A NEW PROOF OF THE ROUTH-HURWITZ STABILITY CRITERION USING THE SECOND METHOD OF LIAPUNOV

By P. C. PARKS

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ABSTRACT. The second method of Liapunov is a useful technique for investigating the stability of linear and non-linear ordinary differential equations. It is well known that the second method of Liapunov, when applied to linear differential equations with real constant coefficients, gives rise to sets of necessary and sufficient stability conditions which are alternatives to the well-known Routh–Hurwitz conditions. In this paper a direct proof of the Routh–Hurwitz conditions themselves is given using Liapunov's second method. The new proof is 'elementary' in that it depends on the fundamental concept of stability associated with Liapunov's second method, and not on theorems in the complex integral calculus which are required in the usual proofs. A useful by-product of this new proof is a method of determining the coefficients of a linear differential equation with real constant coefficients in terms of its Hurwitz determinants.

1. *Introduction.* The work of Liapunov (1) on the stability of motion is of growing importance, particularly in the field of automatic control and servomechanisms. The 'second method' of Liapunov is a very useful technique for investigating the stability of non-linear systems which often arise in this field (Kalman and Bertram (2)).

To apply Liapunov's second method we must consider a set of first-order differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, \ldots, x_n, t) \quad (i = 1, 2, 3, \ldots, n). \tag{1}$$

Many dynamical systems may be reduced to this form, if necessary by the introduction of additional variables x_i. The f_i are real and continuous functions. The variables $x_1, x_2, x_3, \ldots, x_n$ define a point in the 'state space' and the solutions of (1) give rise to 'trajectories' in this state space. There is no loss of generality in studying only the stability of the solution $x_i = 0$ $(i = 1, 2, 3, \ldots, n)$, the origin of the state space: this is said to be 'asymptotically stable' if the trajectory starting from any point in a finite neighbourhood of the origin remains within a finite neighbourhood of the origin and tends to the origin as $t \to \infty$. Liapunov's second method involves finding sign definite functions $V(x_1, x_2, x_3, \ldots, x_n, t)$ defining closed contour surfaces surrounding the origin and then determining the rate of change of V, $\dot{V}$, along a trajectory. If $\dot{V}$ has the opposite sign to V then, provided the functions V satisfy certain conditions, it follows that the trajectories cross the contour surfaces inwards and tend to the origin as $t \to \infty$. In fact

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt} \\ &= \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x_1, x_2, x_3, \ldots, x_n, t),\end{aligned}$$

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and so useful information about stability may be obtained by finding a suitable function V without having to solve the equations (1). Such functions V are called 'Liapunov functions'. We seek functions V which are positive definite, and which have negative definite derivatives $\dot{V}$.

If the equations (1) are linear with real constant coefficients we may write them in matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{2}$$

where $\mathbf{A}$ is a real constant matrix. Quadratic forms in the variables $x_1, x_2, x_3, \ldots, x_n$, are suitable Liapunov functions. The following theorem was proved by Liapunov himself.

THEOREM 1. *A necessary and sufficient condition for* $x_i = 0$ $(i = 1, 2, 3, \ldots, n)$ *to be an asymptotically stable solution of the matrix equation* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ ($\mathbf{A}$ *being a real constant matrix*) *is that* $\mathbf{x}'\mathbf{P}\mathbf{x}$ *be a positive definite quadratic form where* $\mathbf{P}$ *is a symmetric matrix satisfying the equation*

$$\mathbf{PA} + \mathbf{A}'\mathbf{P} = -\mathbf{Q},$$

$\mathbf{x}'\mathbf{Q}\mathbf{x}$ *being any positive definite quadratic form and* $\mathbf{Q}$ *a real symmetric matrix.* ($\mathbf{A}'$ *is the transpose of the matrix* $\mathbf{A}$.)

We shall make use of a useful extension of Theorem 1 (Kalman and Bertram (2)).

THEOREM 1A. *A necessary and sufficient condition for* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ ($\mathbf{A}$ *a real constant matrix*) *to represent an asymptotically stable system is that* $V = \mathbf{x}'\mathbf{P}\mathbf{x}$ *be a positive definite quadratic form where* $\mathbf{P}$ *is a symmetric matrix, and* $\dot{V} = \mathbf{x}'(\mathbf{PA} + \mathbf{A}'\mathbf{P})\,\mathbf{x}$ *is a negative semi-definite quadratic form not identically zero along any trajectory of* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, *other than* $\mathbf{x} \equiv \mathbf{0}$.

By taking $\mathbf{A}$ to be the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

we may find from Theorems 1 and 1A a host of necessary and sufficient conditions for the stability of the nth order differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \ldots + a_{n-1} \frac{dy}{dt} + a_n y = 0. \tag{3}$$

We may choose any positive definite or positive semi-definite $\mathbf{x}'\mathbf{Q}\mathbf{x}$ and then solve for the $\frac{1}{2}n(n+1)$ elements p_{ij} $(= p_{ji})$ of $\mathbf{P}$ using $\frac{1}{2}n(n+1)$ linear equations derived from equating $\mathbf{PA} + \mathbf{A}'\mathbf{P}$ and $-\mathbf{Q}$ element by element. (These equations are soluble unless a root of the characteristic equation of $\mathbf{A}$ is zero or the sum of a pair of roots is zero.)

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The stability conditions are derived from the Sylvester criterion for $\mathbf{x}'\mathbf{P}\mathbf{x}$ to be positive definite.

$$p_{11} > 0, \quad \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{vmatrix} > 0, \quad \ldots, \quad \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{vmatrix} > 0.$$

The conditions being necessary and sufficient are clearly alternatives to the Routh–Hurwitz stability criterion for equation (3). We state the Hurwitz and Routh stability criteria below.

THEOREM 2 (Hurwitz (3)). *A necessary and sufficient condition for the differential equation*

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + a_2 \frac{d^{n-2}y}{dt^{n-2}} + \ldots + a_{n-1}\frac{dy}{dt} + a_n y = 0$$

to represent an asymptotically stable system is that the determinants $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n$ *are all positive, where*

$$\Delta_r = \begin{vmatrix} a_1 & 1 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & 1 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{2r-1} & a_{2r-2} & a_{2r-3} & a_{2r-4} & \ldots & a_r \end{vmatrix},$$

it being understood that if an element a_s *appears in* Δ_r *with* $s > n$ *it is replaced by zero.*

THEOREM 3 (Routh (4), (5), pp. 221 *et seq.*). *A necessary and sufficient condition for the differential equation*

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + a_2 \frac{d^{n-2}y}{dt^{n-2}} + \ldots + a_{n-1}\frac{dy}{dt} + a_n y = 0$$

to represent an asymptotically stable system is that the elements of the first column of the matrix (C_{ij}) *are all positive, where the first two rows of* (C_{ij}) *are defined in terms of the even and odd coefficients* a_i *as follows:*

$$\begin{aligned}
&C_{11} = 1, \quad C_{1j} = a_{2(j-1)} \quad \textit{for} \quad 2 \leqslant j \leqslant m+1, \\
&C_{1j} = 0 \qquad \textit{for} \quad j > m+1, \quad \textit{if } n = 2m \textit{ is even, or if } n = 2m+1 \textit{ is odd;} \\
&C_{2j} = a_{2j-1} \quad \textit{for} \quad 1 \leqslant j \leqslant m, \\
&C_{2j} = 0 \qquad \textit{for} \quad j > m, \qquad \textit{if } n = 2m, \\
&C_{2j} = a_{2j-1} \quad \textit{for} \quad 1 \leqslant j \leqslant m+1, \\
&C_{2j} = 0 \qquad \textit{for} \quad j > m+1, \quad \textit{if } n = 2m+1,
\end{aligned}$$

and the subsequent elements are defined by the recurrence relation

$$C_{ij} = C_{i-2, j+1} - \frac{C_{i-2,1}C_{i-1,j+1}}{C_{i-1,1}} \quad (i = 3, 4, 5, \ldots, n+1;\ j = 1, 2, 3, \ldots).$$

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The matrix (C_{ij}) has $n+1$ rows and an infinity of columns. The numbers of non-zero elements in each row form the sequence $m+1, m, m, m-1, m-1, \ldots, 3, 3, 2, 2, 1, 1$ if $n = 2m$ and the sequence $m+1, m+1, m, m, m-1, m-1, \ldots, 3, 3, 2, 2, 1, 1$ if $n = 2m+1$. These non-zero elements are sometimes known as the 'Routh array'.

There is a close and well-known connexion between Theorems 2 and 3.

THEOREM 4 (see, for example, Gantmacher (6), pp. 203 *et seq.*).

$$C_{11} = 1, \quad C_{21} = \Delta_1, \quad C_{i1} = \frac{\Delta_{i-1}}{\Delta_{i-2}} \quad (i = 3, 4, \ldots, n+1).$$

This may be proved by successive column subtraction processes on Δ_r to reduce it to lower triangular form in which the elements on the principal diagonal are $C_{21}, C_{31}, \ldots, C_{r+1,1}$ so that

$$\Delta_r = \prod_{i=2}^{r+1} C_{i1}.$$

As the second method of Liapunov provides through Theorem 1 alternative conditions to the Routh–Hurwitz criterion (Theorems 2 and 3) one might ask whether there is a direct proof of Theorems 2 or 3 using the second method of Liapunov, as the existing proofs of Theorems 2 and 3 are complicated and usually involve contour integration theorems of the complex integral calculus. Such a proof will now be presented.

2. *Proof of Theorem 2 by the second method of Liapunov.* We shall represent the differential equation (3) not by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ as described immediately below Theorem 1 A but, tentatively, by the matrix equation $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ -b_n & 0 & 1 & \ldots & 0 & 0 \\ 0 & -b_{n-1} & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & -b_2 & -b_1 \end{bmatrix}.$$

The form of $\mathbf{B}$ was first used by Schwarz (7) and was also quoted by Kalman and Bertram (2). We should like the characteristic equation of $\mathbf{B}$

$$(-1)^n \det(\mathbf{B} - \lambda\mathbf{I}) = 0 \tag{4}$$

and the equation obtained by substituting the solution $y = e^{\lambda t}$ in (3), i.e.

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0 \tag{5}$$

to be identical.

There is a correlation between the terms in the Routh array of equation (4) and the coefficients of powers of λ in successive principal minors of $(-1)^n(\mathbf{B} - \lambda\mathbf{I})$ starting from the top left-hand corner.

THEOREM 5. *The first column of the Routh array of* $(-1)^n \det(\mathbf{B} - \lambda\mathbf{I}) = 0$ *consists of the sequence of elements*

$$1, \quad b_1, \quad b_2, \quad b_1b_3, \quad b_2b_4, \quad b_1b_3b_5, \quad b_2b_4b_6, \quad b_1b_3b_5b_7, \quad \ldots, \tag{6}$$

the final $(n+1)$*th element ending in* b_n.

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Proof. We note first that all the non-zero elements of the Routh matrix (C_{ij}) are uniquely determined by either the $n+1$ non-zero elements in its first two rows, or by the $n+1$ elements of its first column. For, given the non-zero elements of the first two rows we may complete the construction of the matrix in the usual way. On the other hand if we are given the $n+1$ elements of the first column then using the general rule of construction in the rearranged form

$$C_{i-2,j+1} = C_{ij} + \frac{C_{i-2,1}C_{i-1,j+1}}{C_{i-1,1}}$$

we may construct in turn the elements $C_{n-1,2}, C_{n-2,2}, \ldots, C_{12}, C_{n-3,3}, C_{n-4,3}, \ldots, C_{13}, C_{n-5,4}, C_{n-6,4}, \ldots, C_{14}$ and so on until the entire Routh array of non-zero elements has been reconstructed.

Let us consider a Routh matrix of which the first column is $1, b_1, b_2, b_1b_3, b_2b_4, b_1b_3b_5, \ldots$ as defined above. Let the general term of the matrix C_{ij} be written as $C_{i1}C'_{ij}$, where $C'_{i1} = 1$ $(i = 1, 2, \ldots, n+1)$, and the remaining non-zero C'_{ij} are unknowns. From the general rule of construction for the matrix (Theorem 3)

$$C'_{ij}C_{i1} = C_{i-2,1}C'_{i-2,j+1} - \frac{C_{i-2,1}C_{i-1,1}C'_{i-1,j+1}}{C_{i-1,1}},$$

or

$$C'_{ij}b_{i-1} = C'_{i-2,j+1} - C'_{i-1,j+1}$$

as

$$\frac{C_{i,1}}{C_{i-2,1}} = b_{i-1}.$$

Now let us consider the polynomials in λ obtained by expanding successive principal minors M_r of $\det(\lambda\mathbf{I} - \mathbf{B})$. The first four polynomials are

$$\begin{aligned} M_1 &= \lambda, \\ M_2 &= \lambda^2 + b_n, \\ M_3 &= \lambda^3 + \lambda(b_n + b_{n-1}), \\ M_4 &= \lambda^4 + \lambda^2(b_n + b_{n-1} + b_{n-2}) + b_n b_{n-2} \end{aligned}$$

and the following general recurrence relationship exists

$$\begin{aligned} M_r &= \lambda M_{r-1} + b_{n-r+2}M_{r-2} \quad (r = 3, 4, \ldots, n-1), \\ M_n &= (\lambda + b_1)M_{n-1} + b_2 M_{n-2}. \end{aligned}$$

Now the coefficients of powers of λ in $M_1, M_2, M_3, \ldots$ occur in a similar pattern to the non-zero elements in the $n, n-1, n-2, \ldots$th rows of the Routh matrix. Let us label these coefficients in the same way, i.e. d_{ij}, where d_{i1} $(= 1)$ is the coefficient of the highest power of λ in M_{n+1-i} $(i = 2, \ldots, n)$. From the recurrence relationship between M_r, M_{r-1} and M_{r-2} above we may deduce that in general

$$d_{ij} = d_{i+1,j} + b_{i+1}d_{i+2,j-1}$$

which, replacing i by $i-2$ and j by $j+1$, is

$$d_{ij}b_{i-1} = d_{i-2,j+1} - d_{i-1,j+1}.$$

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Thus the d_{ij} and C'_{ij} obey the same recurrence relationship. Moreover

$$C'_{i1} = d_{i1} = 1 \quad (i = 2, \ldots, n) \quad \text{and} \quad C'_{n-1,2} = d_{n-1,2} = b_n.$$

Hence, $C'_{ij} = d_{ij}$ wherever d_{ij} is defined and it follows that the non-zero elements of the second row of the Routh matrix consists of the coefficients of powers of λ in M_{n-1} multiplied by b_1. Extending the general process one more step the non-zero elements of the first row of the Routh matrix consist of the coefficients of powers of λ in $\lambda M_{n-1} + b_2 M_{n-2}$. These are precisely the odd and even coefficients respectively of

$$(-1)^n \det(\mathbf{B} - \lambda \mathbf{I}) = (\lambda M_{n-1} + b_2 M_{n-2} + b_1 M_{n-1}).$$

We have shown therefore that if the first column of a Routh matrix consists of the elements 1, b_1, b_2, $b_1 b_3$, $b_2 b_4$, $b_1 b_3 b_5$, ..., then the non-zero elements of the first two rows of this matrix are identical with the first two rows of the Routh matrix of $(-1)^n \det(\mathbf{B} - \lambda \mathbf{I}) = 0$. By the uniqueness of construction of the matrix it follows, conversely, that the first column of the Routh matrix of $(-1)^n \det(\mathbf{B} - \lambda \mathbf{I}) = 0$ is

$$1, \quad b_1, \quad b_2, \quad b_1 b_3, \quad b_2 b_4, \quad b_1 b_3 b_5, \quad \ldots.$$

The Routh matrix and the expansion of $\det(\lambda \mathbf{I} - \mathbf{B})$ are shown in full for the special case $n = 5$ in Appendix 1.

We now consider the question of representing equation (5) by $(-1)^n \det(\mathbf{B} - \lambda \mathbf{I}) = 0$ by a suitable choice of the b_i.

THEOREM 6. *The equations* $(-1)^n \det(\mathbf{B} - \lambda \mathbf{I}) = 0$ *and*

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n = 0 \tag{5}$$

can be made identical by choosing

$$b_1 = \Delta_1, \quad b_2 = \frac{\Delta_2}{\Delta_1}, \quad b_3 = \frac{\Delta_3}{\Delta_1 \Delta_2}, \quad b_r = \frac{\Delta_{r-3} \Delta_r}{\Delta_{r-2} \Delta_{r-1}} \quad (r = 4, 5, \ldots, n), \tag{7}$$

where $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n$ *are the Hurwitz determinants of* (5), *provided*

$$\Delta_i \neq 0 \quad (i = 1, 2, \ldots, n-1).$$

Proof. If we choose

$$b_1 = \Delta_1, \quad b_2 = \frac{\Delta_2}{\Delta_1}, \quad b_3 = \frac{\Delta_3}{\Delta_1 \Delta_2}, \quad \ldots, \quad b_r = \frac{\Delta_{r-3} \Delta_r}{\Delta_{r-2} \Delta_{r-1}} \quad (r = 4, 5, \ldots, n)$$

then from Theorem 5 the first column of the Routh matrix of $(-1)^n \det(\mathbf{B} - \lambda \mathbf{I}) = 0$ is

$$1, \quad \Delta_1, \quad \frac{\Delta_2}{\Delta_1}, \quad \frac{\Delta_3}{\Delta_2}, \quad \frac{\Delta_4}{\Delta_3}, \quad \ldots, \quad \frac{\Delta_n}{\Delta_{n-1}}.$$

However, from Theorem 4 this is the first column of the Routh matrix of equation (5). From the uniqueness of construction of the two Routh matrices it follows that the first two rows of each are identical and hence that $(-1)^n \det(\mathbf{B} - \lambda \mathbf{I}) = 0$ and equation (5) are identical.

We have thus constructed a matrix $\mathbf{B}$ such that $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ and equation (3) represent the same dynamical system. The elements $b_1, b_2, b_3, \ldots$ of $\mathbf{B}$ are given in terms of the Hurwitz determinants of (3) by the sequence (7).

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We now apply the Liapunov function $V = \mathbf{x}'\mathbf{P}\mathbf{x}$ with

$$\mathbf{P} = \begin{bmatrix} b_1 b_2 \dots b_n & 0 & \dots & 0 & 0 \\ 0 & b_1 b_2 \dots b_{n-1} & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & b_1 b_2 & 0 \\ 0 & 0 & \dots & 0 & b_1 \end{bmatrix}$$

(Kalman and Bertram (2)), to the equation $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$. $\dot{V} = \mathbf{x}'(\mathbf{P}\mathbf{B} + \mathbf{B}'\mathbf{P})\mathbf{x} = -2b_1^2 x_n^2$ is negative semi-definite and by examining the set of equations $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ we find $\dot{V}$ cannot be identically zero unless $x_i = 0$ $(i = 1, 2, \dots, n)$. Hence from Theorem 1A a necessary and sufficient condition for asymptotic stability is that $\mathbf{x}'\mathbf{P}\mathbf{x}$ be positive definite, that is all the terms in the main diagonal of $\mathbf{P}$ must be positive. This leads to the necessary and sufficient conditions

$$b_1 > 0, \quad b_2 > 0, \quad b_3 > 0, \quad \dots, \quad b_n > 0,$$

or

$$\Delta_1 > 0, \quad \frac{\Delta_2}{\Delta_1} > 0, \quad \frac{\Delta_3}{\Delta_1 \Delta_2} > 0, \quad \dots, \quad \frac{\Delta_{n-3}\Delta_n}{\Delta_{n-2}\Delta_{n-1}} > 0,$$

or $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$, …, $\Delta_n > 0$, which establishes the Hurwitz criterion (Theorem 2), provided no Δ_i is zero $(i = 1, 2, \dots, n-1)$.

3. *Concluding note.* The matrix $\mathbf{B}$, besides providing a new proof of the Hurwitz stability criterion and a direct Liapunov–Routh–Hurwitz link, enables us to express the coefficients of equation (3) in terms of the Hurwitz determinants. This is known as the 'inverse problem of stability' (Jarominek (8), Cremer and Effertz (9)). The matrix $(-1)^n(\mathbf{B} - \lambda\mathbf{I})$ may be expanded by means of principal minors of increasing size starting from the bottom right-hand corner to obtain the characteristic equation $(-1)^n \det(\mathbf{B} - \lambda\mathbf{I}) = 0$. (There is again a recurrence relation between principal minors of order r, $r-1$ and $r-2$.) The resulting coefficients of powers of λ which are obtained in terms of the b_i $(i = 1, 2, 3, \dots, n)$ may be expressed in terms of the Δ_i $(i = 1, 2, 3, \dots, n)$ using the sequence (7). Appendix 2 gives a table to find such expressions for $n = 1, 2, 3, \dots, 8$. This table may be extended for higher values of n without difficulty.

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APPENDIX 1

The correlation between the Routh array of $(-1)^n \det(\mathbf{B} - \lambda\mathbf{I}) = 0$ *and the expansion of the determinant itself*

This correlation is demonstrated for the particular case $n = 5$. The Routh matrix is

$$\begin{bmatrix} 1 & b_2 + b_3 + b_4 + b_5 & b_2(b_4 + b_5) + b_3 b_5 & 0 & 0 & \dots \\ b_1 & b_1(b_3 + b_4 + b_5) & b_1 b_3 b_5 & 0 & 0 & \dots \\ b_2 & b_2(b_4 + b_5) & 0 & 0 & 0 & \dots \\ b_1 b_3 & b_1 b_3 b_5 & 0 & 0 & 0 & \dots \\ b_2 b_4 & 0 & 0 & 0 & 0 & \dots \\ b_1 b_3 b_5 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}.$$

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The expansion of

$$\begin{vmatrix} \lambda & -1 & 0 & 0 & 0 \\ b_5 & \lambda & -1 & 0 & 0 \\ 0 & b_4 & \lambda & -1 & 0 \\ 0 & 0 & b_3 & \lambda & -1 \\ 0 & 0 & 0 & b_2 & \lambda+b_1 \end{vmatrix}$$

starting from the top left-hand corner leads to the set of polynomials

$$M_1 = \lambda \equiv d_{51}\lambda,$$

$$M_2 = \lambda^2 + b_5 \equiv d_{41}\lambda^2 + d_{42},$$

$$M_3 = \lambda^3 + \lambda(b_4+b_5) \equiv d_{31}\lambda^3 + d_{32}\lambda,$$

$$M_4 = \lambda^4 + \lambda^2(b_3+b_4+b_5) + b_3 b_5 \equiv d_{21}\lambda^4 + d_{22}\lambda^2 + d_{23},$$

$$M_5 = \lambda^5 + \lambda^3(b_2+b_3+b_4+b_5) + \lambda[b_2(b_4+b_5)+b_3b_5] + b_1\{\lambda^4 + \lambda^2(b_3+b_4+b_5) + b_3b_5\}.$$

APPENDIX 2

A table by means of which coefficients of the equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = 0$$

may be found in terms of its Hurwitz determinants

$n \backslash r$	1	2	3	4	5	6	7	8
1	b_1							
2	—	b_2						
3	—	b_3	$b_1 b_3$					
4	—	b_4	$b_1 b_4$	$b_2 b_4$				
5	—	b_5	$b_1 b_5$	$b_5 \sum_{i=2}^{3} b_i$	$b_1 b_3 b_5$			
6	—	b_6	$b_1 b_6$	$b_6 \sum_{i=2}^{4} b_i$	$b_1 b_6 \sum_{i=3}^{4} b_i$	$b_2 b_4 b_6$		
7	—	b_7	$b_1 b_7$	$b_7 \sum_{i=2}^{5} b_i$	$b_1 b_7 \sum_{i=3}^{5} b_i$	$b_7\left(b_2 b_4 + b_5 \sum_{i=2}^{3} b_i\right)$	$b_1 b_3 b_5 b_7$	
8	—	b_8	$b_1 b_8$	$b_8 \sum_{i=2}^{6} b_i$	$b_1 b_8 \sum_{i=3}^{6} b_i$	$b_8\left(b_2 b_4 + b_5 \sum_{i=2}^{3} b_i + b_6 \sum_{i=2}^{4} b_i\right)$	$b_8\left(b_1 b_3 b_5 + b_1 b_6 \sum_{i=3}^{4} b_i\right)$	$b_2 b_4 b_6 b_8$

Instructions. For given value of n the coefficient a_r is found by adding the elements in the rth column of the array in the table as far as and including the element in the nth row; on substituting

$$b_1 = \Delta_1, \quad b_2 = \frac{\Delta_2}{\Delta_1}, \quad b_3 = \frac{\Delta_3}{\Delta_1 \Delta_2}, \quad b_r = \frac{\Delta_{r-3}\,\Delta_r}{\Delta_{r-2}\Delta_{r-1}} \quad (r = 4, 5, 6, \ldots)$$

the required expression for a_r is obtained.

The table may be easily extended noting that if $r \geqslant 3$ the element in the rth column and kth row is the product of b_k and the sum of the elements in the $(r-2)$th column as far as and including the element in the $(k-2)$th row. (The first column consists of the single element b_1 and the second column has b_k as its element in the kth row for $k \geqslant 2$.) This follows from the recurrence relationship when expanding successive principal minors of $(-1)^n \det(\mathbf{B} - \lambda\mathbf{I})$ starting from the bottom right-hand corner.

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