) $g_{k-1}(\pm j\infty) = 1 - (-1)^k (-1)^{k-1} = 2$ (9b)

Property 1-(*ii*) is direct from (8). Next, let $\Delta \arg f_k(s)$ denote the net increment of the argument of $f_k(s)$ on the imaginary axis, namely

$$\Delta \arg f_k(s) = \arg f_k(+j\infty) - \arg f_k(-j\infty) \tag{10}$$

We define $\Delta \arg(1+\mu_k s)$ and $\Delta \arg g_{k-1}(s)$ in the same manner as (10). By the principle of the argument, we obtain the following properties.

Property 2:

(*i*) $\Delta \arg(1+\mu_k s) = \begin{cases} \pi & \text{if} \quad \mu_k > 0 \\ -\pi & \text{if} \quad \mu_k < 0 \end{cases}$ (11)

(*ii*) $\Delta \arg g_{k-1}(s) = 0$

(*iii*) If $\mu_k \neq 0$,

$\Delta \arg f_k(s) = \mathrm{sign}(\mu_k)\pi + \Delta \arg f_{k-1}(s)$ (12)

(*iv*) kth order polynomial $f_k(s)$ satisfies

$\Delta \arg f_k(s) = \{(k - R_k) - R_k\}\pi$ (13)

if $f_k(s)$ has R_k zeros in the open right half-plane of the s-plane and $(k - R_k)$ zeros in the open left half-plane of the s-plane.

(*iv*) $f_k(s)$ is a kth order Hurwitz polynomial, if and only if

$\Delta \arg f_k(s) = k\pi$. (14)

4. ROUTH STABILITY TEST

4.1 Main Result

Now, we are ready to state the main theorem which leads to the Routh stability test. From Eq. (7) and *Property 2* we can prove the following *Theorem 1* and *Corollary 1.*

Theorem 1: A real coefficient polynomial $f_n(s)$ in (1) is a Hurwitz polynomial if and only if

$$\mu_k > 0, \quad k = n,\ n-1, \cdots,\ 2, \tag{15}$$

where μ_k is the constant generated by the order reduction formula in Eq. (3).

Corollary 1:

(*i*) If $\mu_k > 0$, the number of stable zeros of $f_k(s)$ increases by one than that of $f_{k-1}(s)$, and the number of unstable zeros of $f_k(s)$ is equal to that of $f_{k-1}(s)$.

(*ii*) If $\mu_k < 0$, the numbers of stable zeros of $f_k(s)$ is equal to that of $f_{k-1}(s)$, and the number of unstable zeros of $f_k(s)$ increases by one than that of $f_{k-1}(s)$.

(*iii*) If every μ_k is nonzero, the number of negative μ_k s among $\{\mu_n, \mu_{n-1}, \cdots, \mu_2, \mu_1\}$ gives the number of unstable zeros of $f_n(s)$.

4.2 Sign Changes in The Routh Table and Unstable Zeros

Now we discuss the structure of the Routh table. Using the coefficients of $f_k(s)$ in (2) and $f_{k-1}(s)$ in (4), the $(n$-k+1$)$th row and the (n-k+2)th row of the Routh table are written as

(n-k+1)th row $\Leftrightarrow$ $\alpha_0 \quad \alpha_2 \quad \alpha_4 \quad \alpha_6 \quad \alpha_8 \quad \cdots$

(n-k+2)th row $\Leftrightarrow$ $\beta_0 \quad \beta_2 \quad \beta_4 \quad \beta_6 \quad \beta_8 \quad \cdots$

Since the order reduction formula in (3) propagates the coefficients from $f_k(s)$ to $f_{k-1}(s)$ as $\beta_{2i} = \alpha_{2i+1}$ (see Eq. (4b), the alternative expression of the (n-k+2)th row of the Routh table is given by

(n-k+2)th row $\Leftrightarrow$ $\alpha_1 \quad \alpha_3 \quad \alpha_5 \quad \alpha_7 \quad \alpha_9 \quad \cdots$

If we denote the ith row jth column element of the Routh table by $R_{i,j}$, the constant μ_k in (3b) can be expressed as

$$2\mu_k = \frac{\alpha_0 \text{ in } f_k(s)}{\beta_0 \text{ in} f_{k-1}(s)} = \frac{\alpha_0 \text{ in} f_k(s)}{\alpha_1 \text{ in} f_k(s)} = \frac{R_{n-k+1,1}}{R_{n-k+2,1}} \tag{16}$$

Therefore, the number of times of sign changes in the first column elements of the Routh table coincides with the number of negative μ_ks among $\{\mu_n, \mu_{n-1}, \cdots, \mu_2, \mu_1\}$. Then, *Corollary 1-(iii)* leads to the following result.

Theorem 2: If every μ_k is nonzero, the number of times of the sign changes in the first column elements of the Routh table for *nth* order polynomial $f_n(s)$ coincides with the number of unstable zeros of $f_n(s)$.

Example 1:

Consider the following sixth order polynomial.

$$f_6(s) = (s^2 - s + 3)(s^2 + s + 1)^2$$
$$= s^6 + s^5 + 4s^4 + 5s^3 + 8s^2 + 5s + 3$$

Clearly, $f_6(s)$ has two unstable zeros and four stable zeros. The corresponding reduced order polynomials $f_k(s)$ and the parameter μ_k and the Routh table are shown in the Table 1. In the Table 1, wee can see that the reduced order polynomials $f_k(s)$ and $f_{k-1}(s)$ always have the same even polynomial part when k is odd and the same odd polynomial part when k is even. Each two consecutive rows of the Routh table consist of the coefficients of the even polynomial and the odd polynomial of $f_k(s)$. There exist two negative μ_ks and two times sign changes in the first column elements of the Routh table. These results are consistent with the fact that the given polynomial $f_6(s)$ has two unstable zeros.

5. CONCLUSION AND DISCUSSION

We have shown a simple proof of the Routh stability criterion. The polynomial order reduction formula in (3) is the direct representation of the Routh algorithm. Based on this formula and the principle of the argument, we have shown that the number of the sign changes in the first column elements of the Routh table coincides with the number of unstable zeros of a given polynomial. In this paper we have not discussed the singular case such that the application of the order reduction formula in (3) become impossible due to $\alpha_1 = 0$ in $f_k(s)$. The singular case occurs when a given polynomial has the zeros on the borderline of stability or on the mirror image position with respect to the borderline of stability. How to count the number of the stable and the unstable zeros of the polynomials in singular case is stated in [3][5][7].

Table 1.

The relations between the reduced order polynomials $f_k(s)$ and the Routh table

$\mu_k = \frac{\alpha_0}{2\alpha_1} = \frac{\alpha_0}{2\beta_0}$	Reduced order polynomial $f_k(s)$	Routh table
$\mu_6 = 1/2$	$f_6(s) = (s^6 + 4s^4 + 8s^2 + 3) + (s^5 + 5s^3 + 5s)$	1 4 8 3
$\mu_5 = -1/2 < 0$	$f_5(s) = (s^5 + 5s^3 + 5s) + (-s^4 + 3s^2 + 3)$	1 5 5
$\mu_4 = -1/16 < 0$	$f_4(s) = (-s^4 + 3s^2 + 3) + (8s^3 + 8s)$	−1 3 3
$\mu_3 = 1$	$f_3(s) = (8s^3 + 8s) + (4s^2 + 3)$	8 8
$\mu_2 = 1$	$f_2(s) = (4s^2 + 3) + 2s$	4 3
$\mu_1 = 1/3$	$f_1(s) = 2s + 3$	2
		3

Table 2.

The reduced order complex polynomials $f_k(s)$ and the complex Routh test

$\mu_k = \dfrac{1}{2\,\mathrm{Re}\left(\dfrac{\alpha_1}{\alpha_0}\right)} = \dfrac{1}{2\mathrm{Re}\left(\dfrac{\beta_0}{\alpha_0}\right)}$	Complex Routh Table Reduced Order Complex Polynomial $f_k(s)$					
$\mu_4 = 1/4$	$f_4(s) =$	s^4	$+ (2+3j)s^3$	$+ (-5+6j)s^2$	$-(14+3j)s$	$-8-6j$
$\mu_3 = 1/2$	$f_3(s) =$	$2s^3$	$+ (2+6j)s^2$	$-14s$	$-8-6j$	
$\mu_2 = -5/3 < 0$	$f_2(s) =$	$(2+6j)s^2$	$-6s$	$-8-6j$		
$\mu_1 = 51/26$	$f_1(s) =$	$(-24+6j)s$	$-8-6j$			

Finally, we discuss the Routh test for the complex coefficient polynomials. Extending *Theorem 1 and Theorem 2* to the polynomials with complex coefficients is not difficult. Suppose that $f_n(s)$ in (1) is an *nth* order complex coefficients polynomial. Then the order reduction formula in (3) and the order augmentation formula in (7) can be modified as in below.

$$f_{k-1}(s)=f_k(s)-\mu_k s\left\{f_k(s)-\lambda_k(-1)^k\hat{f}_k(-s)\right\}$$
$$=\beta_0 s^{k-1}+\beta_1 s^{k-2}+\cdots+\beta_{k-2}s+\beta_{k-1} \qquad (17)$$

$$f_k(s)=(1+\mu_k s)f_{k-1}(s)$$
$$\times\left\{1-(-1)^k\lambda_k\frac{\mu_k s}{1+\mu_k s}\frac{\hat{f}_{k-1}(-s)}{f_{k-1}(s)}\right\} \qquad (18)$$

where

$$\hat{f}_k(-s)=\bar{\alpha}_0(-s)^k+\bar{\alpha}_1(-s)^{k-1}+\cdots+\bar{\alpha}_k \qquad (19)$$

$$\lambda_k=\frac{\alpha_0}{\bar{\alpha}_0}\quad\left(\text{hence } |\lambda_k|=1\right) \qquad (20)$$

$$\mu_k=\frac{\alpha_0}{\alpha_1+\lambda_k\bar{\alpha}_1}=\frac{1}{2\,\mathrm{Re}\left(\dfrac{\alpha_1}{\alpha_0}\right)}=\frac{\alpha_0}{\beta_0+\lambda_k\bar{\beta}_0}$$
$$=\frac{1}{2\,\mathrm{Re}\left(\dfrac{\beta_0}{\alpha_0}\right)} \qquad (21)$$

By (17)-(21), Eq. (4b) and (4c) are modified to

$$\beta_i=-\frac{\begin{vmatrix}\alpha_0 & \alpha_{i+1}\\ \alpha_1+\lambda_k\bar{\alpha}_1 & \alpha_{i+2}-(-1)^i\lambda_k\bar{\alpha}_{i+2}\end{vmatrix}}{\alpha_1+\lambda_k\bar{\alpha}_1},$$
$$i=0,\ 1,\ 2,\cdots,\ k-2 \qquad (22a)$$

$$\beta_{k-1}=\alpha_k \qquad (22b)$$

Then by the principle of the argument, we can arrive at the result that the *nth* order complex coefficients polynomial $f_n(s)$ is a Hurwitz polynomial if and only if

$\mu_k>0$, $k=n,\ n-1,\cdots,\ 2$, . We note that the statements in *Corollary 1* are still valid for μ_ks in (21). The algorithm in Eq. (17)-(22) is equivalent to the complex Routh algorithm in [9] which has been derived by using Cauchy index and Strum chain for complex coefficient polynomial [10]. The algorithm in Eq. (17)-(22) is also equivalent to the stability criterion in the form of continued fraction for complex polynomial in [8] which is related with the passive ladder network realization of lossless positive real function. When we discuss the stability of time-delay systems, the coefficients α_is and β_is in (22) and μ_ks in (21) become the real rational functions of e^{-jsh} and we have that time-delay system is stable if [p. 226, 11][12]

$$\mu_k(e^{-j\omega h})>0,\quad 0\le\omega h\le 2\pi \qquad (23)$$
$$k=n,\ n-1,\cdots,\ 2,$$

Example 2:

Consider the following forth order complex polynomial.

$$f_4(s)=s^4+(2+3j)s^3+(-5+6j)s^2-(14+3j)s-8-6j$$
$$=(s+1+2j)(s-2+j)(s+2)(s+1)$$

The complex Routh algorithm in (17)-(22) generates the reduced order polynomials in the Table 2. There are three positive μ_ks and one negative μ_k. Hence $f_4(s)$ has one unstable zero, which agrees with the result of the factorization of $f_4(s)$.

6. REFERENCES

[1] H. Chapellat, M. Mansour, and S. P. Bhattacharyya, "Elementary proofs of some classical stability criteria, "*IEEE Trans. on Education*, vol. 33, no. 3, pp. 232-239 (Aug.1990).

[2] P. P. Vaidyanathan and S. K. Mitra, "A unified structural interpretation of some well-known stability-test procedures for linear systems", *Proc. IEEE*, vol. 75, no. 4 (April 1987).

[3] Ming-Tzu Ho, A. Datta, and S. P. Bhattacharyya, "An elementary derivation of the Routh-Hurwitz criterion," *IEEE Trans. on Automat. Contr.* vol. 43, no. 3, pp. 405-409 (March 1998).

[4] M. Mansour, "Robust stability in systems described by rational functions", *Control and Dynamics Systems*, Vol. 51, Academic Press, 1992

[5] P. C. Parks and V. Hahn, *Stability Theory*, Prentice-Hall, 1992.

[6] R. H. Raible, "A simplification of Jury's tabular form", *IEEE Trans. Automatic Control*, vol. 19, pp. 248-250, 1974.

[7] L. H. Keel and S. P. Bhattacharyya, "A new proof of the Jury test", *Automatica*, vol. 35, pp. 251-258, 1999

[8] Wall, H. S., "Analytic Theory of Continued Fractions" Chapter 10, Van Nostrand, New York, 1948

[9] N.Hamada and S. Takahashi, "Generalized Routh algorithm by Lyapunov's second method", *Electronics and Communications in Japan, Vol.* 57, No. 3, March, 1974

[10] F. R. Gantmacher, *The theory of matrices*, Chelsea, 1959.

[11] N. K. Bose, *"Applied multidimensional systems theory"* , Van Nostrand, New York, 1982.

[12] P. Agathoklis and S. Foda "Stability and the Lyapunov equation for delay differential systems," *Int. J. Control*, Vol. 49, No. 2, pp. 417–432, 1989.