UNIVERSITÉ DE BORDEAUX **L2 parcours international, 2021/2022**
Probability and Statistics

REINFORCED RANDOM WALK

We consider two distinct paths denoted α etβ connecting the same starting and arrival points. Successive individuals are travelling along these paths. Let us imagine for instance that these two paths are used by ants: it is well-known that ants leave behind them pheromons that tend to attract other ants. Therefore, the most used path will be more likely chosen by the following ants, which will tend to amplify the phenomena... Similarly, if we consider hikers that have to chose between two paths, they will more often choose the path that contains the less grass and vegetation, but it also means that they will contribute to the rarefaction of the vegetation on this path. This phenomenon is called a reinforcement phenomenon.

For all $n \geqslant 1$ we denote X_n the random variable taking values in $\{\alpha, \beta\}$ that indicates the path made by the n-th individual. We denote A_n and B_n the strictly positive random variables indicating the attractivity of paths α and β during the $n+1$-ith passage. We take $A_0 = B_0 = 1$, a function $r : \mathbb{R}^{+*} \to \mathbb{R}^{+*}$ called reinforcement function and $(U_n)_{n\geqslant 1}$ a sequence of independent and identically distributed random variables with uniform law on $[0, 1]$. We model $(X_n)_{n\geqslant 1}$ by

$$(X_{n+1}, A_{n+1}, B_{n+1}) = \begin{cases} (\alpha, r(A_n), B_n) & \text{si } U_{n+1} \leqslant \frac{A_n}{A_n+B_n} \\ (\beta, A_n, r(B_n)) & \text{si } U_{n+1} > \frac{A_n}{A_n+B_n} \end{cases}.$$

We are interested in the number of passages through the path α and through the path β at time instant n, given by

$$Y_n = \sum_{k=1}^{n} \mathbb{1}_{X_k=\alpha} \quad \text{and} \quad Z_n = \sum_{k=1}^{n} \mathbb{1}_{X_k=\beta} = n - Y_n.$$

We have the following asymptotic behaviours for X_n and Y_n.

Théorème 0.1

- **No-reinforcement.** *If $r(x) = x$ for all $x > 0$, then*

$$\mathbb{P}(\lim_{n\to+\infty} \frac{Y_n}{n} = \frac{1}{2}) = 1$$

  and

$$\mathbb{P}(\lim_{n\to+\infty} \frac{Z_n}{n} = \frac{1}{2}) = 1$$

- **Linear reinforcement.** *If $r(x) = x + 1$ for all $x > 0$, then for all $\omega \in \Omega$,*

$$\lim_{n\to\infty} \frac{Y_n(\omega)}{n}$$

  exists but depends on ω (thus is a random value). Moreover

$$\frac{Y_n}{n} \xrightarrow{\mathcal{L}} \mathcal{U}([0,1])$$

which means that for n large enough, the random variable $\frac{Y_n}{n}$ behaves like a random variable with uniform law on $[0,1]$.

- **Geometrical reinforcement.** *If $r(x) = \rho x$ for all $x > 0$ where $\rho > 1$ is a constant, then the random sequence $(X_n)_{n \geqslant 1}$ is constant from a (random) range, and*

$$\mathbb{P}(\lim_{n \to +\infty} X_n = \alpha) = \mathbb{P}(\lim_{n \to +\infty} X_n = \beta) = 1/2.$$

Write a Scilab program able to simulate the trajectories of $(X_n)_{n \geqslant 1}$, $(Y_n)_{n \geqslant 1}$ and to illustrate the convergence properties in the different situations described above.