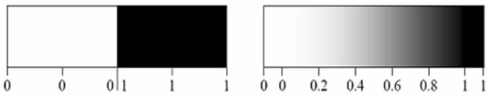
Unit 5

FUZZY LOGIC

- Fuzzy logic is **the logic** underlying **approximate**, rather than exact, **modes of reasoning**.
- It is an extension of multivalued logic: **Everything**, including truth, **is a matter of degree**.
- It contains as special cases **not only** the classical two-value logic and multivalue logic systems, **but also** probabilistic logic.
- A proposition ***p*** has a **truth value**
  - 0 or 1 in two-value system,
  - element of a set T in multivalue system,
  - **Range over the fuzzy subsets of T** in fuzzy logic.

- Boolean logic uses sharp distinctions.
- Fuzzy logic reflects how people think.



(*a*) Boolean Logic. (*b*) Multi-valued Logic.

- Fuzzy logic is a set of mathematical principles for knowledge representation and reasoning based on degrees of membership.

TYPES AND MODELING OF UNCERTAINTY

Stochastic Uncertainty:

- **The probability of hitting the target is 0.8**

Lexical Uncertainty:

- **"Tall Men", "Hot Days", or "Stable Currencies"**
- **We will probably have a successful business year.**
- **The experience of expert A shows that B is Likely to Occur. However, expert C is convinced This Is Not True.**

Example: One finds in desert two bottles of fluids with the following labels:

- ✓ **bottle 1: there is a probability of 5% that this bottle is poisoned.**
- ✓ **bottle 2: this bottle contains a liquid which belongs to the set of drinkable water with membership function value of 0.95.**

FUZZY vs PROBABILITY

- Fuzzy ≠ Probability
- Probability deals with uncertainty an likelihood
- Fuzzy logic deals with ambiguity an vagueness

- Let L=set of all liquids
  - £ be the subset ={all drinkable liquids}
- Suppose you had been in desert (you must drink!) and you come up with two bottles marked C and A.
- **Bottle C is labeled $\mu_£(C)$=0.95 and bottle A is labeled Pr[A∈ £]=0.95**

• C could contain swamp water, but would not contain any poison. Membership of 0.95 means that the contents of C are fairly similar to perfectly drinkable water.

• The probability that A is drinkable is 0.95, means that over a long run of experiments, the context of A are expected to be drinkable in about 95% of the trials. In other cases it may contain poison.

NEED OF FUZZY LOGIC

- Based on intuition and judgment.
- No need for a mathematical model.
- Provides a smooth transition between members and nonmembers.
- Relatively simple, fast and adaptive.
- Less sensitive to system fluctuations.
- Can implement design objectives, difficult to express mathematically, in linguistic or descriptive rules.

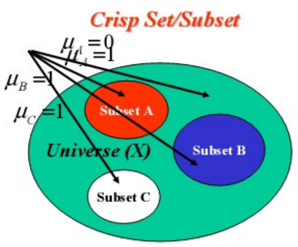
CLASSICAL SETS (CRISP SETS)

Conventional or crisp sets are Binary. An element either belongs to the set or does not.

{True, False}

{1, 0}

CRISP SETS



OPERATIONS ON CRISP SETS

- UNION: $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- INTERSECTION: $A \cap B = \{x | x \in A \text{ and } x \in B\}$
- COMPLEMENT: $\bar{A} = \{x | x \notin A, x \in X\}$
- DIFFERENCE: $A|B \text{ or } (A - B) = \{x | x \in A \text{ and } x \notin B\}$
$$= A - (A \cap B)$$

PROPERTIES OF CRISP SETS

The various properties of crisp sets are as follows:

1. Commutativity
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
2. Associativity
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$
3. Distributivity
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
4. Idempotency
$$A \cup A = A$$
$$A \cap A = A$$
5. Transitivity
$$\text{If } A \subseteq B \subseteq C, \text{ then } A \subseteq C$$

6. Identity
$$A \cup \phi = A, \quad A \cap \phi = \phi$$
$$A \cap X = A, \quad A \cap X = X$$
7. Involution (double negation)
$$\bar{\bar{A}} = A$$
8. Law of excluded middle
$$A \cup \bar{A} = X$$
9. Law of contradiction
$$A \cap \bar{A} = \phi$$
10. DeMorgan's law
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

FUZZY SETS

Rules of thumb frequently stated in "fuzzy" linguistic terms.

John is *tall*.

If someone is *tall* **and** *well-built*

then his basketball skill is good.

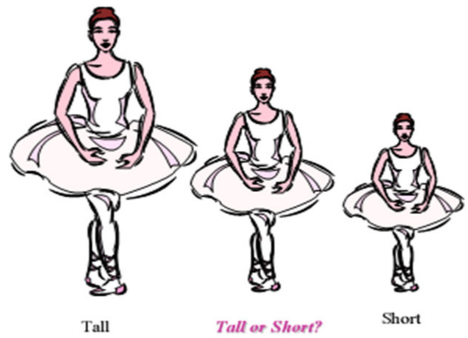
Fuzzy Sets

$0 \le \mu S(x) \le 1$ ------ $\mu S(x)$ (or $\mu(S, x)$) is the **degree** of membership of x in set S

$\mu S(x) = 0$ x is not at all in S

$\mu S(x) = 1$ x is fully in S.

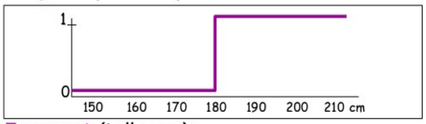
If $\mu S(x) = 0$ or 1, then the set S is **crisp.**



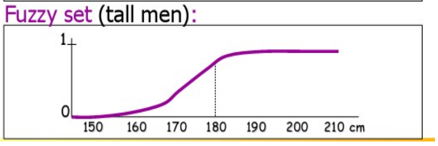
Fuzzy set

Is a function f: domain → [0,1]

Crisp set (tall men):

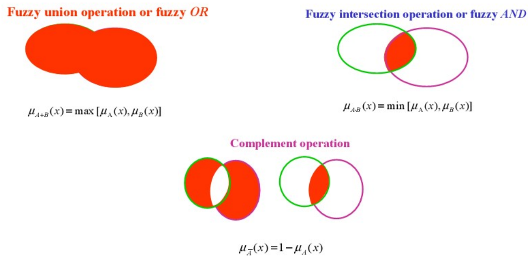


Fuzzy set (tall men):



OPERATIONS ON FUZZY SETS

- Union: $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$
- Intersection: $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$
- Complement: $\mu_{\neg A}(x) = 1 - \mu_A(x)$



PROPERTIES OF FUZZY SETS

The same as for crisp sets

- Commutativity
- Associativity
- Distributivity
- Idempotency
- Identity
- De Morgan's Laws
- …

1. Commutativity

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}$$
$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$$

2. Associativity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C}$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}$$

3. Distributivity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C})$$

4. Idempotency

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A}$$
$$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

5. Identity

$$\underset{\sim}{A} \cup \phi = \underset{\sim}{A} \text{ and } \underset{\sim}{A} \cup U = U \text{(universal set)}$$
$$\underset{\sim}{A} \cap \phi = \phi \text{ and } \underset{\sim}{A} \cap U = \underset{\sim}{A}$$

6. Involution (double negation)

$$\overline{\overline{\underset{\sim}{A}}} = \underset{\sim}{A}$$

7. Transitivity

$$\text{If } \underset{\sim}{A} \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{C}, \text{ then } \underset{\sim}{A} \subseteq \underset{\sim}{C}$$

8. Demorgan's law

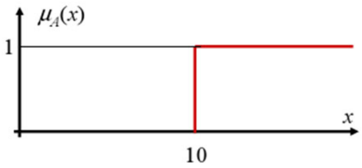
$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$$
$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$$

MEMBERSHIP FUNCTIONS

CRISP MEMBERSHIP FUCNTIONS

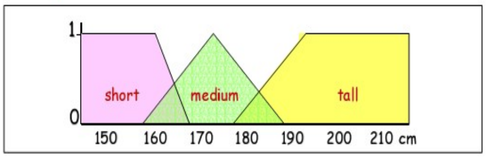
- Crisp membership functions (μ) are either one or zero.
- Consider the example: Numbers greater than 10. The membership curve for the set A is given by

$A = \{x \mid x > 10\}$



REPRESENTING A DOMAIN IN FUZZY LOGIC

Fuzzy sets (men's height):



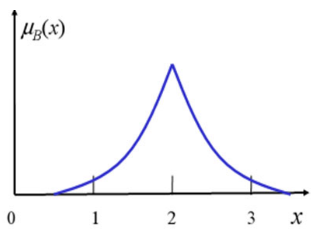
FUZZY MEMBERSHIP FUCNTIONS

- Categorization of element x into a set A
  described through a membership function $\mu_A(x)$
- Formally, given a fuzzy set A of universe X
  $\mu_A(x)$: X → [0,1], where
  - $\mu_A(x) = 1$ if x is totally in A — $\mu_{Tall}(200) = 1$
  - $\mu_A(x) = 0$ if x is totally not in A — $\mu_{Tall}(160) = 0$
  - $0 < \mu_A(x) < 1$ if x is partially in A — $0 < \mu_{Tall}(180) < 1$
- (Discrete) Fuzzy set A is represented as:
  A = $\{\mu_A(x_1)/x_1, \mu_A(x_2)/x_2, \dots, \mu_A(x_n)/x_n\}$

Tall = {0/160, 0.2/170, 0.8/180, 1/190}

The set B of numbers approaching 2 can be represented by the membership function

$$\mu_B(x) = e^{-|x-2|}$$



FUZZINESS vs PROBABILITY

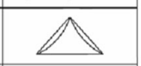
- When first exposed to fuzzy logic, humans associate membership functions with density functions.
- This is not so, since:
  - Probability density is an abstraction from empirical frequency.
    - ⇒ an aggregate property.
      - how often events occur in different ways.
      - ways that are quite crisp and mutually exclusive after occurrence.
  - Fuzzy relations, by contrast, are properties of single events that are always there, and not different from occurrence to occurrence.

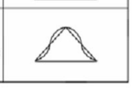
LINGUISTIC VARIABLES

- A linguistic variable is a fuzzy variable.
  - The linguistic variable speed ranges between 0 and 300 km/h and includes the fuzzy sets slow, very slow, fast, …
  - Fuzzy sets define the linguistic values.
- Hedges are qualifiers of a linguistic variable.
  - All purpose: very, quite, extremely
  - Probability: likely, unlikely
  - Quantifiers: most, several, few
  - Possibilities: almost impossible, quite possible

LINGUISTIC HEDGES (LINGUISTIC QUANTIFIERS)

- Hedges modify the shape of a fuzzy set.

Hedge	Mathematical Expression	Graphical Representation
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^2$	
Extremely	$[\mu_A(x)]^3$	

Hedge	Mathematical Expression	Graphical Representation
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2[\mu_A(x)]^2$ if $0 \le \mu_A \le 0.5$; $1-2[1-\mu_A(x)]^2$ if $0.5 < \mu_A \le 1$	

TRUTH TABLES

Truth tables define logic functions of two propositions. Let X and Y be two propositions, either of which can be true or false.

The operations over the propositions are:

1. Conjunction (∧): X AND Y.
2. Disjunction (∨): X OR Y.
3. Implication or conditional (⇒): IF X THEN Y.
4. Bidirectional or equivalence (⇔): X IF AND ONLY IF Y.

LINGUISTIC VARIABLE

- **A linguistic variable associates words or sentences with a measure of belief functions, also called membership function.**
- **The set of values that it can take is called term set.**
- **Each value in the set is a fuzzy variable defined over a base variable.**
- **The base variable defines the Universe of discourse for all the fuzzy variables in the term set.**

A linguistic variable is a quintuple [*X*, *T*(*X*), *U*, *G*, *M*] where

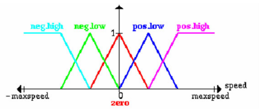
- *X* is the name of the variable,
- *T*(*X*) is the term set, i.e. the set of names of linguistic values of *X*,
- *U* is the universe of discourse,
- *G* is the grammar to generate the names and
- *M* is a set of semantic rules for associating each *X* with its meaning.

LINGUISTIC VARIABLE

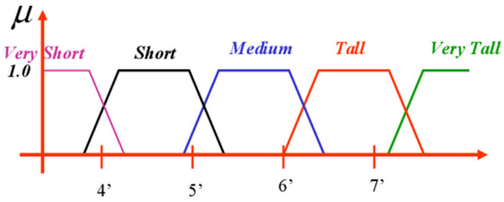
- Let x be a linguistic variable with the label "speed".
- Terms of x, which are fuzzy sets, could be "positive low", "negative high" from the term set T:

T = {PostiveHigh, PositiveLow, NegativeLow, NegativeHigh, Zero}

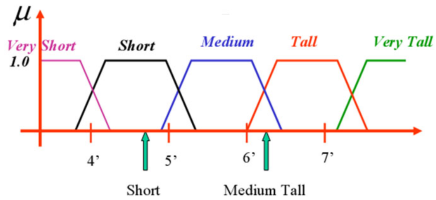
- Each term is a fuzzy variable defined on the base variable which might be the scale of all relevant velocities.



MEMBERSHIP FUCNTIONS



$$\mu = [\mu_{vs}, \mu_s, \mu_m, \mu_t, \mu_{vt}]$$

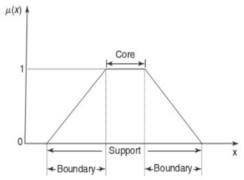


$$\mu = [0, 1, 0, 0, 0]$$

$$\mu = [0, 0, 0.5, 0.5, 0]$$

FEATURES OF MEMBERSHIP FUNCTIONS

- CORE: $\mu_{\underset{\sim}{A}}(x) = 1$
- SUPPORT: $\mu_{\underset{\sim}{A}}(x) > 0$
- BOUNDARY: $0 < \mu_{\underset{\sim}{A}}(x) < 1$



FUZZY RULES

A fuzzy rule is defined as the conditional statement of the form

If x is A
THEN y is B

where x and y are linguistic variables and A and B are linguistic values determined by fuzzy sets on the universes of discourse X and Y.

- The decision-making process is based on rules with sentence conjunctives **AND, OR** and **ALSO**.
- Each rule corresponds to a fuzzy relation.
- Rules belong to a **rule base.**
- Example: If (Distance x to second car is **SMALL) OR** (Distance y to obstacle is **CLOSE**) **AND** (speed v is **HIGH**) **THEN** (perform **LARGE** correction to steering angle θ) **ALSO** (make **MEDIUM** reduction in speed v).
- Three antecedents (or premises) in this example give rise to two outputs (consequences).

FUZZY RULE FORMATION

IF height is tall
THEN weight is heavy.

Here the fuzzy classes height and weight have a given range (i.e., the universe of discourse).

range (height) = [140, 220]
range (weight) = [50, 250]

FORMATION OF FUZZY RULES

Three general forms are adopted for forming fuzzy rules. They are:

- Assignment statements,
- Conditional statements,
- Unconditional statements.

Assignment Statements

y = small
Orange color = orange
$a = s$
Paul is not tall and not very short
Climate = autumn
Outside temperature = normal

Unconditional Statements

Goto sum.
Stop.
Divide by a.
Turn the pressure low.

Conditional Statements

IF y is very cool THEN stop.
IF A is high THEN B is low ELSE B is not low.
IF temperature is high THEN climate is hot.

DECOMPOSITION OF FUZZY RULES

A compound rule is a collection of several simple rules combined together.

- Multiple conjunctive antecedent,
- Multiple disjunctive antecedent,
- Conditional statements (with ELSE and UNLESS).

DECOMPOSITION OF FUZZY RULES

Multiple Conjunctive

IF x is $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ THEN y is B_m.
Assume a new fuzzy subset $\underset{\sim}{A}_m$ defined as

$$\underset{\sim}{A}_m = \underset{\sim}{A}_1 \cap \underset{\sim}{A}_2 \cap \cdots \cap \underset{\sim}{A}_n$$

and expressed by means of membership function

$$\mu_{\underset{\sim}{A}_m}(x) = \min[\mu_{\underset{\sim}{A}_1}(x), \mu_{\underset{\sim}{A}_2}(x), \ldots \mu_{\underset{\sim}{A}_n}(x)].$$

Multiple disjunctive antecedent

IF x is $\underset{\sim}{A}_1$ OR x is $\underset{\sim}{A}_2, \ldots$ OR x is $\underset{\sim}{A}_n$ THEN y is $\underset{\sim}{B}_m$.
This can be written as
IF x is $\underset{\sim}{A}_n$ THEN y is $\underset{\sim}{B}_m$,
where the fuzzy set $\underset{\sim}{A}_m$ is defined as

$$\underset{\sim}{A}_m = \underset{\sim}{A}_1 \cup \underset{\sim}{A}_2 \cup \underset{\sim}{A}_3 \cup \cdots \cup \underset{\sim}{A}_n$$

Conditional Statements (With Else and Unless)

IF $\underset{\sim}{A}_1$ (THEN $\underset{\sim}{B}_1$) UNLESS $\underset{\sim}{A}_2$
can be decomposed as
IF $\underset{\sim}{A}_1$ THEN $\underset{\sim}{B}_1$
OR
IF $\underset{\sim}{A}_2$ THEN NOT $\underset{\sim}{B}_1$
IF $\underset{\sim}{A}_1$ THEN ($\underset{\sim}{B}_1$) ELSE IF $\underset{\sim}{A}_2$ THEN ($\underset{\sim}{B}_2$)
can be decomposed into the form
IF $\underset{\sim}{A}_1$ THEN $\underset{\sim}{B}_1$
OR
IF NOT $\underset{\sim}{A}_1$ AND IF $\underset{\sim}{A}_2$ THEN $\underset{\sim}{B}_2$

AGGREGATION OF FUZZY RULES

Aggregation of rules is the process of obtaining the overall consequents from the individual consequents provided by each rule.

- Conjunctive system of rules.
- Disjunctive system of rules.

AGGREGATION OF FUZZY RULES

Conjunctive system of rules

Conjunctive system of rules: For a system of rules to be jointly satisfied, the rules are connected by "and" connectives. Here, the aggregated output, y, is determined by the fuzzy intersection of all individual rule consequents, y_i, where $i = 1$ to n, as

$$y = y_1 \text{ and } y_2 \text{ and } \ldots \text{ and } y_n$$

or

$$y = y_1 \cap y_2 \cap y_3 \cap \cdots \cap y_n.$$

This aggregated output can be defined by the membership function

$$\mu_y(y) = \min[\mu_{y_1}(y), \ \mu_{y_2}(y), \ \ldots, \mu_{y_n}(y)] \text{ for } y \in Y.$$

Disjunctive system of rules

Disjunctive system of rules: In this case, the satisfaction of at least one rule is required. The rules are connected by "or" connectives. Here, the fuzzy union of all individual

rule contributions determines the aggregated output, as

$$y = y_1 \text{ or } y_2 \text{ or } \ldots \text{ or } y_n$$

or

$$y = y_1 \cup y_2 \cup y_3 \cup \cdots \cup y_n.$$

Again it can be defined by the membership function

$$\mu_y(y) = \max[\mu_{y_1}(y),\ \mu_{y_2}(y),\ \ldots \mu_{y_n}(y)] \text{ for } y \in Y.$$

FUZZY RULE - EXAMPLE

Rule 1: If height is short then weight is light.

Rule 2: If height is medium then weight is medium.

Rule 3: If height is tall then weight is heavy.

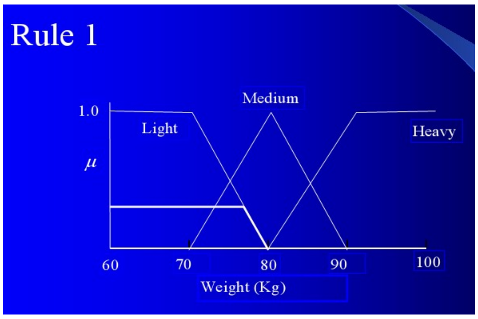
Problem: Given

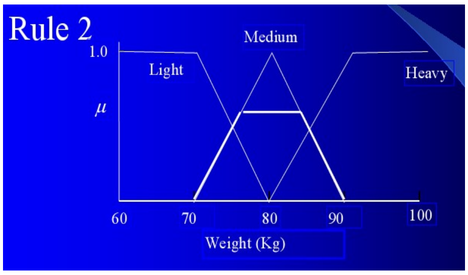
(a) membership functions for short, medium-height, tall, light, medium-weight and heavy;
(b) The three fuzzy rules;
(c) the fact that John's height is 6'1"

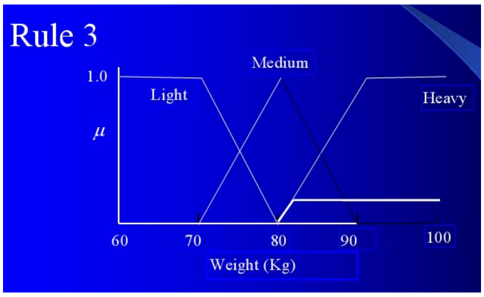
estimate John's weight.

Solution:
(1) From John's height we know that
John is short (degree 0.3)
John is of medium height (degree 0.6).
John is tall (degree 0.2).

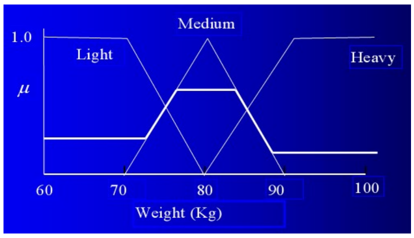
(2) Each rule produces a fuzzy set as output by truncating the consequent membership function at the value of the antecedent membership.







- The cumulative fuzzy output is obtained by OR-ing the output from each rule.
- Cumulative fuzzy output (weight at 6'1").



1. De-fuzzify to obtain a numerical estimate of the output.
2. Choose the middle of the range where the truth value is maximum.
3. John's weight = 80 Kg.

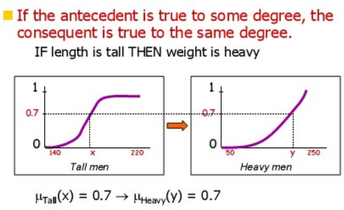
FUZZY REASONING

There exist four modes of fuzzy approximate reasoning, which include:

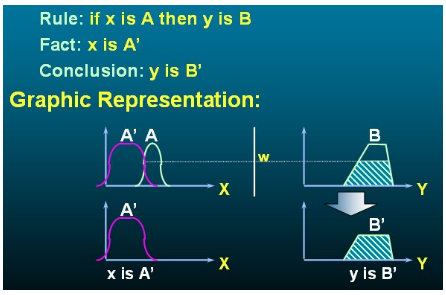
1. Categorical reasoning,
2. Qualitative reasoning,
3. Syllogistic reasoning,
4. Dispositional reasoning.

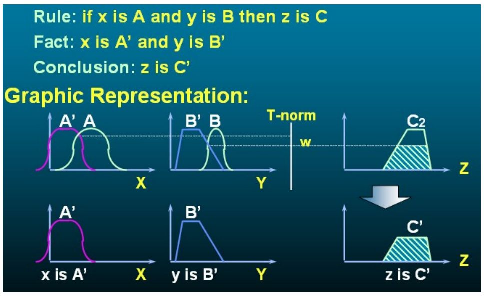
REASONING WITH FUZZY RULES

- In classical systems, rules with true antecedents fire.
- In fuzzy systems, truth (i.e., membership in some class) is relative, so all rules fire (to some extent).



SINGLE RULE WITH SINGLE ANTECEDANT





MULTIPLE ANTECEDANTS

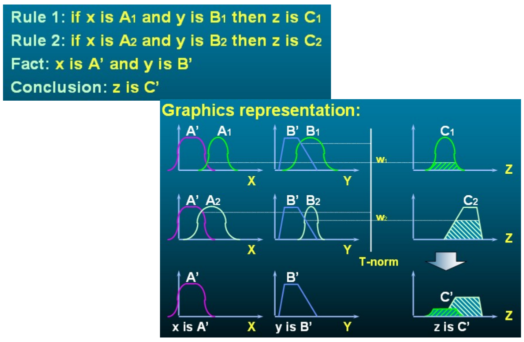
IF x is A AND y is B THEN z is C
IF x is A OR y is B THEN z is C

Use unification (OR) or intersection (AND) operations to calculate a membership value for the whole antecedent.

AND: $\mu_C(z) = \min(\mu_A(x), \mu_B(y))$
OR: $\mu_C(z) = \max(\mu_A(x), \mu_B(y))$

E.g. If rain is heavy AND wind is strong THEN weather is bad
$((\mu_{heavy}(rain) = 0.7) \wedge (\mu_{strong}(wind) = 0.4)) \rightarrow (\mu_{bad}(weather) = 0.4)$

MULTIPLE RULE WITH MULTIPLE ANTECEDANTS



MULTIPLE CONSEQUENTS

IF x is A THEN y is B AND z is C

Each consequent is affected equally by the membership in the antecedent class(es).

E.g., IF x is tall THEN x is heavy AND x has large feet.

$\mu_{Tall}(x) = 0.7 \rightarrow \mu_{Heavy}(y) = 0.7 \quad \wedge \quad \mu_{LargeFeet}(y) = 0.7$

Various applications (fuzzy logic)

- Approximate reasoning
- Expert system
- Computational linguistic
- Database interface
- Pattern recognition
- Image processing
- Decision making
- Industrial process control