7.4 Summary

In this chapter, we have discussed the basic definitions, properties and operations on classical sets and fuzzy sets. Fuzzy sets are the tools that convert the concept of fuzzy logic into algorithms. Since fuzzy sets allow partial membership, they provide computer with such algorithms that extend binary logic and enable it to take human-like decisions. In other words, fuzzy sets can be thought of as a media through which the human thinking is transferred to a computer. One difference between fuzzy sets and classical sets is that the former do not follow the law of excluded middle and law of contradiction. Hence, if we want to choose fuzzy intersection and union operations which satisfy these laws, then the operations will not satisfy distributivity and idempotency. Except the difference of set membership being an infinite valued quantity instead of a binary valued quantity, fuzzy sets are treated in the same mathematical form as classical sets.

7.5 Solved Problems

1. Find the power set and cardinality of the given set $X = \{2, 4, 6\}$. Also find cardinality of power set.

Solution: Since set X contains three elements, so its cardinal number is

$$n_X = 3$$

The power set of X is given by

$$P(X) = \{\phi, \{2\}, \{4\}, \{6\}, \{2,4\}, \{4,6\}, \{2,6\}, \{2,4,6\}\}$$

The cardinality of power set $P(X)$, denoted by $n_{P(X)}$, is found as

$$n_{P(X)} = 2^{n_X} = 2^3 = 8$$

2. Consider two given fuzzy sets

$$\underset{\sim}{A} = \left\{\frac{1}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0.2}{8}\right\}$$

$$\underset{\sim}{B} = \left\{\frac{0.5}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{1}{8}\right\}$$

Perform union, intersection, difference and complement over fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

Solution: For the given fuzzy sets we have the following

(a) Union

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} = \left\{\frac{1}{2} + \frac{0.4}{4} + \frac{0.5}{6} + \frac{1}{8}\right\}$$

(b) Intersection

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} = \left\{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.1}{6} + \frac{0.2}{8}\right\}$$

(c) Complement

$$\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x) = \left\{\frac{0}{2} + \frac{0.7}{4} + \frac{0.5}{6} + \frac{0.8}{8}\right\}$$

$$\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x) = \left\{\frac{0.5}{2} + \frac{0.6}{4} + \frac{0.9}{6} + \frac{0}{8}\right\}$$

(d) Difference

$$\underset{\sim}{A}|\underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \left\{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0}{8}\right\}$$

$$\underset{\sim}{B}|\underset{\sim}{A} = \underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \left\{\frac{0}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{0.8}{8}\right\}$$

3. Given the two fuzzy sets

$$\underset{\sim}{B}_1 = \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$$

$$\underset{\sim}{B}_2 = \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\right\}$$

find the following:

(a) $\underset{\sim}{B}_1 \cup \underset{\sim}{B}_2$; (b) $\underset{\sim}{B}_1 \cap \underset{\sim}{B}_2$; (c) $\overline{\underset{\sim}{B}_1}$;
(d) $\overline{\underset{\sim}{B}_2}$; (e) $\underset{\sim}{B}_1|\underset{\sim}{B}_2$; (f) $\overline{\underset{\sim}{B}_1 \cup \underset{\sim}{B}_2}$;
(g) $\overline{\underset{\sim}{B}_1 \cap \underset{\sim}{B}_2}$; (h) $\underset{\sim}{B}_1 \cap \overline{\underset{\sim}{B}_1}$; (i) $\underset{\sim}{B}_1 \cup \overline{\underset{\sim}{B}_1}$;
(j) $\underset{\sim}{B}_2 \cap \overline{\underset{\sim}{B}_2}$; (k) $\underset{\sim}{B}_2 \cup \overline{\underset{\sim}{B}_2}$

Solution: For the given fuzzy sets, we have the following:

(a) $\underset{\sim}{B}_1 \cup \underset{\sim}{B}_2 = \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$

(b) $\underset{\sim}{B}_1 \cap \underset{\sim}{B}_2 = \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\right\}$

(c) $\overline{\underset{\sim}{B}_1} = \left\{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$

(d) $\overline{\underset{\sim}{B}_2} = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$

(e) $\underset{\sim}{B}_1|\underset{\sim}{B}_2 = \underset{\sim}{B}_1 \cap \overline{\underset{\sim}{B}_2} = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$

(f) $\overline{\underset{\sim}{B}_1 \cup \underset{\sim}{B}_2} = \left\{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$

(g) $\overline{\underset{\sim}{B}_1 \cap \underset{\sim}{B}_2} = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$

(h) $\underset{\sim}{B}_1 \cap \overline{\underset{\sim}{B}_1} = \left\{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$

(i) $\underset{\sim}{B}_1 \cup \overline{\underset{\sim}{B}_1} = \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$

(j) $\underset{\sim}{B}_2 \cap \overline{\underset{\sim}{B}_2} = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\right\}$

(k) $\underset{\sim}{B}_2 \cup \overline{\underset{\sim}{B}_2} = \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$

4. It is necessary to compare two sensors based upon their detection levels and gain settings. The table of gain settings and sensor detection levels with a standard item being monitored providing typical membership values to represent the detection levels for each sensor is given in Table 1.

Table 1

Gain setting	Detection level of sensor 1	Detection level of sensor 2
0	0	0
10	0.2	0.35
20	0.35	0.25
30	0.65	0.8
40	0.85	0.95
50	1	1

Now given the universe of discourse $X = \{0, 10, 20, 30, 40, 50\}$ and the membership functions for the two sensors in discrete form as

$$\underset{\sim}{D}_1 = \left\{\frac{0}{0} + \frac{0.2}{10} + \frac{0.35}{20} + \frac{0.65}{30} + \frac{0.85}{40} + \frac{1}{50}\right\}$$

$$\underset{\sim}{D}_2 = \left\{\frac{0}{0} + \frac{0.35}{10} + \frac{0.25}{20} + \frac{0.8}{30} + \frac{0.95}{40} + \frac{1}{50}\right\}$$

find the following membership functions:

(a) $\mu_{\underset{\sim}{D}_1 \cup \underset{\sim}{D}_2}(x)$; (b) $\mu_{\underset{\sim}{D}_1 \cap \underset{\sim}{D}_2}(x)$; (c) $\mu_{\overline{\underset{\sim}{D}_1}}(x)$;
(d) $\mu_{\overline{\underset{\sim}{D}_2}}(x)$; (e) $\mu_{\underset{\sim}{D}_1 \cup \overline{\underset{\sim}{D}_1}}(x)$; (f) $\mu_{\underset{\sim}{D}_1 \cap \overline{\underset{\sim}{D}_1}}(x)$;
(g) $\mu_{\underset{\sim}{D}_2 \cup \overline{\underset{\sim}{D}_2}}$; (h) $\mu_{\underset{\sim}{D}_2 \cap \overline{\underset{\sim}{D}_2}}(x)$; (i) $\mu_{\underset{\sim}{D}_1 | \underset{\sim}{D}_2}(x)$;
(j) $\mu_{\underset{\sim}{D}_2 | \underset{\sim}{D}_1}(x)$

Solution: For the given fuzzy sets we have

(a) $\mu_{\underset{\sim}{D}_1 \cup \underset{\sim}{D}_2}(x)$

$$= \max\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\underset{\sim}{D}_2}(x)\} = \left\{\frac{0}{0} + \frac{0.35}{10} + \frac{0.35}{20} + \frac{0.8}{30} + \frac{0.95}{40} + \frac{1}{50}\right\}$$

(b) $\mu_{\underset{\sim}{D}_1 \cap \underset{\sim}{D}_2}(x)$

$$= \min\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\underset{\sim}{D}_2}(x)\} = \left\{\frac{0}{0} + \frac{0.2}{10} + \frac{0.25}{20} + \frac{0.65}{30} + \frac{0.85}{40} + \frac{1}{50}\right\}$$

(c) $\mu_{\overline{\underset{\sim}{D}_1}}(x)$

$$= 1 - \mu_{\underset{\sim}{D}_1}(x) = \left\{\frac{1}{0} + \frac{0.8}{10} + \frac{0.65}{20} + \frac{0.35}{30} + \frac{0.15}{40} + \frac{0}{50}\right\}$$

(d) $\mu_{\overline{\underset{\sim}{D}_2}}(x)$

$$= 1-\mu_{\underset{\sim}{D}_2}(x)$$
$$= \left\{\frac{1}{0}+\frac{0.65}{10}+\frac{0.75}{20}+\frac{0.2}{30}+\frac{0.05}{40}+\frac{0}{50}\right\}$$

(e) $\mu_{\underset{\sim}{D}_1\cup\overline{\underset{\sim}{D}_1}}(x)$

$$= \max\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\overline{\underset{\sim}{D}_1}}(x)\}$$
$$= \left\{\frac{1}{0}+\frac{0.8}{10}+\frac{0.65}{20}+\frac{0.65}{30}+\frac{0.85}{40}+\frac{1}{50}\right\}$$

(f) $\mu_{\underset{\sim}{D}_1\cap\overline{\underset{\sim}{D}_1}}(x)$

$$= \min\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\overline{\underset{\sim}{D}_1}}(x)\}$$
$$= \left\{\frac{0}{0}+\frac{0.2}{10}+\frac{0.35}{20}+\frac{0.35}{30}+\frac{0.15}{40}+\frac{0}{50}\right\}$$

(g) $\mu_{\underset{\sim}{D}_2\cup\overline{\underset{\sim}{D}_2}}(x)$

$$= \max\{\mu_{\underset{\sim}{D}_2}(x), \mu_{\overline{\underset{\sim}{D}_2}}(x)\}$$
$$= \left\{\frac{1}{0}+\frac{0.65}{10}+\frac{0.75}{20}+\frac{0.8}{30}+\frac{0.95}{40}+\frac{1}{50}\right\}$$

(h) $\mu_{\underset{\sim}{D}_2\cap\overline{\underset{\sim}{D}_2}}(x)$

$$= \min\{\mu_{\underset{\sim}{D}_2}(x), \mu_{\overline{\underset{\sim}{D}_2}}(x)\}$$
$$= \left\{\frac{0}{0}+\frac{0.35}{10}+\frac{0.25}{20}+\frac{0.2}{30}+\frac{0.05}{40}+\frac{0}{50}\right\}$$

(i) $\mu_{\underset{\sim}{D}_1|\underset{\sim}{D}_2}(x)$

$$= \mu_{\underset{\sim}{D}_1\cap\overline{\underset{\sim}{D}_2}}(x) = \min\{\mu_{\underset{\sim}{D}_1}(x), \mu_{\overline{\underset{\sim}{D}_2}}(x)\}$$
$$= \left\{\frac{0}{0}+\frac{0.2}{10}+\frac{0.35}{20}+\frac{0.2}{30}+\frac{0.05}{40}+\frac{0}{50}\right\}$$

(j) $\mu_{\underset{\sim}{D}_2|\underset{\sim}{D}_1}(x)$

$$= \mu_{\underset{\sim}{D}_2\cap\overline{\underset{\sim}{D}_1}}(x) = \min\{\mu_{\underset{\sim}{D}_2}(x), \mu_{\overline{\underset{\sim}{D}_1}}(x)\}$$
$$= \left\{\frac{0}{0}+\frac{0.35}{10}+\frac{0.25}{20}+\frac{0.35}{30}+\frac{0.15}{40}+\frac{0}{50}\right\}$$

5. Design a computer software to perform image processing to locate objects within a scene. The two fuzzy sets representing a plane and a train image are:

$$\text{Plane} = \left\{\frac{0.2}{\text{train}}+\frac{0.5}{\text{bike}}+\frac{0.3}{\text{boat}}+\frac{0.8}{\text{plane}}+\frac{0.1}{\text{house}}\right\}$$
$$\text{Train} = \left\{\frac{1}{\text{train}}+\frac{0.2}{\text{bike}}+\frac{0.4}{\text{boat}}+\frac{0.5}{\text{plane}}+\frac{0.2}{\text{house}}\right\}$$

Find the following:

(a) Plane ∪ Train; (b) Plane ∩ Train;
(c) $\overline{\text{Plane}}$; (d) $\overline{\text{Train}}$;
(e) Plane|Train; (f) $\overline{\text{Plane} \cup \text{Train}}$;
(g) $\overline{\text{Plane} \cap \text{Train}}$; (h) Plane ∪ $\overline{\text{Plane}}$;
(i) Plane ∩ $\overline{\text{Plane}}$; (j) Train ∪ $\overline{\text{Train}}$;
(k) Train ∪ $\overline{\text{Train}}$

Solution: For the given fuzzy sets we have the following:

(a) Plane ∪ Train

$$= \max\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$
$$= \left\{\frac{1.0}{\text{train}}+\frac{0.5}{\text{bike}}+\frac{0.4}{\text{boat}}+\frac{0.8}{\text{plane}}+\frac{0.2}{\text{house}}\right\}$$

(b) Plane ∩ Train

$$= \min\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$
$$= \left\{\frac{0.2}{\text{train}}+\frac{0.2}{\text{bike}}+\frac{0.3}{\text{boat}}+\frac{0.5}{\text{plane}}+\frac{0.1}{\text{house}}\right\}$$

(c) $\overline{\text{Plane}} = 1-\mu_{\text{Plane}}(x)$

$$= \left\{\frac{0.8}{\text{train}}+\frac{0.5}{\text{bike}}+\frac{0.7}{\text{boat}}+\frac{0.2}{\text{plane}}+\frac{0.9}{\text{house}}\right\}$$

(d) $\overline{\text{Train}} = 1-\mu_{\text{Train}}(x)$

$$= \left\{\frac{0}{\text{train}}+\frac{0.8}{\text{bike}}+\frac{0.6}{\text{boat}}+\frac{0.5}{\text{plane}}+\frac{0.8}{\text{house}}\right\}$$

(e) Plane|Train

$$= \text{Plane} \cap \overline{\text{Train}}$$
$$= \min\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Train}}}(x)\}$$
$$= \left\{\frac{0}{\text{train}}+\frac{0.5}{\text{bike}}+\frac{0.3}{\text{boat}}+\frac{0.5}{\text{plane}}+\frac{0.1}{\text{house}}\right\}$$

(f) $\overline{\text{Plane} \cup \text{Train}}$

$$= 1 - \max\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$
$$= \left\{\frac{0}{\text{train}}+\frac{0.5}{\text{bike}}+\frac{0.6}{\text{boat}}+\frac{0.2}{\text{plane}}+\frac{0.8}{\text{house}}\right\}$$

(g) $\overline{\text{Plane} \cap \text{Train}}$

$$= 1 - \min\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$
$$= \left\{\frac{0.8}{\text{train}}+\frac{0.8}{\text{bike}}+\frac{0.7}{\text{boat}}+\frac{0.5}{\text{plane}}+\frac{0.9}{\text{house}}\right\}$$

(h) Plane ∪ $\overline{\text{Plane}}$

$$= \max\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Plane}}}(x)\}$$
$$= \left\{\frac{0.8}{\text{train}}+\frac{0.5}{\text{bike}}+\frac{0.7}{\text{boat}}+\frac{0.8}{\text{plane}}+\frac{0.9}{\text{house}}\right\}$$

(i) Plane ∩ $\overline{\text{Plane}}$

$$= \min\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Plane}}}(x)\}$$
$$= \left\{\frac{0.2}{\text{train}}+\frac{0.5}{\text{bike}}+\frac{0.3}{\text{boat}}+\frac{0.2}{\text{plane}}+\frac{0.1}{\text{house}}\right\}$$

(j) Train ∪ $\overline{\text{Train}}$

$$= \max\{\mu_{\text{Train}}(x), \mu_{\overline{\text{Train}}}(x)\}$$
$$= \left\{\frac{1.0}{\text{train}}+\frac{0.8}{\text{bike}}+\frac{0.6}{\text{boat}}+\frac{0.5}{\text{plane}}+\frac{0.8}{\text{house}}\right\}$$

(k) Train ∩ $\overline{\text{Train}}$

$$= \min\{\mu_{\text{Train}}(x), \mu_{\overline{\text{Train}}}(x)\}$$
$$= \left\{\frac{0}{\text{train}}+\frac{0.2}{\text{bike}}+\frac{0.4}{\text{boat}}+\frac{0.5}{\text{plane}}+\frac{0.2}{\text{house}}\right\}.$$

6. For aircraft simulator data the determination of certain changes in its operating conditions is made on the basis of hard break points in the mach region. We define two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ representing the condition of "near" a mach number of 0.65 and "in the region" of a mach number of 0.65, respectively, as follows

$\underset{\sim}{A}$ = near mach 0.65

$$= \left\{\frac{0}{0.64}+\frac{0.75}{0.645}+\frac{1}{0.65}+\frac{0.5}{0.655}+\frac{0}{0.66}\right\}$$

$\underset{\sim}{B}$ = in the region of mach 0.65

$$= \left\{\frac{0}{0.64}+\frac{0.25}{0.645}+\frac{0.75}{0.65}+\frac{1}{0.655}+\frac{0.5}{0.66}\right\}$$

For these two sets find the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$; (b) $\underset{\sim}{A} \cap \underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$;
(d) $\overline{\underset{\sim}{B}}$; (e) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$; (f) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}$

Solution: For the two given fuzzy sets we have the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$

$$= \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$
$$= \left\{\frac{0}{0.64}+\frac{0.75}{0.645}+\frac{1}{0.65}+\frac{1}{0.655}+\frac{0.5}{0.66}\right\}$$

(b) $\underset{\sim}{A} \cap \underset{\sim}{B}$

$$= \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$
$$= \left\{\frac{0}{0.64}+\frac{0.25}{0.645}+\frac{0.75}{0.65}+\frac{0.5}{0.655}+\frac{0}{0.66}\right\}$$

(c) $\overline{\underset{\sim}{A}} = 1-\mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{1}{0.64}+\frac{0.25}{0.645}+\frac{0}{0.65}+\frac{0.5}{0.655}+\frac{1}{0.66}\right\}$$

(d) $\overline{\underset{\sim}{B}} = 1-\mu_{\underset{\sim}{B}}(x)$

$$= \left\{\frac{1}{0.64}+\frac{0.75}{0.645}+\frac{0.25}{0.65}+\frac{0}{0.655}+\frac{0.5}{0.66}\right\}$$

(e) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$

$$= 1 - \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$
$$= \left\{\frac{1}{0.64}+\frac{0.25}{0.645}+\frac{0}{0.65}+\frac{0}{0.655}+\frac{0.5}{0.66}\right\}$$

(f) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}$

$$= 1 - \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$
$$= \left\{\frac{1}{0.64}+\frac{0.75}{0.645}+\frac{0.25}{0.65}+\frac{0.5}{0.655}+\frac{1}{0.66}\right\}$$

7. For the two given fuzzy sets

$$\underset{\sim}{A} = \left\{\frac{0.1}{0}+\frac{0.2}{1}+\frac{0.4}{2}+\frac{0.6}{3}+\frac{1}{4}\right\}$$
$$\underset{\sim}{B} = \left\{\frac{1}{0}+\frac{0.5}{1}+\frac{0.7}{2}+\frac{0.3}{3}+\frac{0}{4}\right\}$$

find the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$; (b) $\underset{\sim}{A} \cap \underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$;
(d) $\overline{\underset{\sim}{B}}$; (e) $\underset{\sim}{A} \cup \overline{\underset{\sim}{A}}$; (f) $\underset{\sim}{A} \cap \overline{\underset{\sim}{A}}$;
(g) $\underset{\sim}{B} \cup \overline{\underset{\sim}{B}}$; (h) $\underset{\sim}{B} \cap \overline{\underset{\sim}{B}}$; (i) $\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}$;
(j) $\underset{\sim}{A} \cup \overline{\underset{\sim}{B}}$; (k) $\underset{\sim}{B} \cap \overline{\underset{\sim}{A}}$; (l) $\underset{\sim}{B} \cup \overline{\underset{\sim}{A}}$;
(m) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$; (n) $\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$

Solution: For the given sets we have:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{1}{0} + \frac{0.5}{1} + \frac{0.7}{2} + \frac{0.6}{3} + \frac{1}{4}\right\}$$

(b) $\underset{\sim}{A} \cap \underset{\sim}{B} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.1}{0} + \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

(c) $\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{0.9}{0} + \frac{0.8}{1} + \frac{0.6}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(d) $\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x)$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.7}{3} + \frac{1}{4}\right\}$$

(e) $\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.9}{0} + \frac{0.8}{1} + \frac{0.6}{2} + \frac{0.6}{3} + \frac{1}{4}\right\}$$

(f) $\underset{\sim}{A} \cap \overline{\underset{\sim}{A}} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.1}{0} + \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(g) $\underset{\sim}{B} \cup \overline{\underset{\sim}{B}} = \max\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{1}{0} + \frac{0.5}{1} + \frac{0.7}{2} + \frac{0.7}{3} + \frac{1}{4}\right\}$$

(h) $\underset{\sim}{B} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

(i) $\underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.2}{1} + \frac{0.3}{2} + \frac{0.6}{3} + \frac{1}{4}\right\}$$

(j) $\underset{\sim}{A} \cup \overline{\underset{\sim}{B}} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0.1}{0} + \frac{0.5}{1} + \frac{0.4}{2} + \frac{0.7}{3} + \frac{1}{4}\right\}$$

(k) $\underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \min\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.9}{0} + \frac{0.5}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

(l) $\underset{\sim}{B} \cup \overline{\underset{\sim}{A}} = \max\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{1}{0} + \frac{0.8}{1} + \frac{0.7}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(m) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = 1 - \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

(n) $\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0}{0} + \frac{0.5}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0}{4}\right\}$$

8. Let U be the universe of military aircraft of interest' as defined below:

$$U = \{a10,\ b52,\ c130,\ f2,\ f9\}$$

Let $\underset{\sim}{A}$ be the fuzzy set of bomber class aircraft:

$$\underset{\sim}{A} = \left\{\frac{0.3}{a10} + \frac{0.4}{b52} + \frac{0.2}{c130} + \frac{0.1}{f2} + \frac{1}{f9}\right\}$$

Let $\underset{\sim}{B}$ be the fuzzy set of fighter class aircraft:

$$\underset{\sim}{B} = \left\{\frac{0.1}{a10} + \frac{0.2}{b52} + \frac{0.8}{c130} + \frac{0.7}{f2} + \frac{0}{f9}\right\}$$

Find the following:

(a) $\underset{\sim}{A} \cup \underset{\sim}{B}$; (b) $\underset{\sim}{A} \cap \underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$; (d) $\overline{\underset{\sim}{B}}$;
(e) $\underset{\sim}{A} | \underset{\sim}{B}$; (f) $\underset{\sim}{B} | \underset{\sim}{A}$; (g) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$;
(h) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}$; (i) $\overline{\underset{\sim}{A}} \cup \underset{\sim}{B}$; (j) $\overline{\underset{\sim}{B}} \cup \underset{\sim}{A}$

Solution: We have

(a) $\underset{\sim}{A} \cup \underset{\sim}{B} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.3}{a10} + \frac{0.4}{b52} + \frac{0.8}{c130} + \frac{0.7}{f2} + \frac{1}{f9}\right\}$$

(b) $\underset{\sim}{A} \cap \underset{\sim}{B} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.1}{a10} + \frac{0.2}{b52} + \frac{0.2}{c130} + \frac{0.1}{f2} + \frac{0}{f9}\right\}$$

(c) $\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{0.7}{a10} + \frac{0.6}{b52} + \frac{0.8}{c130} + \frac{0.9}{f2} + \frac{0}{f9}\right\}$$

(d) $\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x)$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.2}{c130} + \frac{0.3}{f2} + \frac{1}{f9}\right\}$$

(e) $\underset{\sim}{A} | \underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0.3}{a10} + \frac{0.4}{b52} + \frac{0.2}{c130} + \frac{0.1}{f2} + \frac{1}{f9}\right\}$$

(f) $\underset{\sim}{B} | \underset{\sim}{A} = \underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \min\{\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)\}$

$$= \left\{\frac{0.1}{a10} + \frac{0.2}{b52} + \frac{0.8}{c130} + \frac{0.7}{f2} + \frac{0}{f9}\right\}$$

(g) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = 1 - \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.7}{a10} + \frac{0.6}{b52} + \frac{0.2}{c130} + \frac{0.3}{f2} + \frac{0}{f9}\right\}$$

(h) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = 1 - \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.8}{c130} + \frac{0.9}{f2} + \frac{1}{f9}\right\}$$

(i) $\overline{\underset{\sim}{A}} \cup \underset{\sim}{B} = \max\{\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)\}$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.8}{c130} + \frac{0.9}{f2} + \frac{1}{f9}\right\}$$

(j) $\overline{\underset{\sim}{B}} \cup \underset{\sim}{A} = \max\{\mu_{\overline{\underset{\sim}{B}}}(x), \mu_{\underset{\sim}{A}}(x)\}$

$$= \left\{\frac{0.9}{a10} + \frac{0.8}{b52} + \frac{0.2}{c130} + \frac{0.3}{f2} + \frac{1}{f9}\right\}$$

9. Consider two fuzzy sets

$$\underset{\sim}{A} = \left\{\frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$$

$$\underset{\sim}{B} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.2}{3} + \frac{1}{4}\right\}$$

Find the algebraic sum, algebraic product, bounded sum and bounded difference of the given fuzzy sets.

Solution: We have

(a) Algebraic sum

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x) = [\mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)] - [\mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)]$$

$$= \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\} - \left\{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4}\right\}$$

$$= \left\{\frac{0.28}{1} + \frac{0.44}{2} + \frac{0.52}{3} + \frac{0}{4}\right\}$$

(b) Algebraic product

$$\mu_{\underset{\sim}{A}\cdot\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$$

$$= \left\{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4}\right\}$$

(c) Bounded sum

$$\mu_{\underset{\sim}{A}\oplus\underset{\sim}{B}}(x) = \min[1, \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)]$$

$$= \min\left\{1, \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}\right\}$$

$$= \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}$$

(d) Bounded difference

$$\mu_{\underset{\sim}{A}\odot\underset{\sim}{B}}(x) = \max[0, \mu_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{B}}(x)]$$

$$= \max\left\{0, \left\{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right\}\right\}$$

$$= \left\{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right\}$$

10. The discretized membership functions for a transistor and a resistor are given below:

$$\mu_{\underset{\sim}{T}}=\left\{\frac{0}{0}+\frac{0.2}{1}+\frac{0.7}{2}+\frac{0.8}{3}+\frac{0.9}{4}+\frac{1}{5}\right\}$$

$$\mu_{\underset{\sim}{R}}=\left\{\frac{0}{0}+\frac{0.1}{1}+\frac{0.3}{2}+\frac{0.2}{3}+\frac{0.4}{4}+\frac{0.5}{5}\right\}$$

Find the following: (a) Algebraic sum; (b) algebraic product; (c) bounded sum; (d) bounded difference.

Solution: We have

(a) Algebraic sum

$$\begin{aligned}\mu_{\underset{\sim}{T}+\underset{\sim}{R}}(x) &= [\mu_{\underset{\sim}{T}}(x)+\mu_{\underset{\sim}{R}}(x)]-[\mu_{\underset{\sim}{T}}(x)\cdot\mu_{\underset{\sim}{R}}(x)]\\ &=\left\{\frac{0}{0}+\frac{0.3}{1}+\frac{1.0}{2}+\frac{1.0}{3}+\frac{1.3}{4}+\frac{1.5}{5}\right\}\\ &\quad-\left\{\frac{0}{0}+\frac{0.02}{1}+\frac{0.21}{2}+\frac{0.16}{3}+\frac{0.36}{4}+\frac{0.5}{5}\right\}\\ &=\left\{\frac{0}{0}+\frac{0.28}{1}+\frac{0.79}{2}+\frac{0.84}{3}+\frac{0.94}{4}+\frac{1}{5}\right\}\end{aligned}$$

(b) Algebraic product

$$\begin{aligned}\mu_{\underset{\sim}{T}\cdot\underset{\sim}{R}}(x) &= \mu_{\underset{\sim}{T}}(x)\cdot\mu_{\underset{\sim}{R}}(x)\\ &=\left\{\frac{0}{0}+\frac{0.02}{1}+\frac{0.21}{2}+\frac{0.16}{3}+\frac{0.36}{4}+\frac{0.5}{5}\right\}\end{aligned}$$

(c) Bounded sum

$$\begin{aligned}\mu_{\underset{\sim}{T}\oplus\underset{\sim}{R}}(x) &= \min\{1,\mu_{\underset{\sim}{T}}(x)+\mu_{\underset{\sim}{R}}(x)\}\\ &=\min\left\{1,\left\{\frac{0}{0}+\frac{0.3}{1}+\frac{1.0}{2}+\frac{1.0}{3}+\frac{1.3}{4}+\frac{1.5}{5}\right\}\right\}\\ &=\left\{\frac{0}{0}+\frac{0.3}{1}+\frac{1.0}{2}+\frac{1.0}{3}+\frac{1.0}{4}+\frac{1.0}{5}\right\}\end{aligned}$$

(d) Bounded difference

$$\begin{aligned}\mu_{\underset{\sim}{T}\odot\underset{\sim}{R}}(x) &= \max\{0,\mu_{\underset{\sim}{T}}(x)-\mu_{\underset{\sim}{R}}(x)\}\\ &=\max\left\{0,\left\{\frac{0}{0}+\frac{0.1}{1}+\frac{0.4}{2}+\frac{0.6}{3}+\frac{0.5}{4}+\frac{0.5}{5}\right\}\right\}\\ &=\left\{\frac{0}{0}+\frac{0.1}{1}+\frac{0.4}{2}+\frac{0.6}{3}+\frac{0.5}{4}+\frac{0.5}{5}\right\}\end{aligned}$$

7.6 Review Questions

1. Define classical sets and fuzzy sets.
2. State the importance of fuzzy sets.
3. What are the methods of representation of a classical set?
4. Discuss the operations of crisp sets.
5. List the properties of classical sets.
6. What is meant by characteristic function?
7. Write the function theoretic form representation of crisp set operations.
8. Justify the following statement: "Partial membership is allowed in fuzzy sets."
9. Discuss in detail the operations and properties of fuzzy sets.
10. Represent the fuzzy sets operations using Venn diagram.
11. What is the cardinality of a fuzzy set? Whether a power set can be formed for a fuzzy set?
12. Apart from basic operations, state few other operations involved in fuzzy sets.

13. Compare and contrast classical logic and fuzzy logic.
14. Why the excluded middle law does not get satisfied in fuzzy logic?
15. Describe the importance of fuzzy sets and its application in engineering sector.

7.7 Exercise Problems

1. Find the cardinality of the given set:

$$A=\{1,3,5,7,9\}$$

2. Consider set $X = \{2,4,6,8,10\}$. Find its power set, cardinality and cardinality of power set.

3. Show the following fuzzy sets satisfy DeMorgan's law:

(a) $\mu_A(x)=\frac{1}{1+5x}$

(b) $\mu_B(x)=\left(\frac{1}{1+5x}\right)^{1/2}$

4. Consider two fuzzy sets

$$\underset{\sim}{A}=\left\{\frac{1}{2.0}+\frac{0.65}{4.0}+\frac{0.5}{6.0}+\frac{0.35}{8.0}+\frac{0}{10.0}\right\}$$

$$\underset{\sim}{B}=\left\{\frac{0}{2.0}+\frac{0.35}{4.0}+\frac{0.5}{6.0}+\frac{0.65}{8.0}+\frac{1}{10.0}\right\}$$

Find the following:

(a) $\underset{\sim}{A}\cup\underset{\sim}{B}$; (b) $\underset{\sim}{A}\cap\underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$; (d) $\overline{\underset{\sim}{B}}$;
(e) $\overline{\underset{\sim}{A}}\cap\overline{\underset{\sim}{B}}$; (f) $\overline{\underset{\sim}{A}}\cup\overline{\underset{\sim}{B}}$; (g) $\overline{\underset{\sim}{A}\cup\underset{\sim}{B}}$;
(h) $\overline{\underset{\sim}{A}\cap\underset{\sim}{B}}$; (i) $\underset{\sim}{A}\cup\overline{\underset{\sim}{A}}$; (j) $\underset{\sim}{A}\cap\overline{\underset{\sim}{A}}$;
(k) $\underset{\sim}{B}\cup\overline{\underset{\sim}{B}}$; (l) $\underset{\sim}{B}\cap\overline{\underset{\sim}{B}}$

5. We want to compare two liquid level controllers for their control levels and flow speed. The following values of flow speed and liquid control levels were recorded with a standard liquid flow monitor:

Flow speed	Control level 1	Control level 2
0	0	0
20	0.5	0.45
40	0.35	0.55
60	0.75	0.65
80	0.95	0.9
100	1	1

Given the universe of discourse is $X = \{0,20,40,60,80,100\}$ and the membership functions

$$\underset{\sim}{L}_1=\left\{\frac{0}{0}+\frac{0.5}{20}+\frac{0.35}{40}+\frac{0.75}{60}+\frac{0.95}{80}+\frac{1}{100}\right\}$$

$$\underset{\sim}{L}_2=\left\{\frac{0}{0}+\frac{0.45}{20}+\frac{0.55}{40}+\frac{0.65}{60}+\frac{0.9}{80}+\frac{1}{100}\right\}$$

find the following memberships using standard set operations:

(a) $\mu_{\underset{\sim}{L}_1\cup\underset{\sim}{L}_2}(x)$; (b) $\mu_{\underset{\sim}{L}_1\cap\underset{\sim}{L}_2}(x)$; (c) $\mu_{\overline{\underset{\sim}{L}_1}}(x)$;
(d) $\mu_{\overline{\underset{\sim}{L}_2}}(x)$; (e) $\mu_{\overline{\underset{\sim}{L}_1\cup\underset{\sim}{L}_2}}(x)$; (f) $\mu_{\overline{\underset{\sim}{L}_1\cap\underset{\sim}{L}_2}}(x)$;
(g) $\mu_{\underset{\sim}{L}_1\cap\overline{\underset{\sim}{L}_2}}$; (h) $\mu_{\underset{\sim}{L}_1\cup\overline{\underset{\sim}{L}_2}}(x)$; (i) $\mu_{\underset{\sim}{L}_1\cup\overline{\underset{\sim}{L}_1}}(x)$;
(j) $\mu_{\underset{\sim}{L}_2\cup\overline{\underset{\sim}{L}_2}}(x)$

6. Consider two membership functions as follows:

For fuzzy set $\underset{\sim}{A}$: $\mu_{\underset{\sim}{A}}(x)=\frac{|(60-x)|}{8}+1$

For fuzzy set $\underset{\sim}{B}$: $\mu_{\underset{\sim}{B}}(x)=\frac{|(40-x)|}{8}+1$

Find the following:

(a) $\underset{\sim}{A}\cup\underset{\sim}{B}$; (b) $\underset{\sim}{A}\cap\underset{\sim}{B}$; (c) $\overline{\underset{\sim}{A}}$; (d) $\overline{\underset{\sim}{B}}$;
(e) $\overline{\underset{\sim}{A}\cup\underset{\sim}{B}}$; (f) $\overline{\underset{\sim}{A}\cap\underset{\sim}{B}}$