Appendix W

The Root-Locus Design Method

W.5 Root Locus Time Delay

Time delays often arise in control systems, both from delays in the process itself and from delays in the processing of sensed signals. Chemical plants often have processes with a time delay representing the time material takes to be transported via pipes or other conveyer. In measuring the attitude of a spacecraft en route to Mars, there is a significant time delay for the sensed quantity to arrive back on Earth due to the speed of light. There is also a small time delay in any digital control system due to the cycle time of the computer and the fact that data is processed at discrete intervals. Time delay *always* reduces the stability of a system; therefore, it is important to be able to analyze its effect. In this section we discuss how to use the root locus for such analysis. Although an exact method of analyzing time delay is available in the frequency-response methods to be described in Chapter 6, knowing several different ways to analyze a design allows the control designer to do a better job.

Time delays always reduce the stability of a system.

Consider the problem of designing a control system for the temperature of the heat exchanger described in Chapter 2. The transfer function between the control A_s and the measured output temperature T_m is described by two first-order terms plus a time delay T_d of 5 sec. The time delay results because the temperature sensor is physically located downstream from the exchanger, so that there is a delay in its reading. The transfer function is

An example of a root locus with time delay

$$G(s) = \frac{e^{-5s}}{(10s + 1)(60s + 1)},$$ (W.1)

where the e^{-5s} term arises from the time delay.[1]

The corresponding root-locus equations with respect to proportional gain K are

$$
\begin{aligned}
1 + KG(s) &= 0, \\
1 + K\frac{e^{-5s}}{(10s + 1)(60s + 1)} &= 0, \\
600s^2 + 70s + 1 + Ke^{-5s} &= 0.
\end{aligned}
$$ (W.2)

How would we plot the root locus corresponding to Eq. (W.2)? Since it is not a polynomial, we cannot proceed with the methods used in previous examples. Instead, there are two basic approaches we will describe: approximation and direct application of the phase condition.

In the first approach we reduce the given problem to one we have previously solved by approximating the nonrational function e^{-5s} with a rational function. Since we are concerned with control systems and hence typically with low frequencies, we want an approximation that will be good for small s.[2] The most common means for finding such an approximation is attributed to H. Padé. It

Padé approximant

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[1] Time delay is often referred to as "transportation lag" in the process industries.

[2] The nonrational function e^{-5s} is analytic for all finite values of s and so may be approximated by a rational function. If nonanalytic functions such as $\sqrt{s}$ were involved, great caution would be needed in selecting an approximation valid near $s = 0$.

consists of matching the series expansion of the transcendental function e^{-5s} with the series expansion of a rational function whose numerator is a polynomial of degree p and whose denominator is a polynomial of degree q. The result is called a (p, q) **Padé approximant**[3] to e^{-5s}. We will initially compute the approximants to e^{-s}, and in the final result we will substitute $T_d s$ for s to allow for any desired delay.

To illustrate the process, we begin with the $(1, 1)$ approximant. In this case we wish to select b_0, b_1, and a_0 so that the error

$$e^{-s} - \frac{b_0 s + b_1}{a_0 s + 1} = \varepsilon \tag{W.3}$$

is small. For the Padé approximant, we expand both e^{-s} and the rational function given by Eq. (W.3) into a McLauren series and match as many of the initial terms as possible. The two series are

$$e^{-s} = 1 - s + \frac{s^2}{2} - \frac{s^3}{3!} + \frac{s^4}{4!} - \cdots,$$
$$\frac{b_0 s + b_1}{a_0 s + 1} = b_1 + (b_0 - a_0 b_1)s - a_0(b_0 - a_0 b_1)s^2 + a_0^2(b_0 - a_0 b_1)s^3 + \cdots.$$

Matching coefficients of the first four terms, we get the following equations to solve:

$$\begin{aligned}
b_1 &= 1, \\
b_0 - a_0 b_1 &= -1, \\
-a_0(b_0 - a_0 b_1) &= \frac{1}{2}, \\
a_0^2(b_0 - a_0 b_1) &= -\frac{1}{6}.
\end{aligned}$$

There are an infinite number of equations but only three parameters. We match the first three coefficients and then substitute $T_d s$ for s. The resulting $(1, 1)$ Padé approximant is

$$e^{-T_d s} \cong \frac{1 - (T_d s/2)}{1 + (T_d s/2)}. \tag{W.4}$$

If we assume $p = q = 2$, we have five parameters, and a better match is possible. In this case we have the $(2, 2)$ approximant, which has the transfer function

$$e^{-T_d s} \cong \frac{1 - T_d s/2 + (T_d s)^2/12}{1 + T_d s/2 + (T_d s)^2/12}. \tag{W.5}$$

The comparison of these approximants is best seen from their pole-zero configurations as plotted in Fig. W.1. The locations of the poles are in the LHP and the zeros are in the RHP at the reflections of the poles.

In some cases a very crude approximation is acceptable. For small delays the $(0, 1)$ approximant can be used, which is simply a first-order lag given by

$$e^{-T_d s} \cong \frac{1}{1 + T_d s}. \tag{W.6}$$

Contrasting methods of approximating delay

To illustrate the effect of a delay and the accuracy of the different approximations, root loci for the heat exchanger are drawn in Fig. W.2 for four cases. Notice that, for low gains and up to the point where the loci cross the imaginary axis, the approximate curves are very close to the exact. However, the $(2, 2)$ Padé curve follows the exact curve much further than does the first-order lag, and its increased accuracy would be useful if the delay were larger. All analyses of the delay show its destabilizing effect and how it limits the achievable response time of the system.

While the Padé approximation leads to a rational transfer function, in theory it is not necessary for plotting a root locus. A direct application of the phase condition can be used to plot portions of

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[3]The (p, p) Padé approximant for a delay of T seconds is most commonly used and is computed by the MATLAB command `[num,den]=pade(T, p)`.
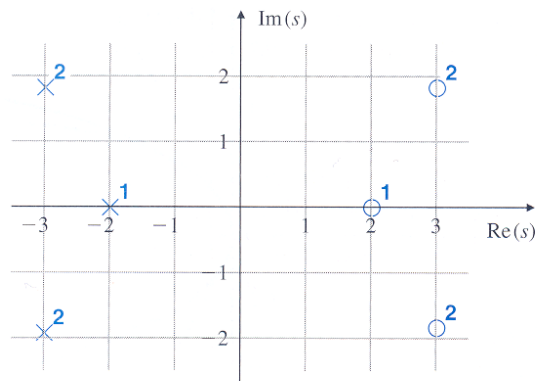
Figure W.1: Poles and zeros of the Padé approximants to e^{-s}, with superscripts identifying the corresponding approximants; for example, x^1 represents the $(1, 1)$ approximant
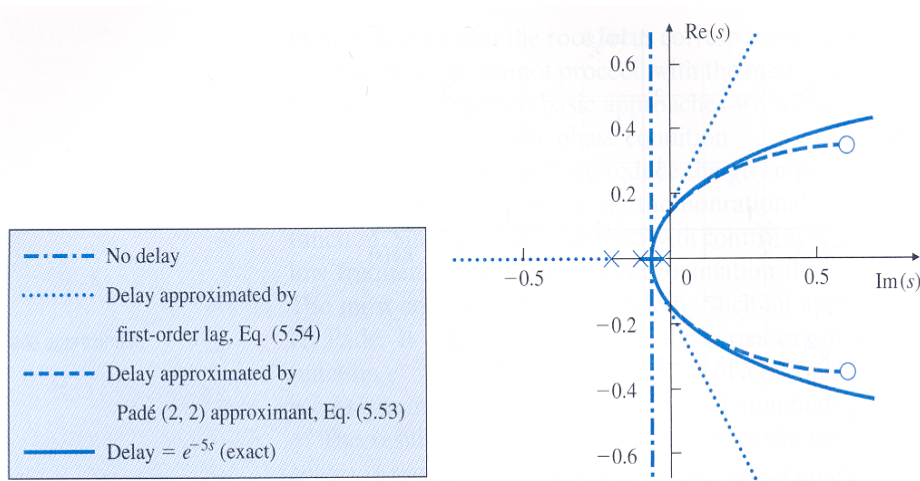

Figure W.2: Root loci for the heat exchanger with and without time delay

an exact locus of a system with time delay. The phase-angle condition does not change if the transfer function of the process is nonrational, so we still must search for values of s for which the phase is $180° + 360°l$. If we write the transfer function as

$$G(s) = e^{-T_d s}\bar{G}(s),$$

the phase of $G(s)$ is the phase of $\bar{G}(s)$ minus $\lambda\omega$ for $s = \sigma + j\omega$. Thus we can formulate a root-locus problem as searching for locations where the phase of $\bar{G}(s)$ is $180° + T_d\omega + 360°(l-1)$. To plot such a locus, we would fix ω and search along a horizontal line in the s-plane until we found a point on the locus, then raise the value of ω, change the target angle, and repeat. Similarly, the departure angles are modified by $T_d\omega$, where ω is the imaginary part of the pole from which the departure is being computed. Unfortunately, MATLAB does not provide a program to plot the root locus of systems with delay, and we must be satisfied here with Padé approximants. Since it is possible to plot the frequency response (or Bode plot) of delay exactly and easily, if the designer feels that the Padé approximant is not satisfactory, the expedient approach is to use the frequency-response design methods described in Chapter 6.